Symmetry Analysis of a Nonlinear System of Coupled Burgers Equations

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Abstract

In this dissertation, we study a system of coupled Burgers equations that model polydispersive sedimentation from Lie symmetry standpoint. We perform Lie symmetry analysis on the system and obtain symmetry reductions. Travelling wave solutions are constructed using the translation symmetries in time and space. Furthermore, we compute conservation laws of the system using two methods; the multiplier method and the conservation theorem due to Ibragimov.

Keywords: Coupled Burgers equations; Lie symmetry analysis; symmetry reductions; travelling wave solutions; conservation laws.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Saumu Athman Abdallah, 24 October 2019

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1. Introduction

Many physical phenomena that appear in nature are modelled using nonlinear partial differential equations (NLPDEs). To find meaningful information about such models, it is paramount to find exact solutions to such NLPDEs. To date, there are no general methods of finding exact closed form solutions of NLPDEs. However, several methods have been introduced by researchers to find some particular solutions of the aforementioned equations. These methods include Bäcklund transformation method Gu (1990), Hirota bilinear method Hirota (2004), ansatz method Hu and Zhang (2001), the simplest equation method Kudryashov (2005), the $(G'/G)-$expansion method Wang et al. (2008), Kudryashov method Kudryashov (2012) and Lie symmetry analysis method Olver (1993); Ibragimov (1999).

Lie group analysis was developed by Sophus Lie (1842-1899) in the nineteenth century. Lie developed an interest in applying Galois theory of algebraic equations to differential equations (DEs) and thus came up with the theory of continuous groups, now named as Lie groups. The important idea of this approach is that if a DE is invariant under a Lie group of transformations, then such an equation can be reduced to a simpler form that could be solved. Additionally, similarity or invariant solutions of the equation could be found. Recently, many books have been written on this subject, such as Ovsiannikov (1982); Olver (1993); Bluman and Kumei (1989); Ibragimov (1994–1996) and this method has been extensively used to find exact closed form solutions of certain NLPDEs.

Conservation laws are important in understanding various physical phenomena of the real world. They have many applications in various areas, such as in physics, biology and engineering Leveque (1992); Olver (1993); Ibragimov (2007); Sjöberg (2007); Bluman et al. (2010). Conservation laws explain the integrability of DEs and the effectiveness of numerical methods. Recently, they have been used to construct exact solutions of certain PDEs. Thus, it is an important problem to determine all conservation laws of given DEs.

The German mathematician Emmy Noether (1882–1935) found a connection between symmetries and conservation laws. This is stated in a theorem known today as Noether theorem Noether (1918) and it deals only with DEs which have a Lagrangian. However, there are DEs that do not have a Lagrangian and so substitute methods were introduced to find conservation laws. Recently, researchers have developed methods of finding conservation laws that do not depend on the existence of the variational principle. Among these are, the multiplier approach and the method due to Ibragimov Olver (1993); Anco and Bluman (2002); Ibragimov (2007).

This project aims to study the system of coupled Burgers equations that model a system of polydispersive sedimentation. Lie's method is employed to reduce coupled Burgers equations to second order ordinary differential equations and thereafter, construct group-invariant solutions of the system. Also, we derive conservation laws for the system.

The system of coupled Burgers equations has many applications in fluid dynamics, and finding its solution will aid in understanding a lot of physical world problems. The solutions and conservation laws for the case of a one-dimensional coupled Burgers system are derived from the symmetry point of view. These results will be of great importance to researchers since they have not been reported before until now, to the best of our knowledge.

The layout of this dissertation is as follows:

In Chapter 2, we give a brief description of some preliminaries concerning Lie symmetry methods and conservation laws.

In Chapter 3, we give an illustrative example of the Burgers equation. We find group-invariant solutions of the equation and derive its conservation laws using the multiplier approach.

In Chapter 4, we study coupled Burgers equations. We compute Lie point symmetries and after that, we perform symmetry reductions. Also, travelling wave solutions were constructed. Furthermore, we derive conservation laws for the system by using two approaches; the multiplier approach and Ibragimov's approach.

In Chapter 5, we give a summary of the work done in the dissertation and mention some future work.

References are given at the end.

2. Preliminaries

This Chapter presents, very briefly, some concepts of Lie's theory, which we will utilise in our dissertation Ibragimov (1994–1996). We also give some methods for deriving conservation laws for nonlinear partial differential equations (PDEs) Anco and Bluman (2002); Ibragimov (2007).

2.1 One-parameter local Lie groups

We work in Euclidean space $\mathbb{R}^n$, with coordinates $x = (x^1, x^2, \ldots, x^n)$ being independent variables and $\mathbb{R}^m$, with $u = (u^1, u^2, \ldots, u^m)$ representing dependent variables. Consider the transformation

$$T_a : \bar{x}^i = f^i(x, u, a), \;\; \bar{u}^\alpha = g^\alpha(x, u, a), \tag{2.1.1}$$

where a is a continuous parameter, which lies in an open interval, $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and g^α are differentiable functions.

2.1.1 Definition. A set $\mathcal{G}$ of transformations (2.1.1) is a continuous one-parameter local Lie group if the following three conditions hold:

(i) (Closure) If $T_a, T_b \in \mathcal{G}$ for $a, b \in \mathcal{D}' \subset \mathcal{D}$, then $T_b T_a = T_c \in \mathcal{G}$, where $c = \Phi(a, b) \in \mathcal{D}$.

(ii) (Identity) $T_0 \in \mathcal{G}$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$.

(iii) (Inverse) There exists $T_a \in \mathcal{G}$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in \mathcal{G}$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$.

It should be noted that the associativity property follows from (i).

2.2 Infinitesimal and extended transformations

Consider a system of kth-order $(k \geq 1)$ PDEs, viz.,

$$E_\alpha(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \;\; \alpha = 1, 2, \ldots, m, \tag{2.2.1}$$

where $u_{(1)}$ $u_{(2)}$ $\ldots$ $u_{(k)}$ are collection of all first-, second-, up to kth-order partial derivatives: $u_{(1)} = \{u_i^\alpha\}$, $u_{(2)} = \left\{u_{ij}^\alpha\right\}$, $\ldots$, $u_{(k)} = \left\{u_{i_1 \ldots i_k}^\alpha\right\}$ and

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_i(u_j^\alpha) = D_i D_j(u^\alpha), \ldots, \tag{2.2.2}$$

where D_i is the total derivative operator with respect to x^i, defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \quad i = 1, \ldots, n. \tag{2.2.3}$$

In Lie group analysis the variables $x, u, u_{(1)} \ldots$ are treated as functionally independent variables, connected only by relations (2.2.2) Ibragimov (1994–1996).

Consider a one-parameter Lie group $\mathcal{G}$ of transformations

$$\bar{x}^i = f^i(x, u, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{u}^\alpha = g^\alpha(x,u,a), \quad g^\alpha|_{a=0} = u^\alpha. \tag{2.2.4}$$

According to Lie's theory, the construction of $\mathcal{G}$ is equivalent to the determination of the corresponding infinitesimal transformations:

$$\bar{x}^i \approx x^i + a\,\xi^i(x,u),\ \ \bar{u}^\alpha \approx u^\alpha + a\,\eta^\alpha(x,u), \tag{2.2.5}$$

which are obtained by the Taylor series expansion of functions f^i and g^α in a of (2.1.1) about $a = 0$ subject to the initial conditions

$$f^i\Big|_{a=0} = x^i,\ \ g^\alpha|_{a=0} = u^\alpha.$$

Consequently,

$$\xi^i(x,u) = \frac{\partial f^i}{\partial a}\bigg|_{a=0},\ \ \eta^\alpha(x,u) = \frac{\partial g^\alpha}{\partial a}\bigg|_{a=0}. \tag{2.2.6}$$

Introducing the symbol X of the infinitesimal transformations, one can write (2.2.5) as

$$\bar{x}^i \approx (1 + a\,X)x,\ \ \bar{u}^\alpha \approx (1 + a\,X)u,$$

where the differential operator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{2.2.7}$$

is named as the infinitesimal generator or operator of the group $\mathcal{G}$. If the group $\mathcal{G}$ is admitted by (2.2.1), we say that X is an admitted operator of (2.2.1).

The transformed derivatives are obtained from (2.1.1) with the help of change of variables formulae

$$D_i = D_i(f^j)\bar{D}_j, \tag{2.2.8}$$

where $\bar{D}_j$ is the total derivative with respect to $\bar{x}^i$ and so

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha),\ \ \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha),\ldots.$$

Thus, we have

$$D_i(g^\alpha) = D_i(f^j)\bar{D}_j(u^\alpha) = D_i(f^j)\bar{u}_j^\alpha \tag{2.2.9}$$

and

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta\frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha = \frac{\partial g^\alpha}{\partial x^i} + u_i^\beta\frac{\partial g^\alpha}{\partial u^\beta}. \tag{2.2.10}$$

Now the quantities $\bar{u}_j^\alpha$ can be written as

$$\bar{u}_i^\alpha = \Psi_i^\alpha(x,u,u_{(1)},a),\ \ \Psi^\alpha|_{a=0} = u_i^\alpha. \tag{2.2.11}$$

The transformations (2.2.4) and (2.2.11) form a one-parameter group $\mathcal{G}^{[1]}$, which is called the first prolongation group. Let us now define the infinitesimal transformation of the first derivatives as

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha, \tag{2.2.12}$$

then (2.2.5) and (2.2.12) define the infinitesimal transformation of the group $\mathcal{G}^{[1]}$. Likewise, higher-order prolongations of $\mathcal{G}$, viz., $\mathcal{G}^{[2]}$ and $\mathcal{G}^{[3]}$ are defined. From (2.2.9), we have

$$D_i(f^j)(\bar{u}_j^\alpha) = D_i(g^\alpha).$$

Using (2.2.5) and (2.2.12), the above equation becomes

$$D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) = D_i(u^\alpha + a\eta^\alpha),$$

which gives

$$u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j = u_i^\alpha + aD_i\eta^\alpha.$$

Hence

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \tag{2.2.13}$$

which is the first prolongation formula. Likewise Ibragimov (1994–1996),

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \tag{2.2.14}$$

$$\vdots$$

$$\zeta_{i_1,i_2,\ldots,i_k}^\alpha = D_{i_k}(\zeta_{i_1,i_2,\ldots,i_{k-1}}^\alpha) - u_{i_1,i_2,\ldots,i_{k-1}\,j}^\alpha D_{i_k}(\xi^j). \tag{2.2.15}$$

The prolonged generators are given by

$$\begin{aligned} X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} \quad \text{(sum on } i, \alpha), \\ &\vdots \\ X^{[k]} &= X^{[k-1]} + \zeta_{i_1,\ldots,i_k}^\alpha \frac{\partial}{\partial u_{i_1,\ldots,i_k}^\alpha}, \ k \geq 1, \end{aligned} \tag{2.2.16}$$

where X is given by (2.2.7).

