Growth/Decay-Fragmentation Equations

Tumelo Donald Sereo (tumelo@aims.ac.za)
African Institute for Mathematical Sciences (AIMS)

Supervised by: Prof Jacek Banasiak
University of Pretoria, South Africa

22 September 2020

Submitted in partial fulfillment of a structured masters degree at AIMS South Africa

Abstract

In this project, we find explicit solutions of the fragmentation equations with decay terms. The fragmentation describes how large clusters undergoes splitting processes while the decay terms describe how the clusters shrink due to dissolution and chemical reactions. We consider the fragmentation equations with decay with coefficient given by power laws. We transform them to simpler form that we will try to solve. We apply the method of characteristics to reduce the partial differential equation (PDE) into ordinary differential equation (ODE) and reduce the fragmentation/decay equations to a fragmentation like equations. We investigate the resolvability of these equations by the power series method.

Thitokgang ya tiro e, ke go batlisisa ka botlalo tharololo e e tlhapileng ya ikhweishene ya kgaogano mmogo le tshenyego. Kgaogano e tlhalosa gore ditlhopha tse digolo di kgaogana jang go fitlhelela di nna tse dinnye, mme fa tshenyego e bontsha ka fa ditlhopha di fokotsegang ka teng ka ntlha yago nyelela ka iketlo le go fetoga ga dikhemikhale. Mo tirong e, retla elatlhoko ikhweishene ya kgaogano e na le melawana ya tiriso, eleng molao wa maatla. Re tla fetola ikhweishene ya rona go mokgwa o bonolo o re tla lekang go o ranola. Go fitlhelela se, re dirisa mokgwa wa tshwanelo go ngotla PDE goya go ODE le go fokotsa ikhweishene ya kgaogano mmogo le tshenyego go e e maleba go tshwana le ekhweishene ya kgaogano ele nngwe. Re tla batlisisa ka botlalo tharabololo ya diikhweishene tse, re dirisa mokgwa tsweletso wa maatla go laola tirwana e.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Tumelo Donald Sereo, 22 September 2020

Contents

1. Introduction

1.1 Background Theory

The fragmentation process is the theory of describing how large clusters of matter (animate or inanimate) split into smaller parts. Fragmentation describe physical phenomena ranging from chemical engineering, polymer and monomer science, dissolution, combustion, population studies in biology as well as erosion. These phenomena are modelled and described using the fragmentation equation which was first provided by (Smoluchowski, 1916). The English translation of Smoluchowski's work is given in (Chandrasekhar, 1943). As mentioned earlier fragmentation plays an important role in different physical cases, so we are going to provide a brief discussion concerning that process. In chemical and biological applications, fragmentation is accompanied by decay of clusters, for example, the polymer degradation. The monomers combine together into pairs, then again be trimers (monomer triples) and they eventually form a polymer which in later stage can undergo the process of degradation due to external forces that leads to the breaking of bonds in the polymers (Montroll and Simha, 1940). The population ecology is one of the most recent study which investigate how animals combine to form larger groups and split into smaller groups. These type of processes are modelled using the fragmentation equations describing into details the distribution size of clusters to density-dependent rates of group division (Gueron and Levin, 1995).

There are many ways to analyse these pure fragmentation equation and fragmentation equation with decay, which include, the abstract methods discussed in Banasiak et al. (2019). A lot of work has been achieved, as these equations are linear, for example the exact solution of these equations has been obtained in recent investigations from Oukouomi and Doungmo (2014) using the Laplace transform method, and investigate the dynamics of the system, Huang et al. (1991) presented series of solutions to understand the fragmentation with mass loss and McGrady and Ziff (1988) provided analytical solutions. Moreover, solutions were found with the development of a numerical scheme for the approximate solution, see (Elminyawi et al., 1991).

In this project, we focus on pure fragmentation equations and fragmentation equation with decay. We consider the rate equation developed in Edwards et al. (1990) to study the fragmentation processes looking into different models and considering cases allowing for explicit solutions or comprehensive analysis. The rate equation is given as

$$\frac{\partial}{\partial t}n(x,t) = -a(x)n(x,t) + \int_x^\infty a(y)b(x|y)n(y,t)dy + \frac{\partial}{\partial x}[c(x)n(x,t)]. \tag{1.1.1}$$

This equation describes a fragmentation equation with decay, where $n(x,t)$ is the density function, denoting the density of cluster of size $x > 0$ at time $t \geq 0$. The term $a(x)$ describe overall rate of fragmentation clusters of size x, $b(x|y)$ gives the number of clusters of size x produced by fragmentation of size y clusters, that is a daughter distribution function and $c(x)$ denotes a continuous mass loss rate.

It follows that if $c(x) = 0$, then Equation (1.1.1) describes pure fragmentation. The term $b(x,y)$ is assumed to be non-negative with $b(x,y) = 0$ for $x > y$ and to satisfy the local mass conservation law

$$y = \int_0^y xb(x|y)dx, \quad y > 0, \tag{1.1.2}$$

but is otherwise arbitrary.

The above condition expresses the fact that the mass of all daughter clusters must be equal to the mass the of parent clusters. As mentioned earlier, Equation (1.1.1) has been analysed in many papers including the abstract analysis. In general, explicit solutions are not possible for arbitrary coefficient $a(x)$, $b(x|y)$ and $c(x)$, that is the reason in many papers they assume power laws. We consider the general dimensionless power laws rates

$$a(x) = x^{\alpha}, \quad b(x, y) = g(y)x^{\nu}, \quad c(x) = \epsilon x^{\gamma}, \tag{1.1.3}$$

where parameters γ,$\alpha \in \mathbb{R}$ and $\alpha = 1 + \nu$ in (Huang et al., 1996). Then when we use the mass conservation law given by Equation (1.1.2), we obtain:

$$g(y) = \frac{(2+\nu)}{y^{1+\nu}}, \tag{1.1.4}$$

with an arbitrary parameter $\nu \in$ (-2, 0], see Banasiak and Noutchie (2010) for more details and $\nu > -2$ to ensure that Equation (1.1.2) exists.

1.2 Objectives

In this project, our aim is to re-examine the power series solution proposed in Huang et al. (1996) for both pure fragmentation equation and the fragmentation Equation (1.1.1) with a constant decay rate. We will also provide a recursive formula for solutions of more general fragmentation with decay where the decay rate is $c(x) = \epsilon x$.

1.3 Outline

This project is formed by four chapters. In Chapter 2, we cover all the mathematical preliminaries required to solve our problem. In Chapter 3, we transform the rate equation and find the explicit solution and we give the interpretation to the solutions obtained. Finally, we draw a conclusion in Chapter 4 and discuss the future work.

2. Mathematical Preliminaries

In this chapter, we introduce the methods and mathematical definitions which we are going to use to solve the fragmentation equation.

2.1 Characteristics Method

Let $x(t)$ be a curve on $\mathbb{R}^2$. If we have a function $u(x,t)$ in $\mathbb{R}^2$, then the derivative of $u(x,t)$ along $x(t)$ is given by

$$\frac{d}{dt}u(x,t) = u_t(x,t) + \frac{dx}{dt}u_x(x,t), \tag{2.1.1}$$

after applying chain rule.

Suppose we want to solve the initial value problem partial differential equation (PDE) of the form

$$u_t(x,t) + cu_x(x,t) = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{2.1.2}$$

$$u(x,0) = u_0(x), \tag{2.1.3}$$

where c is a constant.

Now, if we compare Equations (2.1.1) and (2.1.2), we see that Equation (2.1.2) can be considered as an ordinary differential equation (ODE) along the curve $x = x(t)$ with

$$\frac{du}{dt} = 0 \quad \text{and} \quad \frac{dx}{dt} = c. \tag{2.1.4}$$

The above system of Equations (2.1.4) is called the characteristic equations, which we are going to use to solve Equation (2.1.2). Now, it is easy to integrate Equations (2.1.4) and get

$$u = \text{constant} \quad \text{along any line} \quad x - ct = \xi, \tag{2.1.5}$$

where ξ is a constant of integration.

