Towards an Arbitrage Theorem in an Infinite Horizon

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Abstract

The purpose of this essay is to understand how mathematical tools can been used towards tackling the existence problem of arbitrage opportunity in a complete market with a focus on the infinite time horizon. We start by working in the continuous finite time horizon and concluded by considering the case of an infinite horizon. Although, the no-arbitrage property seems to hold in the infinite horizon, one needs to pay close attention to a family of filtrations, and the corresponding change of measures should agree when moving from a finite to an infinite horizon. Our result shows that arbitrage opportunities cannot be eliminated by just constructing a single martingale measure on the canonical filtration over the positive real line. However, we are able to show that by considering a countable and infinite family of martingale measures over the infinite horizon, a sufficient no-arbitrage condition holds.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Zainab Olufunke Olukoga, 24 October 2019

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1. Introduction and background

1.1 Introduction

Arbitrage is a risk-free investment strategy that occurs due to an imbalance in market prices. It is a way of making money without initial investment and the possibility of making no loss. The concept of arbitrage opportunity, also called the no-arbitrage theorem, has been in play as far back as the evolution of mathematical finance. This reason somewhat justifies why it is regarded as one of the fundamental theorems of asset pricing while the completeness of a market is the second.

Various mathematical works of literature with a focus on different planning horizons have discussed the arbitrage concept in a way to control this opportunity from occurring. Schachermayer (1992) presented proof of the fundamental theorem of asset pricing in discrete finite-time. In doing so, he gave an elementary proof to the work of (R.Dalang, A.Morton, and W.Willinger, 1990) using orthogonality arguments. Also, on the discrete finite-time, Jacod and Shiryaev (1998) gave simpler proofs to these theorems, and one of the new results discovered was the condition for a local martingale to be a martingale. Using Kenneth Arrow's invention of the general equilibrium model of security markets in 1953, De Vries (2000) worked on the arbitrage theorem in discrete infinite time. The mentioned works of literature agree to the fact that in a discrete finite-time setting, the existence of no-arbitrage opportunity is possible if and only if there is an equivalent martingale measure. As mentioned by De Vries (2000), the Black-Scholes model is a useful model for the arbitrage theorem in a continuous finite-time horizon. These are just a few of the various studies that have been carried out on the no-arbitrage property.

This project is set to show how the no-arbitrage condition can be characterized for a complete market model in the finite horizon and how it can also be extended to the infinite horizon which in turn gives a deeper understanding of the mathematical formulation of a financial market model. A notable feature of a local martingale that was used is that when they are bounded from below, they become a supermartingale and hence, a martingale.

The layout for the rest of the chapters of this project is as follows; In the concluding part of Chapter 1, we discussed definitions, propositions, and remarks peculiar to the core of this project, such as probability theory and some concepts of a financial market. In Chapter 2, we discussed the no-arbitrage theorem, standard market, and complete market in a continuous finite-time horizon. The Infinite-time planning horizon was introduced in chapter 3. Examples on the application of the no-arbitrage theorem for the finite and infinite horizons were discussed in Chapter 4. Finally, we gave conclusion in Chapter 5.

In the rest of this chapter, we review some concepts and results in probability theory and financial markets useful for the comprehension of this essay. They form the backbone for the mathematics applied in this work.

The content of this section, including notations used, follows the treatment of topics as found in classical references such as Oksendal (2013), Jacod and Protter (2003), Karatzas et al. (1998) and Fries (2007). The proofs can also be found therein. We present a quick overview of these concepts.

1.2 Mathematical preliminaries

We introduce some important definitions and results in probability theory that will be used in our construction of the arbitrage theory. The proofs to these results are omitted; however, we will point the reader to the specific references where they can be found.

The modern treatment of arbitrage relies heavily on mathematical tools such as σ-algebra, Borel spaces, measurability, and some mathematical properties like the martingale property. Also, the development of stochastic integral allows for the concept of arbitrage to be understood by a broader audience. We present the tools as follows.

1.2.1 Definition. Let Ω be a non-empty set.

1. A σ-algebra (or σ-field) $\mathcal{F}$ on Ω is a collection of subsets of Ω satisfying
   (i.) $\Omega \in \mathcal{F}$;
   (ii.) $\forall A \in \mathcal{F}, A^c \in \mathcal{F}$;
   (iii.) If $(A_i)_{i\in I}$ is a countable collection of sets in $\mathcal{F}$, then $\bigcup\limits_{i\in I} A_i \in \mathcal{F}$.
2. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two σ-algebras on Ω, $\mathcal{F}_1$ is said to be a sub-σ-algebra of $\mathcal{F}_2$ if $\mathcal{F}_1 \subset \mathcal{F}_2$.

For this essay, we will focus on a particular class of σ-algebra, which is the σ-algebra generated by a random variable.

1.2.2 Remark. We give the following remark.

(i.) Given any family $\mathcal{B}$ of subset of Ω, we denote by

$$\sigma(\mathcal{B}) := \cap\{\mathcal{G} : \mathcal{G}, \sigma\text{-algebra of } \Omega, \mathcal{B} \subset \mathcal{G}\}$$

the smallest σ-field of Ω containing $\mathcal{B}$, $\sigma(\mathcal{B})$ is called the σ-field generated by $\mathcal{B}$.

When $\mathcal{B}$ is a collection of all open sets of a topological space Ω, $\sigma(\mathcal{B})$ is called the Borel σ-algebra on Ω and the elements of $\sigma(\mathcal{B})$ are called Borel sets.

(ii.) If $X : \Omega \longrightarrow \mathbb{R}^n$ is a function, then the σ-algebra generated by X, denoted by $\sigma(X)$, is the smallest σ-algebra on Ω containing all the sets of the form

$$\{X^{-1}(U) : U \subset \mathbb{R}^n, \quad \text{open}\}.$$

Throughout this essay, the σ-algebra of interest will be $\sigma(X)$, where X is a random variable.

Let $(\Omega, \mathcal{F})$ be a measurable space, we want to define a metric on Ω such that $(\Omega, \mathcal{F})$ becomes a complete separable metric space in a way that $\mathcal{F}$ is then the Borel σ-algebra. The metric of interest is called a probability measure on Ω which satisfies the folowing.

1.2.3 Definition. Let $\mathcal{F}$ be a σ-field on Ω. A probability measure is an application $P : \mathcal{F} \longrightarrow [0,1]$ satisfying

(i.) $P(\Omega) = 1 - P(\emptyset) = 1$;

(ii.) If $(A_i)_{i\in I}$ is a countable collection of elements of $\mathcal{F}$ pairwise disjoints, then

$$P\left(\bigcup_{i\in I} A_i\right) = \sum_{i\in I} P(A_i).$$

The triple $(\Omega, \mathcal{F}, P)$ is called a probability space.

Given a probability space $(\Omega, \mathcal{F}, P)$, the event $A \subset \Omega$ is said to be P-null or negligible if $P(A) = 0$. A property is said to be true almost surely if the set on which this property is not true is negligible.

To use a probability space as the basis of our financial market model, we introduce the concept of a complete probability space.

1.2.4 Definition. A probability space $(\Omega, \mathcal{F}, P)$ is said to be a complete probability space if for all $B \in \mathcal{F}$ with $P(B) = 0$ and all $A \subset B$ one has $A \in \mathcal{F}$.

An important concept to use on a probability space for a financial market is defined below.

1.2.5 Definition. Let $(\Omega, \mathcal{F}, P)$ and $(\Omega', \mathcal{F}', P')$ be two probability spaces. A function $X : \Omega \longrightarrow \Omega'$ is said to be $(\mathcal{F}, \mathcal{F}')$-measurable if and only if

$$X^{-1}(A) := \{\omega \in \Omega : X(\omega) \in A\} \subset \mathcal{F}, \qquad \forall\, A \in \mathcal{F}'.$$

A random variable X is a measurable function $X : (\Omega, \mathcal{F}) \longrightarrow (\Omega', \mathcal{F}')$.

