A Galois connection in the quasi-metrization of an ordered metric space

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24 October 2019

Submitted in partial fulfillment of a structured masters degree at AIMS South Africa

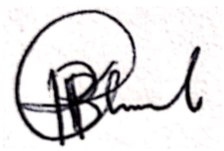
African Institute for Mathematical Sciences
SOUTH AFRICA

Abstract

In this essay, we investigate the quasi-metrization in the context of partially ordered metric spaces. Moreover, we outline an interesting Galois connection between $\mathcal{M}(X)$, the set all of pairs $(m, \leq)$, where m is a metric on X and $\leq$ a partial order on X, and $\mathcal{Q}(X)$ the set of all T_0-quasi-metrics on X.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



Precious Blessing Khumalo, 24 October 2019

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1. Introduction

An elementary but fundamental problem in general topology is to find which topological spaces are induced by metrics.

This work aims at generalizing this result for topological spaces equipped with a preorder $\leq$. In this case one would like to prove the existence of a certain function $p : X \times X \to [0, \infty)$ which contains both the topology and the order, where the order is obtained using the condition $p(x, y) = 0 \iff x \leq y$ [1].

For motivations given by the applications, it is important to establish if one can just work with quasi-pseudo-metrizable preordered spaces instead of more general topological preordered spaces.

In this work, we will investigate quasi-pseudometrizability in the setting of ordered metric spaces. Our question of interest here is the following:

Problem 1.0.1. Given a partially ordered metric space $(X, m, \leq)$, when does there exist a T_0-quasi-metric d on X such that whenever $x, y \in X$

$$d^s(x, y) := \max\{d(x, y), d(y, x)\} = m(x, y)$$

and

$$d(x, y) = 0 \iff x \leq y \ \ ?$$

In a more refined way, the problem can be rephrased as follow:

The obvious link: If d is a T_0-quasi-metric on a set X, then $(X, d^s, \leq_d)$ is a partially ordered metric space.

Question: Is there a link in the opposite direction, i.e. if $(X, m, \leq)$ is a partially ordered metric space, can we manufacture a T_0-quasi-metric d on X, using m and $\leq$, which is related to m and $\leq$ in a "reasonable" way?

The rest of this essay is organised as follows. In Chapter 2, we give the building blocks by stating terminologies that we will be using in the essay. We first give terminologies for quasi-pseudometric and ordered space, also give examples in each case then we conclude the chapter by showing interdependence between the two.

In Chapter 3, we discuss the uniqueness and existence of the inducing T_0-quasi-pseudometric space using the interval condition (or linear condition). Proposition 3.2.3 gives the summary of results in this chapter.

In Chapter 4, we construct a quasi-pseudometric in the general setting where the order is arbitrary. Proposition 4.2.3 gives the necessary and sufficient conditions under which the constructed quasi-pseudometric answers Problem 1.0.1.

In Chapter 5, we use the results found in the previous two chapters to establish a Galois connection which is defined in Definition 5.1.1, in the general setting and in our context.

[1] We shall say later that this refers to the specialisation order.

2. The buildings blocks

In this chapter we give some definitions, fix upon some notations and give results that we will later refer to in the essay. We use James (2012) and Willard (2004) as references for background results from topology.

To present this essay as a self-contained paper, we also give terminologies on order theory; the material is mainly taken from Schröder (2003).

We start this section by stating definitions in the theory of asymmetric topology.

2.1 T_0-quasi-metric spaces and their topologies

Definition 2.1.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a **quasi-pseudometric** on X if:

i) $d(x, x) = 0$ whenever $x \in X$,

ii) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

Given the nonempty set X and d quasi-pseudometric, we say the pair (X, d) is a quasi-pseudometric space.

Moreover, if

iii) $d(x, y) = 0 = d(y, x) \Longrightarrow x = y$, then d is said to be a T_0-**quasi-metric**. The latter condition is referred as the T_0-condition.

Given the nonempty set X and d a T_0-quasi-metric, we say the pair (X, d) is a T_0-quasi-pseudometric space or di-space.

A quasi-pseudometric d on a set X which has the following properties:

iv) $d(x, y) = d(y, x)$ whenever $x, y \in X$ is called a **pseudo-metric** on X;

v) $d(x, y) = 0 \Longrightarrow x = y$ whenever $x \in X$ is called a **quasi-metric** on X.

Finally, a T_0-quasi-metric which satisfies iv) is a **metric**.

If the function d takes the values $[0, \infty]$ (i.e. in particular ∞), then we say d is an *extended quasi-pseudometric* with the value ∞.

Definition 2.1.2. Let (X, d) be a quasi-pseudometric space. The **conjugate** (or ***dual***) of d is the function denoted d^{-1} and defined, whenever $x, y \in X$ by

$$d^{-1}(x, y) = d(y, x).$$

Remark 2.1.3. Considering the function d^s defined Problem 1.0.1, whenever $x, y \in X$,

$$d^s(x, y) = \max\{d(x, y), d(y, x)\},$$

defines a metric on X whenever d is a T_0-quasi-metric.

Proof.

Let $x, y, z \in X$.

ia)

$$\begin{aligned} d^s(x,x) &= \max\{d(x,x), d(x,x)\} \\ &= \max\{0,0\} \\ &= 0, \end{aligned}$$

i.e. $d^s(x,x) = 0$ whenever $x \in X$.

ib)

$$\begin{aligned} d^s(x,y) = 0 &\iff 0 \le \max\{d(x,y), d(y,x)\} = 0 \\ &\iff d(x,y) = 0 \text{ and } d(y,x) = 0 \\ &\iff x = y \text{ since } d \text{ is } T_0, \end{aligned}$$

i.e. $d^s(x,y) = 0 \implies x = y$ whenever $x, y \in X$.

ii)

$$\begin{aligned} d^s(x,y) &= \max\{d(x,y), d(y,x)\} \\ &= \max d(y,x), d(x,y) \\ &= d^s(y,x) \end{aligned}$$

i.e. $d^s(x,y) = d^s(y,x)$ whenever $x, y \in X$.

iii)

$$d(x,z) \le d(x,y) + d(y,z) \le d^s(x,y) + d^s(y,z)$$

and

$$d(z,x) \le d(z,y) + d(y,x) \le d^s(z,y) + d^s(y,x).$$

Combining the above inequalities, we get

$$d^s(z,x) \le d(z,y) + d(y,x) \le d^s(z,y) + d^s(y,x).$$

□

We conclude this introductory part by saying a few words on the topology of quasi-pseudometric spaces. The paper by Künzi (2001) gives useful information that is used in this section.

Definition 2.1.4. Let (X, d) be a quasi-pseudometric space. Then for each $x \in X$ and $\epsilon > 0$, the set

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$$

denotes the *open ϵ-ball centered at x, of radius ε with respect to d*. It should be noted that the collection

$$\{B_d(x, \epsilon) : x \in X, \epsilon > 0\}$$

yields a base for the topology $\tau(d)$ induced by d on X. In a similar manner, for each $x \in X$ and $\epsilon \geq 0$, we define

$$C_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\},$$

known as the *closed ϵ-ball centered at x, of radius ϵ with respect to d*. The collection

$$\{B_{d^{-1}}(x, \epsilon) : x \in X, \epsilon > 0\}$$

also yields a base for the topology $\tau(d^{-1})$ induced by d^{-1} on X. The set $C_d(x, \epsilon)$ is $\tau(d^{-1})$-closed, but not $\tau(d)$-closed in general.