As a result, this technique is used to reduce a given first order PDE into an ODE, which can be solved by integration.

Now, when we apply our initial condition (2.1.3), we get that $(\xi, 0)$ is our point on x-axis which the characteristic curve $x - ct = \xi$ will intercept at $t = 0$. Then the solution for Equation (2.1.2) is

$$u(x,t) = u_0(x - ct).$$

2.1.1 Example. Consider the initial value problem:

$$u_t + 2u_x = x, \quad x \in \mathbb{R}, t > 0, \tag{2.1.6}$$

$$u(x,0) = v_0(x).$$

Then the characteristic equations are:

$$\frac{du}{dt} = x \text{ and } \frac{dx}{dt} = 2, \text{ with } x(0) = \xi. \tag{2.1.7}$$

Integrating the second equation of (2.1.7), we obtain:

$$x - 2t = \xi.$$

By applying the following change of variable, $u(x,t) = V(\xi,t)$ to the first equation of (2.1.7) and by differentiating, we have :

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial V(\xi,t)}{\partial t} = x. \tag{2.1.8}$$

Since we know that $x = \xi + 2t$, we can re-write Equation (2.1.8) as

$$\frac{\partial V(\xi,t)}{\partial t} = \xi + 2t,$$

which results in

$$V(\xi,t) = \xi t + t^2 + C, \tag{2.1.9}$$

where C is an arbitrary constant of integration.

Now we have $V(\xi,0) = C = v_0(\xi)$. Therefore,

$$V(\xi,t) = \xi t + t^2 + v_0(\xi), \tag{2.1.10}$$

and the solution for Equation (2.1.6) is obtained as

$$u(x,t) = tx - t^2 + v_0(x - 2t).$$

This idea can be extended to the following type of first order linear PDEs:

$$a(x,t)\frac{\partial u(x,t)}{\partial t} + c(x,t)\frac{\partial u(x,t)}{\partial x} + u(x,t) = f(x,t), \tag{2.1.11}$$

where a, c and f are continuous functions in $\mathbb{R}^2$.

In particular, we obtain the homogeneous equation with $f(x,t) = 0$.

2.2 Power Series

Suppose we have the fragmentation equation given as

$$\frac{\partial}{\partial t}n(x,t) = -a(x)n(x,t) + \int_x^\infty a(y)b(x|y)n(y,t)dy, \quad n(x,0) = n_0(x). \tag{2.2.1}$$

Then, using the integrating factor in Equation (2.2.1), we can always express $n(x,t)$ in the form

$$n(x,t) = e^{-a(x)t}f(x,t). \tag{2.2.2}$$

The derivative of $n(x,t)$ is

$$\frac{\partial n(x,t)}{\partial t} = -a(x)e^{-a(x)t}f(x,t) + e^{-a(x)t}\frac{\partial f(x,t)}{\partial t}. \tag{2.2.3}$$

Substituting Equations (2.2.3) and (2.2.2) into Equation (2.2.1), we get:

$$\frac{\partial f(x,t)}{\partial t} = \int_x^{\infty} a(y)b(x|y)e^{t[a(x)-a(y)]}f(y,t)dy. \tag{2.2.4}$$

To solve Equation (2.2.4), we simplify it by using the fundamental theorem of calculus (FTC) which states that, if f is a continuous function on $[a,b]$ and g defined as

$$g(x) = \int_x^b f(t)dt,$$

where $a < x < b$, then the derivative of g with respect to the lower limit is

$$\frac{dg(x)}{dx} = \frac{d}{dx}\int_x^b f(t)dt = -f(x).$$

Now, we differentiate Equation (2.2.4) with respect to x and apply the FTC, we get:

$$\frac{\partial^2 f(x,t)}{\partial t \partial x} = \frac{\partial}{\partial x}\left(\int_x^{\infty} a(y)b(x|y)e^{t[a(x)-a(y)]}f(y,t)dy\right), \tag{2.2.5}$$

and after a little algebra, Equation (2.2.5) can be written in the form of

$$\frac{\partial^2 f(x,t)}{\partial t \partial x} + \frac{\partial f(x,t)}{\partial t} + mf(x,t) = 0. \tag{2.2.6}$$

To solve this type of equation, we use the power series method, that is, we assume that the solution of Equation (2.2.6) is

$$f(x,t) = \sum_{n=0}^{\infty} C_n(x)t^n, \tag{2.2.7}$$

where C_n represents the coefficient of the series. We substitute Equation (2.2.7) into (2.2.6) to obtained the coefficient C_n.

3. A Fragmentation Model: Analysis

In this chapter, we introduce the rate equation which describe the fragmentation process of a cluster. We shall present the closed form solutions of the fragmentation equation with decay and investigate the other possible forms of solutions. We transform the equation into more simple form which we can solve and examine again the solution obtained in (Huang et al., 1996). Furthermore, we perform our case and observe to what extent can we arrive in order to obtain the exact solution.

3.1 Transformation

Let us recall the equation

$$\frac{\partial}{\partial t}n(x,t) = -a(x)n(x,t) + \int_x^\infty a(y)b(x|y)n(y,t)dy + \frac{\partial}{\partial x}[c(x)n(x,t)], \tag{3.1.1}$$

we defined in Chapter 1, with the initial condition

$$n(x,0) = n_0(x), \tag{3.1.2}$$

as well as the power laws defined by

$$a(x) = x^\alpha, \quad b(x,y) = g(y)x^\nu, \quad c(x) = \epsilon x^\gamma. \tag{3.1.3}$$

In this note, we substitute (3.1.3) into Equation (3.1.1), and we get:

$$\frac{\partial n(x,t)}{\partial t} = -x^\alpha n(x,t) + (2+\nu)x^\nu \int_x^\infty y^{\alpha-1-\nu}n(y,t)dy + \epsilon\frac{\partial}{\partial x}[x^\gamma n(x,t)], \tag{3.1.4}$$

and the binary fragmentation processes occur if the value of $\nu = 0$, that is, the parent cluster break up only into two daughter clusters.

We simplify Equation (3.1.4) by transforming it according to the following change of variables:

$$x = u^{1/\alpha} \quad \text{and} \quad n(x,t) = x^\nu w(u,t). \tag{3.1.5}$$

With this aim in mind, we substitute x into n and find the derivative of n with respect to t from Equation (3.1.5), hence we obtain

$$\frac{\partial n(x,t)}{\partial t} = u^{\nu/\alpha}\frac{\partial w(u,t)}{\partial t}. \tag{3.1.6}$$

We perform the differentiation in the last term of Equation (3.1.4) to get

$$\begin{aligned}\frac{\partial}{\partial x}[x^\gamma n(x,t)] &= \gamma x^{\gamma-1}n(x,t) + x^\gamma\frac{\partial}{\partial x}[n(x,t)] \\ &= \gamma x^{\gamma-1}n(x,t) + x^\gamma\left[\nu x^{\nu-1}w(u,t) + x^\nu\frac{\partial w(u,t)}{\partial u}\frac{du}{dx}\right].\end{aligned} \tag{3.1.7}$$

The derivative of u with respect to x is $\frac{du}{dx} = \alpha x^{\alpha-1}$, so substituting it in Equation (3.1.7), we have

$$\frac{\partial}{\partial x}[x^\gamma n(x,t)] = u^{\nu/\alpha}\left[(\gamma+\nu)w(u,t)u^{\frac{\gamma+\alpha-1}{\alpha}-1} + \alpha u^{\frac{\gamma+\alpha-1}{\alpha}}\frac{\partial w(u,t)}{\partial u}\right]. \tag{3.1.8}$$

Substituting Equations (3.1.6) and (3.1.8) into Equation (3.1.4), we get:

$$u^{\nu/\alpha}\frac{\partial w(u,t)}{\partial t} = -\,uu^{\nu/\alpha}w(u,t) + (\nu+2)u^{\nu/\alpha}\int_{x(u)}^{\infty} y^{\alpha-\nu-1}n(y,t)dy + \epsilon u^{\nu/\alpha}\left[(\gamma+\nu)w(u,t)u^{\frac{\gamma+\alpha-1}{\alpha}-1} + \alpha u^{\frac{\gamma+\alpha-1}{\alpha}}\frac{\partial w(u,t)}{\partial u}\right]. \quad (3.1.9)$$

We require Equation (3.1.9) to be in terms of the new variables introduced in (3.1.5). Let us consider the following change of variables, in the integral, we replace x by y and u by a variable s, then from Equation (3.1.5), we get the following:

$$n(y,t) = y^{\nu}w(u,t) \quad \text{and} \quad y = s^{1/\alpha}. \quad (3.1.10)$$

We already know that y is in the interval $x \leq y < \infty$, then we can say $x \leq s^{1/\alpha} < \infty$, which also implies that

$$x^{\alpha} \leq s < \infty,$$

that is, the interval for the new variable s is $u \leq s < \infty$.