2.3 Group admitted by differential equations

We now present some basic definitions and a theorem that are essential in the symmetry analysis method:

2.3.1 Definition. The operator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{2.3.1}$$

is a symmetry of the kth-order PDE system (2.2.1), if

$$X^{[k]} E_\alpha \Big|_{E_\alpha = 0} = 0. \tag{2.3.2}$$

2.3.2 Definition. Equation (2.3.2) is known as the determining equation of (2.2.1).

2.3.3 Definition. A group $\mathcal{G}$ is called a symmetry group of (2.2.1), if (2.2.1) is form-invariant, i.e.,

$$E_\alpha(\bar{x}, \bar{u}, \bar{u}_{(1)}, \ldots, \bar{u}_{(k)}) = 0. \tag{2.3.3}$$

2.3.4 Definition. A function $h(x,u)$ is said to be an invariant of the group $\mathcal{G}$ given by (2.1.1), if

$$h(\bar{x},\bar{u}) = h(x,u). \tag{2.3.4}$$

2.3.5 Theorem. *A function $h(x,u)$ is invariant under the symmetry X, if*

$$Xh \equiv \xi^i(x,u)\frac{\partial h}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial h}{\partial u^\alpha} = 0\,. \tag{2.3.5}$$

It follows from the above theorem that every one-parameter group of point transformations (2.1.1) has $n-1$ functionally independent invariants. One can take, as bases invariants, the left-hand side $n-1$ first integrals

$$J_1(x,u) = c_1, \cdots, J_{n-1}(x,u) = c_{n-1}$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^m}{\eta^m(x,u)}.$$

2.4 Lie algebra

In this subsection we provide the definition of a Lie algebra of operators.

2.4.1 Definition. A vector space L of operators Ibragimov (1994–1996)

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

is said to be a Lie algebra if the following holds: If

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u^\alpha},\ X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

belong to L, then

$$[X_1, X_2] = X_1(X_2) - X_2(X_1), \tag{2.4.1}$$

their commutator, also belongs to L.

It is clearly evident that the commutator obeys the properties of bilinearity, skew-symmetry and Jacobi's identity.

2.5 System of two partial differential equations

In this Section we consider the following system of two second-order PDEs with (t,x) being independent variables, and (u,v) being dependent variables:

$$E(t,x,u,v,u_t,v_t,u_x,v_x,u_{tt},\ldots,v_{xx}) = 0, \tag{2.5.1}$$

$$F(t,x,u,v,u_t,v_t,u_x,v_x,u_{tt},\ldots,v_{xx}) = 0. \tag{2.5.2}$$

Thus, the vector field (2.3.1), with $(x^1, x^2, u^1, u^2) \to (t, x, u, v)$ and $(\xi^1, \xi^2, \eta^1, \eta^2) \to (\tau, \xi, \phi, \psi)$, in this particular case, becomes

$$X = \tau\,\partial_t + \xi\,\partial_x + \phi\,\partial_u + \psi\,\partial_v, \tag{2.5.3}$$

where the coefficients τ, ξ, ϕ and ψ depend on (t, x, u, v).

According to the definition (2.3.1), this vector field is a symmetry of (2.5.1)–(2.5.2) if

$$X^{[2]}\,E\Big|_{E=0,F=0} = 0, \quad X^{[2]}\,F\Big|_{E=0,F=0} = 0, \tag{2.5.4}$$

where

$$\begin{aligned} X^{[2]} = X &+ \zeta^1_1\frac{\partial}{\partial u_t} + \zeta^1_2\frac{\partial}{\partial u_x} + \zeta^2_1\frac{\partial}{\partial v_t} + \zeta^2_2\frac{\partial}{\partial v_x} + \zeta^1_{11}\frac{\partial}{\partial u_{tt}} + \zeta^1_{12}\frac{\partial}{\partial u_{tx}} + \zeta^1_{22}\frac{\partial}{\partial u_{xx}} \\ &+ \zeta^2_{11}\frac{\partial}{\partial v_{tt}} + \zeta^2_{12}\frac{\partial}{\partial v_{tx}} + \zeta^2_{22}\frac{\partial}{\partial v_{xx}} \end{aligned} \tag{2.5.5}$$

is the second prolongation of X, and

$$\begin{aligned} &\zeta^1_1 = D_t(\phi) - u_t D_t(\tau) - u_x D_t(\xi),\ \ \zeta^1_2 = D_x(\phi) - u_t D_x(\tau) - u_x D_x(\xi), \\ &\zeta^2_1 = D_t(\psi) - v_t D_t(\tau) - v_x D_t(\xi),\ \ \zeta^2_2 = D_x(\psi) - v_t D_x(\tau) - v_x D_x(\xi), \\ &\zeta^1_{11} = D_t(\zeta^1_1) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi),\ \ \zeta^1_{12} = D_x(\zeta^1_1) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi), \\ &\zeta^1_{22} = D_x(\zeta^1_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi),\ \ \zeta^2_{11} = D_t(\zeta^2_1) - v_{tt} D_t(\tau) - v_{tx} D_t(\xi), \\ &\zeta^2_{12} = D_x(\zeta^2_1) - v_{tt} D_x(\tau) - v_{tx} D_x(\xi),\ \ \zeta^2_{22} = D_x(\zeta^2_2) - v_{tx} D_x(\tau) - v_{xx} D_x(\xi). \end{aligned}$$

Here the total derivatives D_t, D_x are given by

$$\begin{aligned} D_t &= \partial_t + u_t\partial_u + v_t\partial_v + u_{tt}\partial_{u_t} + v_{tt}\partial_{v_t} + u_{tx}\partial_{u_x} + v_{tx}\partial_{v_x} + \ldots, \\ D_x &= \partial_x + u_x\partial_u + v_x\partial_v + u_{tx}\partial_{u_t} + v_{tx}\partial_{v_t} + u_{xx}\partial_{u_x} + v_{xx}\partial_{v_x} + \ldots. \end{aligned}$$

Expanding the above values of ζ's, we obtain

$$\begin{aligned} \zeta^1_1 &= \phi_t + \phi_u u_t + \phi_v v_t - u_t\tau_t - u_t^2\tau_u - u_t v_t\tau_v - u_x\xi_t - u_x u_t\xi_u - u_x v_t\xi_v, \\ \zeta^1_2 &= \phi_x + u_x\phi_u + v_x\phi_v - u_t\tau_x - u_t u_x\tau_u - u_t v_x\tau_v - u_x\xi_x - u_x^2\xi_u - u_x v_x\xi_v, \\ \zeta^2_1 &= \psi_t + u_t\psi_u + v_t\psi_v - v_t\tau_t - v_t u_t\tau_u - v_t^2\tau_v - v_x\xi_t - v_x u_t\xi_u - v_x v_t\xi_v, \\ \zeta^2_2 &= \psi_x + u_x\psi_u + v_x\psi_v - v_t\tau_x - v_t u_x\tau_u - v_t v_x\tau_v - v_x\xi_x - v_x u_x\xi_u - v_x^2\xi_v \end{aligned}$$

and

$$\begin{aligned} \zeta^1_{22} = &- 2u_t u_x\tau_{uv} v_x - u_t v_x^2\tau_{vv} - u_t\tau_{xx} - u_x\xi_{xx} + 2\phi_{vx} v_x - 2\xi_{ux} u_x^2 - 2u_{xx}\xi_x \\ &+ \phi_u u_{xx} + \phi_v v_{xx} - 2u_{tx}\tau_x + 2\phi_{ux} u_x - 2u_{tx}\tau_v v_x - 2u_t\tau_{ux} u_x - 2u_{tx}\tau_u u_x \\ &- 2u_x^2\xi_{uv} v_x - 2u_{xx}\xi_v v_x - 2u_x\xi_{vx} v_x + \phi_{xx} + \phi_{uu} u_x^2 + 2u_x\phi_{uv} v_x - u_x\xi_v v_{xx} \\ &- u_t\tau_v v_{xx} - u_t\tau_u u_{xx} - 2u_t\tau_{vx} v_x - 3\xi_u u_x u_{xx} + \phi_{vv} v_x^2 - u_t u_x^2\tau_{uu} - u_x v_x^2\xi_{vv} \\ &- u_x^3\xi_{uu}, \\ \zeta^2_{22} = &- 2v_t u_x\tau_{uv} v_x - v_t v_x^2\tau_{vv} - 2v_x^2 u_x\xi_{uv} - 2v_{tx}\tau_v v_x + 2u_x\psi_{uv} v_x - 2v_t\tau_{ux} u_x \\ &- v_t\tau_u u_{xx} - 2\xi_{vx} v_x^2 - v_t\tau_v v_{xx} - v_x^3\xi_{vv} - v_x u_x^2\xi_{uu} - 2v_t\tau_{vx} v_x + \psi_{xx} + 2\psi_{ux} u_x \\ &- 2v_{xx}\xi_u u_x - 3\xi_v v_x v_{xx} - 2v_{tx}\tau_u u_x + 2\psi_{vx} v_x + \psi_v v_{xx} - 2v_{tx}\tau_x - 2v_{xx}\xi_x \\ &+ \psi_u u_{xx} - v_t\tau_{xx} - v_t u_x^2\tau_{uu} - v_x\xi_u u_{xx} - 2v_x\xi_{ux} u_x + \psi_{uu} u_x^2 + \psi_{vv} v_x^2 - v_x\xi_{xx}. \end{aligned}$$

Likewise, one can obtain the expanded values of ζ^1_{11}, ζ^1_{12}, ζ^2_{11} and ζ^2_{12}.