In what follows, unless otherwise stated, $(\Omega, \mathcal{F}, P)$ denotes a probability space and X a random variable,

$$X : \Omega \longrightarrow \mathbb{R}^n,\ n > 1.$$

1.2.6 Remark. Every random variable induces a probability measure on $\mathbb{R}^n$ denoted μ_X and defined by

$$\mu_X(B) := P(X^{-1}(B)),\ \forall\, B \text{ open sets of } \mathbb{R}^n.$$

μ_X is called the distribution function of X.

The following proposition gives a useful characterization of the independence of two random variables in terms of their expected values. The proof can be found in Oksendal (2013, p. 9).

1.2.7 Proposition. Two random variables X_1 and X_2 are independents if and only if for any two measurable positive functions f_1 and f_2, the following equality holds

$$\mathbb{E}(f_1(X_1)f_2(X_2)) = \mathbb{E}(f_1(X_1))\mathbb{E}(f_2(X_2)).$$

A useful definition for the construction of a financial market model is given below.

1.2.8 Definition. A stochastic process is a family of random variables $(X_t)_{t\geq 0}$. That is, for all $t > 0$, the application

$$\begin{array}{rccc} X_t & : \Omega & \longrightarrow & \mathbb{R} \\ & \omega & \longmapsto & X_t(\omega) \end{array} \text{ is measurable.}$$

If $(X_t)_{t\geq 0}$ is a stochastic process, then for all $t \geq 0$, the applicaion $t \longmapsto X_t(\omega)$ is called **sample path**.

1.2.9 Definition. A stochastic process X is called progressively measurable with respect to the filtration $\{\mathcal{F}_t\}$ if, for each $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R}^D)$, the set $\{(s,\omega); 0 \leq s \leq t, \omega \in \Omega, X_s(\omega) \in A\}$ belongs to the product σ-field $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$; in other words, the mapping

$$((s,\omega) \mapsto X_s(\omega) : ([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}_t) \to (\mathbb{R}^D, \mathcal{B}(\mathbb{R}^D)))$$

is measurable, for each $t \geq 0$. Karatzas and Shreve (1998, p. 4).

1.3 Mathematical finance preliminaries

Some of the vital finance concepts mentioned in this work are presented in this section.

1.3.1 Definition. A financial market consists of;

(i.) a probability space $(\Omega, \mathcal{F}, P)$;

(ii.) a positive constant T called the terminal time;

(iii.) a D-dimensional Brownian motion $\{W(t), \mathcal{F}(t); 0 \leq t \leq T\}$ defined on $(\Omega, \mathcal{F}, P)$, where $\{\mathcal{F}(t)\}_{0\leq t\leq T}$ is the augmentation (by the null sets in $\mathcal{F}^W(T)$) of the filtration $\{\mathcal{F}^W(t)\}_{0\leq t\leq T}$ generated by $W(.)$;

(iv.) a progressively measurable risk-free rate process $r(\cdot)$ satisfying $\int_0^T |r(t)|dt < \infty$ almost surely;

(v.) a progressively measurable, N-dimensional mean rate of return process $b(\cdot)$ satisfying

$$\int_0^T ||b(t)||dt < \infty \text{ almost surely;}$$

(vi.) a progressively measurable $(N \times D)$-matrix valued volatility process $\sigma(\cdot)$ satisfying

$$\sum_{n=1}^{N} \sum_{d=1}^{D} \int_0^T \sigma_{nd}^2(t)dt < \infty \text{ almost surely;}$$

(vii.) a vector of positive constant initial stock prices $S(0) = (S_1(0), \ldots, S_N(0))$;

We denote this financial market as $\mathcal{M} = (r(\cdot), b(\cdot), \sigma(\cdot), S(0))$.

The coefficients of the market model are taken to be measurable functions,

$$r : [0,T] \times \mathbb{R}^K \to \mathbb{R};\ b : [0,T] \times \mathbb{R}^K \to \mathbb{R}^N;\ \sigma : [0,T] \times \mathbb{R}^K \to L(\mathbb{R}^D, \mathbb{R}^K).$$

Generally, assets traded in the market are classified as riskless or risky, and their values are used together to construct a portfolio process for the market. We proceed to give a formal definition of the portfolio process.

1.3.2 Definition. A portfolio process $(\pi_0(\cdot), \pi(\cdot))$ for a financial market $\mathcal{M}$ consists of an $\{\mathcal{F}(t)\}$-progressively measurable, real-valued process $\pi_0(\cdot)$ and an $\{\mathcal{F}(t)\}$-progressively measurable $\mathbb{R}^N$-valued process $\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_N(\cdot))'$ such that

$$\int_0^T |\pi_0(t) + \pi'(t)\underset{\sim}{1}|[|r(t)|dt] < \infty, \tag{1.3.1}$$

$$\int_0^T |\pi'(t)(b(t) - r(t)\underset{\sim}{1})|dt < \infty, \tag{1.3.2}$$

$$\int_0^T ||\sigma'(t)\pi(t)||^2 dt < \infty \tag{1.3.3}$$

holds almost surely. Where $\underset{\sim}{1}$ denotes the N-dimensional vector with every component equal to one.

The level of risk an investor is willing to incur is dependent on the type of portfolio he wishes to invest. The riskier a portfolio is, the more the gains or losses expected to be made at a maturity time.

1.3.3 Definition. The gains process $G(\cdot)$ associated with $(\pi_0(\cdot), \pi(\cdot))$ is

$$G(t) := \int_0^t [\pi_0(s) + \pi'(s)\underset{\sim}{1}](r(s)ds) + \int_0^t \pi'(s)[b(s) - r(s)\underset{\sim}{1}]ds + \int_0^t \pi'(s)\sigma(s)dW(s),$$

for $0 \le t \le T$.

The portfolio is said to be **self-financed** if

$$G(t) = \pi_0(t) + \pi'(t)\underset{\sim}{1} \quad \forall\, t \in [0, T].$$

An excess yield can also be use to characterize the gains process. This will be seen in subsequent chapters.

1.3.4 Definition. The N-dimensional vector of excess yield (over the interest rate) process is defined by

$$R(t) := \int_0^t [b(u) - r(u)\underset{\sim}{1}]du + \int_0^t \sigma(u)dW(u), \quad 0 \le t \le T.$$

For an investor with various sources of income and expenses, we define a cumulative income process.

1.3.5 Definition. Let $\mathcal{M}$ be a financial market, a cumulative income process $\Gamma(t),\ 0 \le t \le T$ is a semimartingale. i.e. the sum of a finite variation process RCLL (right-continuous with left hand limits) and a local martingale.

We define the wealth process associated with the cumulative income and portfolio process.

1.3.6 Definition. The wealth process associated with $(\Gamma(\cdot), (\pi_0(\cdot), \pi(\cdot)))$ is

$$X(t) := \Gamma(t) + G(t).$$

$G(\cdot)$ is the gains process.

The portfolio is said to be Γ-**financed** if

$$X(t) = \Gamma(t) + \pi_0(t) + \pi'(t)\underset{\sim}{1} \quad \forall t \in [0, T].$$

1.3.7 Proposition. Since Γ is self-financed, then the discounted wealth process is given by

$$\frac{X(t)}{S_0(t)} = \Gamma(0) + \int_{(0,t]} \frac{d\Gamma(u)}{S_0(u)} + \int_0^t \frac{1}{S_0(u)}\pi'(u)dR(u),\ 0 \le t \le T$$

$R(\cdot)$ is excess yield process. It is worthy to mention that $\pi_0(\cdot)$ and $S_0(\cdot)$ represent the money market in a portfolio and a stock process respectively.

In order to simplify the characterization for arbitrage opportunities in the market, a portfolio process should satisfy a further condition of being tame.

1.3.8 Definition. An $\mathcal{F}_t$-adapted process, $\mathbb{R}^N$-valued process $\pi(\cdot)$ satisfying (1.3.1), (1.3.2) and (1.3.3) is said to be tame if the discounted gains semimartingale

$$\frac{G(t)}{S_0(t)} = M_0^{\pi}(t) := \int_0^t \frac{1}{S_0(u)}\pi'(u)dR(u),\ 0 \le t \le T$$

is almost surely bounded from below by a real constant that does not depend on t (but possibly depends on $\pi(\cdot)$).