Hence a quasi-pseudometric space is a natural example of a bitopological space since it comes simultaneously with the topology $\tau(d)$ which is called *forward topology*, and the topology $\tau(d^{-1})$ which is called *backward topology*.

Definition 2.1.5. A map $f : (X, d) \to (Y, e)$ between the quasi-pseudometric spaces (X, d) and (Y, e) is called *isometric* provided that $d(x, y) = e(f(x), f(y))$ whenever $x, y \in X$.

A T_0-quasi-metric space (X, d) is said to be *bicomplete* if (X, d^s) is a complete metric space.

A T_0-quasi-metric space (X, d) is said to be *sup-separable* if (X, d^s) is a separable metric space.

2.1.1 Examples. The following are examples of quasi-pseudometric spaces.

A fundamental example in the theory of quasi-metric spaces is the "truncated difference".

Example 2.1.6. (See Künzi (2001)) Let x and y be non negative real numbers. We shall write $x \dot{-} y$ for $max\{x - y, 0\}$. It should be noted that $u(x, y) = x \dot{-} y$ with $x, y \in [0, \infty)$ defines the standard T_0-quasi-metric on $[0, \infty)$. Thus $([0, \infty), u)$ is a T_0-quasi-metric space.

Proof.

For $x, y \in [0, \infty)$, let u be defined as

$$u(x, y) = x \dot{-} y = \max\{x - y, 0\}.$$

i) $d(x, x) = x \dot{-} x = \max\{x - x, 0\} = \max\{0, 0\} = 0.$

ii) For the T_0-condition, we have

$$d(x, y) = 0 \Rightarrow x - y \leq 0 \Rightarrow x \leq y,$$

$$d(y,x) = 0 \Rightarrow y - x \leq 0 \Rightarrow y \leq x,$$

therefore,

$$x = y.$$

iii) For the triangle inequality, without loss of generality, we distinguish the following three cases:

1. Case 1: $x \leq y \leq z$

$$d(x,y) = 0;\ d(y,z) = 0;\ d(x,z) = 0,$$

since

$$0 \leq 0 + 0,$$

therefore,

$$d(x,z) \leq d(x,y) + d(y,z).$$

2. Case 2: $z \leq x \leq y$

$$d(x,y) = 0;\ d(y,z) = y - z;\ d(x,z) = x - z,$$

since

$$x - z \leq x - y + y - z \leq 0 + y - z,$$

therefore,

$$d(x,z) \leq d(x,y) + d(y,z).$$

3. Case 3: $x \leq z \leq y$

$$d(x,y) = 0;\ d(y,z) = y - z;\ d(x,z) = 0,$$

since

$$0 \leq 0 + y - z,$$

therefore,

$$d(x,z) \leq d(x,y) + d(y,z).$$

□

Another very important example in this theory is the ***Sorgenfrey line***.

Example 2.1.7. (See Künzi (2001))

$$d(x,y) = \begin{cases} \min\{1, y-x\} & \text{, if } x \leq y, \\ 1 & \text{, otherwise.} \end{cases}$$

In this case d induces a T_1-topology $\mathcal{T}$ on $\mathbb{R}$ whose base consists of all left balls centred at $x \in \mathbb{R}$. The left balls are of the form $B_d(x,\epsilon) = [x, x+\epsilon)$ where $x \in \mathbb{R}$ and $0 < \epsilon < 1$ (note that for any $x \in \mathbb{R}$ and $\epsilon \geq 1$, $B_d(x,\epsilon) = \mathbb{R}$). The topological space $(\mathbb{R}, \mathcal{T})$ is called the ***Sorgenfrey line***, and is a well-known object in topology and a source of many counter-examples. The associated metric d^s is the discrete metric.

2.1.2 Asymmetric normed spaces. Important examples of quasi-metrics are provided by asymmetric norms,

Definition 2.1.8. Let X be a nonempty real vector space. A function $n : X \to [0,\infty)$ is called an asymmetric norm if:

i) $n(x) = n(-x) = 0 \Rightarrow x = 0$,

ii) $n(\alpha x) = \alpha n(x)$, $\alpha \geq 0$

iii) $n(x+y) \leq n(x) + n(y)$.

For a nonempty real vector space X and an asymmetric norm n, we call the pair (X, n) an asymmetric normed space.

The following is an example of the use of an asymmetric norm on $\mathbb{R}$.

Example 2.1.9. (See Conradie and Künzi (2018)) An asymmetric norm n induces a T_0-quasi-metric d_n on X defined by

$$d_n(x,y) = n(x-y) \quad \text{for all } x, y \in X.$$

The associated partial order is defined by

$$x \leq_n y \quad \text{if and only if } n(x-y) = 0.$$

2.2 Ordered sets

In this section, we provide some background from order theory. The material can be read in Schröder (2003).

Definition 2.2.1. Let X be a nonempty set. A binary relation $\leq \subseteq X \times X$ on a set X is called a **preorder** if it is:

i) reflexive, that is, $x \leq x$, whenever $x \in X$,

ii) transitive, that is, if $x \leq y$ and $y \leq z$, then $x \leq z$ whenever $x, y, z \in X$.

A preordered set is a set equipped with a preorder, that is, a set on which a preorder is given.

Moreover, if the preorder $\leq$ is

iii) antisymmetric, that is, if $x \leq y$ and $y \leq x$, then $x = y$, whenever $x, y \in X$,

it is called a **partial order**.

A set A together with a partial ordering $\leq$ is called a p*artially ordered set* or *poset*. The poset is denoted as $(A, \leq)$.

Example 2.2.2. The inclusion relation $\subseteq$ is a partial ordering on the power set of a set A. Indeed, since for every set S, we have $S \subseteq S$, $\subseteq$ is reflexive. If $S \subseteq R$ and $R \subseteq S$ then $R = S$, which means $\subseteq$ is anti-symmetric. It is transitive as $R \subseteq S$ and $S \subseteq T$ implies $R \subseteq T$. Hence, $\subseteq$ is a partial ordering on $\mathcal{P}(S)$, and $(\mathcal{P}(S), \subseteq)$ is a poset.

Definition 2.2.3. A partial order will be called a **linear order** (or **total order**) if it is

iv) decisive, that is, $x \leq y$ or $y \leq x$, whenever $x, y \in X$.

A linearly ordered set also called a chain is a set equipped with a linear order, that is, a set on which a linear order is defined.

Remark 2.2.4. Moreover, a set that is partially ordered by $\leq$ but not linearly ordered may have linearly ordered subsets. Such subsets are also called chains or $\leq$-chains. For example, let us order $\mathcal{P}(\{a, b, c\})$ with respect to the subset relation. Since

$$\emptyset \subseteq \{a\} \subseteq \{a, b\} \subseteq \{a, b, c\},$$

the set $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$ is a chain of length 3 in $\mathcal{P}(\{a, b, c\})$.