2.6 Conservation laws

2.6.1 Fundamental operators and their relationship.

Consider a system of PDEs of pth-order, namely

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(p)}) = 0, \ \alpha = 1, \dots, m. \tag{2.6.1}$$

Recall the Euler operator

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{r\geq 1}(-1)^r D_{i_1} \dots D_{i_r} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_r}}, \ \alpha = 1, \dots, m, \tag{2.6.2}$$

and the Lie-Bäcklund operator

$$X = \xi^j \frac{\partial}{\partial x^j} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \ \xi^j, \eta^\alpha \in \mathcal{A}. \tag{2.6.3}$$

Here $\mathcal{A}$ represents the space of differential functions (Ibragimov, 1994–1996). The operator (2.6.3) in its extended form is

$$X = \xi^j \frac{\partial}{\partial x^j} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{r\geq 1} \zeta^\alpha_{i_1 i_2 \dots i_r} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_r}}, \tag{2.6.4}$$

with

$$\begin{aligned} \zeta^\alpha_j &= D_j(W^\alpha) + \xi^l u^\alpha_{jl}, \\ &\vdots \\ \zeta^\alpha_{i_1 \dots i_r} &= D_{i_1} \dots D_{i_r}(W^\alpha) + \xi^l u^\alpha_{l i_1 \dots i_r}, \ r > 1. \end{aligned} \tag{2.6.5}$$

Here $W^\alpha = \eta^\alpha - \xi^j u^\alpha_l$ is the Lie characteristic function. In characteristic form the Lie-Bäcklund operator (2.6.4) is

$$X = \xi^j D_j + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{r\geq 1} D_{i_1} \dots D_{i_r}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_r}} \tag{2.6.6}$$

and the associated Noether operators are given by

$$N^j = \xi^j + W^\alpha \frac{\delta}{\delta u^\alpha_j} + \sum_{r\geq 1} D_{i_1} \dots D_{i_r}(W^\alpha) \frac{\delta}{\delta u^\alpha_{j i_1 i_2 \dots i_r}}, \ j = 1, \dots, n \tag{2.6.7}$$

where

$$\frac{\delta}{\delta u^\alpha_j} = \frac{\partial}{\partial u^\alpha_j} + \sum_{r\geq 1}(-1)^r D_{l_1} \dots D_{l_r} \frac{\partial}{\partial u^\alpha_{j l_1 l_2 \dots l_r}}, \ j = 1, \dots, n, \quad \alpha = 1, \dots, m. \tag{2.6.8}$$

The Lie-Bäcklund, Euler-Lagrange and Noether operators are related by the identity (Ibragimov, 2007)

$$X + D_j(\xi^j) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_j N^j. \tag{2.6.9}$$

The vector $T = (T^1, T^2, \dots, T^n)$, where $T^l \in \mathcal{A}$, for $l = 1, \dots, n$, is a conserved vector of (2.6.1) if

$$D_j T^j|_{(2.6.1)} = 0 \tag{2.6.10}$$

and it defines a local conservation law.

2.6.2 Multiplier Method.

The multiplier $\Lambda^\alpha(x, u, u_{(1)}, \dots)$ of the PDE system (2.6.1) satisfy the equation (Olver, 1993; Anco and Bluman, 2002)

$$\Lambda^\alpha E_\alpha = D_j T^j \tag{2.6.11}$$

identically, with the right hand side being a divergence expression. Multipliers Λ^α are computed by solving the determining equations (Anco and Bluman, 2002)

$$\frac{\delta}{\delta u^\alpha}(\Lambda^\alpha E_\alpha) = 0. \tag{2.6.12}$$

2.6.3 Conservation theorem due to Ibragimov.

This approach (Ibragimov, 2007) enables us to compute a conservation law for each Lie point symmetry of the PDE system. Let us consider the PDE system

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(r)}) = 0 \tag{2.6.13}$$

whose adjoint equations are

$$E^\star_\alpha(x, u, v, \dots, v_{(r)}, u_{(r)}) \equiv \frac{\delta}{\delta u^\alpha}(v^\gamma E_\gamma) = 0 \tag{2.6.14}$$

with $v = (v^1, \dots, v^m)$ being a new dependent variable.

The formal Lagrangian $\mathcal{L}$ of the system (2.6.13) and its adjoint equations (2.6.14) is given by Ibragimov (2007)

$$\mathcal{L} = v^\gamma E_\gamma(x, u, u_{(1)}, \dots, u_{(r)}). \tag{2.6.15}$$

2.6.4 Theorem. *Ibragimov (2007)*

Every infinitesimal symmetry

$$X = \xi^j(x, u)\frac{\partial}{\partial x^j} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha} \tag{2.6.16}$$

of system (2.6.13) leads to a conservation laws

$$D_j T^j|_{E_{\alpha=0}} = 0 \tag{2.6.17}$$

that is given by

$$T^i = \xi^i \mathcal{L} + \mathcal{W}^\alpha \left[\frac{\partial \mathcal{L}}{\partial u^\alpha_i} - D_j \frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}}\right) + \dots\right] \tag{2.6.18}$$

$$+ D_j(\mathcal{W}^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} - D_k \frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} + \dots\right] + D_j D_k(\mathcal{W}^\alpha)\frac{\partial \mathcal{L}}{\partial u_{ijk}} + \dots, \tag{2.6.19}$$

where $\mathcal{W}^\alpha = \eta^\alpha - \xi^j u^\alpha_j, \alpha = 1, \dots, m$ and $\mathcal{L}$ is the formal Lagrangian (2.6.15).

3. Burgers equation: an illustrative example

We study the viscous Burgers equation in this Chapter. First, we derive its symmetries and then construct group-invariant solutions and thereafter, find conservation laws by using the multiplier method.

3.1 Introduction

The viscous Burgers equation (Burgers, 1948) is given by

$$u_t - uu_x - u_{xx} = 0, \tag{3.1.1}$$

where (t, x) are independent variables and u the dependent variable. It describes the motion of weakly nonlinear waves in gases when dissipative effects are sufficiently small to be considered in the first approximation only. When dissipation tends to zero, this equation gives an adequate description of waves in a non-viscous medium.

3.2 Exact solutions of Burgers equation

3.2.1 Computation of symmetries of (3.1.1).

The vector field given by

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u} \tag{3.2.1}$$

is a symmetry of Burgers equation (3.1.1) if

$$X^{[2]}(u_t - uu_x - u_{xx})\Big|_{(3.1.1)} = 0. \tag{3.2.2}$$

Using the definition of $X^{[2]}$ from Chapter 2 we get

$$\left(\tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{22}\frac{\partial}{\partial u_{xx}}\right)(u_t - uu_x - u_{xx})\,|_{u_{xx}=u_t-uu_x} = 0,$$

which gives

$$-\eta u_x + \zeta_1 - u\zeta_2 - \zeta_{22}|_{u_{xx}=u_t-uu_x} = 0, \tag{3.2.3}$$

where ζ_1 and ζ_2 are defined by (2.2.13) and ζ_{22} is given by (2.2.15). Substituting the values of ζ_1, ζ_2 and ζ_{22} in (3.2.3) we obtain the following determining equation:

$$\begin{aligned}
&\eta_t - \eta u_x + (\eta_u - \tau_t)u_t - \xi_t u_x - \tau_u u_t^2 - \xi_u u_t u_x - u\eta_x - (\eta_u - \xi_x)uu_x + \tau_x uu_t + \xi_u uu_x^2 \\
&+ \tau_u uu_x u_t - \eta_{xx} - (2\eta_{xu} - \xi_{xx})u_x + \tau_{xx}u_t + 2\tau_{xu}u_x u_t - (\eta_{uu} - 2\xi_{xu})u_x^2 + \xi_{uu}u_x^3 \\
&+ \tau_{uu}u_t u_x^2 - (\eta_u - 2\xi_x)u_{xx} + 3\xi_u u_x u_{xx} + \tau_u u_t u_{xx} + 2u_{xt}(\tau_x + \tau_u u_x)|_{u_{xx}=u_t-uu_x} = 0.
\end{aligned}$$

