Portfolio Optimisation in a Stable Regime Switching Market Framework

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Abstract

Over the years, stable distributions have gained popularity when it comes to modelling asset returns. As opposed to normal distribution which was previously used for financial modelling, stable distributions have been used to explain and capture the observed fat tails and skewness that we often encounter in financial datasets. In this paper, we discuss methods that have been efficiently used to estimate the parameters of the stable distribution. The main objective of this paper is to construct an optimal finance portfolio of risky and risk-free assets under a regime-switching framework. These assets follow a stable distribution. The approach to portfolio optimization problem was presented here in financial market with regime-switching framework by accommodating the observed shifts in market regimes that take place in each period. A portfolio was optimized using simulations.

Keywords: Portfolio Optimization, Regime-switching Models, Parameter Estimation, Stable Distribution

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Zizipho Maqokolo, 24 October 2019

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1. Introduction and Literature Review

Asset returns are a key aspect in financial modelling. Asset returns are assumed to have a normal distribution, however this assumption of normality is rejected. Because it underestimates the risk on a financial investment portfolio. Mandelbrot (1960) first studied the empirical distribution of financial assets which led to the rejection of normal distribution. Since then the literature has gradually deviated from this assumption of normality to stable distribution. Stable distributions are said to arise in the study of heavy-tailed distributions.

When considering multivariate α-stable distribution, there is a need to estimate parameters of this distribution in portfolio optimization. Estimating a spectral measure is important since it gives information about the dependence structure between the assets that make up the portfolio. A proposed method of estimating the spectral measure is using the empirical characteristic function method in a multivariate setting. Using maximum likelihood estimation to estimate spectral measure may be difficult since stable distribution densities do not have closed form representation.

In various fields of research including finance (for example Carr et al. (2002)), electrical engineering (Stuck and Kleiner (1974)) and physics (Janicki and Weron (1994)) among other fields, often the distribution of datasets deviates from normality and exhibit excess heavy-tailed behaviour. Stable distributions were introduced by Lévy (1924) to address this phenomenon which cannot be explained by the normal distribution. Over the years, these distributions gained popularity because of their rich properties including the generalization of the classical Central Limit Theorem (CLT).

In finance, the asset returns were assumed to follow a normal distribution. However, there is overwhelming empirical evidence from the seminal works of Mandelbrot (1960) and Fama (1965) that this assumption of normality should be rejected due to excess kurtosis found in empirical analyses of the return distribution. Fama (1965) explains that the motivation for not using the normal distribution to model financial data is that it does not capture the large fluctuations seen in asset returns. Hence, Mandelbrot (1960) introduced the α-stable distribution to model financial asset returns.

Nolan (2018) describes α-stable distributions (also referred to as stable distributions) as distributions used to model heavy tailed asset returns. This class of distributions is characterized by four parameters, α, β, σ and μ, which, respectively, denote the index of stability, skewness, scale and shift of the distribution. Although stable distributions have gained popularity and are applied in many fields, they possess some set backs. One of the major drawback is the absence of closed-form expression for their probability density function with exceptions of the Gaussian distribution (for $\alpha = 2$), Lévy distribution (for $\alpha = \frac{1}{2}$, $\beta = 1$), and Cauchy distribution (for $\alpha = 1$, $\beta = 0$). In Chapter 2, more of the theory and mathematics behind these family of stable distributions will be explored in detail.

1.1 Parameter Estimation

As briefly explained above, the absence of closed forms for the density of stable distributions is a major drawback in terms of estimating parameters for stable distributions. Therefore, the methods of estimation have been employed that rely on other properties of the distributions. Hence, in this section we discuss the literature of methods used to estimate parameters proceeding Chapter 2.

Following the work of Mandelbrot (1960) on the application of stable distributions to model asset returns, Zolotarev (1964) developed the representation of stable laws using integrals. The results to Zolotarev's work have then been used to develop parameter estimation approaches. Hence, there are a number of methods that have been proposed, developed and widely used to efficiently estimate the parameters of the stable distributions (see (Nolan et al., 2001; Xu et al., 2011; Reuss et al., 2016)).

The quantiles method was first introduced by Fama and Roll (1971) to estimate the parameters of symmetric stable distributions, that is for $\beta = 0$. Fama and Roll (1971)'s sample quantile method was later revised by McCulloch (1986) to introduce a generalization of the quantile method by incorporating asymmetric ($\beta \neq 0$) stable distributions to provide consistent estimators of parameters for $0.6 \leq \alpha \leq 2$. However, for $\alpha < 0$, the estimators are said to be asymptotically biased.

In Ma and Nikias (1995), fractional lower order moments (FLOM) method is introduced and used to model noise in impulsive signal environments as symmetric stable distributions. However, there was a need to cover asymmetric cases. Hence, Kuruoglu (2001) introduced a remedy for the FLOM method. This remedy estimates the parameters using logarithmic moments method for stable distributions in order to avoid the challenge of solving the function $Sinc$ which cannot be done when using FLOM methods as described by Kateregga et al. (2017).

Maximum Likelihood Estimation (MLE) is one of the methods that have been widely used in applications of modern finance due to its properties of generality and asymptotic efficiency (Yu, 2004). DuMouchel et al. (1973) proposed maximum likelihood method to approximate consistent estimates which is further studied by Nolan et al. (2001). Brorsen and Yang (1990) modified this method to consider the symmetric stable distributions parameters by expressing the likelihood function as an integral, which Nolan et al. (2001) refer to as direct integration. MLE can be difficult to implement due to the likelihood function that is unbounded over the parameter space (see Yu (2004)). However, the Fourier transform of the likelihood function is always bounded hence it can be expressed in a closed form. Hence, the Fourier transform of the density function is given by the characteristic function as a one-to-one correspondence between the density function and its Fourier transform (Kateregga et al., 2017). This leads to the method based on the use of characteristic functions.

Empirical Characteristic Function (ECF) approach was introduced by Press (1972) as a method based on the transformations of characteristic function. In Chapter 2, we shall discuss this method in more detail, to estimate the spectral measure Γ.

1.2 Regime-Switching Models

In finance, markets often change behaviour abruptly. Most of these changes are related to incidents such as financial crises (for example, recession or economic expansion), currency volatility or sudden shifts in government policies. When these changes occur in the financial markets, the market is said to go through *regime changes*. **?** define a regime change as a significant change in behaviour of financial markets. Moreover, as explained in Lange and Rahbek (2009), these regimes reflect changes in the underlying financial and economic mechanism through the observed time period. To identify these regime changes in the markets and capture the behaviour of most financial time series including fat tails and skewness, a plausible approach called regime switching model is used and described by Ang and Timmermann (2012). Regime switching models are widely used in finance and economics to model interest rates (Garcia and Perron, 1996), exchange rates (Engel and Hamilton, 1990) and stock returns (Kim et al., 1998).