Remark 2.2.5. Given a partial order $\leq$ on a set X, we can always define the dual order $\leq' = \geq$ of $\leq$ as :

$$x \leq y \iff y \geq x \iff y \leq' x$$

whenever $x, y \in X$. The poset $(X', \leq')$ where $X' = X$ is called the dual of the former one. To avoid any confusion, we point out that for a poset $(X, \leq_X)$,

$$x \leq_X y \iff y \geq_X x \iff y \leq'_{X'} x$$

whenever $x, y \in X = X'$.

Definition 2.2.6. Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two posets. A function, $f : X \to Y$, is *monotonic* (or *order-preserving*) if and only if for all $a, b \in X$,

$$\text{if} \quad a \leq_X b \quad \text{then} \quad f(a) \leq_Y f(b).$$

Remark 2.2.7. Furthermore, a function f of one poset $(X, \leq_X)$ to another $(Y, \leq_Y)$ is *order-reversing* if it is an *order-preserving* function of $(X, \leq_X)$ to the dual $(Y', \leq'_{Y'} = \geq_Y)$ of $(Y, \leq_Y)$.

It is important to show the interdependence (the dependence) between quasi-pseudometrics and partial orders.

Definition 2.2.8. Let (X, d) be a T_0-quasi-metric space. We associate to the quasi-pseudometric d the binary relation denoted $\leq_d$ and defined by

$$x \leq_d y \Longleftrightarrow d(x, y) = 0.$$

We can show that $\leq_d$ is a partial order; we call it **the specialization order of** d.

Proof.

$$x \leq_d y \Longleftrightarrow d(x, y) = 0.$$

i) Since $d(x, x) = 0$, then $x \leq_d x$.

ii) If $x \leq_d y$ and $y \leq_d x$, then

$d(x, y) = 0 = d(y, x)$ and since d is T_0, then $x = y$.

ii) If $x \leq_d y$ and $y \leq_d z$, then $d(x, y) = 0 = d(y, z)$ and since $d(x, z) \leq d(x, y) + d(y, z)$, we have that $d(x, z) = 0$, i.e $x \leq_d z$..

□

Thus we can associate a partial order to every T_0-quasi-metric. We can also find a T_0-quasi-metric associated with a partial order in the following definition:

Definition 2.2.9. Let X be a set and $\leq$ be a partial order on X. The map $d_{\leq}$ defined on $X \times X$ defined

$$d_{\leq}(x, y) = \begin{cases} 0 & , \text{ if } x \leq y, \\ \infty & , \text{ otherwise}, \end{cases}$$

is an extended T_0-quasi-metric on X.

3. The interval conditions

In this chapter, we look at the simplest case of our problem, i.e. the case where the order on the ordered metric space is linear. We give an exposé of the investigation by Gaba and Künzi (2015, 2016); Gaba (2016).

We begin by giving the following definition.

Definition 3.0.1. (See Gaba and Künzi (2016)) Let $(X, m, \leq)$ be a partially ordered metric space. We say that a T_0-quasi-metric d **induces** $(X, m, \leq)$ if

$$\leq_d = \leq \quad \text{and} \quad d^s = m.$$

When such a T_0-quasi-metric d exists, we shall say that the ordered metric space $(X, m, \leq)$ is **induced** by d or that d is an **inducing** T_0-quasi-metric for the ordered metric space $(X, m, \leq)$.

3.1 Uniqueness of the inducing quasi-metric

We start with answering a common question in mathematics, that of uniqueness. We recall a result from Gaba and Künzi (2015) but propose an alternative proof.

Proposition 3.1.1. *(Compare (Gaba, 2016, Proposition 1.1.1.))*

Let $(X, m, \leq)$ be a linearly ordered metric space. If there exists a T_0-quasi-metric d which induces $(X, m, \leq)$, then d is unique in the sense that if d_1 and d_2 induce $(X, m, \leq)$, then $d_1 = d_2$ or $d_1 = d_2^{-1}$.

Proof. Suppose that there exist d_1, d_2 which induce $(X, m, \leq)$.

Then,

$$d_1^s(x, y) = m(x, y) = d_2^s(x, y)$$

i.e.

$$\max\{d_1(x, y), d_1(y, x)\} = \max\{d_2(x, y), d_2(y, x)\} = m(x, y),$$

and

$$x \leq_{d_1} y \iff x \leq y \iff x \leq_{d_2} y;$$

or

$$x \leq_{d_1} y \iff x \leq y \iff x \leq_{d_2^{-1}} y.$$

This equality gives rise to four cases,

1. <u>Case 1</u>, $d_1(x, y) = d_2(x, y)$;

2. Case 2, $d_1(y,x) = d_2(y,x)$;
3. Case 3, $d_1(y,x) = d_2(x,y)$;
4. Case 4, $d_1(x,y) = d_2(y,x)$.

Case 1 and Case 2 give us that $d_1 = d_2$, so the proof is complete.

Case 3 and Case 4 give us that $d_1 = d_2^{-1}$, so the proof is complete.

The original proof, more constructive, can be read in Gaba (2016), and goes as follows:

The map d defined by

$$d(x,y) = \begin{cases} 0 & \text{, if } x \leq y, \\ m(x,y) & \text{, otherwise,} \end{cases}$$

is the only choice for d since $x \leq y \implies d(x,y) = 0$ and $x > y$ certainly implies that $d^s(x,y) = \max\{d(x,y), d(y,x)\} = m(x,y)$. Since for $x > y$, $d(y,x) = 0$, it follows that $m(x,y) = \max\{d(x,y), 0\} = d(x,y)$, because $m(x,y) > 0$. □

Remark 3.1.2. The second proof has an advantage as it points out that there is only one candidate. The candidate however need not be a quasi-pseudometric, since it does not always satisfy the triangle inequality.

3.2 d is a T_0-quasi-metric

In this subsection, we show the derived necessary and sufficient conditions for the existence of an inducing T_0-quasi-metric d. We do that by showing the candidate map d defined above satisfies the triangle inequality.

If we can find that such a T_0-quasi-metric d exists, then for any $x, y, z \in X$, we must have

$$d(x,z) \leq d(x,y) + d(y,z).$$

Given a linear order $\leq$, and $x, y, z \in X$, without loss of generality, we can assume that x, y and z are pairwise distinct (i.e. $x \neq y \neq z, x \neq z$). There are six ways of arranging x, y, z with respect to this linear order, namely:

$$z < x < y;\ \ y < z < x;\ \ x < y < z;\ \ x < z < y;\ \ y < x < z;\ \ z < y < x.$$

Computing the triangle inequality for each of this arrangement gives the following independent conditions, for all $x, y, z \in X$,

$$(\mathcal{C}) \quad \begin{cases} z < x < y \implies m(x,z) \leq m(y,z) & (\mathcal{C}_1), \\ y < z < x \implies m(x,z) \leq m(x,y) & (\mathcal{C}_2). \end{cases}$$

Writing the conditions $(\mathcal{C})$ in a natural way, we obtain that for all $x, y, z \in X$,

$$\boxed{(\mathcal{C}_1) \iff (x < y < z \implies m(y,x) \leq m(z,x))}.$$

$$\boxed{(\mathcal{C}_2) \iff (x < y < z \implies m(z,y) \leq m(z,x))}.$$

Remark 3.2.1. It was pointed out in (Gaba, 2016, Remark 1.1.2) the conditions $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ are not equivalent.