Now replacing u_{xx} by $u_t - uu_x$ in the above equation we obtain

$$\begin{aligned}
&\eta_t - \eta u_x + (\eta_u - \tau_t)u_t - \xi_t u_x - \tau_u u_t^2 - \xi_u u_t u_x - u\eta_x - (\eta_u - \xi_x)uu_x + \tau_x uu_t + \xi_x uu_x^2 \\
&+ \tau_u uu_t u_x - \eta_{xx} - (2\eta_{xu} - \xi_{xx})u_x + \tau_{xx}u_t - (\eta_{uu} - 2\xi_{xu})u_x^2 + 2\tau_{xu}u_x u_t + \xi_{uu}u_x^3
\end{aligned}$$

$$+\tau_{uu}u_tu_x^2 - \eta_u(u_t - uu_x) + 2(u_t - uu_x)\xi_x + 3u_x(u_t - uu_x)\xi_u + u_t(u_t - uu_x)\tau_u$$
$$+ 2u_{xt}(\tau_x + \tau_u u_x) = 0.$$

We can split the above equation on derivatives of u, since coefficient functions τ, ξ and η depend only on t, x and u. Thus, we obtain

$$\tau_x = 0, \tag{3.2.4}$$
$$\tau_u = 0, \tag{3.2.5}$$
$$\xi_u = 0, \tag{3.2.6}$$
$$\eta_{uu} = 0, \tag{3.2.7}$$
$$2\xi_x - \tau_t = 0, \tag{3.2.8}$$
$$\xi_{xx} - 2\eta_{xu} - \xi_x u - \xi_t - \eta = 0, \tag{3.2.9}$$
$$\eta_t - \eta_x u - \eta_{xx} = 0. \tag{3.2.10}$$

Equations (3.2.4) and (3.2.5) imply that

$$\tau = B(t) \tag{3.2.11}$$

with B an arbitrary function of t. From (3.2.6) we get

$$\xi = A(t, x) \tag{3.2.12}$$

with A an arbitrary function of t and x. Integrating equation (3.2.7) we get

$$\eta = D(t, x)u + E(t, x)$$

with D and E arbitrary functions of t and x. Substituting values of ξ and η in (3.2.9), we obtain

$$A_{xx} - Du - E - A_t - A_x u - 2D_x = 0. \tag{3.2.13}$$

Splitting equation (3.2.13) on powers u yields

$$u^1 \quad : \quad D + A_x = 0, \tag{3.2.14}$$
$$u^0 \quad : \quad A_{xx} - E - A_t - 2D_x = 0. \tag{3.2.15}$$

Now substituting value of η in (3.2.10) we get

$$D_t u + E_t - D_x u^2 - E_x u - D_{xx} u - E_{xx} = 0.$$

Splitting the above equation on powers of u yields

$$u^2 \quad : \quad D_x = 0, \tag{3.2.16}$$
$$u^1 \quad : \quad D_t - E_x - D_{xx} = 0, \tag{3.2.17}$$
$$u^0 \quad : \quad E_t - E_{xx} = 0. \tag{3.2.18}$$

From (3.2.16) we have $D = D(t)$. Thus from equation (3.2.14) we have $A_x = -D(t)$, and upon integrating yields

$$A = -D(t)x + F(t) \tag{3.2.19}$$

with F an arbitrary function of t. Equations (3.2.15) and (3.2.17) give, respectively

$$A_t = -E \quad \text{and} \quad E_x = D'(t). \tag{3.2.20}$$

Now integrating the second equation with respect to x, we get

$$E = D'(t)x + G(t) \tag{3.2.21}$$

with G an arbitrary function of t. Using the value of E into equation (3.2.18) we get

$$D''(t)x + G'(t) = 0. \tag{3.2.22}$$

Splitting equation (3.2.22) on x we have

$$x^1 \quad : \quad D''(t) = 0, \tag{3.2.23}$$
$$x^0 \quad : \quad G'(t) = 0. \tag{3.2.24}$$

Therefore

$$D(t) = \; C_1 t + C_2 \tag{3.2.25}$$

and

$$G(t) = \; C_3 \tag{3.2.26}$$

with C_1, C_2, C_3 arbitrary constants, and from (3.2.21) we have

$$E = C_1 x + C_3. \tag{3.2.27}$$

The first equation of (3.2.20) simplifies to

$$A_t = -C_1 x - C_3. \tag{3.2.28}$$

Now substituting equations (3.2.19) and (3.2.25) into (3.2.8), we have

$$B_t = -2C_1 t - 2C_2. \tag{3.2.29}$$

Integrating (3.2.29) on t we get

$$B = -C_1 t^2 - 2C_2 t + C_4, \tag{3.2.30}$$

where C_4 is an arbitrary constant. Hence $\tau = -C_1 t^2 - 2C_2 t + C_4$. Using equation (3.2.19) we have

$$A_t = -D'(t)x + F'(t). \tag{3.2.31}$$

This means that

$$A_t = -C_1 x + F'(t). \tag{3.2.32}$$

Finally equations (3.2.28) and (3.2.32) imply that

$$-C_1 x - C_3 = -C_1 x + F'(t), \tag{3.2.33}$$

which gives $F'(t) = -C_3$. Integrating yields

$$F(t) = -C_3 t + C_5, \tag{3.2.34}$$

where C_5 is an arbitrary constant. Thus we obtain

$$\begin{aligned} \tau &= -C_1 t^2 - 2C_2 t + C_4, \\ \xi &= -C_1 tx - C_2 x - C_3 t + C_5, \\ \eta &= C_1(tu + x) + C_2 u + C_3. \end{aligned}$$

Hence Lie point symmetries of Burgers equation (3.1.1) are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}, \\ X_4 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \\ X_5 &= t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (x + tu)\frac{\partial}{\partial u}. \end{aligned}$$

3.2.2 Constructing group-invariant solutions of (3.1.1).

We now compute group-invariant solutions of Burgers equation (3.1.1).

Case 1. We firstly consider the symmetry operator

$$X_1 = \frac{\partial}{\partial t}. \tag{3.2.35}$$

The characteristic equations associated with the operator (3.2.35) are

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0},$$

which gives two invariants

$$J_1 = x, \ J_2 = u.$$

Thus, $u = f(x)$ is the group-invariant solution (f an arbitrary function). Substituting u into (3.1.1) yields

$$f''(x) + f(x)f'(x) = 0,$$

whose solution is

$$f(x) = C_1 \tanh\left(\frac{1}{2}C_1 x + C_2\right),$$

with C_1, C_2 arbitrary constants. Thus, under X_1, group-invariant solution of (3.1.1) is

$$u(t,x) = C_1 \tanh\left(\frac{1}{2}C_1 x + C_2\right).$$

Case 2. We consider

$$X_3 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}. \tag{3.2.36}$$

The characteristic equations for (3.2.36) yield two invariants $J_1 = t$, $J_2 = u + x/t$. Consequently, the group-invariant solution of (3.1.1) under X_3 is $J_2 = \phi(J_1)$, where ϕ is an arbitrary function. That is

$$u(t,x) = \phi(t) - \frac{x}{t}. \tag{3.2.37}$$

Substituting this value of u into (3.1.1) yields a first-order ODE $\phi' + \phi/t = 0$, whose solution is $\phi(t) = C/t$ with C an arbitrary constant. Thus, the group-solution under X_3 is

$$u(t,x) = \frac{C - x}{t}. \tag{3.2.38}$$

Case 3. Next we consider

$$X_4 \;=\; 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}.$$

The associated Lagrangian equations to X_4 yield two invariants $J_1 = x/\sqrt{t}$ and $J_2 = \sqrt{t}u$. Thus,

$$u(t,x) = \frac{1}{\sqrt{t}}f(\lambda), \quad \lambda = \frac{x}{\sqrt{t}}. \tag{3.2.39}$$

The substitution of u into (3.1.1) gives

$$f'' + ff' + \frac{1}{2}(\lambda f' + f) = 0 \tag{3.2.40}$$

and integrating it once gives

$$f' + \frac{1}{2}f^2 + \frac{1}{2}\lambda f = C, \tag{3.2.41}$$

with C an arbitrary constant. Here, if we take $C = 0$, (3.2.41) reduces to a Bernoulli equation for f and when solved gives

$$f(\lambda) = \frac{2}{\sqrt{\pi}}\left[\frac{e^{\frac{-\lambda^2}{4}}}{A + erf\left(\frac{\lambda}{2}\right)}\right] \tag{3.2.42}$$

with A an arbitrary constant and

$$erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds \tag{3.2.43}$$

the error function. Therefore, group-invariant solution for (3.1.1) under X_4 is

$$u(t,x) = \frac{2}{\sqrt{\pi t}}\left(\frac{e^{\frac{-x^2}{4t}}}{A + erf\left(\frac{x}{2\sqrt{t}}\right)}\right). \tag{3.2.44}$$

Case 4. Finally, we consider the Lie symmetry operator

$$X_5 = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} - (x+tu)\frac{\partial}{\partial u}.$$

Likewise, we can construct group-invariant solution of (3.1.1) under X_5 and it is given by

$$u(t,x) = \frac{C_1}{t}\tanh\left\{\frac{C_1 x}{2t} + C_2\right\} - \frac{x}{t},$$

where C_1 is constant.