Hamilton (1989) introduced Markov regime-switching model as a way to incorporate changes in the dynamics of the assets and accommodate the observed shifts in the market which was further analysed in Kim et al. (1998). Alizadeh et al. (2008) used Markov regime-switching to examine hedging performance for oil future markets. Schwendener (2010) explains that Markov regime-switching model depends on a discrete non-observed random variable S_t which denotes the regime at time t. This discrete random variable is referred to as *Markov chain*. Reuss et al. (2016) explained that the switches between regimes are modelled as a Markov chain.

Škrinjarić and Šego (2016) highlighted the importance of regime-switching models in portfolio optimization as they lead to a good portfolio performance and minimize risks.

1.3 Markov Chains

As previously mentioned, Markov-switching models with regimes are based on the assumption that a regime is governed by the discrete Markov chain, hence it is used to determine the change from one state of regime to another within a finite number of possible states. Thus, in this section, we briefly discuss Markov chain.

Schwendener (2010) defines Markov chain as a discrete-time stochastic process that consists of a finite number of states where the probability of moving to a next state conditional on the present state does not depend on the past states, that is the probability distribution of the state S_t at time t is dependent solely on the previous state S_{t-1}. The transitions of the Markov chain are determined by the transition probability matrix **P**.

Definition 1.3.1. (Reuss et al., 2016) Assume an n-step Markov chain with transition probabilities p_{ij} for $i, j = 1, 2, .., n$. This transition probability matrix **P** consists of the probabilities of transitioning from state i at time $t-1$ to state j at time t where probabilities are defined as

$$p_{ij} = P(S_t = j | S_{t-1} = i) \tag{1.3.1}$$

for $i, j \in \{1, 2, ..., n\}$, each entry in the transition matrix is non-negative $0 \leq p_{ij} \leq 1$ and each row sums up to unity, that is, $\sum_{j=1}^{n} p_{ij} = 1$. Hence, the transition probability matrix is given by

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}.$$

1.4 Portfolio Selection

In this section, we look into portfolio analysis to construct an optimal portfolio based on risk and return proceeding Chapter 3.

Firstly, we need to understand what is meant by a portfolio. Reuss et al. (2016) defined portfolio as a collection of assets and a way to allocate wealth invested on a number of risky assets. Markowitz (1952) proposed modern portfolio theory by introducing mean-variance approach for optimal portfolio selection. Similarly, Markowitz (1952) assumed that the returns of assets follow a normal distribution and are characterised by mean and variance. The performance of a portfolio is measured by mean and variance where mean is the expected return on the investment and variance is the measure of risk on the investment (Markowitz, 1952).

Speidell et al. (1989) defined the concept of portfolio optimization as a way to allocate the total wealth among a number of assets in order to minimize portfolio risk and maximize expected return. This is a classical mean-variance framework that has been used over the years to pursue a portfolio optimization problem in order to perform a portfolio selection. The mathematics behind this portfolio theory model will be discussed further in Chapter 2.

1.5 Objectives of the Essay

The objective of this research essay is to construct an optimal finance portfolio of assets that will maximize returns or minimize risks. In this essay, we aim to minimize risk. The asset returns in the portfolio are assumed to follow α-stable distribution which captures heavy-tail returns and asymmetries. The portfolio is optimized in financial market with regime-switching framework by accommodating the observed shifts in market regimes that take place in each period. We assume that each regime is modelled by a multivariate α-stable distribution where we consider asset return as a random vector that depends on the state of a non-observable Markov chain with state space.

1.6 Outline of the Essay

This paper is organised as follows: Chapter 1 gives an overview and extensions of the literature on stable distributions, methods used to estimate parameters, Markov chains in the context of regime switching modelling and also focuses on portfolio optimization in relation to stable distributions and regime switching. In Chapter 2, we discuss the mathematics behind stable distributions, parameter estimation, Markov chains and portfolio selection. We focus on our model having stable parameters and portfolio optimization problem with parameters subject to Markov regime-switching leading to a portfolio objective function to be optimized in Chapter 3. In Chapter 4, we perform analyses and interpret the results. Chapter 5 concludes and summarises the results.

2. Mathematical Background

2.1 Alpha-stable Distributions

Stable distributions have been studied extensively under both univariate and multivariate framework. In this research, we focus on the latter.

2.1.1 Stable Distribution Parametrization. A stable univariate distribution is characterised by four parameters denoted by α, σ, β and μ as described by Reuss et al. (2016). Hence, the distribution is denoted by $S_\alpha(\sigma, \beta, \mu)$ where α is an *index of stability* describing the thickness of the distribution, β is a *skewness* parameter, σ is a *scale* parameter and μ is a *location (shift)* parameter. These parameters are defined within the following bounds $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\sigma > 0$ and $\mu \in \mathbb{R}$ as given in Nolan (2018) and Reuss et al. (2016).

Definition 2.1.2. Let X follow an α-stable distribution also expressed as

$$X \sim S_\alpha(\sigma, \beta, \mu). \tag{2.1.1}$$

We define its characteristic function as

$$\varphi_X(t) = \mathbb{E}[\exp(itX)] = \begin{cases} \exp\{-\sigma^\alpha |t|^\alpha (1 - i\beta \text{sign}(t) \tan\frac{\pi\alpha}{2}) + i\mu t\} & \text{for } \alpha \neq 1 \\ \exp\{-\sigma|t|(1 + i\beta\frac{2}{\pi}\text{sign}(t)\ln|t|) + i\mu t\} & \text{for } \alpha = 1 \end{cases} \tag{2.1.2}$$

where

$$\text{sign}(t) := \begin{cases} -1 & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ 1 & \text{if } t > 0. \end{cases} \tag{2.1.3}$$

In a multivariate setting, the skewness and scale parameters are determined by a *spectral measure* Γ. In this case, the distribution is characterised by the *index of stability* α, the *spectral measure* Γ, the *shift* parameter μ and is denoted by $S_\alpha(\Gamma, \mu)$. The spectral measure is an important parameter because modelling financial portfolios. It captures the dependence structure between assets that make up the portfolio (see Nolan et al. (2001)).