Since $\alpha = 1+\nu$, we see clearly that $\nu = \alpha - 1$. Now, if we differentiate y in Equation (3.1.10) with respect to s, we obtain:

$$dy = \frac{1}{\alpha}s^{(\frac{1}{\alpha}-1)}ds.$$

Therefore, if we substitute Equations (3.1.10) in the integral term of Equation (3.1.9) and the interval of s, we get

$$\int_{x}^{\infty} y^{\alpha-\nu-1}n(y,t)dy = \frac{1}{\alpha}\int_{u}^{\infty} w(s,t)ds. \quad (3.1.11)$$

We now take back this term into Equation (3.1.9) and have the transformed equation given as

$$\frac{\partial w(u,t)}{\partial t} = -uw(u,t) + m\int_{u}^{\infty} w(s,t)ds + \tau w(u,t)u^{\mu-1} + \beta u^{\mu}\frac{\partial w(u,t)}{\partial u}, \quad (3.1.12)$$

where

$$m = \frac{\nu+2}{\alpha}, \quad \tau = \epsilon(\gamma+\nu), \quad \mu = \frac{\gamma+\alpha-1}{\alpha}, \quad \beta = \epsilon\alpha \quad \text{and} \quad \alpha = 1+\nu.$$

3.2 Solution Representation

The explicit solution is obtained in Huang et al. (1991) in the absence of mass loss, that is when $\epsilon = 0$ in Equation (3.1.4). We perform the investigation of Equation (3.1.12) obtained in Section 3.1 and examine under what values of τ and μ can we get the exact solution.

Case 1: We set $\tau = 0$ and $\mu = 0$, the simplest choice of the parameters.

Then we get $\gamma = -\nu$ where $0 \leq \gamma < 1$ and we recall that $\alpha = 1+\nu$ with $0 < \alpha \leq 1$.

In this case, Equation (3.1.12) becomes

$$\frac{\partial w(u,t)}{\partial t} = -uw(u,t) + m\int_{u}^{\infty} w(s,t)ds + \beta\frac{\partial w(u,t)}{\partial u}. \quad (3.2.1)$$

Applying the method of characteristics we discussed earlier in Chapter 2, we see that our characteristics equations are:

$$\frac{\partial w(u,t)}{\partial t} = -uw(u,t) + m\int_u^\infty w(s,t)ds, \tag{3.2.2}$$

$$\frac{du}{dt} = -\beta, \quad u(0) = \xi. \tag{3.2.3}$$

Then Equation (3.2.3) results on

$$u + \beta t = \xi, \tag{3.2.4}$$

where $\xi > 0$, is a constant of integration.

Now we let $w(u,t) = V(\xi,t)$, with new variable $\eta = s + \beta t$. Now since $u \leq s \leq \infty$, we have the new interval as $\xi \leq \eta \leq \infty$, then Equation (3.2.2) becomes

$$\frac{\partial V(\xi,t)}{\partial t} - \beta t V(\xi,t) = -\xi V(\xi,t) + m\int_\xi^\infty V(\eta,t)d\eta. \tag{3.2.5}$$

The integrating factor $I.F$ of Equation (3.2.5) is

$$I.F = e^{-\beta t^2/2}.$$

Then Equation (3.2.5) becomes

$$\frac{\partial}{\partial t}\Big(e^{-\beta t^2/2}V(\xi,t)\Big) = -\xi e^{-\beta t^2/2}V(\xi,t) + m\int_\xi^\infty e^{-\beta t^2/2}V(\eta,t)d\eta. \tag{3.2.6}$$

Now we let $\bar{V}(\xi,t) = e^{-\beta t^2/2}V(\xi,t)$, then we have

$$\frac{\partial \bar{V}(\xi,t)}{\partial t} = -\xi\bar{V}(\xi,t) + m\int_\xi^\infty \bar{V}(\eta,t)d\eta. \tag{3.2.7}$$

The solution of Equation (3.2.7) is obtained from the integrating factor, which is of the form

$$\bar{V}(\xi,t) = e^{-\xi t}f(\xi,t). \tag{3.2.8}$$

We get the derivative of $\bar{V}(\xi,t)$ as

$$\frac{\partial \bar{V}(\xi,t)}{\partial t} = -\xi e^{-\xi t}f(\xi,t) + e^{-\xi t}\frac{\partial f(\xi,t)}{\partial t}, \tag{3.2.9}$$

then, substituting Equations (3.2.8) and (3.2.9) into (3.2.7), we obtain

$$\frac{\partial f(\xi,t)}{\partial t} = m\int_\xi^\infty e^{\xi t}e^{-\eta t}f(\eta,t)d\eta. \tag{3.2.10}$$

A formal differentiation of Equation (3.2.10) with respect to ξ gives

$$\begin{aligned}\frac{\partial^2 f(\xi,t)}{\partial t\partial\xi} &= m\Big[t\int_\xi^\infty e^{\xi t}e^{-\eta t}f(\eta,t)d\eta - e^{\xi t}e^{-\xi t}f(\xi,t)\Big] \\ &= t\frac{\partial f(\xi,t)}{\partial t} - mf(\xi,t),\end{aligned}$$

which we can re-write as

$$\frac{\partial^2 f(\xi,t)}{\partial t \partial \xi} - t\frac{\partial f(\xi,t)}{\partial t} + mf(\xi,t) = 0. \tag{3.2.11}$$

Now, we apply the series expansion in the form

$$f(\xi,t) = \sum_{n=0}^{\infty} C_n(\xi)t^n. \tag{3.2.12}$$

We then have:

$$\frac{\partial f(\xi,t)}{\partial t} = \sum_{n=1}^{\infty} nC_n(\xi)t^{n-1} \quad \text{and} \quad \frac{\partial^2 f(\xi,t)}{\partial t \partial \xi} = \sum_{n=1}^{\infty} nC_n'(\xi)t^{n-1}. \tag{3.2.13}$$

The expansion transforms Equation (3.2.11) into

$$\sum_{n=1}^{\infty} nC_n'(\xi)t^{n-1} - \sum_{n=0}^{\infty} nC_n(\xi)t^n + m\sum_{n=0}^{\infty} C_n(\xi)t^n = 0.$$

We then make powers of t to match by making them k, that is $k = n-1$ and $k = n$. Then our equation takes the form

$$C_1'(\xi) + mC_0(\xi) + \sum_{k=1}^{\infty} \left[(k+1)C_{k+1}'(\xi) - kC_k(\xi) + mC_k(\xi)\right]t^k = 0. \tag{3.2.14}$$

Equation (3.2.14) is satisfied if and only if

$$C_1'(\xi) + mC_0(\xi) = 0 \quad \text{and} \quad (k+1)C_{k+1}'(\xi) - kC_k(\xi) + mC_k(\xi) = 0, \quad k \geq 1. \tag{3.2.15}$$

Solving second equation in (3.2.15) by integrating and little algebra, we get

$$C_{k+1}(\xi) = \frac{m-k}{k+1}\int_{\xi}^{\infty} C_k(\eta)d\eta, \quad \text{for} \quad k \in \mathbb{N}_0, \tag{3.2.16}$$

and $f(\xi,0) = C_0(\xi)$.