3.3 Conservation laws of Burgers equation

In this Section we use multiplier method to compute conservation laws of Burgers equation (3.1.1). Recall Euler-Lagrange operator from (2.6.2). We look for zero-order multiplier $\Lambda = \Lambda(t,x,u)$. The determining equation for computing Λ is

$$\frac{\delta}{\delta u}\left[\Lambda\left\{u_t - uu_x - u_{xx}\right\}\right] = 0. \tag{3.3.1}$$

Expanding equation (3.3.1) gives

$$\Lambda_u(u_t - uu_x - u_{xx}) - D_t(\Lambda) - \Lambda u_x + D_x(u\Lambda) - D_x^2(\Lambda) = 0. \tag{3.3.2}$$

Applying the total derivatives (2.2.3) to equation (3.3.2) gives

$$-2\Lambda_u u_{xx} - \Lambda_t + \Lambda_x u - \Lambda_{xx} - 2\Lambda_{xu}u_x - \Lambda_{uu}u_x^2 = 0. \tag{3.3.3}$$

Splitting (3.3.3) on derivatives of u, we obtain

$$u_{xx} : \Lambda_u = 0, \tag{3.3.4}$$
$$u_x^2 : \Lambda_{uu} = 0, \tag{3.3.5}$$
$$u_x : \Lambda_{xu} = 0, \tag{3.3.6}$$
$$\text{rest} : u\Lambda_x - \Lambda_t - \Lambda_{xx} = 0. \tag{3.3.7}$$

Equations (3.3.5) and (3.3.6) are already satisfied by equation (3.3.4), thus integrating (3.3.4) gives

$$\Lambda = A(t,x) \tag{3.3.8}$$

with $A(t,x)$ an arbitrary function of t, x. Now substituting this value of Λ into equation (3.3.7), we get

$$A_x u - A_t - A_{xx} = 0.$$

Splitting on u, we have

$$u : A_x = 0, \tag{3.3.9}$$
$$u^0 : A_t + A_{xx} = 0. \tag{3.3.10}$$

Equation (3.3.9) implies that $A = A(t)$. Thus, substituting this value of A into equation (3.3.10), we get

$$A'(t) = 0. \tag{3.3.11}$$

Integrating equation (3.3.11) gives $A(t) = C_1$ where C_1 is an arbitrary constant. Thus, the multiplier is given by $\Lambda = C_1$. A multiplier Λ for Burgers equation (3.1.1) has the property that

$$\Lambda(u_t - uu_x - u_{xx}) = D_t T^t + D_x T^x, \tag{3.3.12}$$

where $T^t = T^t(t, x, u, u_x)$ and $T^x = T^x(t, x, u, u_x)$. Here, since the multiplier $\Lambda = C_1$, we take $C_1 = 1$ and solve equation (3.3.12). Expanding equation (3.3.12), we have

$$u_t - uu_x - u_{xx} = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} u_{xx}.$$

Splitting this equation on derivatives of u, we obtain

$$u_{tx} \; : \; T^t_{u_x} = 0, \tag{3.3.13}$$

$$u_{xx} \; : \; T^x_{u_x} = -1, \tag{3.3.14}$$

$$\text{rest} \; : \; u_t - uu_x = T^t_t + T^t_u u_t + T^x_x + T^x_u u_x. \tag{3.3.15}$$

Equation (3.3.13) gives

$$T^t = A(t, x, u)$$

with A an arbitrary function of t, x, u. From (3.3.14), we get

$$T^x = -u_x + B(t, x, u)$$

with B an arbitrary function of t, x, u. Substituting the above values of T^t and T^x into equation (3.3.15), we get

$$u_t - uu_x = A_t + A_u u_t + B_x + B_u u_x. \tag{3.3.16}$$

Splitting equation (3.3.16) on derivatives of u yields

$$u_t : A_u = 1, \tag{3.3.17}$$

$$u_x : B_u = -u, \tag{3.3.18}$$

$$\text{rest} : A_t + B_x = 0. \tag{3.3.19}$$

Equation (3.3.17) gives

$$A = u + C(t, x)$$

with C an arbitrary function of t, x. Equation (3.3.18) gives

$$B = -\frac{1}{2}u^2 + D(t, x)$$

with D an arbitrary function of t, x. Substituting the values of A and B in (3.3.19), we obtain $C_t + D_x = 0$. We can take C and D as zero because they contribute to trivial part of conservation law. Thus, conservation law of Burgers equation (3.1.1) is

$$T^t = u,$$

$$T^x = -u_x - \frac{1}{2}u^2.$$

Remark: Since the multiplier $\Lambda = C_1$, one can conclude that Burgers equation is itself in conserved form.

3.4 Concluding remarks

In this Chapter, we studied the Burgers equation (3.1.1). Firstly, we computed Lie point symmetries of Burgers equation and then used them to find group-invariant solutions. Secondly, conservation laws were constructed using the multiplier method.

4. The coupled Burgers equations

4.1 Introduction

Coupled Burgers equations can describe a model of polydispersive sedimentation. These equations were derived in (Esipov, 1995) while studying the movement of two particle groups in fluid and how gravity affects them. When the particles spread and mix with the fluid, suspensions or colloids form depending on the size of particles. A continuity equation for small particles describing the conservation of species with concentration $c(t, x)$ and flux $J(t, x)$ is given by

$$\frac{\partial c}{\partial t} + \text{div}J = 0 \tag{4.1.1}$$

with

$$J = V(c)c - D(c)\nabla c, \tag{4.1.2}$$

where $V(c)$ represents Stokes velocity and $D(c)$ diffusion gradient. Fluids containing small particles form colloids and from (4.1.1) and (4.1.2), they represent the one-dimensional Burgers-like equation

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}[V(c)c] + \frac{\partial}{\partial x}\left[D(c)\left(\frac{\partial c}{\partial x}\right)\right]. \tag{4.1.3}$$

The Burgers equation describes many important phenomena such as hydrodynamic turbulence, vorticity transport and in shock waves where the velocity is balanced by diffusivity. For bimodal distribution of particle sizes consisting of two concentration (c_1, c_2) which satisfy the continuity equation (4.1.1) have their corresponding fluxes expressed as (Esipov, 1995)

$$J_i = V_i(c_1, c_2) - D_i(c_1, c_2)\frac{\partial c_i}{\partial x}. \tag{4.1.4}$$

The hindered velocities of the particle groups are given by

$$v_i = V_i\left(1 - \sum_{j=1}^{j=n} k_{ij}c_j\right) \quad \text{for } 1 \leq i \leq n, \tag{4.1.5}$$

where n is the number of particle groups. A method of computing the constants k_{ij} is described in (Batchelor, 1982). Using expansion at the two concentrations results in a system of two PDEs

$$\begin{aligned} \frac{\partial c_1}{\partial t} &= V_1\frac{\partial}{\partial x}\left[(1 - k_{11}c_1 - k_{12}c_2)c_1\right] + D_1\frac{\partial^2 c_1}{\partial x^2}, \\ \frac{\partial c_2}{\partial t} &= V_2\frac{\partial}{\partial x}\left[(1 - k_{21}c_1 - k_{22}c_2)c_2\right] + D_2\frac{\partial^2 c_2}{\partial x^2}, \end{aligned} \tag{4.1.6}$$

known as coupled Burgers equations, which is a generalization of the mono-dispersive Burgers-like equation (4.1.3).

In our work, we study a particular case of (4.1.6) with concentrations $c_1 = u(t, x)$ and $c_2 = v(t, x)$, $V_1 = V_2 = -1$, $D_1 = D_2 = 1$, and $k_{11} = k_{22} = 1$, $k_{12} = k_{21} = -1$. Substituting these values into (4.1.6), we obtain the coupled Burgers equations, namely

$$E = u_t + u_x - u_{xx} - 2uu_x + (uv)_x = 0, \tag{4.1.7}$$

$$F = v_t + v_x - v_{xx} - 2vv_x + (uv)_x = 0. \tag{4.1.8}$$

This system predicts the phase shift phenomena and its solution describes the motion of particles at the interface (Smith, 1966).

The results of this Chapter have been submitted for publication Khalique and Abdallah (2019).

4.2 Symmetries and symmetry reductions of coupled Burgers equations

4.2.1 Lie point symmetries of coupled Burgers equations.

To find Lie point symmetries of coupled Burgers equations (4.1.7)–(4.1.8), we employ the Lie classical method. The infinitesimal transformations of the Lie group with parameter a are given by

$$\begin{aligned}\bar{t} &= t + a\,\tau(t,x,u,v),\\ \bar{x} &= x + a\,\xi(t,x,u,v),\\ \bar{u} &= u + a\,\phi(t,x,u,v),\\ \bar{v} &= v + a\,\psi(t,x,u,v).\end{aligned}$$

The vector field is (2.5.3), namely

$$X = \tau(t,x,u,v)\frac{\partial}{\partial t} + \xi(t,x,u,v)\frac{\partial}{\partial x} + \phi(t,x,u,v)\frac{\partial}{\partial u} + \psi(t,x,u,v)\frac{\partial}{\partial v}. \tag{4.2.1}$$

Since the equations are of second order, we use the second prolongation $X^{[2]}$ (2.5.5) and solve the determining equations

$$X^{[2]}\,E\Big|_{E=0} = 0, \tag{4.2.2}$$

$$X^{[2]}\,F\Big|_{F=0} = 0, \tag{4.2.3}$$

to derive the Lie point symmetries. The expression of $X^{[2]}$ for the system is given by

$$X^{[2]} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial u} + \psi\frac{\partial}{\partial v} + \zeta_1^1\frac{\partial}{\partial u_t} + \zeta_1^2\frac{\partial}{\partial v_t} + \zeta_2^1\frac{\partial}{\partial u_x} + \zeta_2^2\frac{\partial}{\partial v_x} + \zeta_{22}^1\frac{\partial}{\partial u_{xx}} + \zeta_{22}^2\frac{\partial}{\partial v_{xx}}, \tag{4.2.4}$$

where ζ_1^1, ζ_1^2, ζ_2^1, ζ_2^2, ζ_{22}^1 and ζ_{22}^2 are as given in Chapter 2. The equations (4.2.2) and (4.2.3) yield

$$\zeta_1^1 + \zeta_2^1 - \zeta_{22}^1 - 2u_x\phi - 2u\zeta_2^1 + v_x\phi + u\zeta_2^2 + u_x\psi + v\zeta_2^1\Big|_{(4.1.7)-(4.1.8)} = 0, \tag{4.2.5}$$

$$\zeta_1^2 + \zeta_2^2 - \zeta_{22}^2 - 2v_x\psi - 2v\zeta_2^2 + v_x\phi + u\zeta_2^2 + u_x\psi + v\zeta_2^1\Big|_{(4.1.7)-(4.1.8)} = 0. \tag{4.2.6}$$