Lemma 2.1.3. (Nolan et al., 2001) Consider a portfolio of assets as a multivariate stable vector $\mathbf{X} = (X_1, ..., X_d)$ in $\mathbb{R}^d$ then the characteristic function is given as

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \mathbb{E}[\exp\{i\langle \mathbf{X}, \mathbf{t}\rangle\}] = \exp(-I_{\mathbf{X}}(t) + i\langle \boldsymbol{\mu}, \mathbf{t}\rangle), \tag{2.1.4}$$

where the exponent function is

$$I_{\mathbf{X}}(\mathbf{t}) = \int_{S^d} \psi_\alpha(\langle \mathbf{t}, \mathbf{s}\rangle) d\Gamma(\mathbf{s}) \tag{2.1.5}$$

where S^d is the unit sphere in $\mathbb{R}^d$; Γ is the spectral measure of the vector $\mathbf{X}$; $\boldsymbol{\mu}$ is a shift vector in $\mathbb{R}^d$; for $\mathbf{t}, \mathbf{s} \in \mathbb{R}$, $\langle \mathbf{t}, \mathbf{s}\rangle$ is the inner product and

$$\psi_\alpha(u) = \begin{cases} |u|^\alpha (1 - i\text{sign}(u)\tan\frac{\pi\alpha}{2}), & \text{for } \alpha \neq 1 \\ |u|(1 + i\frac{2}{\pi}\text{sign}(u)\ln|u|), & \text{for } \alpha = 1 \end{cases}. \tag{2.1.6}$$

Proof. (See Byczkowski et al. (1993)). □

There are properties that make stable distribution adaptable and useful to capture skewness and heavy-tail phenomena of financial data as explained in seminal works of (Fama, 1965) and (Reuss et al., 2016). These properties are as follows:

Theorem 2.1.4. *Generalized Central Limit Theorem (Nolan et al., 2001) Let $X_1, X_2, X_3, ...$ be a sequence of independent and identically distributed random variables then there is a random variable Z that follows a stable distribution is defined by*

$$\frac{1}{b_n}\left(\sum_{i=1}^{n} X_i - a_n\right) \longrightarrow Z$$

where "$\longrightarrow$" denotes weak convergence in distribution, b_n is a positive constant and a_n is real.

Proof. See Nolan (2018) where CLT is proved and from that generalized CLT is derived. □

Remark 2.1.5. Stable distributions use generalized Central Limit Theorem (CLT) which is an extension of the classical CLT to include random variables with infinite variance. This property states that the sum of normalized independent and identically distributed random variables with infinite variance converges to a stable distribution (Reuss et al., 2016).

Remark 2.1.6. Stable distribution is a limiting distribution in probability to other distributions.

Corollary 2.1.7. *Stability of Summation* (Reuss et al., 2016; Samoradnitsky, 2017)

If $X_1, X_2, ...$ are independent and identically distributed stable random variables, where $\alpha = 1$, then [1]

$$X_1 + X_2 + ... + X_n := n^{\frac{1}{\alpha}} X_1 + \frac{2}{\pi}\sigma\beta n \ln(n). \tag{2.1.7}$$

Remark 2.1.8. Stable distributions have stability of summation property which means that the finite sum of independent random variables, where each variable follows stable distribution, is also a stable distribution. *Note:* $\sum_{i=1}^{n} X_i$ is distributed by $S_\alpha\left(\sigma n^{\frac{1}{\alpha}}, \beta, \mu n\right)$.

As previously mentioned in Chapter 1, the density of stable distribution cannot be expressed in a closed form. With exception of the following:

- *Gaussian (Normal) distribution* $N(\mu, \sigma^2)$ where the *index of stability* $\alpha = 2$ is given by $S_2(\sigma, \beta, \mu)$ with standard deviation $\frac{\sigma}{2}$ and mean μ (β becomes irrelevant) and its density is given as

$$\frac{1}{2\sigma\sqrt{\pi}} e^{\frac{-(x-\mu)^2}{4\sigma^2}},$$

- *Cauchy distribution* with the *index of stability* $\alpha = 1$ and *skewness* $\beta = 0$ is given by $S_1(\sigma, 0, \mu)$ and its density is given as

$$\frac{1}{\pi}\frac{\sigma}{(x-\mu)^2 + \sigma^2},$$

[1] For $\alpha \neq 1$, $X_1 + X_2 + ... + X_n := n^{\frac{1}{\alpha}} X_1 + \mu(n - n^{\frac{1}{\alpha}})$

- *Lévy distribution* with the *index of stability* $\alpha = 1/2$ and *skewness* $\beta = \pm 1$ is given by $S_{\frac{1}{2}}(\sigma, -1, \mu)$ or $S_{\frac{1}{2}}(\sigma, +1, \mu)$ and its density is given as

$$\sqrt{\frac{\sigma}{2\pi}} \frac{1}{(x-\mu)^{\frac{3}{2}}} e^{-\frac{\sigma}{2(x-\mu)}}$$

(see Xu et al. (2011) and Kateregga et al. (2017)).

Figure 2.1 shows the standardized densities of subclasses for stable distribution.

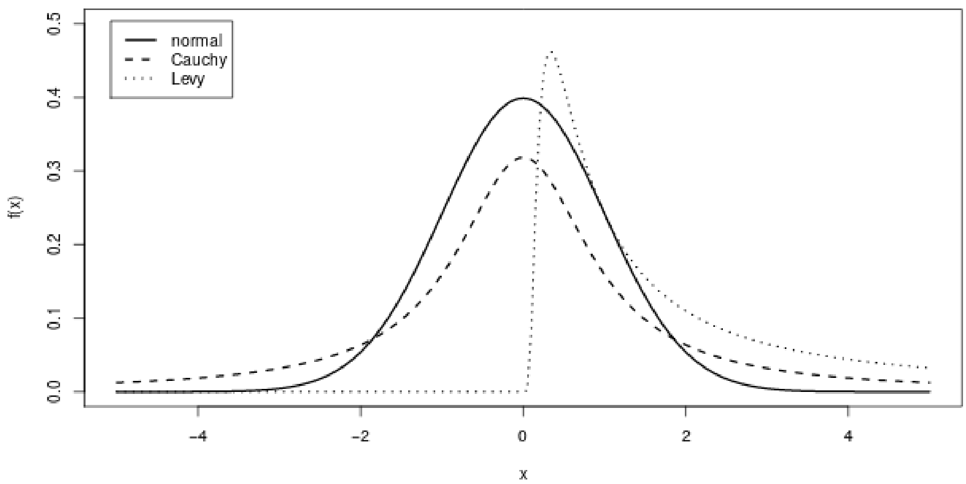

Figure 2.1: Plots of standardized densities of Gaussian (straight line), Cauchy (dashed line) and Lévy distribution (dotted line) as shown in Nolan (2018)

2.2 Parameter Estimation Methods

There are many parameter estimation methods however in this section, we focus on Empirical Characteristic Function method since it performs better estimation for multivariate dependence structure given through the spectral measure Γ (Nolan et al., 2001).