Lemma 3.2.2. *((Gaba and Künzi, 2015, Lemma 1))(Interval condition) Let $(X, m, \leq)$ be a partially ordered metric space. If there exists a T_0-quasi-metric d which induces $(X, m, \leq)$, then for any $x, y, z \in X$ such that $x \leq y \leq z$, we have that*

$$\max\{m(x,y), m(y,z)\} \leq m(x,z).$$

Proof. Suppose that a T_0-quasi-metric d on X exists and induces $(X, m, \leq)$ a linearly ordered metric space. Let $x, y, z \in X$, and assume that $x \leq y \leq z$.

Using Proposition 3.1.1, we have,

If $x \leq y$ then $d(x,y) = 0$ and $d(y,x) = m(x,y)$. Therefore $y \geq x$ implies that $d^s(x,y) = \max\{d(x,y), d(y,x)\} = \max\{d(x,y), 0\} = d(x,y) = m(x,y)$.

If $y \leq z$ then $d(y,z) = 0$ and $d(z,y) = m(y,z)$. Therefore $y \geq z$ implies that $d^s(y,z) = \max\{d(y,z), d(z,y)\} = \max\{d(y,z), 0\} = d(y,z) = m(y,z)$.

If $x \leq z$ then $d(x,z) = 0$ and $d(z,x) = m(x,z)$. Therefore $x \geq z$ implies that $d^s(x,z) = \max\{d(x,z), d(z,x)\} = \max\{d(x,z), 0\} = d(x,z) = m(x,z)$.

Thus

$$m(y,z) = d(z,y) \leq d(z,x) + d(x,y) = m(x,z) + 0 = m(x,z).$$

and,

$$m(x,y) = d(y,x) \leq d(y,z) + d(z,x) = 0 + m(x,z) = m(x,z).$$

We have,

$$m(x,y) \leq m(x,z) \text{ and } m(y,z) \leq m(x,z)$$

therefore $\max\{m(x,y), m(y,z)\} \leq m(x,z)$ □

The interval condition 3.2.2 actually characterizes linearly ordered metric spaces that are induced by a T_0-quasi-metric.

Proposition 3.2.3. *((Gaba and Künzi, 2015, Proposition 2)) Suppose that $\leq$ is a linear order on a metric space (X, m). Then there exists a T_0-quasi-metric d on X that induces the linearly ordered metric space $(X, m, \leq)$ if and only if m satisfies the following condition: For any $x, y, z \in X$ we have that $x \leq y \leq z$ implies that $m(y,z) \leq m(x,z)$ and $m(x,y) \leq m(x,z)$.*

The above proposition gives an overview of the results in this section "Interval conditions" as it gives the necessary and sufficient condition to which the T_0-quasi-metric space d induces a linearly ordered metric space $(X, m, \leq)$. Applications of this proposition are given in the following examples.

Example 3.2.4. ((Gaba, 2016, Example 1.1.2.)) Let $X = \{a, b, c\}$ be equipped with the linear order $\leq$ such that $a < b < c$ and define the metric m by: $m(a, b) = 2$, $m(b, c) = 3$, $m(a, c) = 4$ and $m(a, a) = 0$ whenever $x \in X$. Alternatively the metric m can be defined by a matrix,

$$M = (m_{i,j})_{\substack{1 \leq i \leq 3 \\ 1 \leq j \leq 3}} = \begin{pmatrix} 0 & 2 & 4 \\ 2 & 0 & 3 \\ 4 & 3 & 0 \end{pmatrix}$$

where $m_{i,j}$ is always the j-th entry of m_i.

We can see that m satisfies conditions in Proposition 3.2.3 $2 = m(a, b) \leq m(a, c) = 4$ and $3 = m(b, c) \leq 4 = m(a, c)$. The interval condition is indeed satisfied since

$$3 = \max\{m(a, b), m(b, c)\} \leq m(a, c) = 4.$$

We then know that the T_0-quasi-metric d given by $d(a, b) = d(b, c) = d(a, c) = 0$ see proof by Gaba 3.1.1, $d(b, a) = 2, d(c, b) = 3, d(c, a) = 4$ and $d(x, x) = 0$ whenever $x \in X$, induces $(X, m, \leq)$.

The following example shows that linearity is a necessary condition.

Example 3.2.5. ((Gaba, 2016, Example 1.1.3.)) Let $X = \{a, b, c\}$ be equipped with the linear order $\leq$ such that $a < b < c$ and the metric m such that $m(a, b) = 2$, $m(b, c) = 2$, $m(a, c) = 1$ and $m(a, a) = 0$ whenever $x \in X$. Alternatively the metric m can be defined by a matrix,

$$M = (m_{i,j})_{\substack{1 \leq i \leq 3 \\ 1 \leq j \leq 3}} = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

where $m_{i,j}$ is always the j-th entry of m_i.

Then any inducing T_0-quasi-metric d on X would satisfy

$$2 = m(b, a) = d(b, a) \leq d(b, c) + d(c, a) = 0 + m(c, a) = 1$$

which is a contradiction. So $(X, m, \leq)$ is not induced by any T_0-quasi-metric on X. Observe also that since $m(a, b) = 2 > 1 = m(a, c)$ and the interval condition is not satisfied and this can be seen from the above contradiction. This conclusion can be drawn from the inspection of the conditions in Proposition 3.2.3 that m needs to satisfy: $2 = m(a, b) \leq m(a, c) = 1$ and $2 = m(b, c) \leq m(a, c) = 1$.

4. The general case

In this chapter, we present a general theory to solve our problem. We are now interested in the case where the order on the metric space X is not necessarily linear but arbitrary.

In this part, we describe the main construction of a inducing T_0-quasi-metric and we outline the major results, as presented in Gaba and Künzi (2015).

4.1 A construction

We next describe the main construction of Gaba and Künzi (2015):

Let $(X, m, \leq)$ be a partially ordered metric space. Given $x, y \in X$, we set $r(x, y) = m(x, y)$ if $x \not\leq y$ and $r(x, y) = 0$ if $x \leq y$.

For each $x, y \in X$, set

$$D_{(m,\leq)}(x, y) = \inf \left\{ \sum_{i=0}^{n-1} r(x_i, x_{i+1}) : x_0, x_1, \ldots, x_n \in X, x_0 = x, x_n = y, n \in \mathbb{N} \right\}.$$

Then $D_{(m,\leq)}$ is a quasi-pseudometric on X, but it need not be a T_0-quasi-metric (see (Gaba and Künzi, 2016, Example 7)). Let us note that $D^{-1}_{(m,\leq)} = D_{(m,\geq)}$. The following results are proved in Gaba and Künzi (2015).