Substituting the values of $\zeta's$ from Chapter 2, in equations (4.2.5) and (4.2.6), replacing u_{xx} by $u_t + u_x - 2uu_x + uv_x + vu_x$, and v_{xx} by $v_t + v_x - 2vv_x + uv_x + vu_x$, and splitting on the derivatives of u and v, we get the set of determining equations:

$$\tau_x = 0, \tag{4.2.7}$$

$$\tau_u = 0, \tag{4.2.8}$$

$$\tau_v = 0, \tag{4.2.9}$$

$$\xi_u = 0, \tag{4.2.10}$$

$$\xi_v = 0, \tag{4.2.11}$$

$$2\xi_x - \tau_t = 0, \tag{4.2.12}$$

$$\psi_{uu} = 0, \tag{4.2.13}$$

$$\psi_{uv} = 0, \tag{4.2.14}$$

$$\psi_{vv} = 0, \quad (4.2.15)$$
$$\phi_{uu} = 0, \quad (4.2.16)$$
$$\phi_{uv} = 0, \quad (4.2.17)$$
$$\phi_{vv} = 0, \quad (4.2.18)$$
$$v\phi_x + (1 + u - 2v)\psi_x - \psi_{xx} + \psi_t = 0, \quad (4.2.19)$$
$$(-1 + 2u - v)\phi_x - u\psi_x + \phi_{xx} - \phi_t = 0, \quad (4.2.20)$$
$$2\phi - 6(u - v)\phi_v + 2u\psi_v - 2u\phi_u - 4\phi_{xv} + u\tau_t = 0, \quad (4.2.21)$$
$$2\psi - 2v\psi_v + 2v\phi_u + 6u\psi_u - 6v\psi_u - 4\psi_{xu} + v\tau_t = 0, \quad (4.2.22)$$
$$2\phi - 4\psi + 2v\phi_v - 2u\psi_u - 4\psi_{xv} + \tau_t + u\tau_t - 2v\tau_t - 2\xi_t + \tau_{tx} = 0, \quad (4.2.23)$$
$$4\phi - 2\psi + 2v\phi_v - 2u\psi_u + 4\phi_{xu} - \tau_t + 2u\tau_t - v\tau_t + 2\xi_t - \tau_{tx} = 0. \quad (4.2.24)$$

We now solve the above equations to find the values of τ, ξ, ϕ and ψ. From equations (4.2.7), (4.2.8) and (4.2.9), we get

$$\tau \equiv \tau(t) = a(t), \quad (4.2.25)$$

where $a(t)$ is an arbitrary function of t. Equations (4.2.10) and (4.2.11) imply that

$$\xi = \xi(t, x). \quad (4.2.26)$$

Substituting the value of τ in (4.2.12) and integrating on x yields

$$\xi = \frac{1}{2}a_t x + b(t), \quad (4.2.27)$$

where $b(t)$ is an arbitrary function of t. Equation (4.2.13) implies that

$$\psi = P(t, x, v)u + Q(x, t, v). \quad (4.2.28)$$

Substituting ψ in (4.2.14) and (4.2.15), yields

$$\psi = uP_1(t, x) + vQ_1(t, x) + Q_2(t, x), \quad (4.2.29)$$

where P_1, Q_1 and Q_2 are arbitrary functions of t and x.

Equation (4.2.16) implies that

$$\phi = R(t, x, v)u + S(t, x, v). \quad (4.2.30)$$

Substituting equation (4.2.17) and (4.2.18), we obtain

$$\phi = uR_1(t, x) + vS_1(t, x) + S_2(t, x). \quad (4.2.31)$$

Substituting the value of ψ and ϕ in (4.2.19) yields

$$v(uR_{1x}+vS_{1x}+S_{2x})+(1+u-2v)(uP_{1x}+vQ_{1x}+Q_{2x})-uP_{1xx}-vQ_{1xx}-Q_{2xx}+uP_{1t}+vQ_{1t}+Q_{2t} = 0. \quad (4.2.32)$$

Splitting (4.2.32) on u and v, yields

$$u^2 : P_{1x} = 0, \quad (4.2.33)$$
$$v^2 : S_{1x} - 2Q_{1x} = 0, \quad (4.2.34)$$
$$uv : R_{1x} + Q_{1x} - 2P_{1x} = 0, \quad (4.2.35)$$

$$u : P_{1x} + Q_{2x} - P_{1xx} + P_{1t} = 0, \tag{4.2.36}$$
$$v : S_{2x} + Q_{1x} - 2Q_{2x} - Q_{1xx} + Q_{1t} = 0, \tag{4.2.37}$$
$$1 : Q_{2x} - Q_{2xx} + Q_{2t} = 0. \tag{4.2.38}$$

Now substituting the value of ψ and ϕ in equation (4.2.20) gives

$$(-1+2u-v)(uR_{1x}+vS_{1x}+S_{2x})-u(uP_{1x}+vQ_{1x}+Q_{2x})+uR_{1xx}+vS_{1xx}+S_{2xx}-uR_{1t}-vS_{1t}-S_{2t} = 0. \tag{4.2.39}$$

Separating equation (4.2.39) on u and v, we get

$$u^2 : P_{1x} - 2R_{1x} = 0, \tag{4.2.40}$$
$$v^2 : S_{1x} = 0, \tag{4.2.41}$$
$$uv : R_{1x} + Q_{1x} - 2S_{1x} = 0, \tag{4.2.42}$$
$$u : R_{1x} + Q_{2x} - 2S_{2x} - R_{1xx} + R_{1t} = 0, \tag{4.2.43}$$
$$v : S_{1x} + S_{2x} - S_{1xx} + S_{1t} = 0, \tag{4.2.44}$$
$$1 : S_{2x} - S_{2xx} + S_{2t} = 0. \tag{4.2.45}$$

From (4.2.33), we have $P_1(t,x) = P_1(t)$. Thus, equation (4.2.36) gives $Q_2(t,x) = -xP_1'(t) + K(t)$.

Equation (4.2.38) gives

$$P_1'(t) + xP_1''(t) - K'(t) = 0. \tag{4.2.46}$$

Splitting (4.2.46) on powers of x we obtain,

$$x^1 : P_1''(t) = 0, \tag{4.2.47}$$
$$x^0 : P_1'(t) - K'(t) = 0. \tag{4.2.48}$$

Integrating (4.2.47) and (4.2.48) on t yields,

$$P_1(t) = A_1t + A_2, \tag{4.2.49}$$
$$K(t) = A_1t + A_3, \tag{4.2.50}$$

whereA_1, A_2 and A_3 are arbitrary constants. Substituting back to the expression of $P_1(t,x)$ and $Q_2(t,x)$, we have

$$P_1(t,x) = A_1t + A_2, \tag{4.2.51}$$
$$Q_2(t,x) = -xA_1 + A_1t + A_3. \tag{4.2.52}$$

Similarly, from equation (4.2.41) we have $S_1(t,x) = S_1(t)$ and from (4.2.40), $R_1(t,x) = R_1(t)$. Thus, equation (4.2.44) gives $S_2(t,x) = -xS_1'(t) + H(t)$.

Integrating $S_2(t,x)$ on t and x and substituting the integrals into (4.2.45) gives

$$S_1'(t) + xS_1''(t) - H'(t) = 0. \tag{4.2.53}$$

Splitting (4.2.53) on powers of x we obtain

$$x^1 : \; S_1''(t) = 0, \tag{4.2.54}$$
$$x^0 : S_1'(t) - H'(t) = 0, \tag{4.2.55}$$

which implies that

$$S_1(t) = A_4 t + A_5, \tag{4.2.56}$$

$$H(t) = A_4 t + A_6, \tag{4.2.57}$$

and so rewriting the expressions for $S_1(t,x)$ and $S_2(t,x)$, we have

$$S_1(t,x) = A_4 t + A_5, \tag{4.2.58}$$

$$S_2(t,x) = -xA_4 + tA_4 + A_6, \tag{4.2.59}$$

where A_4, A_5 and A_6 are arbitrary constants.

From equation (4.2.43) , integrating $R_1(t,x)$ on t gives

$$R_1(t,x) = (A_1 - 2A_4)t + A_7. \tag{4.2.60}$$

Substituting equations (4.2.29), (4.2.31) and (4.2.25) in equation (4.2.21), yields

$$2(uR_1 + vS_1 + S_2) - 6(u-v)S_1 + 2uQ_1 - 2uR_1 + ua'(t) = 0. \tag{4.2.61}$$

Splitting on u and v, we get

$$u : 2Q_1 - 6S_1 + a'(t) = 0, \tag{4.2.62}$$

$$v : 8S_1 = 0, \tag{4.2.63}$$

$$1 : 2S_2 = 0. \tag{4.2.64}$$

Substituting equations (4.2.29), (4.2.31) and (4.2.25) in equation (4.2.22), we obtain

$$2(uP_1 + vQ_1 + Q_2) - 2vQ_1 + 2vR_1 + 6uP_1 - 6vP_1 + va'(t) = 0. \tag{4.2.65}$$