2.2.1 Empirical Characteristic Function. This section is based on the works of Nolan et al. (2001) and Press (1972) where Empirical Characteristic Function (ECF) method is introduced as a method based on the transformations of characteristic function.

2.2.2 Estimation of the spectral measure Γ. In this research, we estimate $\mathbf{\Gamma}$ using ECF method for multivariate case.

Lemma 2.2.3. (Nolan et al., 2001) Consider an independent and identically distributed sample $X_1, ..., X_n$ of random vectors drawn from stable distribution with spectral measure $\mathbf{\Gamma}$. With $\hat{\varphi}_n(\mathbf{t})$ and $\hat{I}_n(\mathbf{t})$ given as the empirical counterparts of φ and I where $\hat{\varphi}_n(\mathbf{t})$ is the sample characteristic function is then given by

$$\hat{\varphi}_n(\mathbf{t}) = \frac{1}{n} \sum_{i=1}^{n} \exp\left(i\langle \mathbf{t}, \mathbf{X}_i \rangle\right)$$

and

$$\hat{I}_n(\mathbf{t}) = -\ln \hat{\varphi}_n(\mathbf{t}).$$

Proof. (See (Byczkowski et al., 1993; Pivato and Seco, 2003)) □

According to Nolan et al. (2001), for the spectral measure estimation, we consider a discrete measure to approximate the exact spectral measure using the following form

$$\Gamma(\cdot) = \sum_{j=1}^{n} \gamma_j \delta_{\mathbf{s}_j}(\cdot) \tag{2.2.1}$$

where $\gamma_j = \Gamma(A_j)$ are the weights at point s_j in the unit sphere S^d and $\delta_{\mathbf{s}_j}$ are the point masses at s_j, for $j = 1, ..., n$. Hence, to estimate $\Gamma(\cdot)$, the characteristic function $\varphi_{\mathbf{X}}(\mathbf{t})$ as shown in Equation (2.1.4) is transformed to a d-dimensional stable random vector characteristic function given as

$$\varphi_{\mathbf{X}}(\mathbf{t}) = -\sum_{j=1}^{n} \psi_\alpha(\langle \mathbf{t}, \mathbf{s}_j \rangle)\gamma_j. \tag{2.2.2}$$

2.3 Portfolio Selection

As previously discussed, mean-variance approach was introduced for optimal portfolio selection. Consider the portfolio optimization problem formulated under Markowitz' mean-variance framework. Suppose the portfolio has a set of assets, characterized by expected mean and covariances. For an optimal portfolio to give minimum risk for an expected return, optimal weight for each asset in the portfolio needs to be determined (Jorion, 1992).

According to Markowitz (1952), the expected portfolio return given by $\mathbb{E}[R_p]$ is computed as follows:

$$\mathbb{E}[R_p] = \sum_{i=1}^{n} w_i \mathbb{E}[R_i] \tag{2.3.1}$$

where w_i is the weight which is the proportion of asset i in the portfolio (with $0 \leq w_i \leq 1$ for all i) and the total of portfolio weights is given as

$$\sum_{i=1}^{n} w_i = 1. \tag{2.3.2}$$

Note that the portfolio return is a linear combination of the returns of the individual assets. However, the variance of a portfolio is not simple as the expected return so then this variance is expressed by defining the covariance matrix as suggested in Markowitz (1952). Hence, the covariance of the return between asset i and j is defined as

$$\sigma_{ij} = \mathbb{E}[(R_i - \mathbb{E}[R_i])(R_j - \mathbb{E}[R_j])]. \tag{2.3.3}$$

This covariance σ_{ij} may then be expressed in terms of correlation coefficient ρ:

$$\sigma_{ij} = \sigma_{ij} = \rho_{ij}\sigma_i\sigma_j \tag{2.3.4}$$

where ρ_{ij} is the correlation coefficient between the return of assets i and j, for $\rho_{ij} \in [-1, 1]$ and σ_i is the standard deviation of asset i (see Markowitz (1952)). Hence the variance of a portfolio is given by

$$\mathbb{V}[R_p] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \tag{2.3.5}$$

where w_i is the weight of asset i.

Therefore, the portfolio optimization problem in n-assets case is reduced using mean-variance approach, which minimizes the variance of portfolio over w, that is,

$$\underset{w}{\text{Minimize}}\ w'Vw,$$

subject to

$$\begin{cases} w'\mathbf{1} = 1, \\ \mu' w = \mathbb{E}[R_p] \end{cases}$$

where $\mathbf{1}$ is a vector of ones, μ and w are defined as

$$\mu = \begin{bmatrix} \mathbb{E}[R_1] \\ \mathbb{E}[R_2] \\ \vdots \\ \mathbb{E}[R_n] \end{bmatrix} \text{ and } w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

as discussed in Yaman and Dalkılıç (2000).

However, the classical mean-variance approach is not valid for the portfolio optimization problem since it takes into account the variance of the portfolio and the assumption in this paper is that the portfolio of assets follow a stable distribution (with $1 < \alpha < 2$). Recall that this class of distributions does not have finite variance by generalized CLT, hence the portfolio optimization problem in this paper is modified to cater for the finite variance assumption being dropped.

For asset returns that are assumed to follow a multivariate stable distribution, we are constructing a portfolio optimization problem based on Value at Risk (VaR) and Conditional Value at Risk (CVaR) as opposed Markowitz' classical mean-variance approach for portfolio selection.

Suppose there is a portfolio of n assets. Now, to determine the joint distribution of the returns in the portfolio. Assume that the asset return at time t is given as a multivariate stable vector

$$\mathbf{R}_t = (R_{t,1}, R_{t,2}, ..., R_{t,n})'$$

and the weights of these assets are given as $w_1, w_2, ..., w_n$ with $0 \leq w_i \leq 1$, for all $i = 1, 2, ..., n$ and

$$\sum_{i=1}^{n} w_i = 1. \tag{2.3.6}$$

Then the portfolio is a linear combination of the individual assets which is distributed as a univariate stable distribution and is given as

$$R_{t,p} = \sum_{i=1}^{n} w_i R_{t,i}. \tag{2.3.7}$$

To determine the multivariate distribution of the portfolio return, we take into consideration the spectral measure which gives the information about dependence structure that exists between assets in the portfolio. In Chapter 3, we look at portfolio optimization problem as an application of our model.

3. Model

In this research, we assume that each portfolio is constructed and optimized in the varied financial market with regime-switching framework. Hence, in this chapter, we focus on a model having stable parameters and portfolio optimization problem with parameters subject to Markov random regime switching leading to a portfolio objective function to be optimized.