Let us go through some cases of k and generate the coefficient C_k. Clearly, we can see that from Equation (3.2.16), we have:

$$\text{For} \quad k = 0: \quad C_1(\xi) = m\int_{\xi}^{\infty} C_0(\eta)d\eta.$$

For $k = 1$, we have the following

$$\begin{aligned} C_2(\xi) &= \frac{m-1}{2}\int_{\xi}^{\infty} C_1(\zeta)d\zeta \\ &= \frac{m(m-1)}{2}\int_{\xi}^{\infty}\left(\int_{\zeta}^{\infty} C_0(\eta)d\eta\right)d\zeta. \end{aligned}$$

In this stage we perform the change of order of iterated integral as follows

$$\begin{aligned} C_2(\xi) &= \frac{m(m-1)}{2}\int_{\xi}^{\infty} C_0(\eta)\left(\int_{\xi}^{\eta} d\zeta\right)d\eta \\ &= \frac{m(m-1)}{2}\int_{\xi}^{\infty} C_0(\eta)(\eta-\xi)d\eta. \end{aligned}$$

We continue changing the order of integrals and get:

$$\text{For} \quad k=2: \quad C_3(\xi) = \frac{m-2}{3}\int_\xi^\infty C_2(\eta)d\eta = \frac{m(m-1)(m-2)}{2\times 3}\int_\xi^\infty C_0(\eta)(\eta-\xi)^2 d\eta.$$

$$\text{For} \quad k=3: \quad C_4(\xi) = \frac{m-3}{4}\int_\xi^\infty C_3(\eta)d\eta = \frac{m(m-1)(m-2)(m-3)}{2\times 3\times 4}\int_\xi^\infty C_0(\eta)(\eta-\xi)^3 d\eta.$$

Then, inductively, we can conclude that:

$$C_k(\xi) = \frac{\prod_{j=1}^{k}(m-j+1)}{k!}\int_\xi^\infty C_0(\eta)(\eta-\xi)^{k-1}d\eta. \tag{3.2.17}$$

We substitute Equation (3.2.17) into Equation (3.2.12) to get

$$f(\xi,t) = C_0(\xi) + \sum_{k=1}^{\infty}\frac{\prod_{j=1}^{k}(m-j+1)t^k}{k!}\int_\xi^\infty C_0(\eta)(\eta-\xi)^{k-1}d\eta. \tag{3.2.18}$$

Therefore, performing back the substitution, we observe that Equation (3.2.8) becomes

$$\bar{V}(\xi,t) = e^{-\xi t}\Big[C_0(\xi) + \sum_{k=1}^{\infty}\frac{\prod_{j=1}^{k}(m-j+1)t^k}{k!}\int_\xi^\infty C_0(\eta)(\eta-\xi)^{k-1}d\eta\Big]. \tag{3.2.19}$$

Since $e^{-\beta t^2/2}V(\xi,t) = \bar{V}(\xi,t)$, we have

$$V(\xi,t) = e^{\beta t^2/2}e^{-\xi t}\Big[C_0(\xi) + \sum_{k=1}^{\infty}\frac{\prod_{j=1}^{k}(m-j+1)t^k}{k!}\int_\xi^\infty C_0(\eta)(\eta-\xi)^{k-1}d\eta\Big]. \tag{3.2.20}$$

Then

$$w(u,t) = e^{\beta t^2/2}e^{-\xi t}\Big[C_0(\xi) + \sum_{k=1}^{\infty}\frac{\prod_{j=1}^{k}(m-j+1)t^k}{k!}\int_\xi^\infty C_0(\eta)(\eta-\xi)^{k-1}d\eta\Big], \tag{3.2.21}$$

where $\xi = u + \beta t$.

Finally Equation (3.1.5) yields that

$$n(x,t) = x^\nu e^{\beta t^2/2}e^{-\xi t}\Big[C_0(\xi) + \sum_{k=1}^{\infty}\frac{\prod_{j=1}^{k}(m-j+1)t^k}{k!}\int_\xi^\infty C_0(\eta)(\eta-\xi)^{k-1}d\eta\Big], \tag{3.2.22}$$

with $u = x^\alpha$. The interpretation of this case is given in Section 3.3.

Case 2: $\tau = 0$ and $\mu = 1$.

We then have $\epsilon = 0$ or $\gamma = -\nu$ and $\gamma = 1$. Applying these into Equation (3.1.12) results in

$$\frac{\partial w(u,t)}{\partial t} = -uw(u,t) + m\int_u^\infty w(s,t)ds + \beta u\frac{\partial w(u,t)}{\partial u}, \tag{3.2.23}$$

which has the characteristics equations as

$$\frac{\partial w(u,t)}{\partial t} = -uw(u,t) + m\int_u^\infty w(s,t)ds, \tag{3.2.24}$$

$$\frac{du}{dt} = -\beta u. \tag{3.2.25}$$

Then Equation (3.2.25) yields

$$u = \xi e^{-\beta t}. \tag{3.2.26}$$

By letting $w(u,t) = V(\xi,t)$, we get:

$$\frac{\partial V(\xi,t)}{\partial t} = -\xi e^{-\beta t} V(\xi,t) + m \int_{\xi}^{\infty} V(\eta,t) d\eta. \tag{3.2.27}$$

We now apply the integrating factor approach, that is

$$I.F = \exp\Big(\int_{t}^{\infty} \xi e^{-\beta x} dx \Big) = e^{-\xi A(t)}, \quad \text{where} \quad A(t) = \frac{e^{-\beta t}}{\beta}.$$

Multiplying Equation (3.2.27) by $I.F$, we get

$$e^{-\xi A(t)} \Big(\frac{\partial V(\xi,t)}{\partial t} + \xi e^{-\beta t} V(\xi,t) \Big) = m e^{-\xi A(t)} \int_{\xi}^{\infty} V(\eta,t) d\eta. \tag{3.2.28}$$

Equation (3.2.28) can be written in the form

$$\frac{\partial}{\partial t} \Big(V(\xi,t) e^{-\xi A(t)} \Big) = m e^{-\xi A(t)} \int_{\xi}^{\infty} V(\eta,t) d\eta. \tag{3.2.29}$$

We set $f(\xi,t) = V(\xi,t)e^{-\xi A(t)}$, then Equation (3.2.29) becomes

$$\begin{aligned} \frac{\partial f(\xi,t)}{\partial t} &= m e^{-\xi A(t)} \int_{\xi}^{\infty} e^{\eta A(t)} f(\eta,t) d\eta \\ &= m \int_{\xi}^{\infty} e^{A(t)(\eta-\xi)} f(\eta,t) d\eta. \end{aligned} \tag{3.2.30}$$

Now differentiating Equation (3.2.30) with respect to ξ, we get

$$\begin{aligned} \frac{\partial^2 f(\xi,t)}{\partial t \partial \xi} &= m \Big[-A(t) \int_{\xi}^{\infty} e^{\xi A(t)} e^{-\eta A(t)} f(\eta,t) d\eta - e^{\xi A(t)} e^{-\xi A(t)} f(\xi,t) \Big] \\ &= -A(t) \frac{\partial f(\xi,t)}{\partial t} - m f(\xi,t), \end{aligned}$$

which we can re-write as

$$\frac{\partial^2 f(\xi,t)}{\partial t \partial \xi} + A(t) \frac{\partial f(\xi,t)}{\partial t} + m f(\xi,t) = 0. \tag{3.2.31}$$

We consider two methods to solve Equation (3.2.31).