If $(\pi_0(\cdot), \pi(\cdot))$ is a portfolio process and $\pi(\cdot)$ is tame, then the portfolio process $(\pi_0(\cdot), \pi(\cdot))$ is tame.

1.3.9 Definition. In a financial market $\mathcal{M}$, a given tame, self-financed portfolio process $\pi(\cdot)$ is an Arbitrage Opportunity if the associated gains process $G(\cdot)$ of

$$G(t) = S_0(t)\int_0^t \frac{1}{S_0(u)}\pi'(u)dR(u);\ 0 \le t \le T$$

satisfies $G(T) \ge 0$ almost surely and $G(T) > 0$ with positive probability. A financial market in which no such arbitrage opportunity exists is said to be **viable**.

1.3.10 Definition. Let $\mathcal{M}$ be a standard financial market, and let B be an $\mathcal{F}(T)$-measurable random variable such that $\frac{B}{S_0(T)}$ is almost surely bounded from below and

$$x := E_0\left[\frac{B}{S_0(T)}\right] < \infty \tag{1.3.4}$$

(i.) We say that B is financeable, if there is a tame, x-financed portfolio process $(\pi_0(\cdot), \pi(\cdot))$ whose associated wealth process satisfies $X(T) = B$ i.e.,

$$\frac{B}{S_0(T)} = x + \int_0^T \frac{1}{S_0(u)}\pi'(u)\sigma(u)dW_0(u) \tag{1.3.5}$$

alomost surely

(ii.) We say that the financial market $\mathcal{M}$ is complete if every $\mathcal{F}(T)$-measurable random variable B with $\frac{B}{S_0(T)}$ bounded from below and satisfying (1.3.4), is financeable. Otherwise we say that the market is incomplete.

1.3.11 Definition. An $\mathcal{F}(t)$-adapted, $\mathbb{R}^N$-valued process $\pi(\cdot)$ satisfying (1.3.2) and (1.3.3) is said to be martingale-generating if under the probability measure P_0 of (2.2.6), the local martingale M_0^{π} of (2.2.7) is a martingale. If $(\pi_0(\cdot), \pi(\cdot))$ is a portfolio process and $\pi(\cdot)$ is martingale-generating, we say that the portfolio process $(\pi_0(\cdot), \pi(\cdot))$ is martingale-generating.

This concludes the preliminary aspect. In the next chapter, we construct an arbitrage theory of a complete market on a finite time horizon.

2. Arbitrage theory in finite horizon

2.1 Introduction

The mathematical implications of the assumption of market viability, a standard, and a complete market are used in this chapter to characterize the no-arbitrage condition in a complete market with a finite time horizon. To make this work self-contained, we present additional results relevant to the treatment of the no-arbitrage theorem.

From (1.2.1), in a financial market model, we can interpret the σ-algebra $\mathcal{F}_t$ as information available to investors in the market at time t. This information is referred to as filtration.

2.1.1 Definition. Fries (2007). Let $(\Omega, \mathcal{F})$ denote a measurable space. For a fixed finite time horizon T, a family of sub-σ-algebras $\{\mathcal{F}_t | 0 \leq t \leq T\}$ where

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}_T \quad \text{for } 0 \leq s \leq t \leq T,$$

is called a filtration on $(\Omega, \mathcal{F})$.

Let $(\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{F}_T, P)$. A stochastic process (X_t) is said to be $\mathcal{F}_t$-adapted if $\forall\, 0 \leq t \leq T,\ X_t$ is $\mathcal{F}_t$-measurable (Fries, 2007, p. 20).

We provide a formal specification and essential property of our choice of an asset model, which will be used in the subsequent application.

2.1.2 Definition. Øksendal (2003). Let $(\mathcal{F}_t)_{0 \leq t \leq T}$ be a filtration on $(\Omega, \mathcal{F}, P_T)$. A stochastic process $(M_t)_{0 \leq t \leq T}$ is called $\mathcal{F}_t$-martingale if the following properties hold

(i.) (M_t) is $\mathcal{F}_t$-adapted;

(ii.) $\mathbb{E}(|M_t|) < \infty,\ \forall\, 0 \leq t \leq T$;

(iii.) $\mathbb{E}(M_t | \mathcal{F}_s) = M_s,\ \forall\, 0 \leq s \leq t \leq T$.

The remark below is useful for the classification of a martingale as a submartingale or a supermartingale.

2.1.3 Remark. From the definition above, we give the following remark;

(i.) If the condition (iii) of definition (2.1.2) is replaced by $\mathbb{E}(M_t | \mathcal{F}_s) \geq M_s,\ \forall\, 0 \leq s \leq t \leq T$, then (M_t) is called submartingale.

(ii.) If the condition (iii) of definition (2.1.2) is replaced by $\mathbb{E}(M_t | \mathcal{F}_s) \leq M_s,\ \forall\, 0 \leq s \leq t \leq T$, then (M_t) is called supermartingale.

A necessary stochastic process useful for the construction and analysis of Martingales is the Brownian motion.

2.1.4 Definition. Fries (2007, p. 22). Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t)_{0 \leq t \leq T}$ a filtration on this space. An $\mathcal{F}_t$-adapted stochastic process $(W_t)_{t \geq 0}$ is called Wiener process or standard Brownian motion if:

(i.) $W_0 = 0$;

(ii.) $t \longmapsto W_t$ is almost surely continuous (sample paths of a Brownian motion);

(iii.) the increments $W(t_1) - W(t_0), \ldots, W(t_k) - W(t_{k-1})$ are mutually independent given that $t_0 < t_1 < \ldots < t_k$;

(iv.) $W_t - W_s \sim N(0, t-s)$, for $0 \leq s \leq t$.

The figure below shows some simulated sample paths of standard Brownian motion.

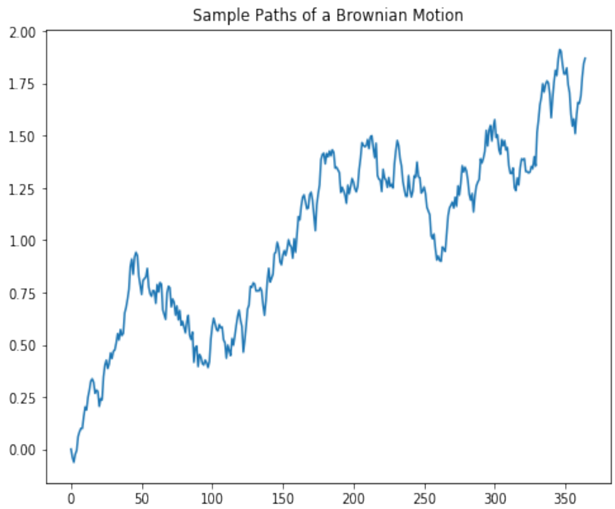

Figure 2.1: Sample Paths of standard Brownian motion.

We mention some stochastic processes categorized as martingales by the presence of the Brownian motion in the following proposition.

2.1.5 Proposition. Pitman and Yor (2018, p. 22). If (W_t) is an $\mathcal{F}_t$ standard Brownian motion, then the following processes are $\mathcal{F}_t$-martingales;

(i.) W_t;

(ii.) $W_t^2 - t$;

(iii.) $\exp\left(\theta W_t - \theta^2 \frac{t}{2}\right)$, for each θ real or complex.

The localized form of a martingale is defined below.

2.1.6 Definition. Øksendal (2003). Let $(\Omega, \mathcal{F}, P)$ be a probability space; let $\mathcal{F}_t = \{\mathcal{F}_t | t \geq 0\}$ be a filtration of $\mathcal{F}$; let $X : [0, +\infty) \times \Omega \longrightarrow \mathcal{S}$ be an $\mathcal{F}_t$-adapted stochastic process on set $\mathcal{S}$. Then X is called an $\mathcal{F}_t$-local martingale if there exists $\mathcal{F}_t$-stopping times $\tau : \Omega \longrightarrow [0, +\infty)$ such that the stopped process

$$X_t^\tau := X_{\min}\{t, \tau\} \text{ is an } \mathcal{F}_t\text{-martingale.}$$

Some aspects of stochastic integral applied in this work as seen in Oksendal (2013) are introduced here.