3.1 Model Setup

3.1.1 Stable regime switching. We assume the presence of regime-switching in our portfolio of asset returns. That is, these returns will be modelled by an unobservable discrete-time Markov-switching. We consider a model with two regimes that is when the market is calm or when it is turbulent. Each regime will be modelled by a stable distribution.

Suppose we model the return, in the univariate setting, as a random variable R_t where

$$R_t = \frac{X_t - X_{t-1}}{X_{t-1}} \tag{3.1.1}$$

that is dependent on the state of an unobservable Markov chain S_t with two states. We assume that this return follows a stable distribution that is

$$R_t \sim S_{\alpha_{S_t}}(\sigma_{S_t}, \beta_{S_t}, \mu_{S_t})$$

where α_{S_t} is the index of stability, β_{S_t} is the skewness parameter, σ_{S_t} is the scale parameter and μ_{S_t} represents the shift of the distribution. The ranges of these parameters are given by $\alpha_{S_t} \in (0, 2]$, $\beta_{S_t} \in [-1, 1]$, $\sigma_{S_t} > 0$ and $\mu_{S_t} \in \mathbb{R}$ (Reuss et al., 2016).

However, in a multivariate setting, to model dependence structure of assets, we consider a random variable $\mathbf{R}_t$ as the asset return that follows a multivariate stable distribution which is given as

$$\mathbf{R}_t \sim S_{\alpha_{S_t}}(\Gamma_{S_t}, \boldsymbol{\mu}_{S_t})$$

where α_{S_t} is the index of stability, Γ_{S_t} is the spectral measure and $\boldsymbol{\mu}_{S_t}$ is the location vector as explained in Reuss et al. (2016).

As a way of modelling stock to accommodate shifts, we will use Markov regime switching model. That is, the regimes that occur in the market are introduced with a state process modelled as Markov chain to determine the transition from one regime to another. For both, univariate and multivariate setting, the transitions of the Markov chain S_t are determined by the transition probability matrix $\mathbf{P}$ which consists of the probabilities of transitioning from regime i at time $t-1$ to regime j at time t where probabilities are defined as

$$p_{ij} = P(S_t = j | S_{t-1} = i)$$

for $i, j \in \{1, 2\}$, $0 \leq p_{ij} \leq 1$ and $\sum_{j=1}^{2} p_{ij} = 1$, $\forall_i$. Hence, the transition probability matrix of our model is given by

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \tag{3.1.2}$$

where $p_{11} = P(S_t = 1 | S_{t-1} = 1)$ and $p_{22} = P(S_t = 2 | S_{t-1} = 2)$.

According to Reuss et al. (2016), the model proposed above accommodate heavy tails and skewness in each regime.

3.1.2 Portfolio Optimization Problem. Recall that the classical mean-variance framework is not applied in this research since the variance is not taken into account because it is not finite for stable distributions.

Hence, for the construction of an optimal portfolio consisting of risk-free and risky assets, the portfolio optimization problem is carried out under certain assumptions and necessary constraints. With assumptions and constraints that are different from the mean-variance framework. The portfolio optimization problem simply maximize the return of the portfolio by managing the risk that the investor is about to take. This portfolio optimization problem is given by

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{maximize}}\ & \boldsymbol{w}'\boldsymbol{\mu}_{\boldsymbol{R}} - \lambda \text{Risk}(\boldsymbol{w}) \\ \text{such that}\ & \boldsymbol{w} \geq 0 \\ & \boldsymbol{w}'\mathbf{1} = \mathbf{1} \end{aligned} \tag{3.1.3}$$

where $\boldsymbol{w}'$ is the transpose of the vector of portfolio weights, $\boldsymbol{\mu}_{\boldsymbol{R}}$ is the vector of expected returns of the portfolio, $\lambda \geq 0$ is the risk aversion parameter which reflects the willingness of the investor to trade at higher expected wealth and Risk($\boldsymbol{w}$) is the risk measure for the portfolio. By the restriction $\boldsymbol{w} \geq 0$, we assume that this leads to short selling of a stock not being allowed, thus show the implications for investors. In these portfolio optimization problems, the risk measure Risk($\boldsymbol{w}$) can be the scale parameter $\sigma(\boldsymbol{w})$, $VaR(\boldsymbol{w})$ or $CVaR(\boldsymbol{w})$. $VaR(\boldsymbol{w})$ and $CVaR(\boldsymbol{w})$ are defined as the risk measures, where $VaR(\boldsymbol{w})$ is the function of the portfolio weights leading to problems with local optima and $CVaR(\boldsymbol{w})$ is a convex function of the portfolio weights to find the global optima. That is setting Risk($\boldsymbol{w}$)=$\sigma(\boldsymbol{w})$, which imply that the scale parameter of the portfolio is used as the risk measure (Reuss et al., 2016).

Note: VaR means Value at Risk and $CVaR$ means Conditional Value at Risk. Similar to Reuss et al. (2016)'s seminal work, for the stable models we consider VaR and $CVaR$ as finite for $\alpha > 1$.

3.2 Methodology

3.2.1 Parameter Estimation. In our model, we assume that the financial stock returns are continuous random variables, which is the case of discrete returns

$$R_t = \frac{X_t - X_{t-1}}{X_{t-1}} \tag{3.2.1}$$

that is the returns of the portfolio are the weighted sums of the stock prices. Equation 3.2.1 computes the daily returns, where X_t is the adjacent close price of the stock and X_{t-1} is the previous adjacent price of the stock.

For our model, we firstly estimate Markov chain S_t parameters given as the pair ($\boldsymbol{\pi} = (0, 1)$ which is the initial distribution and the transition matrix P). We estimate parameters using Morgan Stanley Capital International (MSCI) daily stock data.

3.2.2 Portfolio Optimization Implementation. Similar to the seminal work of Reuss et al. (2016), for the application of stable distributions to model the heavy tailed data on portfolio selection, we conduct the portfolio optimization case study using Risk($\boldsymbol{w}$)=$CVaR(\boldsymbol{w})$ as explained in section 3.1.2. In our case study, a portfolio of assets will be allocated frequently based on the solution of the portfolio optimization problem given in Equation 3.1.3.

3.2.3 Simulation. In this section, we look at the simulations used for purposes of portfolio optimization implementation. To conduct portfolio optimization case study. Firstly, we determine the parameters of Markov chain using MSCI stock index to serve as a regime indicator. We will split data according to the detected regimes. Then we will estimate the distribution parameters for each regime. A simulation will be carried out on these distribution parameters and then the portfolio optimization will be conducted.