Method 1

Substituting the series expansions

$$f(\xi,t) = \sum_{n=0}^{\infty} C_n(\xi) t^n \quad \text{and} \quad e^{-\beta t} = \sum_{n=0}^{\infty} \frac{(-\beta t)^n}{n!},$$

into Equation (3.2.31), we get:

$$\sum_{n=1}^{\infty} n C_n'(\xi) t^{n-1} + \frac{1}{\beta} \sum_{n=0}^{\infty} \frac{(-\beta t)^n}{n!} \sum_{n=1}^{\infty} n C_n(\xi) t^{n-1} + m \sum_{n=0}^{\infty} C_n(\xi) t^n = 0. \tag{3.2.32}$$

We express the product of the sum as Cauchy product, that is

$$\sum_{n=1}^{\infty} nC_n'(\xi)t^{n-1} + \sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\frac{1}{\beta}\frac{(-\beta)^l}{l!}(n+1-l)C_{n+1-l}(\xi)\right]t^n + m\sum_{n=0}^{\infty}C_n(\xi)t^n = 0. \tag{3.2.33}$$

Equation (3.2.33) can be expanded as

$$\begin{aligned}
&\frac{1}{\beta}\Big[C_1(\xi) + 2C_2(\xi)t + 3C_3(\xi)t^2 + 4C_4(\xi)t^3 + \cdots\Big] - t\Big[C_1(\xi) + 2C_2(\xi)t + 3C_3(\xi)t^2 + 4C_4(\xi)t^3 + \cdots\Big]\\
&+ \frac{\beta t^2}{2!}\Big[C_1(\xi) + 2C_2(\xi)t + 3C_3(\xi)t^2 + 4C_4(\xi)t^3 + \cdots\Big] - \frac{\beta^2 t^3}{3!}\Big[C_1(\xi) + 2C_2(\xi)t + 3C_3(\xi)t^2\\
&+ 4C_4(\xi)t^3 + \cdots\Big] + \frac{\beta^3 t^4}{4!}\Big[C_1(\xi) + 2C_2(\xi)t + 3C_3(\xi)t^2 + 4C_4(\xi)t^3 + \cdots\Big] + \sum_{n=1}^{\infty} nC_n'(\xi)t^{n-1}\\
&+ m\sum_{n=0}^{\infty}C_n(\xi)t^n = 0.
\end{aligned} \tag{3.2.34}$$

Let us now group terms according to the power of t in Equation (3.2.34), that is

$$\begin{aligned}
&\frac{1}{\beta}C_1(\xi) + \left[\frac{2}{\beta}C_2(\xi) - C_1(\xi)\right]t + \left[\frac{3}{\beta}C_3(\xi) - 2C_2(\xi) + \frac{\beta}{2!}C_1(\xi)\right]t^2 + \left[\frac{4}{\beta}C_4(\xi) - 3C_3(\xi) + \frac{2\beta}{2!}C_2(\xi)\right.\\
&\left.- \frac{\beta^2}{3!}C_1(\xi)\right]t^3 + \left[\frac{5}{\beta}C_5(\xi) - 4C_4(\xi) + \frac{3\beta}{2!}C_3(\xi) - \frac{2\beta^2}{3!}C_2(\xi) + \frac{\beta}{4!}C_1(\xi)\right]t^4\\
&+ \cdots + \sum_{k=0}^{\infty}\Big[mC_k(\xi) + (k+1)C_{k+1}'(\xi)\Big]t^k = 0.
\end{aligned} \tag{3.2.35}$$

The equations we get from Equation (3.2.35) are:

$$\frac{1}{\beta}C_1(\xi) + mC_0(\xi) + C_1'(\xi) = 0, \tag{3.2.36}$$

$$\frac{2}{\beta}C_2(\xi) - C_1(\xi) + mC_1(\xi) + 2C_2'(\xi) = 0, \tag{3.2.37}$$

$$\frac{3}{\beta}C_3(\xi) - 2C_2(\xi) + \frac{\beta}{2!}C_1(\xi) + mC_2(\xi) + 3C_3'(\xi) = 0, \tag{3.2.38}$$

$$\frac{4}{\beta}C_4(\xi) - 3C_3(\xi) + \frac{2\beta}{2!}C_2(\xi) - \frac{\beta^2}{3!}C_1(\xi) + mC_3(\xi) + 4C_4'(\xi) = 0, \tag{3.2.39}$$

$$\frac{5}{\beta}C_5(\xi) - 4C_4(\xi) + \frac{3\beta}{2!}C_3(\xi) - \frac{2\beta^2}{3!}C_2(\xi) + \frac{\beta}{4!}C_1(\xi) + 5C_5'(\xi) + mC_4(\xi) = 0, \tag{3.2.40}$$

and as we proceed we get the following

$$kC_k'(\xi) + \frac{k}{\beta}C_k(\xi) - (k-1)C_{k-1}(\xi) + mC_{k-1}(\xi) = 0, \quad \text{for} \quad k = 1, 2. \tag{3.2.41}$$

and

$$kC_k'(\xi) + \frac{k}{\beta}C_k(\xi) - (k-1)C_{k-1}(\xi) + mC_{k-1}(\xi) + \sum_{j=1}^{k-1}(-1)^{j+1}\frac{(k-1-j)\beta^j}{(j+1)!}C_{k-1-j}(\xi) = 0, \tag{3.2.42}$$

$$\text{for} \quad k > 2.$$

By solving Equation (3.2.41), we get:

$$C_k'(\xi) + \frac{1}{\beta}C_k(\xi) = -\frac{(1-k+m)}{k}C_{k-1}(\xi), \tag{3.2.43}$$

and multiply Equation (3.2.43) by integrating factor, we obtain

$$\frac{d}{d\xi}\left(C_k(\xi)e^{\frac{1}{\beta}\xi}\right) = \frac{(1-k+m)}{k}e^{\frac{1}{\beta}\xi}C_{k-1}(\eta)d\eta, \quad \text{for} \quad k = 1, 2. \tag{3.2.44}$$

Let $D_k(\xi) = C_k(\xi)e^{\frac{1}{\beta}\xi}$. Equation (3.2.44) becomes

$$D_k(\xi) = \frac{(1-k+m)}{k}\int_\xi^\infty D_{k-1}(\eta)d\eta. \tag{3.2.45}$$

For $k = 1$, we have

$$D_1(\xi) = m\int_\xi^\infty D_0(\eta)d\eta, \tag{3.2.46}$$

this is the equal to

$$C_1(\xi) = m\int_\xi^\infty e^{\frac{1}{\beta}(\eta-\xi)}C_0(\eta)d\eta, \tag{3.2.47}$$

in terms of $C_k(\xi)$. We also observe that for $k = 2$, we get:

$$D_2(\xi) = \frac{(m-1)}{2}\int_\xi^\infty D_1(\zeta)d\zeta, \tag{3.2.48}$$

Changing the order of integration in Equation (3.2.48) and rewrite it in terms of D_1, we have

$$\begin{aligned} D_2(\xi) &= \frac{(m-1)}{2}\int_\xi^\infty \left[m\int_\zeta^\infty D_0(\eta)d\eta\right]d\zeta \\ &= \frac{m(m-1)}{2}\int_\xi^\infty D_0(\eta)\left[\int_\xi^\eta d\zeta\right]d\eta \\ &= \frac{m(m-1)}{2}\int_\xi^\infty D_0(\eta)(\eta-\xi)d\eta, \end{aligned} \tag{3.2.49}$$

which results on

$$C_2(\xi) = \frac{m(m-1)}{2}\int_\xi^\infty e^{\frac{1}{\beta}(\eta-\xi)}C_1(\eta)(\eta-\xi)d\eta. \tag{3.2.50}$$

Our general C_k is

$$C_k(\xi) = \frac{\prod_{j=1}^k(m-j+1)}{k!}\int_\xi^\infty C_0(\eta)(\eta-\xi)^{k-1}d\eta, \tag{3.2.51}$$

for values $k = 1, 2$.

We have determined two coefficients $C_1(\xi)$ and $C_2(\xi)$, in the above calculations from the general Equation (3.2.41).

Equation (3.2.42) is on the form

$$D_k = \frac{(1-k+m)}{k}\int_{\xi}^{\infty} D_{k-1}(\eta)d\eta + \int_{\xi}^{\infty}\sum_{j=1}^{k-1}(-1)^{j+1}\frac{(k-1-j)\beta^j}{k(j+1)!}D_{k-1-j}(\eta)d\eta, \tag{3.2.52}$$

where $D_k = C_k(\xi)e^{\frac{1}{\beta}\xi}$.