2.1.7 Definition ($\mathbb{L}^2([0,T],\mathbb{R})$)**.** Let f denote a $(\mathcal{B}([0,T]),\mathcal{B}(\mathbb{R}))$-measurable and real-valued, non-negative map.

$$f \in \mathbb{L}^2([0,T],\mathbb{R}) \Longleftrightarrow \int_0^T |f|^2 d\mu \ < \infty.$$

that f is square integrable.

2.1.8 Definition. Let $\mathbb{M}^p([0,T],\mathbb{R})$ be the susbspace of $\mathbb{L}^p([0,T],\mathbb{R})$ such that for any process $(X_t) \in \mathbb{M}^p([0,T],\mathbb{R})$ we have

$$\mathbb{E}\left(\int_0^T |X(t)|^p dt \right) < \infty.$$

We consider a Brownian motion W_t and a stochastic process (X_t) both adapted to a given filtration $(\mathcal{F}_t)$. A stochastic integral can be expressed as follows

$$I_t(X) = \int_0^t X(s)dW(s).$$

2.1.9 Definition. The Itô's integral of the simple process $(X_t)_{t\in\mathbb{R}} \in \mathbb{L}^2([0,T],\mathbb{R})$ is defined by

$$I_t(X) = \int_0^t X(s)dW(s) := \sum_{j=0}^{n-1} \theta_j (W_{t_{j+1}} - W_{t_j}).$$

2.1.10 Lemma. If f is an elementary function in $\mathbb{L}^2([a,b],\mathbb{R})$ and W_t a Brownian motion, then :

(i.)
$$\mathbb{E}\left(\int_a^b f(t)dW_t \right) = 0;$$

(ii.)
$$\mathbb{E}\left(\int_a^b f(t)dW_t \right)^2 = \int_a^b \mathbb{E}(f^2(t))dt.$$

"An important property of the Itô integral is that it is a martingale." (Øksendal, 2003).

Before proceeding to the main results of this chapter, we present the application of some mathematical tools such as kernels, ranges, and orthogonal complement. The proof for a few results are presented herein, but for full proof of all the results, the reader is directed to Karatzas et al. (1998, p. 13-14). We begin with some useful lemmas.

2.1.11 Lemma. Karatzas et al. (1998, p. 13). The mappings $(x,\sigma) \longmapsto \text{proj}_{\mathcal{K}(\sigma)}(x)$ and $(x,\sigma) \longmapsto \text{proj}_{\mathcal{K}^\perp(\sigma)}(x)$ from $\mathbb{R}^D$ to $L(\mathbb{R}^D;\mathbb{R}^N)$, and the mappings $(y,\sigma) \longmapsto \text{proj}_{\mathcal{K}(\sigma')}(y)$ and $(y,\sigma) \longmapsto \text{proj}_{\mathcal{K}^\perp(\sigma')}(y)$ from $\mathbb{R}^N$ to $L(\mathbb{R}^D;\mathbb{R}^N)$ are Borel measurable.

Proof. Following the proof of Karatzas et al. (1998, p. 13), we work on the first of the four mappings above. Let $L(\mathbb{R}^D;\mathbb{R}^N)$ be the space of $N \times D$ matrices. Also, let $\mathbb{Q}^N$ the vectors set of rational numbers be dense in $\mathbb{R}^N$. $\sigma' = \sigma^\intercal$, the transpose of σ.

We define the Borel-measurable function $F : \mathbb{R}^D \times L(\mathbb{R}^D; \mathbb{R}^N) \to \mathbb{R}$ by

$$F(z,\sigma) := \inf_{q\in\mathbb{Q}^N} ||z - \sigma' q||, \quad \forall z \in \mathbb{R}^D, \quad \sigma \in L(\mathbb{R}^D; \mathbb{R}^N).$$

If (z, σ) such that $z \in \mathcal{R}(\sigma')$, it implies that there exists $y \in \mathbb{R}^N$ such that $z = \sigma' y$.

So if $(y_n)_n \subseteq \mathbb{Q}^N$, then y_n tends to y, hence, $z = \sigma'(\lim_n y_n) = \lim_n \sigma'(y_n)$ since a linear map is continuous for a finite dimensional space, then for $q \in \mathbb{Q}^N$,

$$||z - \sigma' q|| = ||\lim \sigma' y_n - \sigma' q|| = \lim_{n\to\infty} ||\sigma' y_n - \sigma' q||$$

and since $\inf_{q\in\mathbb{Q}^N} ||\sigma' y_n - \sigma' q|| = 0$, y_n and q are both dense in $\mathbb{Q}^N$, we will have

$$\inf_{q\in\mathbb{Q}^N} \Big(\lim_{n\to\infty} ||\sigma'(y_n) - \sigma' q||\Big) = \lim_{n\to\infty} \Big(\inf_{q\in\mathbb{Q}^N} ||\sigma'(y_n) - \sigma' q||\Big) = 0,$$

we have $F(z, \sigma) = 0$.

Now if $F(z, \sigma) = 0$, we use the characterization of the $\inf$; so that there exists a sequence $\{q_n\} \subseteq \mathbb{Q}^N$ such that $\lim_{n\to\infty} ||z - \sigma' q_n|| = 0$.

From the decomposition,

$$\begin{aligned}
\mathbb{R}^N &= \mathcal{K}(\sigma') \oplus (\mathcal{K}(\sigma'))^\perp, \\
q_n &= p_n + r_n, \\
p_n &\in \mathcal{K}(\sigma'), \quad r_n \in [\mathcal{K}(\sigma')]^\perp.
\end{aligned}$$

Since $\mathcal{K}(\sigma') \cap [\mathcal{K}(\sigma')]^\perp = \{0\}$, then the linear map σ' is an injection on $[\mathcal{K}(\sigma')]^\perp$ and therefore bijective invertible.

$$\begin{aligned}
& q_n = p_n + r_n \to \sigma'(q_n) = \sigma'(p_n) + \sigma'(r_n) \\
\text{and since} \quad & p_n \in \mathcal{K}(\sigma'), \quad \text{then} \quad \sigma'(p_n) = 0 \\
\text{and} \quad & \sigma'(q_n) = \sigma'(r_n).
\end{aligned}$$

Then, $\lim_{n\to\infty} ||z - \sigma' q_n|| = 0$, but $\lim_{n\to\infty} ||z - \sigma'(q_n)|| = \lim_{n\to\infty} ||z - \sigma'(r_n)|| = 0$,

so $\sigma'(r_n) \to z$ and $r_n = (\sigma')^{-1} \circ (\sigma')(r_n) \to (\sigma')^{-1}(z)$.

Recall that the orthogonal complement is always closed in the metric topology, so there exists some $r \in \mathcal{K}^\perp(\sigma')$ such that $\sigma'(r) = z$.

We just proved that $\{(z, \sigma); z \in \mathcal{R}(\sigma')\} = \{(z, \sigma); F(z, \sigma) = 0\}$.

Hence, $\{(z, \sigma); z \in \mathcal{R}(\sigma')\} = F^{-1}(\{0\})$ is a Borel set.

Therefore,

$$\begin{aligned}
\{(x, \sigma, \xi) \in \mathbb{R}^D \times L(\mathbb{R}^{\mathbb{D}}; \mathbb{R}^{\mathbb{N}}) \times \mathbb{R}^D, \xi = \mathsf{proj}_{\mathcal{K}(\sigma)}(x)\} &= \{(x, \sigma, \xi), \quad \xi \in \mathcal{K}(\sigma), \quad (x - \xi) \perp \mathcal{K}(\sigma)\} \\
&= \{(x, \sigma, \xi), \quad \sigma\xi = 0, \quad x - \xi \in \mathcal{R}(\sigma')\}
\end{aligned}$$

is a Borel set.