Essentially, this portfolio optimization case study will be performed in a similar manner as that found in Reuss et al. (2016). Reuss et al. (2016) summarizes how this case study will be performed in a following manner:

1. Estimate regimes of MSCI daily data
2. Split data according to the detected regimes
3. For each regime, estimation of distribution parameters is done
4. According to the model, simulations are performed for the complete investment horizon
5. Simulated scenarios are used for optimization to allocate portfolio accordingly
6. Move to the rebalancing time
7. Calculate portfolio return over the rebalancing period

The results of the conducted portfolio optimization case study will then be analysed in Chapter 4 and follows steps 1 to 7.

3.3 Data

This research uses the MSCI stock index data taken from Yahoo finance database to illustrate the techniques discussed in the section above. This data is sampled from 15 November 2007 to 30 August 2019, to estimate the parameters of the model and detect market regimes in order construct an optimal portfolio of assets by conducting portfolio optimization case study. Our data consists of 2968 daily observations of MSCI Index and 7 variables however we only consider 2 variables (Date and Adjacent Close Price) that we will use throughout our analysis.

4. Results and Discussion

In this chapter, we present and explain results.

For the analysis, useful R software packages were used to estimate stable distribution and detect market regimes. However, the implementation of our portfolio optimization problem is carried out using an algorithm in Python, by formulating to maximize the return of the portfolio using the *cvxpy* package This package is used for modelling language for convex optimization problems.

4.1 Detecting Market Regimes

Firstly, we apply the univariate stable regime-switching model to detect market regimes. We will use the MSCI data as reference index to detect whether the market is in a calm or a turbulent state, by using Hidden Markov Model (HMM). The estimation results for the MSCI are shown below. The two-state Markov chain with transition matrix is given by

$$\mathbf{P}_{MSCI} = \begin{bmatrix} 0.9994 & 0.0006 \\ 0.0008 & 0.9992 \end{bmatrix}.$$

The transition probability represents the possibility that the MSCI stock return movements may stay in the current regime or will move to other regimes. We observe that the probability of staying in regime 1, given that we are currently in regime 1 has a higher probability which is 0.9994. Similarly, the probability of remaining in regime 2, given that we are currently in regime 2 is given be 0.9992. This indicates difficulty in transition to the same regime. As the stock price rise, the probability of transitioning into the high volatility becomes higher.

We estimate the values of the four parameters of the stable distribution for MSCI using all four methods described in Chapter 1 which resulted into Table 4.1.

	McCulloch method	Logarithmic moments	Maximum Likelihood	Empirical Characteristics Function
α	1.43646	1.51840	1.51114	1.44847
β	-0.02881	-0.02250	-0.02750	-0.00689
σ	0.00986	0.01004	0.01025	0.00988
μ	0.00137	0.001328	0.001328	0.00127

Table 4.1: Estimated distribution parameters of MSCI for different methods of estimation

4.2 Analysis

To see the evolution of the financial index MSCI, we plot time series for the adjacent close indices of MSCI as shown in Figure 4.1. From the plot in Figure 4.1, we observe that stock prices have a general increasing trend over time and there is a significant downward trend in 2009 due to financial crisis. It also shows that there was a slow and steady increase momentum between 2010 to 2012. We observe the increase of prices over time with an observable increase after 2015. Figure 4.1 shows that the stock prices display a roughly exponential growth over the period.

The MSCI data will be split according to the detected regimes. For each of these regimes, we estimated the distribution parameters for daily returns as shown in Table 4.2

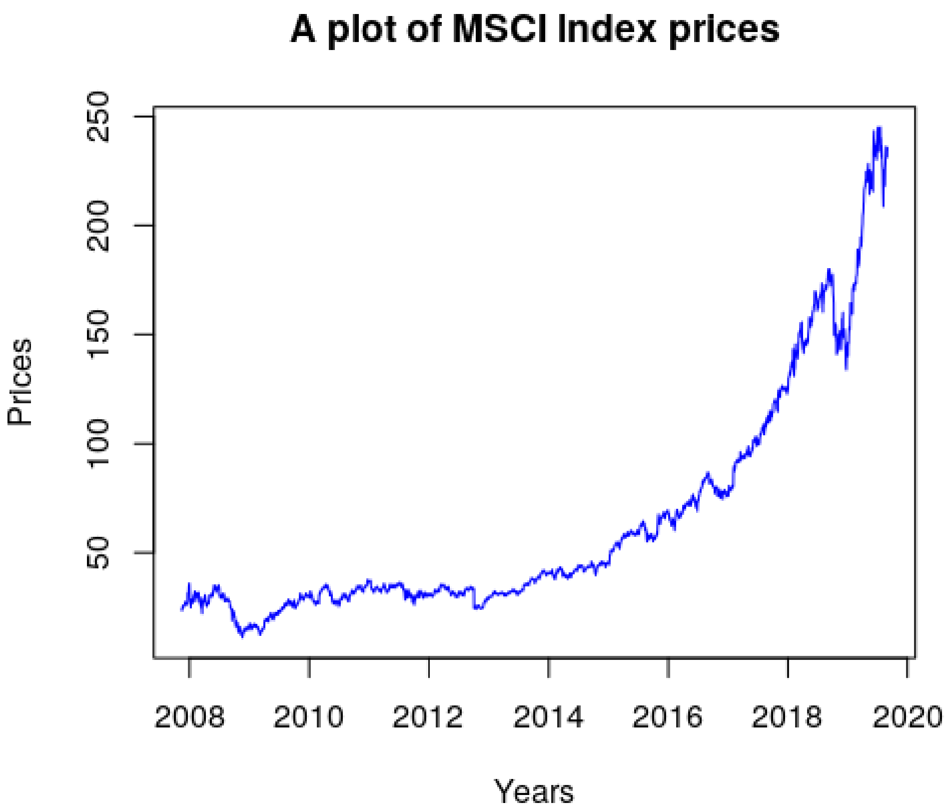

Figure 4.1: Evolution of the financial index MSCI between 2007 and 2019

S_t	α	β	σ	μ
1	1.3868	0.0049	0.0115	0.0010
2	1.6952	-0.1453	0.0083	0.0015

Table 4.2: Estimated distribution parameters of MSCI for the detected regimes

From the values of α in Table 4.2, we observe that this estimation indicates stable distribution behaviour for both regimes. We see that regime $S_t = 2$ has a negative skewness parameter, this corresponds to a positive return on the stock. Regime $S_t = 1$ has lower value for α than regime $S_t = 2$, this indicates heavier tails for regime $S_t = 1$ hence we characterize regime $S_t = 1$ as turbulent regime and regime $S_t = 2$ as calm regime (see Figure 4.4 and 4.3).