For $k=3$, we have

$$D_3(\xi) = \frac{m(m-1)(m-2)}{3!}\int_{\xi}^{\infty} D_0(\eta)(\eta-\xi)^2 d\eta + \frac{\beta m}{3!}\int_{\xi}^{\infty} D_0(\eta)(\eta-\xi)d\eta. \tag{3.2.53}$$

For $k=4$, we have

$$\begin{aligned} D_4(\xi) =& \frac{m(m-1)(m-2)(m-3)}{4!}\int_{\xi}^{\infty} D_0(\eta)(\eta-\xi)^3 d\eta + \Big(\frac{\beta m(m-3)}{4!} \\ &+ \frac{\beta m(m-1)}{2!(4)}\Big)\int_{\xi}^{\infty} D_0(\eta)(\eta-\xi)^2 d\eta - \frac{\beta m(m-1)}{4!}\int_{\xi}^{\infty} D_0(\eta)(\eta-\xi)d\eta. \end{aligned} \tag{3.2.54}$$

From Equation (3.2.42), we can conclude that

$$\begin{aligned} C_k(\xi) &= \frac{(1-k+m)}{k}\int_{\xi}^{\infty} e^{\frac{1}{\beta}(\eta-\xi)}C_{k-1}(\eta)d\eta \\ &+ \int_{\xi}^{\infty}\Big(\sum_{j=1}^{k-1}(-1)^{j+1}\frac{(k-1-j)\beta^j}{k(j+1)!}e^{\frac{1}{\beta}(\eta-\xi)}C_{k-1-j}(\eta)d\eta\Big), \quad \text{for} \quad k>2. \end{aligned} \tag{3.2.55}$$

We present the solution in the form of

$$f(\xi,t) = C_1(\xi) + C_2(\xi) + \sum_{k=3}^{\infty} C_k(\xi)t^k, \tag{3.2.56}$$

where $C_1(\xi)$ and $C_2(\xi)$ are given by Equations (3.2.47) and (3.2.50) respectively, whereas coefficients $C_3(\xi)$, $C_4(\xi)$ and other $C_k(\xi)$ are obtained from recursive formula (3.2.55) with initial condition $f(\xi,0)$.

Now the final step, we perform back substitution to return to the original variables.

Since $f(\xi,t) = V(\xi,t)e^{-\xi A(t)}$, then

$$V(\xi,t) = w(u,t) = e^{\xi A(t)}f(\xi,t), \tag{3.2.57}$$

with $\xi = ue^{\beta t}$ and $n(x,t) = x^{\nu}w(u,t)$, that is

$$n(x,t) = x^{\nu}e^{\xi A(t)}\Big[C_1(\xi) + C_2(\xi) + \sum_{k=3}^{\infty} C_k(\xi)t^k\Big]. \tag{3.2.58}$$

Let us now use an alternative form of the integrating factor for Equation (3.2.27). Suppose the integrating factor is of the form

$$I.F^* = \exp\Big(\int_0^t \xi e^{-\beta s}ds\Big) = e^{-\xi B(t)}, \quad \text{where} \quad B(t) = \frac{1}{\beta}(1-e^{-\beta t}). \tag{3.2.59}$$

Multiply Equation (3.2.27) with $I.F^*$, yields

$$\frac{\partial}{\partial t}\left(V(\xi,t)e^{-\xi B(t)}\right) = me^{-\xi B(t)}\int_{\xi}^{\infty} V(\eta,t)d\eta. \tag{3.2.60}$$

Let $g(\xi,t) = V(\xi,t)e^{-\xi B(t)}$, then Equation (3.2.60) becomes

$$\frac{\partial g(\xi,t)}{\partial t} = m\int_{\xi}^{\infty} e^{B(t)(\eta-\xi)}g(\eta,t)d\eta. \tag{3.2.61}$$

Differentiating Equation (3.2.61) with respect to ξ, we get:

$$\frac{\partial^2 g(\xi,t)}{\partial t\partial \xi} + B(t)\frac{\partial g(\xi,t)}{\partial t} + mg(\xi,t) = 0. \tag{3.2.62}$$

Substituting the series expansions

$$g(\xi,t) = \sum_{n=0}^{\infty} C_n(\xi)t^n \quad \text{and} \quad e^{-\beta t} = \sum_{n=0}^{\infty}\frac{(-\beta t)^n}{n!},$$

into Equation (3.2.62), we get:

$$\sum_{n=1}^{\infty} nC_n'(\xi)t^{n-1} + \frac{1}{\beta}\sum_{n=1}^{\infty} nC_n(\xi)t^{n-1} - \frac{1}{\beta}\sum_{n=0}^{\infty}\frac{(-\beta t)^n}{n!}\sum_{n=1}^{\infty} nC_n(\xi)t^{n-1} + m\sum_{n=0}^{\infty} C_n(\xi)t^n = 0, \tag{3.2.63}$$

which is simply the same as

$$\sum_{n=1}^{\infty} nC_n'(\xi)t^{n-1} + \frac{1}{\beta}\sum_{n=1}^{\infty} nC_n(\xi)t^{n-1} - \sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\frac{1}{\beta}\frac{(-\beta)^l}{l!}(n+1-l)C_{n+1-l}(\xi)\right]t^n$$
$$+ m\sum_{n=0}^{\infty} C_n(\xi)t^n = 0, \tag{3.2.64}$$

after using Cauchy product.

Matching the power of t in Equation (3.2.64) by letting $k = n-1$ and $k = n$, results on

$$\sum_{k=0}^{\infty}(k+1)C_{k+1}'(\xi)t^k + \frac{1}{\beta}\sum_{n=0}^{\infty}(k+1)C_{k+1}(\xi)t^k - \sum_{k=0}^{\infty}\left[\sum_{l=0}^{k}\frac{1}{\beta}\frac{(-\beta)^l}{l!}(k+1-l)C_{k+1-l}(\xi)\right]t^k$$
$$+ m\sum_{k=0}^{\infty} C_k(\xi)t^k = 0. \tag{3.2.65}$$

We write Equation (3.2.65) in compact form as

$$\sum_{k=0}^{\infty}\Bigg[(k+1)C_{k+1}'(\xi) + \frac{1}{\beta}(k+1)C_{k+1}(\xi) - \sum_{l=0}^{k}\frac{1}{\beta}\frac{(-\beta)^l}{l!}(k+1-l)C_{k+1-l}(\xi)$$
$$+ mC_k(\xi)\Bigg]t^k = 0, \tag{3.2.66}$$

which is satisfied if and only if

$$(k+1)C'_{k+1}(\xi) + \frac{1}{\beta}(k+1)C_{k+1}(\xi) - \sum_{l=0}^{k} \frac{1}{\beta}\frac{(-\beta)^l}{l!}(k+1-l)C_{k+1-l}(\xi) + mC_k(\xi) = 0. \qquad (3.2.67)$$

Proceeding formally, we can integrate Equation (3.2.67) and get:

$$C_{k+1}(\xi) = \frac{m}{k+1}\int_{\xi}^{\infty} e^{\frac{1}{\beta}(\eta-\xi)}C_k(\eta)d\eta + \int_{\xi}^{\infty}\sum_{l=0}^{k}\frac{1}{\beta}\frac{(-\beta)^l}{(k+1)l!}e^{\frac{1}{\beta}(\eta-\xi)}(k+1-l)C_{k+1-l}(\eta)d\eta. \qquad (3.2.68)$$

The solution is similar to the one given by Equation (3.2.56), so the expansion $B(t)$ give to the same results as $A(t)$. Here we determine the coefficients using recursive formula (3.2.68). Therefore, **Method 1** provide a recursive formula for solution to **Case 2**.