Define $H : \mathbb{R}^D \times L(\mathbb{R}^{\mathbb{D}}; \mathbb{R}^{\mathbb{N}}) \to \mathbb{R}^D$ by

$$H(x, \sigma) := \text{proj}_{\mathcal{K}(\sigma)}(x), \quad \forall\, x \in \mathbb{R}^D, \quad \sigma \in L(\mathbb{R}^{\mathbb{D}}; \mathbb{R}^{\mathbb{N}}).$$

$\mathsf{G}(H) := \{(x, \sigma, \xi); \quad (x, \sigma) \in \mathbb{R}^D \times L(\mathbb{R}^{\mathbb{D}}; \mathbb{R}^{\mathbb{N}}), \quad \xi = Q(x, \sigma)\}$ is a graph of H.

And according to Parthasarathy (2005, p. 21); Let X_1, X_2 be complete separable metric spaces and $E_1 \subseteq X_1, E_2 \subseteq X_2$ two sets, E_1 being a Borel set. Let φ be a measurable one-one map of E_1 into X_2 such that $\varphi(E_1) = E_2$. Then E_2 is a Borel set.

Hence, H must be a Borel-measurable function. □

2.1.12 Corollary. Karatzas et al. (1998, p. 13). The process $\text{proj}_{\mathcal{K}(\sigma'(t))}[b(t) - r(t)\underset{\sim}{1}]$, $0 \le t \le T$ is progressively measurbale.

Proof. The σ-algebra on $L(\mathbb{R}^{\mathbb{D}}; \mathbb{R}^{\mathbb{N}})$ is $\mathcal{F}_t$. From the lemma above, $\text{proj}_{\mathcal{K}(\sigma'(t))}[b(t) - r(t)\underset{\sim}{1}]$ is Borel measurable, then it is said to be progressively measurable if for every time t, the map

$$[0, T] \times L(\mathbb{R}^{\mathbb{D}}; \mathbb{R}^{\mathbb{N}}) \to \mathbb{R}^N$$

defined by

$$(t, \sigma') \mapsto \text{proj}_{\mathcal{K}(\sigma'(t))}[b(t) - r(t)\underset{\sim}{1}] \text{ is } \mathcal{B}[0, T] \otimes \mathcal{F}_t - \text{measurable.}$$

□

2.1.13 Lemma. Karatzas et al. (1998, p. 14). If the financial market $\mathcal{M}$ is viable, then $b(t) - r(t)\underset{\sim}{1} \in \mathcal{R}(\sigma(t))$ for Lebesgue-almost-every $t \in [0, T]$ almost surely. Where $\mathcal{R}(\sigma(t))$ is the range space of $\sigma(t)$.

Proof. Karatzas et al. (1998, p. 14). □

2.1.14 Lemma. Karatzas et al. (1998, p. 14). Consider the mapping $\phi_1 : \{(y, \sigma) \in \mathbb{R}^N \times L(\mathbb{R}^D; \mathbb{R}^N); y \in \mathcal{R}(\sigma)\} \to \mathbb{R}^D$ defined by the prescription that $\phi(y, \sigma)$ is the unique $\xi \in \mathcal{K}^\perp(\sigma)$ such that $\sigma\xi = y$. Consider also the mapping $\phi_2 : \{(x, \sigma) \in \mathbb{R}^D \times L(\mathbb{R}^D; \mathbb{R}^N); x \in \mathcal{R}(\sigma)\} \to \mathbb{R}^N$ defined by the prescription that $\phi_2(x, \sigma)$ is the unique $\eta \in \mathcal{K}^\perp(\sigma')$ such that $\sigma'\eta = x$. Both ϕ_1 and ϕ_2 are Borel measurable. Where $\mathcal{K}^\perp(\sigma)$ is the orthogonal complement of the kernel of σ.

Proof. Karatzas et al. (1998, p. 14). □

2.2 Main result

2.2.1 Theorem. *Karatzas et al. (1998, p. 12). If a financial market $\mathcal{M}$ is viable, then there exists a progressively measurable process $\theta(\cdot)$ with values in $\mathbb{R}^D$, called the market price of risk, such that for Lebesgue-almost-every $t \in [0,T]$ the risk premium $b(t) - r(t)\underset{\sim}{1}$ is related to $\theta(t)$ by the equation*

$$b(t) - r(t)\underset{\sim}{1} = \sigma(t)\theta(t) \quad a.s \tag{2.2.1}$$

Conversely, suppose that there exists a process $\theta(\cdot)$ that satisfies the above requirements, as well as

$$\int_0^T \theta(s)^2 ds < \infty \tag{2.2.2}$$

$$E\left[\exp\left\{-\int_0^T \theta'(s)dW(s) - \frac{1}{2}\int_0^T \theta(s)^2 ds\right\}\right] = 1. \tag{2.2.3}$$

Then the market $\mathcal{M}$ is viable.

Proof. Proving the first part of the theorem above, viability implies the existence of the market price of risk $\theta(\cdot)$. (2.1.13) shows that $b(t) - r(t)\underset{\sim}{1}$ belongs to the range space of $\sigma(t)$ and (2.1.14) shows that $\phi_1(y,\sigma)$ can be uniquely expressed such that $\sigma\xi = y$. Combining these two lemmas, the progressively measurable process

$$\theta(t) := \phi_1(b(t) - r(t)\underset{\sim}{1}, \sigma(t)) \tag{2.2.4}$$

is defined and it satisfies (2.2.1) for Lebesgue-almost-every $t \in [0,T]$. Thus (2.2.4) can also be uniquely expressed as

$$\sigma(t)\theta(t) = b(t) - r(t)\underset{\sim}{1}.$$

Conversely, let the $\mathbb{R}^N$-valued process $\theta(\cdot)$ be progressively measurable and satisfy conditions (2.2.1), (2.2.2) and (2.2.3). We need to show that if the discounted gains process for any tame portfolio is a supermartingale, then the market is viable.

The associated discounted gains process for any tame portfolio $\pi(\cdot)$ can be expressed as;

$$M^\pi(t) := \frac{G(t)}{S_0(t)} = M(t) = \int_0^t \frac{1}{S_0(u)}\pi'(u)dR(u) = \int_0^t \frac{1}{S_0(u)}\pi'(u)\sigma(u)dW_0(u), \quad 0 \le t \le T$$

where $W_0(t)$ is a Brownian motion under a new probability measure P_0 because according to the Girsanov theorem (Karatzas and Shreve, 1991, p. 191), if W_t is a Brownian motion on a sample space and $\mathcal{F}_t$ is the filtration on the Brownian motion, also if the market price of risk $\theta(s)$ is an adapted process, then we can define

$$Z_0(t) := \exp\left\{-\int_0^T \theta'(s)dW(s) - \frac{1}{2}\int_0^T ||\theta(s)||^2 ds\right\}$$

so that $E[Z_0(t)] = 1$ by the assumption in (2.2.3). Then,

$$W_0(t) := W(t) + \int_0^t \theta(s)ds, \quad 0 \le t \le T \tag{2.2.5}$$

is a Brownian motion under the probability measure P_0 given by;

$$P_0(A) := E\left[1_A \cdot \exp\left\{-\int_0^T \theta'(s)dW(s) - \frac{1}{2}\int_0^T \theta(s)^2 ds\right\}\right], \quad A \in \mathcal{F}(T).$$

Thus, we have been able to find a martingale measure P_0 that is equivalent to the original probability measure P. Hence, $M^{\pi}(\cdot)$ is a local martingale under the probability measure P_0 and $\pi(\cdot)$ is a tame portfolio which is bounded from below. Therefore, it is a supermartingale;

$$E_0(M(T)) \leq E_0 M^{\pi}(0) = 0.$$

We can say that the market is viable; hence, there is no arbitrage opportunity since ($[G(T) \geq 0]$ almost surely P_0 and $[G(T) > 0]$ with a positive probability) can not occur at the same time. The certainty that one will not make a loss in a market and also having the opportunity to make a gain simultaneously without taking any risk should be impossible. □

To obtain a more accessible formlation of the no-arbitrage theorem in a finite time horizon, we introduce additional market structures below.