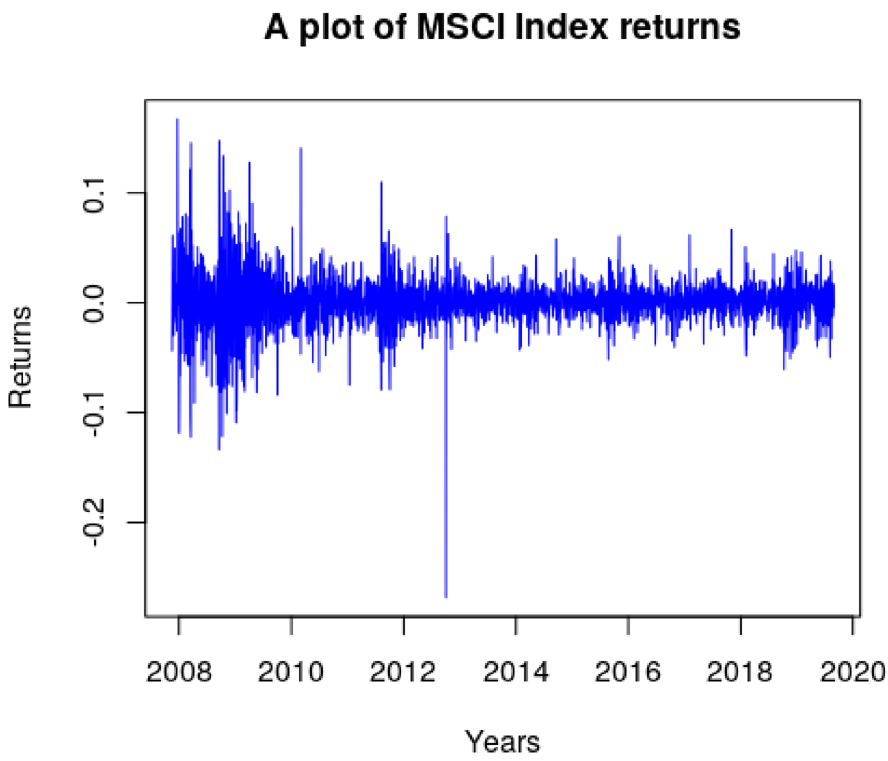

Figure 4.2: The returns of the evolution of the financial index MSCI between 2007 and 2019

In Figure 4.2, for the purpose of de-trending which is removing a trend from a time series dat, we try to analyse returns instead of prices. The stock market returns of the MSCI financial indices in Figure 4.2, corresponds to the growth rate given by Equation 3.2.1. This plot shows daily returns which exhibits higher volatile fluctuations. In particular, the high volatilities during the years 2008 to 2010 are more vividly depicted in Figure 4.2. Between 2012 to 2014, we see a negative spike. In contrast to the prices, the returns fluctuate about a mean close to 0. Furthermore, high oscillations tend to cluster together, reflecting more volatile market periods.

Figure 4.4 and 4.3 give the evolution of the returns after the data was split into regimes.

From Figure 4.4 we can see that volatility varies over time instead of remaining constant. The volatilities also exhibit some persistence, or dependence over time.

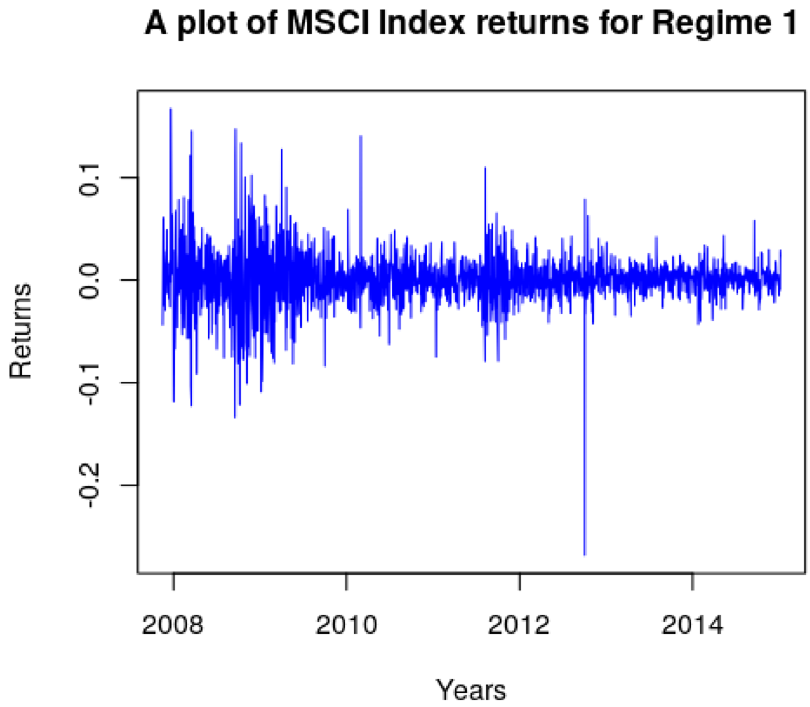

Figure 4.3: The returns of the evolution of the financial index MSCI for regime 1

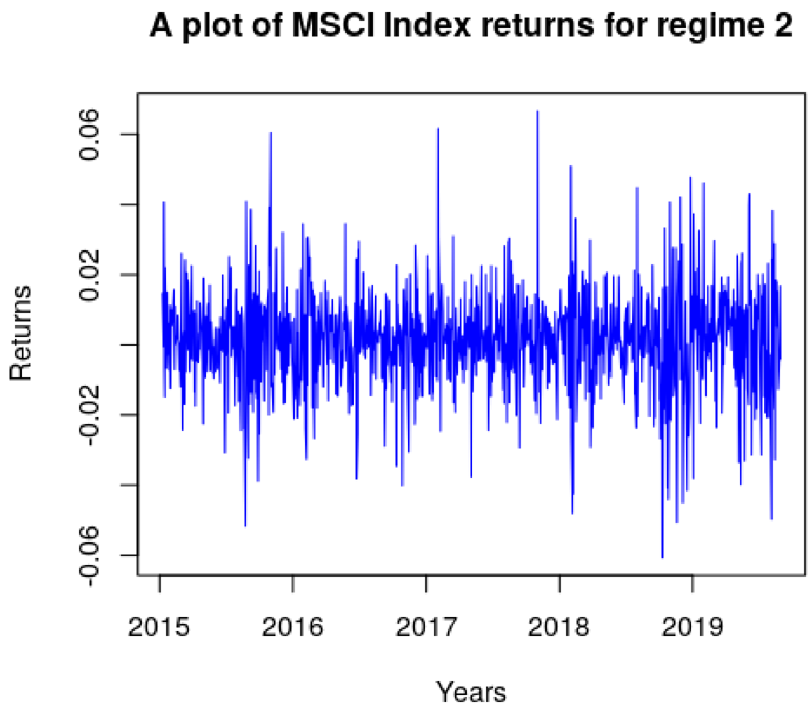

Figure 4.4: The returns of the evolution of the financial index MSCI for regime 2