Lemma 4.1.1. *(Gaba and Künzi, 2015, Remark 3) The distance function $D_{(m,\leq)}$ is a quasi-pseudometric on X.*

Proof. For all $x, y \in X$

$$r(x, y) = \begin{cases} 0 & , \text{ if } x \leq y, \\ m(x, y) & , \text{ otherwise.} \end{cases}$$

i) Since $x \leq x$, then $r(x, x) = 0$ and

$$D_{(m,\leq)}(x, x) = 0.$$

ii) For triangle inequality, let $x, y, z \in X$. We would like to show that

$$D_{(m,\leq)}(x, z) \leq D_{(m,\leq)}(x, y) + D_{(m,\leq)}(y, z).$$

Consider the paths

$$P_1 : x = x_0, \ldots, y = x_k; P_2 : y = y_0, \ldots, z = y_l.$$

Then, we can relabel the two previous paths and get

$$P_3 : x = z_0, \ldots, y = z_k, \ldots, z = z_{k+l}.$$

So by definition of $D_{(m,\leq)}$, we have

$$D_{(m,\leq)}(x,z) \leq \sum_{i=0}^{k} r(x_i, x_{i+1}) + \sum_{i=0}^{l} (y_i, y_{i+1}).$$

The triangle inequality then follows by definition of the infimum.

□

4.2 Example

We give an example of computation of $D_{(m,\leq)}$ on a finite set X.

Example 4.2.1. Let $X = \{1, 2, 3\}$ and equip X with the metric M defined by the matrix

$$N = (M_{i,j})_{\substack{1 \leq i \leq 3 \\ 1 \leq j \leq 3}} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

and the partial order

$$\leq = \{(1,2), (1,3)\} \cup \{(x,x), x \in X\}.$$

Given the partial ordered metric space $(X, m, \leq)$, we compute $r(x, y)$ by

$$r(x,y) = \begin{cases} 0 & \text{, if } x \leq y, \\ m(x,y) & \text{, otherwise.} \end{cases}$$

We see that

$$r(1,2) = 0, r(2,1) = 2, r(1,3) = 0, r(3,1) = 3,$$
$$r(2,3) = m(2,3) = 2, r(3,2) = m(3,2) = 2.$$

And so

$$D_{(m,\leq)}(1,2) = \inf\{r(1,2), r(1,3) + r(3,2)\} = \inf\{0, 2\} = 0.$$

Similarly, we obtain $D_{(m,\leq)}(x,z)$ for any $x, y \in X$.

Lemma 4.2.2. *((Conradie and Künzi, 2018, Lemma 2.1.))*

Let $(X, m, \leq)$ be a partially ordered metric space,

1. $D_{(m,\leq)}(x,y) \leq m(x,y)$ *and hence* $D^s_{(m,\leq)}(x,y) \leq m(x,y)$;
2. $x \leq y \implies D_{(m,\leq)}(x,y) = 0 \implies x \leq_{D_{(m,\leq)}} y$.

Proof. Let $(X, m, \leq)$. Recall that the map r is defined as $r(x, y) = m(x, y)$ if $x \not\leq y$ and $r(x, y) = 0$ if $x \leq y$, whenever $x, y \in X$.

1. For all $x, y \in X$, it is obvious that

$$r(x, y) \leq m(x, y).$$

Moreover, for $x = x_0, x_1 = y$ and by definition of $D_{(m,\leq)}$, we clearly have

$$D_{(m,\leq)}(x, y) \leq r(x, y),$$

and so

$$D_{(m,\leq)} \leq r(x, y) \leq m(x, y) \implies D_{(m,\leq)}(x, y) \leq m(x, y).$$

Furthermore, for all $x, y \in X$, we have,

$$D_{(m,\leq)}(x, y) \leq m(x, y) \text{ and } D_{(m,\leq)}(y, x) \leq m(y, x) = m(x, y),$$

i.e.

$$D^s_{(m,\leq)}(x, y) \leq m(x, y).$$

2. For all $x, y \in X$, if $x \leq y$, then $r(x, y) = 0$ and so $D_{(m,\leq)}(x, y) = 0$. Using the specialization order induces by $D_{(m,\leq)}$, we actually have

$$D_{(m,\leq)}(x, y) = 0 \iff x \leq_{D_{(m,\leq)}} y.$$

□

Then $D_{(m,\leq)}$ is a quasi-pseudometric on X, but it is not necessarily a T_0-quasi-metric (see (Gaba and Künzi, 2016, Example 7)). If there is a T_0-quasi-metric d on X that induces $(X, m, \leq)$, then $D_{(m,\leq)}$ induces the space $(X, m, \leq)$ and $d \leq D_{(m,\leq)}$ (see (Gaba and Künzi, 2015, Remark 4)). Therefore the authors obtained in Gaba and Künzi (2015) the following characterization for the class of induced ordered metric spaces.

Proposition 4.2.3. *(Gaba and Künzi, 2015, Corollary 1) Let $(X, m, \leq)$ be a partially ordered metric space. Then there is a T_0-quasi-metric inducing $(X, m, \leq)$ if and only if (i) $D^s_{(m\leq)} \geq m$ and (ii) for any $a, b \in X$ we have that $D_{(m,\leq)}(a, b) = 0$ implies that $a \leq b$.*

5. A Galois connection

This sections elaborates on how the works of Galois come to play in the present setting.

We continue the investigation of the interdependence between T_0-quasi-metric spaces and partially ordered metric spaces. We then set up a Galois connection between these two classes of spaces.

Figure 5.1: Éveriste Galois. Source: selunec

French mathematician Évariste Galois, 1811-1832. He passed on at a young age of 21, but made a lasting contributions to mathematics in the field of group theory.

Coming back to our problem. That is can we find a T_0-quasi-metric space (X, d) that induces a partially ordered metric space $(X, d^s, \leq_d)$. In this section we establish a Galois connection between T_0-quasi-metric spaces and partially ordered metric spaces.

5.1 The ingredients

Let $(A, \leq_A)$ and $(B, \leq_B)$ be partially ordered sets.

Definition 5.1.1. A **Galois connection** between A and B is a pair (R, S) of maps such that

G1) $R : A \to B$ and $S : B \to A$;

G2) for all $a \in A$, $b \in B$,

$$R(a) \leq_B b \text{ if and only if } a \leq_A S(b).$$

Remark 5.1.2. In this situation, R is called the *lower adjoint* of S and S is called the *upper adjoint* of R. An essential property of a Galois connection is that an upper/lower adjoint of a Galois connection uniquely determines the other, more precisely $R(a)$ is the least element b^* with $a \leq_R S(b^*)$ and $S(b)$ is the largest element a^* with $b \leq_S R(a^*)$.

Simple examples of Galois connections are easy to get:

Example 5.1.3.

Take

$$A = 2\mathbb{N} = \{2n;\ n \in \mathbb{N}\} \text{ and } B = 2\mathbb{N} + 1 = \{2n+1;\ n \in \mathbb{N}\},$$

and equip them with the natural order of $\leq_{\mathbb{N}}$ of $\mathbb{N}$. Define

$$R : A \to B; R(a) = a + 1 \quad \text{and} \quad S : B \to A; S(b) = b - 1.$$

Then

$$R(a) = a + 1 \leq_{\mathbb{N}} b \iff a \leq_{\mathbb{N}} b - 1 = S(b).$$

Therefore (R, S) is a Galois connection between $(2\mathbb{N}, \leq_{\mathbb{N}})$ and $(2\mathbb{N} + 1, \leq_{\mathbb{N}})$.