Splitting on u and v, we get

$$u : 8P_1 = 0, \tag{4.2.66}$$

$$v : 2R_1 - 6P_1 + a'(t) = 0, \tag{4.2.67}$$

$$1 : 2Q_2 = 0. \tag{4.2.68}$$

From equations (4.2.62) and (4.2.67), integrating on t,we get

$$a(t) = (A_1 + A_4)t^2 - 2(A_7 + A_8)t + A_9. \tag{4.2.69}$$

Substituting equations (4.2.29), (4.2.31), (4.2.27) and (4.2.25) in equation (4.2.23), yields

$$2(uR_1+vS_1+S_2)-4(uP_1+vQ_1+Q_2)+2vS_1-2uP_1+a'(t)+ua'(t)-2va'(t)-2\left(\frac{1}{2}xa''(t)+b'(t)\right) = 0. \tag{4.2.70}$$

Splitting (4.2.70) on u and v, we get

$$u : 2R_1 - 6P_1 + a'(t) = 0, \tag{4.2.71}$$

$$v : 4S_1 - 4Q_1 - 2a'(t) = 0, \tag{4.2.72}$$

$$1 : 2S_2 - 4Q_2 + a'(t) - xa''(t) - 2b'(t) = 0. \tag{4.2.73}$$

Similarly, substituting equations (4.2.29), (4.2.31), (4.2.27) and (4.2.25) in equation (4.2.24), yields

$$4(uR_1+vS_1+S_2)-2(uP_1+vQ_1+Q_2)+2vS_1-2uP_1-a'(t)+2ua'(t)-va'(t)+2\left(\frac{1}{2}xa''(t)+b'(t)\right)=0 \tag{4.2.74}$$

Splitting (4.2.74) on u and v, yields

$$u : 4R_1 - 4P_1 + 2a'(t) = 0, \tag{4.2.75}$$
$$v : 6S_1 - 2Q_1 - a'(t) = 0, \tag{4.2.76}$$
$$1 : 4S_2 - 2Q_2 - a'(t) + xa''(t) + 2b'(t) = 0. \tag{4.2.77}$$

From equations (4.2.73) and (4.2.77), we have

$$a'(t) - xa''(t) - 2b'(t) = 0. \tag{4.2.78}$$

Splitting (4.2.78) on powers of x, we get

$$x^1 : a''(t) = 0, \tag{4.2.79}$$
$$x^0 : a'(t) - 2b'(t) = 0. \tag{4.2.80}$$

Using (4.2.79) in (4.2.69), implies that, $2(A_1 + A_4) = 0$ and so $a(t)$ will be given by

$$a(t) = -2(A_7 + A_8) + A_9. \tag{4.2.81}$$

Solving equation (4.2.80) and integrating on t gives

$$b(t) = -(A_7 + A_8)t + A_{10}. \tag{4.2.82}$$

Substituting (4.2.81) and (4.2.82) into equations (4.2.25) and (4.2.27), we finally have

$$\tau = -2c_1t + c_2, \tag{4.2.83}$$
$$\xi = -c_1x - c_1t + c_3, \tag{4.2.84}$$
$$\phi = c_1u, \tag{4.2.85}$$
$$\psi = c_1v, \tag{4.2.86}$$

where c_1, c_2 and c_3 are arbitrary constants. Thus, the infinitesimal symmetries of the coupled Burgers equations (4.1.7)–(4.1.8) are

$$X_1 = \frac{\partial}{\partial t}, \tag{4.2.87}$$
$$X_2 = \frac{\partial}{\partial x}, \tag{4.2.88}$$
$$X_3 = -2t\frac{\partial}{\partial t} - (x+t)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{4.2.89}$$

We note that the first symmetry X_1 represents translation in time and the second symmetry X_2 is translation in space variable x.

4.2.2 One-parameter groups.

We now present the corresponding group of transformations relating to each Lie point symmetry obtained above. This is achieved by using the Lie equations

$$\frac{d\bar{t}}{da} = \tau(\bar{t}, \bar{x}, \bar{u}, \bar{v}),\ \ \bar{t}|_{a=0} = t,\ \ \frac{d\bar{x}}{da} = \xi(\bar{t}, \bar{x}, \bar{u}, \bar{v}),\ \ \bar{x}|_{a=0} = x,$$
$$\frac{d\bar{u}}{da} = \phi(\bar{t}, \bar{x}, \bar{u}, \bar{v}),\ \ \bar{u}|_{a=0} = u,\ \ \frac{d\bar{v}}{da} = \psi(\bar{t}, \bar{x}, \bar{u}, \bar{v}),\ \ \bar{v}|_{a=0} = v.$$

Computing the one-parameter groups, for each X_i, we let T_{a_i} be the corresponding groups. Then the Lie equations give

$$\begin{aligned}
T_{a_1} &: \ (t + a_1, x, u, v),\\
T_{a_2} &: \ (t, x + a_2, u, v),\\
T_{a_3} &: \ \left(te^{-2a_3}, xe^{-a_3} + te^{-3a_3} - te^{-2a_3}, ue^{a_3}, ve^{a_3}\right).
\end{aligned}$$

4.2.3 Commutator table.

The set of all solutions of any determining equations forms a Lie algebra. Here, we present a commutator table for the symmetries X_1, X_2 and X_3. Using the relation (2.4.1), we get

$$\begin{aligned}
[X_1 X_2] &= X_1X_2 - X_2X_1 = 0,\\
[X_1 X_3] &= X_1X_3 - X_3X_1 = -2X_1 - X_2,\\
[X_2 X_3] &= X_2X_3 - X_3X_2 = -X_2.
\end{aligned}$$

The table below summarizes the above results:

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$-2X_1 - X_2$
X_2	0	0	$-X_2$
X_3	$2X_1 + X_2$	X_2	0

4.2.4 Symmetry reductions of coupled Burgers equations.

We now perform symmetry reductions for coupled Burgers equations (4.1.7)–(4.1.8). We consider the following four cases:

Case 1 $X_1 = \partial_t$

Solving the Lagrange equations associated with the operator X_1, we obtain three invariants

$$J_1 = x,\ \ J_2 = u \ \text{ and } J_3 = v.$$

Thus, the group-invariant solution is $u = f(x)$ and $v = g(x)$, where f and g are arbitrary functions. Substituting these values of u and v into (4.1.7)–(4.1.8) yields a system of second-order ordinary differential equations (ODEs)

$$f'(x) - f''(x) - 2f(x)f'(x) + (f(x)g(x))' = 0,$$

$$g'(x) - g''(x) - 2g(x)g'(x) + (f(x)g(x))' = 0.$$

Integration of the above system yields

$$f(x) - f'(x) - f(x)^2 + f(x)g(x) + C_1 = 0,$$
$$g(x) - g'(x) - g(x)^2 + f(x)g(x) + C_2 = 0$$

with C_1, C_2 constants.

Case 2 $X_2 = \partial_x$

The characteristic equations associated with this operator gives three invariants

$$J_1 = t, \;\; J_2 = u, \;\; \text{and } J_3 = v.$$

Hence, the group-invariant solution is $u = \psi(t)$ and $v = \Psi(t)$, where ψ and Ψ are arbitrary functions. Substituting these expressions into (4.1.7)–(4.1.8), we obtain the system of first-order ODEs

$$\psi'(t) = 0, \;\; \Psi'(t) = 0,$$

which on integration gives $\psi(t) = C_1$, $\Psi(t) = C_2$ where C_1 and C_2 are arbitrary constants. Thus, the group-invariant solution to (4.1.7)–(4.1.8) under X_2 is $u(t,x) = C_1$ and $v(t,x) = C_2$.

Case 3 $X_3 = -2t\,\partial_t - (x+t)\,\partial_x + u\,\partial_u + v\,\partial_v$

Solving the associated Lagrange equations for the symmetry X_3, we obtain three invariants

$$z = \frac{x}{\sqrt{t}} - \sqrt{t}, \;\; J_1 = u\sqrt{t}, \;\; J_2 = v\sqrt{t}.$$

Hence, the group-invariant solution is

$$u = \frac{f(z)}{\sqrt{t}} \text{ and } v = \frac{g(z)}{\sqrt{t}},$$

where f and g are arbitrary functions. Using these, system (4.1.7)–(4.1.8) reduces to the system of second-order ODEs

$$-\frac{1}{2}f - \frac{1}{2}zf' - f'' - 2ff' + (fg)' = 0,$$
$$-\frac{1}{2}g - \frac{1}{2}zg' - g'' - 2gg' + (fg)' = 0.$$

Case 4 $X_1 + cX_2 = \partial_t + c\partial_x$ (c a constant)

This symmetry operator will yield travelling wave solutions to the system (4.1.7)–(4.1.8). Associated to it are the three invariants $z = x - ct$, $J_1 = u$, $J_2 = v$. The invariant solutions are given by $u = f(z)$ and $v = g(z)$. Treating z as new independent variable and f and g as new dependent variables, system (4.1.7)–(4.1.8) transforms to the system of second-order ODEs

$$(1-c)f' - f'' - 2ff' + (fg)' = 0,$$
$$(1-c)g' - g'' - 2gg' + (fg)' = 0.$$

Integrating the system once gives,

$$
\begin{aligned}
(1-c)f - f' - f^2 + fg + C_1 &= 0, \\
(1-c)g - g' - g^2 + fg + C_2 &= 0,
\end{aligned}
$$

where C_1 and C_2 are constants. For the particular case, when $C_1 = C_2 = 0$ and $c = 1$, we can integrate the above system to obtain

$$
\begin{aligned}
u(t,x) &= \sqrt{c_1}\tanh\left[\sqrt{c_1}(x-t+c_2)\right], \\
v(t,x) &= \sqrt{c_1}\coth\left[\sqrt{c_1}(x-t+c_2)\right]
\end{aligned}
$$

with c_1, c_2 constants.

4.3 Conservation laws

We compute conservation laws for coupled Burgers equations (4.1.7)–(4.1.8) using two approaches; the multiplier approach and Ibragimov's approach.

4.3.1 Derivation of conservation laws using the multiplier approach.

We shall invoke the general method of multipliers for finding the conserved laws that are admitted by any differential equations.