2.2.2 Definition. Karatzas et al. (1998, p. 17). A financial market model $\mathcal{M}$ is said to be standard if

(i.) it is viable;

(ii.) the number of N stocks is not greater than the dimension D of the underlying Brownian motion;

(iii.) the D-dimensional, progressively measurable market price of risk proces $\theta(\cdot)$ satisfies

$$\int_0^T ||\theta(t)||^2 dt < \infty \text{ almost surely; and}$$

(iv.) the positive local martingale

$$Z_0(t) := \exp\left\{ -\int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds \right\}, \quad 0 \leq t \leq T,$$

is in fact a martingale.

For a standard market, we define the standard martingale measure P_0 on $\mathcal{F}(T)$ by

$$P_0(A) := E[1_A \cdot Z_0(T)], \quad \forall A \in \mathcal{F}(T) \tag{2.2.6}$$

We say that P_0 and P are **equivalent** on $\mathcal{F}(T)$ if a set in $\mathcal{F}(T)$ has P_0-measure zero if and only if it has P-measure zero.

2.2.3 Remark. Karatzas et al. (1998, p. 17) Since we have been able to establish (2.2.5) to be a Brownian motion under P_0 relative to the augmented filtration $\{\mathcal{F}(t)\}$ mentioned in (1.3.1), the discounted gains and discounted wealth processes respectively become;

$$\frac{G(t)}{S_0(t)} = M_0^{\pi}(t) := \int_0^t \frac{1}{S_0(u)}\pi'(u)\sigma(u)dW_0(u) \tag{2.2.7}$$

$$\frac{X(t)}{S_0(t)} = \Gamma(0) + \int_{(0,t]} \frac{d\Gamma(u)}{S_0(u)} + \int_0^t \frac{1}{S_0(u)}\pi'(u)\sigma(u)dW_0(u), \quad 0 \leq t \leq T. \tag{2.2.8}$$

2.2.4 Theorem. *Karatzas et al. (1998, p. 19). Under the standard martingale measure P_0, the process of discounted cumulative wealth minus discounted income*

$$\frac{X(t)}{S_0(t)} - \Gamma(0) - \int_{(0,t]} \frac{d\Gamma(u)}{S_0(u)}, \quad 0 \leq t \leq T \tag{2.2.9}$$

corresponding to any tame $\Gamma(\cdot)$-financed portfolio is a local martingale and bounded from below, hence a supermartingale. In particular,

$$E_0\left[\frac{X(T)}{S_0(T)} - \int_{(0,T]} \frac{d\Gamma(u)}{S_0(u)}\right] \leq \Gamma(0). \tag{2.2.10}$$

The process in (2.2.9) is a martingale under P_0 if and only if equality holds in (2.2.10).

Proof. From (2.2.7) and (2.2.8) we see that (2.2.9) has the stochastic representation;

$$M_0^\pi(t) = \int_0^t \frac{1}{S_0(u)}\pi'(u)\sigma(u)dW_0(u), \quad 0 \leq t \leq T.$$

Thus, $M_0^\pi(t)$ is a local martingale that is bounded from below because $\pi(\cdot)$ is a tame portfolio, and according to Fatou's lemma, a local martingale which is bounded from below is a supermartingale.

Re-arranging (2.2.8) and taking the expection with respect to the measure P_0;

$$E_0\left[\frac{X(t)}{S_0(t)} - \int_{(0,t]} \frac{d\Gamma(u)}{S_0(u)}\right] = E_0\left[\Gamma(0) + \int_0^t \frac{1}{S_0(u)}\pi'(u)\sigma(u)dW_0(u)\right], \quad 0 \leq t \leq T.$$

$$E_0\left[\frac{X(t)}{S_0(t)} - \int_{(0,t]} \frac{d\Gamma(u)}{S_0(u)}\right] = \Gamma(0)$$

Hence, we have shown that equality holds for (2.2.10). Therefore, a supermartingale is a martingale if and only if it has constant expectation.

As mentioned in Karatzas et al. (1998, p. 15): "Within the framework of a viable market, when characterizing the wealth processes that can be achieved through investment, one can assume without loss of generality that the number of N stocks is not greater than the dimension D of the underlying Brownian motion". This will be discussed in the next section.

A complete financial market in a finite horizon can be used to characterize arbitrage. It allows for the construction of a martingale measure, which has a fixed maturity that helps to show the no-arbitrage property.

2.2.5 Proposition. Karatzas et al. (1998, p. 22) A standard financial market $\mathcal{M}$ is complete if and only if for every $\mathcal{F}(T)$-measurable random variable B satisfying

$$E_0\left[\frac{|B|}{S_0(T)}\right] < \infty \tag{2.2.11}$$

and with x defined by (1.3.4), there is a martingale-generating, x-financed portfolio process $(\pi_0(\cdot), \pi(\cdot))$ satisfying (1.3.5).

Proof. Let $\mathcal{M}$ be a complete market and B an $\mathcal{F}(T)$-measurable random variable that satisfies (2.2.11). Since B is $\mathcal{F}(T)$-measurable, then $B^+ = \max\{B, 0\}$ and $B^- = \max\{-B, 0\}$ are $\mathcal{F}(T)$-measurable and the condition (2.2.11) implies that

$$E_0\left[\frac{B^+}{S_0(T)}\right] < \infty \quad \text{and} \quad E_0\left[\frac{B^-}{S_0(T)}\right] < \infty.$$

By completeness of the market $\mathcal{M}$, there exist an x_+-financed portfolio process $(\pi_0^+(\cdot), \pi^+(\cdot))$ and an x_--financed portfolio process $(\pi_0^-(\cdot), \pi^-(\cdot))$ respectively with

$$\frac{B^+}{S_0(T)} = x_+ + \int_0^T \frac{1}{S_0(u)}(\pi^+(u))'\sigma(u)dW_0(u) \tag{2.2.12}$$

$$\frac{B^-}{S_0(T)} = x_- + \int_0^T \frac{1}{S_0(u)}(\pi^-(u))'\sigma(u)dW_0(u). \tag{2.2.13}$$

We shall combine both cases for simplicity using the convention $B^\pm := \max\{\pm B, 0\}$ and $x_\pm$-financed portfolio processes $(\pi_0^\pm(\cdot), \pi^\pm(\cdot))$ with

$$\frac{B^\pm}{S_0(T)} = x_\pm + \int_0^T \frac{1}{S_0(u)}(\pi^\pm(u))'\sigma(u)dW_0(u) \tag{2.2.14}$$

almost surely. Recall that $x_\pm := E_0[\frac{B^\pm}{S_0(T)}]$. Taking expectations in (2.2.14) with respect to the measure P_0, we obtain

$$\begin{aligned}
E_0\left[\frac{B^\pm}{S_0(T)}\right] &= E_0[x_\pm] + E_0\left[\int_0^T \frac{1}{S_0(u)}(\pi^\pm(u))'\sigma(u)dW_0(u)\right] \\
&= E_0\left[E_0\left(\frac{B^\pm}{S_0(T)}\right)\right] + E_0\left[\int_0^T \frac{1}{S_0(u)}(\pi^\pm(u))'\sigma(u)dW_0(u)\right] \\
&= E_0\left[\frac{B^\pm}{S_0(T)}\right] + E_0\left[\int_0^T \frac{1}{S_0(u)}(\pi^\pm(u))'\sigma(u)dW_0(u)\right]
\end{aligned}$$

i.e.