Method 2

Let us consider $e^{-\beta t}$ as a new time:

$$\sigma = e^{-\beta t}, \quad t \geq 0 \quad \text{and} \quad 0 < \sigma \leq 1.$$

Now, we can represent $f(\xi, t)$ as new variables,

$$\begin{aligned} \frac{\partial f(\xi,t)}{\partial t} &= \frac{\partial \bar{f}(\xi,\sigma)}{\partial \sigma}\frac{d\sigma}{dt} \\ &= -\beta\sigma\frac{\partial \bar{f}(\xi,\sigma)}{\partial \sigma}, \\ \frac{\partial^2 f(\xi,t)}{\partial t \partial \xi} &= -\beta\sigma\frac{\partial^2 \bar{f}(\xi,\sigma)}{\partial \sigma \partial \xi}. \end{aligned} \qquad (3.2.69)$$

Substituting σ and Equation (3.2.69) into Equation (3.2.31), we get

$$-\beta\sigma\frac{\partial^2 \bar{f}(\xi,\sigma)}{\partial \sigma \partial \xi} - \sigma^2\frac{\partial \bar{f}(\xi,\sigma)}{\partial \sigma} + m\bar{f}(\xi,\sigma) = 0. \qquad (3.2.70)$$

Now, the series expansion of Equation (3.2.70) is

$$\bar{f}(\xi,\sigma) = \sum_{n=0}^{\infty} C_n(\xi)\sigma^n, \qquad (3.2.71)$$

and we know that $f(\xi, 0) = C_0(\xi)$.

If $t = 0$, this implies that $\sigma = 1$. Then we say

$$f(\xi,0) = \bar{f}(\xi,1) = C_0(\xi).$$

Substituting Equation (3.2.71) into Equation (3.2.70), we get:

$$-\beta\sum_{n=1}^{\infty} nC'_n(\xi)\sigma^n - \sum_{n=1}^{\infty} nC_n(\xi)\sigma^{n+1} + m\sum_{n=0}^{\infty} C_n(\xi)\sigma^n = 0. \qquad (3.2.72)$$

To write this equation in a compact form using single summation, we match the power of σ, we let $k = n+1$ and $k = n$. Then we can re-write (3.2.72) as

$$-\beta \sum_{k=1}^{\infty} kC_k'(\xi)\sigma^k - \sum_{k=2}^{\infty}(k-1)C_{k-1}(\xi)\sigma^k + m\sum_{k=0}^{\infty} C_k(\xi)\sigma^k = 0,$$

and make the index equal, we get

$$mC_0(\xi) + \Big[mC_1(\xi) - \beta C_1'(\xi)\Big]\sigma + \sum_{k=2}^{\infty}\Big[mC_k(\xi) - \beta kC_k'(\xi) - (k-1)C_{k-1}(\xi)\Big]\sigma^k = 0. \quad (3.2.73)$$

Equation (3.2.73) is satisfied if

$$C_0(\xi) = 0, \quad mC_1(\xi) - \beta C_1'(\xi) = 0 \quad \text{and} \quad mC_k(\xi) - \beta kC_k'(\xi) - (k-1)C_{k-1}(\xi) = 0. \quad (3.2.74)$$

Solving the second equation in (3.2.74), we get

$$C_1(\xi) = A_1 \exp\Big(\frac{m\xi}{\beta}\Big), \quad \text{where } A_1 \text{ is a constant.} \quad (3.2.75)$$

Consider the last equation in (3.2.74), we have

$$C_k'(\xi) - \frac{m}{\beta k}C_k(\xi) = -\frac{(k-1)}{\beta k}C_{k-1}(\xi). \quad (3.2.76)$$

The integrating factor is

$$I.F = \exp\Big(-\frac{m\xi}{\beta k}\Big),$$

and Equation (3.2.76) take the form

$$\frac{d}{d\xi}\Big[C_k(\xi)\exp\Big(-\frac{m\xi}{\beta k}\Big)\Big] = -\frac{(k-1)}{\beta k}\exp\Big(-\frac{m\xi}{\beta k}\Big)C_{k-1}(\xi). \quad (3.2.77)$$

By integrating Equation (3.2.77), we get

$$C_k(\xi) = \frac{(k-1)}{\beta k}\int_{\xi}^{\infty} e^{\frac{m}{\beta k}(\xi-\eta)}C_{k-1}(\eta)d\eta + A_k e^{\frac{m\xi}{\beta k}}, \quad \text{where} \quad k > 1. \quad (3.2.78)$$

Unfortunately, this approach provide solutions in terms of the unknown constant A_k. We were not able to determine A_k and relate them to the initial condition $f(\xi, 0)$. As a result, this is open for discussion on future work related to the field.

Considering the integrating factor

$$I.F^* = \exp\Big(\int_0^t \xi e^{-\beta s}ds\Big) = e^{-\xi B(t)}, \quad \text{where} \quad B(t) = \frac{1}{\beta}(1 - e^{-\beta t}), \quad (3.2.79)$$

the new time becomes

$$\sigma = 1 - e^{-\beta t}, \quad t \geq 0 \quad \text{and} \quad 0 \leq \sigma < 1. \quad (3.2.80)$$

Proceeding as previously, we get:

$$\begin{aligned}\frac{\partial g(\xi,t)}{\partial t} &= \frac{\partial \bar{g}(\xi,\sigma)}{\partial \sigma}\frac{d\sigma}{dt}\\ &= \beta(1-\sigma)\frac{\partial \bar{g}(\xi,\sigma)}{\partial \sigma},\\ \frac{\partial^2 g(\xi,t)}{\partial t\partial \xi} &= \beta(1-\sigma)\frac{\partial^2 \bar{g}(\xi,\sigma)}{\partial \sigma\partial \xi}.\end{aligned} \tag{3.2.81}$$

Substituting Equation (3.2.81) into Equation (3.2.62), we get:

$$\beta(1-\sigma)\frac{\partial^2 \bar{g}(\xi,t)}{\partial t\partial \xi} + \sigma(1-\sigma)\frac{\partial \bar{g}(\xi,t)}{\partial t} + m\bar{g}(\xi,t) = 0. \tag{3.2.82}$$

We look for solution of Equation (3.2.70) in the form of series expansion

$$\bar{g}(\xi,\sigma) = \sum_{n=0}^{\infty} C_n(\xi)\sigma^n. \tag{3.2.83}$$

If $t = 0$, then $\sigma = 0$, and we have:

$$g(\xi,0) = \bar{g}(\xi,0) = C_0(\xi).$$

Substituting Equation (3.2.83) into Equation (3.2.82), we get that

$$\beta(1-\sigma)\sum_{n=1}^{\infty} nC_n'(\xi)\sigma^{n-1} + \sigma(1-\sigma)\sum_{n=1}^{\infty} nC_n(\xi)\sigma^{n-1} + m\sum_{n=0}^{\infty} C_n(\xi)\sigma^n = 0. \tag{3.2.84}$$

Expanding Equation (3.2.84) yields

$$\begin{aligned}&\beta\sum_{n=1}^{\infty} nC_n'(\xi)\sigma^{n-1} - \beta\sum_{n=1}^{\infty} nC_n'(\xi)\sigma^{n} + \sum_{n=1}^{\infty} nC_n(\xi)\sigma^{n} - \sum_{n=1}^{\infty} nC_n(\xi)\sigma^{n+1}\\ &+ m\sum_{n=0}^{\infty} C_n(\xi)\sigma^n = 0.\end{aligned} \tag{3.2.85}$$

Let $k = n-1$, $k = n+1$ and $k = n$ in respective sums, that is, we make the power of t to match, we get:

$$\begin{aligned}&\beta\sum_{k=0}^{\infty} (k+1)C_{k+1}'(\xi)\sigma^{k} - \beta\sum_{k=1}^{\infty} kC_k'(\xi)\sigma^{k} + \sum_{k=1}^{\infty} kC_k(\xi)\sigma^{k} - \sum_{k=2}^{\infty} (k-1)C_{k-1}(\xi)\sigma^{k}\\ &+ m\sum_{k=0}^{\infty} C_k(\xi)\sigma^k = 0.\end{aligned} \tag{3.2.86}$$