Example 5.1.4.

Let X be a nonempty set and $(\mathcal{P}(X), \subseteq)$ the power set of X ordered by set inclusion. Then for any fixed $T \subseteq X$, for all A and $B \subseteq X$, we see that

$$R(A) := A \cap T \subseteq B \iff A \subseteq S(B) := T^c \cup B,$$

i.e. (R, S) is a Galois connection between $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(X), \subseteq)$.

Proof. $R(A) := A \cap T$ and $S(B) := T^c \cup B$

Suppose $A \cap T \subseteq B$.

Let $x \in A \cap T \Rightarrow x \in B$

$\iff x \in A \cap T \Rightarrow x \in B$

$\iff (x \in A \text{ and } x \in T) \Rightarrow x \in B$

$\iff x \in A \Rightarrow (x \in T \Rightarrow x \in B)$

$\iff x \in A \Rightarrow (x \notin T \text{ or } x \in B)$

$\iff x \in A \Rightarrow (x \in T^c \text{ or } x \in B)$

Therefore $A \cap T^c \subseteq B$ □

A final example goes as follows:

Example 5.1.5.

Let S_1 and S_2 be two non-empty sets, $f : S_1 \to S_2$ and for any subsets $A \subseteq S_1, B \subseteq S_2$,

$$R(A) = \{f(a);\ a \in A\} \qquad S(B) = \{a;\ f(a) \in B\}.$$

Then (R, S) is a Galois connection between $(\mathcal{P}(S_1), \subseteq)$ and $(\mathcal{P}(S_2), \subseteq)$. Indeed

$$R(A) = \{f(a);\ a \in A\} \subseteq B \iff \forall x \in A, f(a) \in B \iff A \subseteq S(B) = \{a;\ f(a) \in B\}.$$

Remark 5.1.6. Sometimes, the pair (R, S) in Definition 5.1.1 is referred as an **increasing Galois connection**. Now, we can say that the functions R and S establish a **decreasing Galois connection** between $(A, \leq_A)$ and $(B, \leq_B)$ if they establish an increasing Galois connection between $(A, \leq_A)$ and $(B', \leq'_{B'})$, the dual of $(B, \leq_B)$.

Example 5.1.7. Let R be a relation on one set X to another Y. For any $A \subseteq X$ and $B \subseteq Y$, define

$$ub(A) = \{y \in Y; \forall x \in A : xRy\} \qquad lb(B) = \{x \in X; \forall y \in B : xRy\}.$$

Then, it can be easily seen that the mappings

$$A \mapsto ub(A) \qquad \text{and} \qquad B \mapsto lb(B),$$

where $A \subseteq X$ and $B \subseteq Y$, establish a decreasing Galois connection between the posets $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$.

The above construction was first considered by Birkhoff(Birkhoff, 1940, p 122) in 1940 under the name polarities.

5.2 A first connection

Let X be a non-empty set and $\mathcal{X} = \mathbb{R} \cup \{-\infty, +\infty\}$. Denote by R_X the space of functions $d : X^2 \to \mathcal{X}$ and S_X as the power set of X^2, i.e. the space $\mathcal{P}(X^2)$. For fixed $\Gamma \in R_X$, and for $d \in R_X$ and $x, y \in X$, we may naturally define (see Brøndsted (1974), Altman (1982))

$$x \leq_d y \iff d(x, y) \leq \Gamma(x, y)$$

and, for any $\leq \in S_X$ and $x, y \in X$, we define

$$d_{\leq}(x, y) = \begin{cases} \Gamma(x, y) & \text{, if } x \leq y, \\ +\infty & \text{, otherwise.} \end{cases}.$$

We can then show that the mappings

$$d \ \mapsto \leq_d \ \text{and} \ \leq \mapsto \ d_{\leq}$$

establish a decreasing Galois connection between partially ordered sets R_X and S_X, that is for any $d \in R_X$ and $\leq \in S_X$, we have

$$\leq \subseteq \leq_d \iff d \ \leq \ d_{\leq}.$$

Indeed, let X be a fixed non-empty set and we use Definitions 2.2.8 and 2.2.9. Denote, on one part by, A the set of all partial orders on X. We partially order the set A by,

$$\leq_1 \leq_A \leq_2 \iff x \leq_2 y \Rightarrow x \leq_1 y.$$

Now on another part, let B be the set of all quasi-pseudometric space on X. A partial order on B is defined by

$$d_1 \leq_B d_2 \iff d_1(x,y) \leq d_2(x,y) \text{ for all } d_1, d_2 \in B.$$

Proposition 5.2.1. *Let $\leq \in A$ and $d \in B$. Define the maps R, S by,*

$$R : A \to B; \ R(\leq) = d_{\leq}$$

and

$$S : B \to A; \ S(d_{\leq}) = \leq_d .$$

Then the following holds :

1.
$$d_{\leq} \leq_B d \implies \leq \leq_A \leq_d;$$

2.
$$\leq \leq_A \leq_d \implies d_{\leq} \leq_B d;$$

3. *The pair (R, S) is a Galois connection.*

Proof. Let $\leq \in A$ and $d \in B$

1. Suppose that $d_{\leq} \leq_B d$.

   Assume $x \leq_d y \implies d(x,y) = 0 \implies d_{\leq}(x,y) = 0 \implies x \leq y$

   Therefore
$$\leq_d \leq_A \leq .$$

2. Suppose that $\leq_d \leq_A \leq$. Let $x, y \in X$ since $\leq_d \leq_A \leq$,
$$x \leq_d y \implies x \leq y$$
   if
$$d(x,y) = 0 \implies d_{\leq}(x,y) = 0$$
   Assume by contradiction, $d \leq d_{\leq}$ and $d(x,y) = 0$ this implies $d_{\leq}(x,y) \geq 0$, which is a contradiction.

   Therefore
$$d_{\leq} \leq_B d.$$

3. This follows at once from (1) and (2).

□

5.3 The connection in our context

To set up the Galois connection in our context. We define two sets $\mathcal{Q}(X)$ and $\mathcal{M}(X)$, we partially order the sets with $\leq_{\mathcal{Q}(X)}$ and $\leq_{\mathcal{M}(X)}$ respectively in Lemma's 5.3.1 and 5.3.2.

Lemma 5.3.1. *Let X be a fixed non-empty set and let $\mathcal{Q}(X)$ the set of all T_0-quasi-metrics on X. The relation defined on $\mathcal{Q}(X)$ by*

$$d_1 \leq_{\mathcal{Q}(X)} d_2 \iff d_1(x,y) \leq_{[0,\infty)} d_2(x,y) \text{ for all } x, y \in X,$$

whenever $d_1, d_2 \in \mathcal{Q}(X)$ is a partial order on $\mathcal{Q}(X)$.