For the coupled Burgers system (4.1.7)–(4.1.8) we seek all local conservation law multipliers. The determining equations to derive all conservation law multipliers of (4.1.7)–(4.1.8) are

$$
\begin{aligned}
\frac{\delta}{\delta u}\left[\Lambda^1 E + \Lambda^2 F\right] &= 0, \\
\frac{\delta}{\delta v}\left[\Lambda^1 E + \Lambda^2 F\right] &= 0,
\end{aligned}
\tag{4.3.1}
$$

where

$$
\begin{aligned}
\frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} - D_x^3\frac{\partial}{\partial u_{xxx}}, \\
\frac{\delta}{\delta v} &= \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial v_t} - D_x\frac{\partial}{\partial v_x} + D_x^2\frac{\partial}{\partial v_{xx}} - D_x^3\frac{\partial}{\partial v_{xxx}}
\end{aligned}
\tag{4.3.2}
$$

are the Euler operators. Furthermore, D_t and D_x are total derivatives given by

$$
\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tt}\frac{\partial}{\partial u_t} + v_{tt}\frac{\partial}{\partial v_t} + u_{tx}\frac{\partial}{\partial u_x} + v_{tx}\frac{\partial}{\partial v_x} + \dots, \\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + u_{xt}\frac{\partial}{\partial u_t} + v_{xt}\frac{\partial}{\partial v_t} + \dots.
\end{aligned}
$$

Here we seek to compute zeroth-order multipliers, i.e.,

$$\Lambda^m = \Lambda^m(t,x,u,v), \quad m = 1,2.$$

Expanding (4.3.1) and splitting on derivatives of u and v, yields the over-determined systems of seventeen linear PDEs

$$\Lambda^1_u = 0,\ \Lambda^2_v = 0,\ \Lambda^1_{uu} = 0,\ \Lambda^1_{uv} = 0,\ \Lambda^1_{xu} = 0,\ \Lambda^1_{vv} = 0,\ \Lambda^2_{uu} = 0,\ \Lambda^2_{uv} = 0,\ \Lambda^2_{vv} = 0,$$

$$\Lambda^2_{xv}=0,\ \Lambda^1_v-\Lambda^2_u=0,\ \Lambda^2_u+\Lambda^1_v=0,\ \Lambda^2_u-\Lambda^1_v=0,\ \Lambda^1_{xx}+\Lambda^1_x+\Lambda^1_t-2u\Lambda^1_x+v\Lambda^1_x+v\Lambda^2_x=0,$$
$$\Lambda^2_{xx}+\Lambda^2_t+\Lambda^2_x+u\Lambda^2_x+u\Lambda^1_x-2v\Lambda^2_x=0,$$
$$\Lambda^1_v+2\Lambda^1_{xv}-\Lambda^2_u+v\Lambda^1_v-2u\Lambda^1_v+v\Lambda^2_v-u\Lambda^2_u+2v\Lambda^2_u-u\Lambda^1_u=0,$$
$$\Lambda^2_u+2\Lambda^2_{xu}-\Lambda^1_v+u\Lambda^2_u+2u\Lambda^1_v-v\Lambda^1_v-v\Lambda^2_v-2v\Lambda^2_u+u\Lambda^1_u=0.$$

After some simple calculations, the solution of the above system of PDEs is

$$\Lambda^1=C_1,\ \Lambda^2=C_2, \tag{4.3.3}$$

where C_1 and C_2 are arbitrary constants. Thus, we obtain two sets of local conservation law multipliers given by

$$(\Lambda^1_1,\Lambda^1_2)=(1,0),\ (\Lambda^2_1,\Lambda^2_2)=(0,1). \tag{4.3.4}$$

The multipliers have to satisfy the property (2.6.11), given by

$$\Lambda^1E+\Lambda^2F=D_tT^t+D_xT^x$$

and so corresponding to the above multipliers we get conservation laws, whose conserved vectors are

$$T^t_1=u,$$
$$T^x_1=uv-u^2+u-u_x;$$

$$T^t_2=v,$$
$$T^x_2=uv-v^2+v-v_x.$$

Remark: Since the multiplier $\Lambda^1=C_1,\Lambda^2=C_2$, we can conclude that the coupled Burgers equations are themselves conservation laws.

4.3.2 Derivation of conservation laws with Ibragimov approach.

Here we derive conserved vectors for coupled Burgers equations (4.1.7)–(4.1.8) by using the new conservation theorem due to Ibragimov (Ibragimov, 2007).

For coupled Burgers equations, the adjoint equations (Ibragimov, 2007) are of the form

$$E^*\equiv(1-2u+v)f_x+vg_x+f_t+f_{xx}=0,$$
$$F^*\equiv(1-2v+u)g_x+uf_x+g_t+g_{xx}=0 \tag{4.3.5}$$

and the Lagrangian, L, is given by

$$L=f\left\{u_t+u_x-u_{xx}-2uu_x+(uv)_x\right\}+g\left\{v_t+v_x-v_{xx}-2vv_x+(uv)_x\right\}. \tag{4.3.6}$$

We now use the three point symmetries of the system (4.1.7)–(4.1.8) found in Section 3, namely

$$X_1=\frac{\partial}{\partial t},\ X_2=\frac{\partial}{\partial x},\ X_3=-2t\frac{\partial}{\partial t}-(x+t)\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}$$

and derive conservation laws associated with each of these symmetries.

(i) The first symmetry $X_1=\partial_t$ has Lie characteristic functions that are given by $W^1=-u_t$ and $W^2=-v_t$. Hence, applying Ibragimov theorem (Ibragimov, 2007), the conserved vector (T^t_1,T^x_1) is given by

$$T^t_1=u_xfv+v_xfu+u_xf-2u_xfu-u_{xx}f+u_xgv+v_xgu+v_xg-2v_xgv-v_{xx}g,$$

$$T_1^x = -u_t f v - v_t f u - u_t f + 2u_t f u + f u_{xt} - u_t g v - v_t g u - v_t g + 2v_t g v + g v_{xt} - f_x u_t - g_x v_t.$$

This gives energy conservation law for the system (4.1.7)–(4.1.8).

(ii) For the second symmetry $X_2 = \partial_x$, Lie characteristic functions are $W^1 = -u_x$ and $W^2 = -v_x$. Hence, by Ibragimov theorem (Ibragimov, 2007), T_2^t, T_2^x are

$$\begin{aligned} T_2^t &= -f u_x - g v_x, \\ T_2^x &= f u_t + g v_t - f_x u_x - g_x v_x, \end{aligned}$$

which provides linear momentum conservation law for (4.1.7)–(4.1.8).

(iii) The third symmetry $X_3 = -2t\partial_t - (x+t)\partial_x + u\partial_u + v\partial_v$, has Lie characteristic functions $W^1 = u + 2tu_t + (x+t)u_x$, $W^2 = v + 2tv_t + (x+t)v_x$ and consequently the application of Ibragimov's theorem (Ibragimov, 2007) yields

$$\begin{aligned} T_3^t =& -2tv_x f u - 2tu_x f v + 4tu_x f u - tu_x f + xu_x f + 2tu_{xx} f - 2tv_x g u - 2tu_x g v - tv_x g + xv_x g \\ &+ 4tv_x g v + 2tv_{xx} g + f u + g v, \\ T_3^x =& 2tv_t f u + 2tu_t f v - 4tu_t f u + f_x u + tu_t f - xu_t f - 2u_x f - 2tf u_{xt} + 2tv_t g u + 2tu_t g v + tv_t g \\ &- xv_t g - 4tv_t g v + g_x v - 2v_x g - 2tg v_{xt} + 2f u v - 2f u^2 + f u + 2g u v - 2g v^2 + g v + 2tf_x u_t \\ &+ tf_x u_x + xf_x u_x + 2tg_x v_t + tg_x v_x + xg_x v_x. \end{aligned}$$

Remark. Due to the appearance of arbitrary functions $f(t,x)$ and $g(t,x)$, the conservation laws obtained by this method are infinitely many.

4.4 Concluding remarks

In this Chapter, we studied the coupled Burgers equations (4.1.7)–(4.1.8) that model polydispersive sedimentation from the viewpoint of Lie symmetry analysis. Firstly, Lie point symmetries of the system were determined and after that symmetry reductions were performed. Also, travelling wave solutions were constructed with the aid of translation symmetries in time and space. Secondly, conservation laws were derived for the underlying system in two ways; multiplier approach and conservation theorem due to Ibragimov. The derived conservation laws included the conservation of energy and linear momentum for the system (4.1.7)–(4.1.8).

5. Conclusion

In this dissertation, we presented applications of Lie symmetry methods to a partial differential equation and system of partial differential equations.

In Chapter one, Introduction, we gave a very brief history of Lie symmetry methods and described the structure of the dissertation.

Chapter two dealt with some preliminaries concerning Lie symmetry methods and conservation laws, which were utilised in the remainder of the dissertation.

In Chapter three, we studied the Burgers equation as an illustrative example. We calculated some group-invariant solutions using its Lie point symmetries and derived conservation laws by employing the multiplier approach.

Finally, in Chapter four, we investigated a system of coupled Burgers equations that model polydispersive sedimentation. We performed Lie symmetry analysis on the system and obtained symmetry reductions. Travelling wave solutions were also constructed using its translation symmetries in time and space. Moreover, we derived conservation laws of the system in two ways; by employing the multiplier method and the conservation theorem due to Ibragimov.

In future work, we intend to construct an optimal system of one-dimensional sub-algebras of the coupled Burgers equations. Thereafter, we shall obtain group-invariant solutions based on this optimal system of one-dimensional sub-algebras.

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