We now write Equation (3.2.86) in compact form with same indices, we do that to obtain

$$\begin{aligned}&\beta C_1'(\xi) + mC_0(\xi) + \Big[\beta 2C_2'(\xi) - \beta C_1'(\xi) + C_1(\xi) + mC_1(\xi)\Big]\sigma + \sum_{k=2}^{\infty}\Big[\beta(k+1)C_{k+1}'(\xi)\\ &- \beta kC_k'(\xi) + kC_k(\xi) - (k-1)C_{k-1}(\xi) + mC_k(\xi)\Big]\sigma^k = 0,\end{aligned} \tag{3.2.87}$$

which is satisfied if

$$\beta C_1'(\xi) + mC_0(\xi) = 0, \tag{3.2.88}$$
$$\beta 2C_2'(\xi) - \beta C_1'(\xi) + C_1(\xi) + mC_1(\xi) = 0, \tag{3.2.89}$$
$$\beta(k+1)C_{k+1}'(\xi) - \beta k C_k'(\xi) + kC_k(\xi) - (k-1)C_{k-1}(\xi) + mC_k(\xi) = 0. \tag{3.2.90}$$

Solving Equation (3.2.88), results in

$$C_1(\xi) = \frac{m}{\beta}\int_{\xi}^{\infty} C_0(\eta)d\eta. \tag{3.2.91}$$

It follows that Equation (3.2.89) becomes

$$C_2(\xi) = \frac{m}{2\beta}\int_{\xi}^{\infty} C_0(\eta)d\eta + \frac{(m+1)}{2\beta}\int_{\xi}^{\infty} C_1(\zeta)d\zeta.$$

The change of order yields

$$C_2(\xi) = \frac{m}{2\beta}\int_{\xi}^{\infty} C_0(\eta)d\eta + \frac{m(m+1)}{2\beta^2}\int_{\xi}^{\infty} C_0(\eta)(\eta-\xi)d\eta. \tag{3.2.92}$$

This approach, with integrating factor $I.F^*$ (3.2.79), we are able to get the recursive formula given by Equation (3.2.90), subject to the given initial condition $g(\xi, 0)$.

3.3 Discussion

In **case 1**, where $\tau = 0$ and $\mu = 0$, this leads to $\gamma = -\nu$. If we choose the value of ν, for example, if we consider simple case which is $\nu = 0$, we obtain the binary fragmentation which mean the breakup of a parent cluster only yields two daughter clusters. This also leads to $\gamma = 0$, which implies that decay rate do not depend on mass of the cluster since $c(x) = \epsilon$ and $\alpha = 1$ implies that fragmentation rate $a(x) = x$ is proportional to the mass of cluster. Hence, the larger the mass of cluster, the faster fragmentation occurs and the other way round. If we choose $\nu = -\frac{1}{2}$, we get that the number of produced clusters after fragmentation is equal to $\frac{\nu+2}{\nu+1} = 3$, that is we expect three new daughter clusters after each event. Furthermore, this gives that decay term is $\frac{\partial}{\partial x}(\sqrt{x}u)$ with decay rate $c(x) = \sqrt{x}$. As we were re-examining this case from Huang et al. (1996), we provided enough details to obtain the power series solution.

In chemical engineering for example, if we have a cluster in a solute and it is dissolving, by chemical reaction the particles from the surface of the cluster moves to the solution. This means that only particles that are on the surface of the cluster will take part in this process. This shows that $c(x)$ is proportional to the surface of the cluster. In the simple case of spherical cluster, size $\sim$ volume $\sim \frac{4}{3}\pi r^3$ and surface $\sim 4\pi r^2$. We express surface area in terms of size by letting size be $x = \frac{4}{3}\pi r^3$ and surface $s = 4\pi r^2$, which results in $s = 4\pi(\frac{3}{4\pi})^{\frac{2}{3}} x^{\frac{2}{3}}$, then in this case, $c(x) \sim x^{\frac{2}{3}}$. Such examples are describe by **case 1**.

For **case 2**, we observe that if $\tau = 0$, we have $\epsilon(\gamma + \nu) = 0$, this shows that $\epsilon = 0$ or $\gamma = -\nu$. If we consider $\epsilon = 0$, we obtain the decay rate as $c(x) = 0$ and this represent a pure fragmentation process. Considering the parameter $\gamma = -\nu$ and $\mu = 1$, results in $\gamma = 1$. This condition shows that decay rate is $c(x) = \epsilon x$. Assume that we have a cluster of a living cells of size x that can die. If the probability of a cell dying in a unit time is ε, then the average number of dying cells in a unit of time is εx. This means that the rate at which the cluster decrease due to death of cells is $c(x) = \varepsilon x$. In addition, we have the fragmentation rate as $a(x) = 1$.

4. Conclusion

4.1 Conclusion

In this project, we presented closed form solutions of the fragmentation equation with decay. We used power laws coefficient to simplify the fragmentation equation and apply the method of characteristics to reduce partial differential equation (PDE) into an ordinary differential equation (ODE). We considered two cases that enabled us to get closed form solutions, that also describes different physical models. This cases shows that if we revert back to our original variables, we will get other solutions of different models, not only the models with binary fragmentation and constant decay, but models with other features. Therefore these cases can be used to write down the number of different models that describe different physical cases. Then we provided recursive formula for solutions using the power series expansion.

4.2 Future Work

For future, we shall study the models that include cut-off and finds its exact solutions. Cut-off is a mass condition that divide two states, namely, a fragment state and dust state according to their size, see Huang et al. (1996). In addition, we wish to combine fragmentation with coagulation equations, that is (C-F) equation. We aim to provide the analytical investigation of (C-F) equation, provide the methods and techniques we used to solve them.

Acknowledgements

I want to acknowledge AIMS and its funders for supporting this work. My sincere gratitude to my supervisor, Prof Jacek Banasiak from University of Pretoria for his support, guidance and patience in compiling this project. To my tutor, Faraniaina Rasolofoson, thank you for your assistance and advice.

A special thank you to my fellow colleagues, the AIMS 2020 intake, for your support throughout the academic year. To my loving family and friends, I appreciate all the encouragements and the endless support which made my study fruitful and productive to the end. Finally, I thank God for all the strength and protection He gave me.

References

Banasiak, J. and Noutchie, S. O. Controlling number of particles in fragmentation equations. *Physica D: Nonlinear Phenomena*, 239(15):1422–1435, 2010.

Banasiak, J., Lamb, W., and Laurençot, P. *Analytic Methods for Coagulation-Fragmentation Models, Volume I*. CRC Press, 2019.

Chandrasekhar, S. Stochastic problems in physics and astronomy. *Reviews of Modern Physics*, 15(1): 1–89, 1943.

Edwards, B. F., Cai, M., and Han, H. Rate equation and scaling for fragmentation with mass loss. *Physical Review A*, 41(10):5755, 1990.

Elminyawi, I. M., Gangopadhyay, S., and Sorensen, C. Numerical solutions to the Smoluchowski aggregation-fragmentation equation. *Journal of Colloid Interface Science*, 144:315–323, 1991.

Gueron, S. and Levin, S. A. The dynamics of group formation. *Mathematical Biosciences*, 128(1-2): 243–264, 1995.

Huang, J., Edwards, B. F., and Levine, A. D. General solutions and scaling violation for fragmentation with mass loss. *Journal of Physics A: Mathematical and General*, 24(16):3967, 1991.

Huang, J., Guo, X., Edwards, B. F., and Levine, A. D. Cut-off model and exact general solutions for fragmentation with mass loss. *Journal of Physics A: Mathematical and General*, 29(23):7377, 1996.

McGrady, E. and Ziff, R. M. Analytical solutions to fragmentation equations with flow. *AIChE Journal*, 1988.

Montroll, E. W. and Simha, R. Theory of depolymerization of long chain molecules. *The Journal of Chemical Physics*, 8(9):721–726, 1940.

Oukouomi, N. S. C. and Doungmo, G. E. F. Exact solutions of fragmentation equations with general fragmentation rates and separable particles distribution kernels. *Mathematical Problems in Engineering*, pages 1–5, 2014.

Smoluchowski, M. v. Drei vortrage uber diffusion, brownsche bewegung und koagulation von kolloidteilchen. *Physik Zeitschr*, 17:557–585, 1916.