To conduct the portfolio optimization case study, we carried out simulations using the procedure given in section 3.2.3. For the purposes of illustration, we assume that we have a portfolio consists of 5 simulated assets. Figure 4.5 gives a development of the five simulated indices over the months from year 2007 to year 2019. In Figure 4.5, the financial crisis in 2008 stands out, especially for simulated

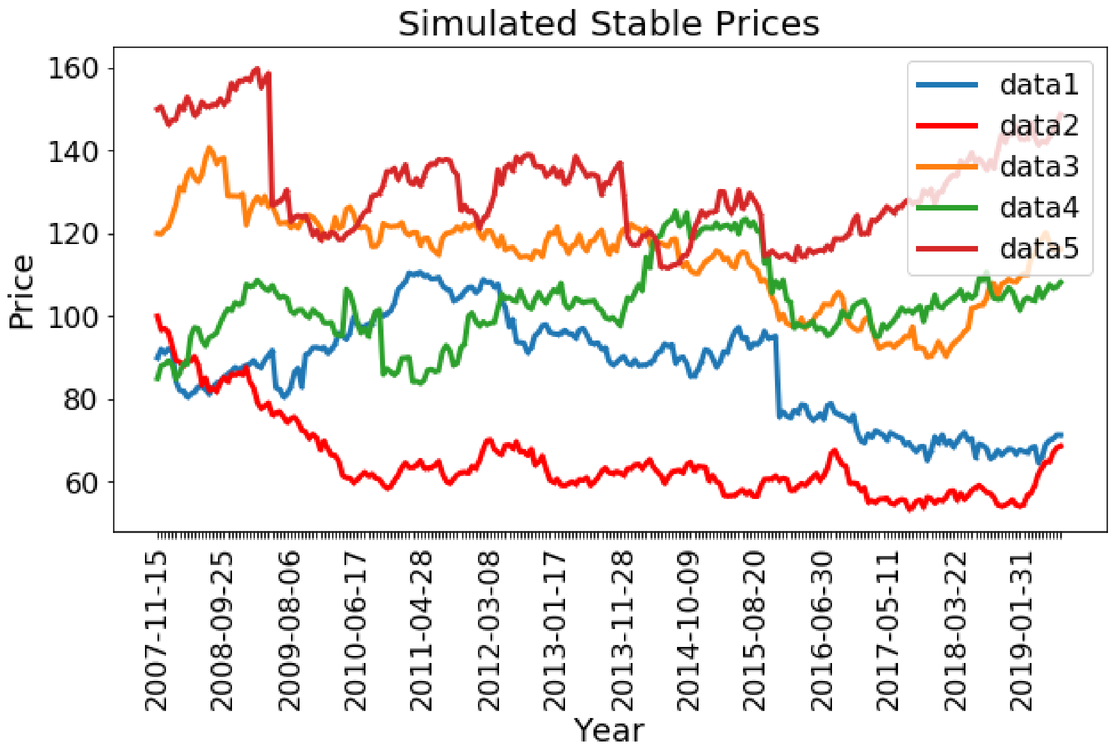

Figure 4.5: Development of the five simulated indices over the months

asset $data5$ where we observe a huge jump, this is an indication that the market might not have been performing well in this period.

As mentioned earlier, to optimize our portfolio in this case, we will only manage the risk by minimizing it. To minimize the risk, we evaluate the risk measure $Risk(w)$ in Equation 3.1.3 by using the two risk quantifiers, VaR and CVaR. We evaluated these risk quantifiers using R package $PerformanceAnalytics$ which is econometric tool designed to analyse risk for financial portfolios. Using this package, we were able to allocate the risk of a portfolio by decomposing the portfolio risk into the risk contribution of each component. We analysed the risk of the portfolio and the components that make up the portfolio using three different methods; Historical, Gaussian and Modified method as shown Figure 4.6 and 4.7. Historical and Gaussian methods are efficient for data that follow a normal distribution whereas Modified method is efficient for data that follow a stable distribution.

Method	price1	price2	price3	price4	price5	Portfolio
Hist	0,067516	0,049654	0,039605	0,058882	0,059643	0,029370
Gaus	0,052229	0,046277	0,035100	0,049615	0,044401	0,022279
Mod	0,142813	0,049431	0,042647	0,085404	0,106232	0,027542

Figure 4.6: CVaR evaluated for each simulated asset and portfolio using different methods

Method	price1	price2	price3	price4	price5	Portfolio
Hist	0,035830	0,040941	0,027251	0,037094	0,028192	0,021743
Gaus	0,041775	0,037165	0,027986	0,039301	0,035363	0,017695
Mod	0,047234	0,039473	0,029625	0,039582	0,039853	0,018026

Figure 4.7: VaR evaluated for each simulated asset and portfolio using different methods

In our analyses of CVaR and VaR, note that CVaR gives an average expected loss whereas VaR gives a range of potential loss. For the analyses, we assumed 95% confidence level as the threshold loss value and the weights of the simulated assets to be 6%, 4%, 40%, 30% and 20% respectively. Since in this research, we assume that asset returns follow stable distribution, so we analyse the results given by the Modified method. In both Figure 4.6 and 4.7, if we look at the Mod row, we notice that the $Portfolio$ column has the least risk as compared to the risk of the simulated prices across all assets, this may be an indication that it is more risk to invest in individual stocks than the portfolio. In Figure 4.6, we see that the loss value for portfolio is 0.027542, this means that we could have about 2.75% potential loss in the worst 5% of returns. Looking at Figure 4.7 under the Mod row and $Portfolio$ column, we have the loss value given as 0.018026, this means that we are 95% confident that we will not incur more than 1.8% loss.

5. Conclusion

Risk and return play an important role in making any investment decision. Portfolio optimization and diversification are the best ways to reduce the risk of an investment. VaR and CVaR are used as risk measures which directly addresses risk diversification, even in portfolios with non-normally distributed assets. Portfolio diversification reduces the risk of the entire portfolio as we have seen in the analysis that the portfolio poses a lower risk than investing in individual stocks.

The goal of this paper was to solve the portfolio optimization problem in a market by constructing an optimal finance portfolio consisting of risk-free and risky assets where risky assets follow a stable distribution. These risky assets were in the form of stock generated using regime-switching parameters with a Markov chain explaining the regime of the market. Hence, regime-switching models were employed in modelling heavy tails in the returns of these risky assets. We presented the application of stable distributions that are used to capture heavy-tails and skewness in financial markets.

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