$$E_0\left[\int_0^T \frac{1}{S_0(u)}(\pi^\pm(u))'\sigma(u)dW_0(u)\right] = 0.$$

We see that the lower-bounded local martingale $\int_0^t \frac{1}{S_0(u)}(\pi^\pm(u))'\sigma(u)dW_0(u)$ has constant expectation to be equal to zero under P_0, therefore it is a supermartingale. Hence, $\pi^\pm$ is martingale-generating. Subtracting (2.2.12) from (2.2.13) we get $\frac{B}{S_0(T)} = x + \int_0^T \frac{1}{S_0(u)}\pi'(u)\sigma(u)dW_0(u)$ which is (1.3.5), $\pi(\cdot) := \pi^+(\cdot) - \pi^-(\cdot)$ is also martingale-generating. Suppose that for any $\mathcal{F}(T)$-measurable random variable B, such that $\frac{B}{S_0(T)}$ is almost surely bounded from below and satisfies (1.3.4), there exists a martingale-generating x-financed portfolio process $(\pi_0(\cdot), \pi(\cdot))$ satisfying (1.3.5). Taking the conditional expectations in (1.3.5) we have that

$$\begin{aligned}
E_0\left[\frac{B}{S_0(T)}\Big|\mathcal{F}(T)\right] &= E_0[x|\mathcal{F}(T)] + E_0\left[\int_0^t \frac{1}{S_0(T)}\pi'(u)\sigma(u)dW_0(u)\Big|\mathcal{F}(T)\right] \\
E_0\left[\frac{B}{S_0(T)}\Big|\mathcal{F}(T)\right] &= x + \int_0^t \frac{1}{S_0(T)}\pi'(u)\sigma(u)dW_0(u)
\end{aligned}$$

Therefore,

$$\int_0^t \frac{1}{S_0(T)}\pi'(u)\sigma(u)dW_0(u) = -x + E_0\left[\frac{B}{S_0(T)}\Big|\mathcal{F}(t)\right], \quad 0 \le t \le T$$

is bounded from below. This follows that $(\pi_0(\cdot), \pi(\cdot))$ is tame, $\frac{B}{S_0(T)}$ is financeable, hence, $\mathcal{M}$ is complete.

For complete markets, the theory is more straightforward and better developed than the theory of incomplete markets. For a standard financial market, the complete and incomplete cases of the financial market are easily specified by the following theorem.

2.2.6 Theorem. *Karatzas et al. (1998, p. 24) A standard financial market $\mathcal{M}$ is complete if and only if the number of stocks N is equal to the dimension D of the underlying Brownian motion and the volatility matrix $\sigma(t)$ is nonsingular for Lebesgue-almost-every $t \in [0, T]$ almost surely.*

Proof. Karatzas et al. (1998, p. 24-27) □

The number of risky assets in the market should correspond to the dimension of Brownian motion associated with these risky assets to enable us to hedge the risk.

2.2.7 Proposition. A complete market on a finite time horizon is arbitrage free.

Proof. See the illustrated example in (4.1.1). □

We proceed to the next chapter on how arbitrage and completeness of a standard market tend to behave in the infinite time horizon.

3. On the arbitrage theory in infinite horizon

3.1 Problem formulation

In this chapter, we introduced the main idea behind the financial market in an infinite time horizon. From the definition of a financial market (1.3.1) in the preliminary chapter, the planning time horizon is finite. Assets in the market take values on $[0, T]$, which is a bounded interval. Aiming at considering financial instruments with an infinite maturity time such as perpetual American options, we need to develop a theory around financial markets on $[0, \infty]$.

We proceed to construct a probability space $(\Omega, \mathcal{F}, P)$ to fit into the infinite horizon case with consideration to the Wiener measure. The probability space will be of a canonical setup.

According to Karatzas et al. (1998, p. 27), we set $\Omega = C([0, \infty)^D$ to be the space of continuous $\mathbb{R}^D$-valued functions, i.e. $\omega = [0, \infty) \to \mathbb{R}^D$ where a coordinate mapping process on Ω is defined to be

$$W(t, \omega) = \omega(t), 0 \leq t < \infty.$$

To define the filtration on an infinite horizon, we have $\mathcal{F}^W(\infty) := \sigma(\bigcup_{0 \leq t < \infty} \mathcal{F}^W(t))$ to be the smallest sigma algebra generated by the Wiener process. If P is the probability measure for which $\{W(t);\ 0 \leq t < \infty\}$ is a D-dimensional Brownian motion, then we can define

$$\mathcal{F} := \sigma(\mathcal{F}^W(\infty) \cup \mathcal{N})$$

to be the completion of $\mathcal{F}^W(\infty)$. The collection of the null sets with respect to the probability measure P of $\mathcal{F}^W(\infty)$ is $\mathcal{N}$ i.e.

$$\mathcal{N} := \{\mathcal{N} \subseteq \Omega,\ \exists\, A \in \mathcal{F}^W(\infty) \text{ with } \mathcal{N} \subseteq A \text{ and } P(A) = 0\}.$$

Even though we are working towards the infinite time, an augmented filtration which is complete and right continuous is required in order to have information about the market. Thus, for each $T \in [0, \infty]$,

$$\mathcal{N}^T := \{\mathcal{N} \subseteq \Omega,\ \exists\, A \in \mathcal{F}^W(T) \text{ with } \mathcal{N} \subseteq A \text{ and } P(A) = 0\}$$

is defined as the collection of the null sets with respect to measure P, hence the augmented filtration becomes;

$$\mathcal{F}^{(T)}(t) := \sigma(\mathcal{F}^W(t) \cup \mathcal{N}^T),\ 0 \leq t \leq T. \tag{3.1.1}$$

To fit into the infinite horizon model, we define a stochastic process as follows;

3.1.1 Definition. Karatzas et al. (1998, p. 28). A stochastic process $Y = \{Y(t); 0 \leq t < \infty\}$ is said to be restrictedly progressively measurable or restrictedly adapted if for every $T \in [0, \infty)$, there exists $\tilde{T} \in [T, \infty)$ such that the restricted process $\{Y(t); 0 \leq t \leq T\}$ is $\{\mathcal{F}^{(\tilde{T})}(t); 0 \leq t \leq T\}$-progressively measurable or adapted respectively.

A financial market can thus be defined on an infinite time horizon.

3.1.2 Definition. Karatzas et al. (1998, p. 28). A financial market $\mathcal{M} = (r(\cdot), b(\cdot), \sigma(\cdot), S(0))$ on the infinite planning horizon $[0, \infty)$ consists of (i.) a D-dimensional Brownian motion $W = \{W(t); 0 \leq t < \infty\}$ that is the coordinate mapping process on the canonical probability space $(\Omega, \mathcal{F}, P)$;

(ii.) restrictedly progressively measurable processes $r(\cdot), b(\cdot)$, and $\sigma(\cdot)$ as described in the defintion of a financial market in the preliminary section satisfying the integrability conditions in the definition for every finite time T;

(iii.) a vector of positive, constant initial stock prices $S(0) = (S_1(0), ..., S_N(0))'$. The prime denotes transposition.

We are now set to define the concept of a standard and complete financial market on an infinite planning horizon.

3.1.3 Definition. Karatzas et al. (1998, p. 28). A financial market $\mathcal{M} = (r(\cdot), b(\cdot), \sigma(\cdot), S(0))$ on an infinite planning horizon is standard and complete if

(i.) the number of stocks N equals the dimension D of the driving Brownian motion;

(ii.) the volatility matrix $\sigma(t)$ is nonsingular for Lebesgue-almost-every $t \in [0, \infty)$ almost surely;

(iii.) the positive local martingale

$$Z_0(t) := \exp\left\{-\int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds\right\},\ 0 \leq t < \infty, \tag{3.1.2}$$

is in fact a $P-$martingale, where

$$\theta(t) := \sigma^{-1}(t)[b(t) - r(t)],\ 0 \leq t < \infty. \tag{3.1.3}$$

Comparing with (2.2.6), we define the martingale measure P_0^T on $\mathcal{F}^W(T)$ for each $T \in [0, \infty)$

$$P_0^T(A) := E[Z_0(T)1_A], \quad \forall A \in \mathcal{F}^W(T). \tag{3.1.4}$$

and with $\theta(\cdot)$ defined by (3.1.3) we set

$$W_0(t) := W(t) + \int_0^t \theta(s)ds,\ 0 \leq t < \infty. \tag{3.1.5}$$

Under P_0^T, the restricted process $\{W_0(t); 0 \leq t \leq T\}$ is a Brownian motion. In addition, for $0 \leq t \leq T$, the probability measure P_0^T is equivalent to P on $\mathcal{F}^{(T)}(t)$; i.e. a set in $\mathcal{F}^{(T)}(t)$ is a P_0^T-null set if and only if it is a P-null set (Karatzas et al., 1998, p. 28).