Proof.

i) Reflexivity: If $d_1 \in \mathcal{Q}(X)$

$$d_1(x,y) \leq d_1(x,y) \text{ for all } x, y \in X \implies d_1 \leq_{\mathcal{Q}(X)} d_1.$$

ii) Transitivity: let $d_1, d_2, d_3 \in \mathcal{Q}(X)$ such that

$$d_1 \leq_{\mathcal{Q}(X)} d_2 \text{ and } d_2 \leq_{\mathcal{Q}(X)} d_3.$$

We have

$$d_1(x,y) \leq d_2(x,y) \text{ and } d_2(x,y) \leq d_3(x,y) \text{ for all } x, y \in X,$$

which implies $d_1(x,y) \leq d_3(x,y)$ for all $x, y \in X$, i.e.

$$d_1 \leq_{\mathcal{Q}(X)} d_3.$$

iii) Antisymmetry: let $d_1, d_2 \in \mathcal{Q}(X)$ such that

$$d_1 \leq_{\mathcal{Q}(X)} d_2 \text{ and } d_2 \leq_{\mathcal{Q}(X)} d_1.$$

We have

$$d_1(x,y) \leq d_2(x,y) \text{ and } d_2(x,y) \leq d_1(x,y) \text{ for all } x, y \in X,$$

which implies $d_1(x,y) = d_2(x,y)$ for all $x, y \in X$, i.e.

$$d_1 = d_2.$$

Thus $d_1 \leq_{\mathcal{Q}(X)} d_2 \iff d_1(x,y) \leq d_2(x,y)$ for all $x, y \in X$ is a partial order on $\mathcal{Q}(X)$. □

Lemma 5.3.2. *Let X be a fixed non-empty set and let $\mathcal{M}(X)$ the set of all ordered pairs $(m, \leq)$, where m is a metric on X and $\leq$ is a partial order on X. The relation defined on $\mathcal{M}(X)$ by*

$$(m_1, \leq_1) \leq_{\mathcal{M}(X)} (m_2, \leq_2) \iff m_1(x,y) \leq m_2(x,y) \text{ and } x \leq_2 y \Longrightarrow x \leq_1 y \text{ for all } x, y \in X,$$

whenever $(m_1, \leq_1), (m_2, \leq_2) \in \mathcal{M}(X)$ is a partial order on $\mathcal{M}(X)$.

Proof.

i) Reflexivity: If $(m_1, \leq_1) \in \mathcal{M}(X)$, Since

$$m_1(x,y) \leq m_1(x,y) \text{ and } x \leq_1 y \Longrightarrow x \leq_1 y \text{ for all } x, y \in X,$$

therefore

$$(m_1, \leq_1) \leq_{\mathcal{M}(X)} (m_1, \leq_1).$$

ii) Transitivity: let $(m_1, \leq_1), (m_2, \leq_2), (m_3, \leq_3) \in \mathcal{M}(X)$ such that

$$(m_1, \leq_1) \leq_{\mathcal{M}(X)} (m_2, \leq_2) \text{ and } (m_2, \leq_2) \leq_{\mathcal{M}(X)} (m_3, \leq_3).$$

We have

$$(m_1, \leq_1) \leq_{\mathcal{M}(X)} (m_2, \leq_2) \iff m_1(x,y) \leq m_2(x,y) \text{ and } x \leq_2 y \Longrightarrow x \leq_1 y \text{ for all } x, y \in X,$$

and

$$(m_2, \leq_2) \leq_{\mathcal{M}(X)} (m_3, \leq_3) \iff m_2(x,y) \leq m_3(x,y) \text{ and } x \leq_3 y \Longrightarrow x \leq_2 y \text{ for all } x, y \in X,$$

On the one side, we can write, for $x, y \in X$,

$$m_1(x,y) \leq m_2(x,y) \text{ and } m_2(x,y) \leq m_3(x,y),$$

so $x, y \in X$,

$$m_1(x,y) \leq m_3(x,y) \text{ for all } x, y \in X. \tag{5.3.1}$$

On the other side, since for $x, y \in X$,

$$x \leq_2 y \Longrightarrow x \leq_1 y \text{ and } x \leq_3 y \Longrightarrow x \leq_2 y,$$

so $x, y \in X$,

$$x \leq_3 y \Longrightarrow x \leq_1 y. \tag{5.3.2}$$

In conclusion, by (5.3.1) and (5.3.2), we arrive at

$$(m_1, \leq_1) \leq_{\mathcal{M}(X)} (m_3, \leq_3).$$

iii) Antisymmetry: let $(m_1, \leq_1), (m_2, \leq_2) \in \mathcal{M}(X)$ such that

$$(m_1, \leq_1) \leq_{\mathcal{M}(X)} (m_2, \leq_2) \text{ and } (m_2, \leq_2) \leq_{\mathcal{M}(X)} (m_1, \leq_1).$$

Following the proof of Lemma 5.3.1, since

$$m_1(x, y) \leq m_2(x, y) \leq m_1(x, y) \text{ for all } x, y \in X \implies m_1 = m_2$$

and

$$x \leq_2 y \Longrightarrow x \leq_1 y \text{ and } x \leq_1 y \Longrightarrow x \leq_2 y \implies \leq_1 = \leq_2,$$

we conclude that

$$(m_1, \leq_1) = (m_2, \leq_2).$$

Thus $\leq_{\mathcal{M}(X)}$ is a partial order on $\mathcal{M}(X)$. □

For our next results, we shall need the following lemma, direct consequence of the definition of $D_{(m,\leq)}$ (see Section 4.1) for an arbitrary partial ordered metric space $(X, m, \leq)$.

Lemma 5.3.3. *((Conradie and Künzi, 2018, Lemma 2.1.))*

Let $(m, \leq) \in \mathcal{M}(X)$. Then for all $x, y \in X$,

1. $D_{(m,\leq)}(x, y) \leq m(x, y)$ *and hence* $D^s_{(m,\leq)}(x, y) \leq m(x, y)$*;*
2. $x \leq y \implies D_{(m,\leq)}(x, y) = 0 \implies x \leq_{D_{(m,\leq)}} y$.

The next proposition is the main result of this essay.

Proposition 5.3.4. *((Conradie and Künzi, 2018, Proposition 2.2.))*

Let $d \in \mathcal{Q}(X)$ and $(m, \leq) \in \mathcal{M}_0(X)$. Define the maps R, S by

$$R : \mathcal{Q}(X) \to \mathcal{M}_0(X); \quad R(d) = (d^s, \leq_d),$$

and

$$S : \mathcal{M}_0(X) \to \mathcal{Q}(X); \quad S((m, \leq)) = D_{(m,\leq)}.$$

Then, the following hold:

1.

$$d \leq_{\mathcal{Q}(X)} S((m,\leq)) \Longrightarrow R(d) \leq_{\mathcal{M}_0(X)} (m,\leq);$$

2.

$$R(d) \leq_{\mathcal{M}_0(X)} (m,\leq) \Longrightarrow d \leq_{\mathcal{Q}(X)} S((m,\leq));$$

3. The pair (R,S) is a Galois connection.

Proof.

Let $d \in \mathcal{Q}(X)$ and $(m,\leq) \in \mathcal{M}_0(X)$.