3.2 Main proposition

The main result of this section is the following proposition below.

3.2.1 Proposition. Karatzas et al. (1998, p. 29). There exists a unique probability measure P_0 on

$$\mathcal{F}^W(\infty) := \sigma(W(s);\ \ 0 \leq s < \infty)$$

such that P_0 agrees with each P_0^T on $\mathcal{F}^W(T)$, for any $T < \infty$. In particular, $\{W_0(t); 0 \leq t < \infty\}$ is a D-dimensional Brownian motion under P_0.

Proof. Karatzas et al. (1998, p. 29-30). □

This proposition presents the idea that to be able to work with a reasonable market model in an infinite horizon, we define a probability measure in such a way that we will be able to get information about our market for every set of finite horizons in the infinite horizon, i.e., a probability measure that will agree on $\mathcal{F}^W(T)$ for every $T < \infty$.

Also, market viability is achieved in this horizon, given that P and P_0 are equivalent in (3.1.1).

4. Applications in asset pricing

We mentioned earlier in the preliminary section of this essay under the definition of a standard market that a set in $\mathcal{F}(T)$ has P_0-measure zero if and only if it has P-measure zero, this means that P and P_0 are equivalent in the finite horizon. However, the case may not be true in an infinite horizon. We present these examples for proper illustration. The following from Karatzas et al. (1998) will be useful for the construction of this illustration.

The discounted stock process;

$$\frac{S_n(t)}{S_0(t)} = S_n(0) \exp\left\{ \int_0^t \sum_{d=1}^{D} \sigma_{nd}(s) dW^{(d)}(s) + \int_0^t \left[b_n(s) - r(s) - \frac{1}{2} \sum_{d=1}^{D} \sigma_{nd}^2(s) \right] ds \right\} \tag{4.0.1}$$

The Brownian motion;

$$W_0(t) := W(t) + \int_0^t \theta(s) ds \tag{4.0.2}$$

Market price of risk;

$$\theta(t)\sigma(t) = b(t) - r(t)\underset{\sim}{1} \tag{4.0.3}$$

4.1 Finite horizon case

4.1.1 Example. Karatzas et al. (1998, p. 30). Let $N = D = 1$, $r(\cdot) \equiv r > 0$, $b(\cdot) \equiv b > r + \frac{1}{2}$, $\sigma(\cdot) \equiv 1$. Then $W_0(t) = W(t) + (b - r)t$ and

$$\frac{S_1(t)}{S_0(t)} = S_1(0) \exp\left[W(t) + \left(b - r - \frac{1}{2} \right) t \right] \tag{4.1.1}$$

$$= S_1(0) \exp\left[W_0(t) - \frac{1}{2} t \right] \tag{4.1.2}$$

By theorem (2.2.6), this is a complete market with one risky asset and one riskless asset.

We define two events;

$$C := \left\{ \frac{S_1(T)}{S_0(T)} = \infty \right\}, \quad D := \left\{ \frac{S_1(T)}{S_0(T)} = 0 \right\} \tag{4.1.3}$$

From the above, we see that $P(C) = 0$, $P_0(C) = 0$, similarly, $P(D) = 0$, $P_0(D) = 0$. P_0 is the martingale measure on $\mathcal{F}^W(T)$. Thus, we conclude that P and P_0 are equivalent on the finite horizon, this can help eliminate the occurrence of arbitrage opportunity by Proposition (2.2.7).

4.2 Infinite horizon case

Using the illustration of the previous section, we consider an example in the infinite horizon as $t \to \infty$.

4.2.1 Example. We define similar events but at $t \to \infty$.

Let

$$C := \left\{ \lim_{t\to\infty} \frac{S_1(t)}{S_0(t)} = \infty \right\}, \quad D := \left\{ \lim_{t\to\infty} \frac{S_1(t)}{S_0(t)} = 0 \right\}, \quad 0 \leq t < \infty. \tag{4.2.1}$$

According to the law of large numbers for Brownian motion Karatzas and Shreve (1991, p. 124), as $t \to \infty$ we have

$$\frac{W(t)}{t} \to 0 \ P\text{-almost surely;} \quad \frac{W_0(t)}{t} \to 0 \ P_0\text{-almost surely.}$$

Where P_0 is defined on $\mathcal{F}^W(\infty)$ and not on $\mathcal{F}^W(T)$.

We see from the above that the probability of event C occuring under measures P and P_0, i.e.

$$P(C) = 1, \quad P_0(C) = 0$$

will not be the same because

$$\begin{aligned}
\lim_{t\to\infty} \frac{S_1(t)}{S_0(t)} &= \lim_{t\to\infty} S_1(0) \exp \left[W(t) + \left(b - r - \frac{1}{2} \right) t \right] \\
&= \lim_{t\to\infty} S_1(0) \exp \left(t \left[\frac{W(t)}{t} + \left(b - r - \frac{1}{2} \right) \right] \right) \\
&= \lim_{t\to\infty} S_1(0) \exp \left(t \left(b - r - \frac{1}{2} \right) \right), \quad \text{since} \quad \frac{W(t)}{t} \to 0, \\
\text{thus, } \lim_{t\to\infty} \frac{S_1(t)}{S_0(t)} &= \infty.
\end{aligned}$$

This also applies to event D,

$$P(D) = 0, \quad P_0(D) = 1.$$

$$\begin{aligned}
\lim_{t\to\infty} \frac{S_1(t)}{S_0(t)} &= \lim_{t\to\infty} S_1(0) \exp \left[W_0(t) - \frac{1}{2} t \right] \\
&= \lim_{t\to\infty} S_1(0) \exp \left(t \left[\frac{W_0(t)}{t} - \frac{1}{2} \right] \right) \\
&= \lim_{t\to\infty} S_1(0) \exp \left(t \left[-\frac{1}{2} \right] \right), \quad \text{since} \quad \frac{W_0(t)}{t} \to 0, \\
\text{thus, } \lim_{t\to\infty} \frac{S_1(t)}{S_0(t)} &= 0.
\end{aligned}$$

Thus, we see that P and P_0 are not equivalent on the infinite horizon.

The example above goes to show that for P and P_0 to be equivalent on the infinite horizon, we need to work with the augmented filtration $\{\mathcal{F}^{(T)}(t)\}_{0\leq t\leq T}$, indexed by $T \in [0, \infty)$ rather than with $\{\mathcal{F}^W(t)\}_{0\leq t<\infty}$. However, for each $T \in [0, \infty)$, P and P_0^T coincides for both events C and D ensuring no arbitrage opportunity in the infinite horizon market.

5. Conclusion

This project work started by understanding how some mathematical tools were used on the arbitrage theorem in the finite horizon. We were able to show the role of a local martingale and existence of equivalent probability measures in $[0, T]$ in order to have a viable market in our main Theorem (2.2.1). Here, we saw that;

(i) for a discounted tame portfolio, a local martingale bounded from below is a supermartingale and ensures that the market is viable;

(ii) two measures are said to be equivalent if P-measure zero is equivalent to P_0-measure zero.

Working towards the arbitrage theorem in an infinite horizon, Proposition (3.2.1) showed that we could only use an augmented filtration for every terminal time $T \in [0, \infty)$ to be able to get information about our market.

We gave an illustration of a complete market to show the importance of equivalent probability measures on both the finite and infinite horizons in Chapter 3. This helped show that market viability cannot be attained if these two probability measures are not equivalent on T which will then be indexed by $T \in [0, \infty)$ for them to be equivalent on $\{\mathcal{F}^W(t)\}_{0 \leq t < \infty}$.

We characterized the no-arbitrage theorem condition in an infinite horizon by a countable and infinite family of complete markets on a finite time horizon.

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