1. Suppose that $d \leq_{\mathcal{Q}(X)} S((m,\leq))$.

   Since $d \leq_{\mathcal{Q}(X)} S((m,\leq))$, then $d(x,y) \leq D_{(m,\leq)}(x,y)$ for all $x,y \in X$. Using Lemma 5.3.3(1), we get for $x,y \in X$

   $$d^s(x,y) \leq D^s_{(m,\leq)}(x,y) \leq m(x,y),$$

   and

   $$x \leq y \implies D_{(m,\leq)}(x,y) = 0 \implies d(x,y) = 0 \implies x \leq_d y.$$

   Therefore $d^s(x,y) \leq m(x,y)$ and $x \leq y \implies x \leq_d y$ for $x,y \in X$, i.e.

   $$R(d) = (d^s, \leq_d) \leq_{\mathcal{M}_0(X)} (m,\leq).$$

2. Suppose that $R(d) \leq_{\mathcal{M}_0(X)} (m,\leq)$. we have,

   $R(d) \leq_{\mathcal{M}_0(X)} (m,\leq) \implies (d^s,\leq_d) \leq_{\mathcal{Q}(X)} (m,\leq)$, and this is equivalent to

   $$d^s(x,y) \leq m(x,y) \text{ and } x \leq y \implies d(x,y) = 0 \quad \text{for all } x,y \in X.$$

   Now, let $x,y \in X$ and taking a path $x = x_0, \ldots, x_n = y$, we can write

   $$d(x,y) \leq \sum_{i=0}^{n-1} d(x_i, x_{i+1}).$$

   Whenever $x_i \leq x_{i+1}$, the term $d(x_i,x_{i+1}) = 0$, so there is no contribution of this term to the sum. Hence the summation can be taken only for the terms for which $d(x_i,x_{i+1}) > 0$.

   So

$$\begin{aligned} d(x,y) &\leq \sum_{i=0}^{n-1} d(x_i, x_{i+1}) \\ &\leq \sum_{d(x_i x_{i+1})>0} d^s(x_i, x_{i+1}) \\ &\leq \sum_{d(x_i x_{i+1})>0} m(x_i, x_{i+1}) \\ &= \sum_{d(x_i x_{i+1})>0} r(x_i, x_{i+1}), \end{aligned}$$

i.e.

$$d(x,y) \leq \sum_{d(x_i x_{i+1})>0} r(x_i, x_{i+1})$$

and by definition of $D_{(m,\leq)}$,

$$d(x,y) \leq D_{(m,\leq)}(x,y),$$

and

$$d \leq_{\mathcal{Q}(X)} D_{(m,\leq)}.$$

3. This follows at once from (1) and (2).

□

Corollary 5.3.5. *((Conradie and Künzi, 2018, Corollary 2.3.)) Let $(m, \leq) \in \mathcal{M}_0(X)$, then for all $x, y \in X$,*

1. *For $d \in \mathcal{Q}(X)$, $(d^s, \leq_d)$ is the least $(m, \leq) \in \mathcal{M}_0(X)$ such that $d \leq_{\mathcal{Q}(X)} D_{(m,\leq)}$.*
2. *For $(m, \leq) \in \mathcal{M}_0(X)$, $D_{(m,\leq)}$ is the largest $d \in \mathcal{Q}(X)$ such that $(d^s, \leq_d) \leq_{\mathcal{M}_0(X)} (m, \leq)$.*

Proof. It follows automatically from Remark 5.1.2. □

Remark 5.3.6. We can say that R and S are weak inverses:

$$R \circ S \leq_{\mathcal{Q}(X)} \mathcal{I}_{\mathcal{M}_0(X)},$$

$$S \circ R \leq_{\mathcal{M}_0(X)} \mathcal{I}_{\mathcal{Q}(X)}$$

i.e

$$(R \circ S)(m, \leq) = R(S(m, \leq)) = R(D_{(m,\leq)}) = (D^s_{(m,\leq)}, \leq_{D_{(m,\leq)}})$$

and

$$(S \circ R)(d) = S(R(d)) = s((d^s, \leq_d)) = D_{(d^s, \leq_d)}$$

5.4 Hulls, Kernels and Pataki connection.

Definition 5.4.1. A **hull (closure) operator** on a partial ordered set $(X, \leq)$ is a map $h : X \to X$ such that for all $x, y \in X$,

a. $x \leq h(x)$ (extensive);

b. $x \leq y \Rightarrow h(x) \leq h(y)$ (non-decreasing);

c. $h(h(x)) = h(x)$ (idempotent).

Definition 5.4.2. A **kernel (interior) operator** on partially ordered $(X, \leq)$ is a map $k : X \to X$ such that for all $x, y \in X$,

a. $k(x) \leq x$ (anti-extensive);

b. $x \leq y \Rightarrow k(x) \leq k(y)$;

c. $k(k(x)) = k(x)$.

Remark 5.4.3. Sometimes interior operators are also called *anti-closure* operators.

The next proposition from Erné et al. (1993) outlines an important property of a Galois connection:

Proposition 5.4.4. *Let (R, S) be a Galois connection between the ordered sets $(A, \leq_A)$ and $(B, \leq_B)$. Then $S \circ R$ is a closure operator on A and $R \circ S$ is an interior operator on B.*

In our context, this translates to :

Corollary 5.4.5. *Define the maps R, S by*

$$R : \mathcal{Q}(X) \to \mathcal{M}_0(X); \quad R(d) = (d^s, \leq_d),$$

and

$$S : \mathcal{M}_0(X) \to \mathcal{Q}(X); \quad S((m, \leq)) = D_{(m,\leq)}.$$

The map

$$S \circ R : \mathcal{Q}(X) \to \mathcal{Q}(X) : d \mapsto D_{(d^s,\leq_d)}$$

is a closure operator and the map

$$R \circ S : \mathcal{M}_0(X) \to \mathcal{M}_0(X) : (m, \leq) \mapsto \left(D^s_{(m,\leq)}, \leq_{D_{(m,\leq)}}\right)$$

is an interior operator.

Proof. Since (R, S) is a Galois connection, the result follows immediately from Proposition 5.4.4. □

The following analogue of Definition 5.1.1 has mainly been suggested to us by Buglyó and Száz (2008).

Definition 5.4.6. If $*$ is a function of one poset X to another Y and $\star$ is a function from X to itself such that

$$x_1^\star \leq_X x_2^\star \Longleftrightarrow x_1^* \leq_Y x_2^*$$

for all $x_1, x_2 \in X$, then we say that the function $*$ and $\star$ establish an increasing **Pataki connection** between X and Y.

Remark 5.4.7. From the above definition, and at the light of Remark 5.1.6, it is easy to infer what one would call a decreasing **Pataki connection** between X and Y.

Example 5.4.8. If T is a function of one poset X to another Y and U is a function from X to itself such that $A \subseteq B \cap E \Longleftrightarrow B^c \cap E \subset A^c \cap E$ for all $A, B, E \in X$,$T(A) = A^c \subset E$ and $U(A) = A \cap E$ is a decreasing **Pataki Connection** between X and Y.

Proof. Let $A, B, E \subset X$ with $X = Y$ Take $A \subseteq B \cap E$ Let $x \in B^c \cap E \Rightarrow x \in E$ and $x \in B^c \Rightarrow x \notin B \Rightarrow x \in A$. □

A close relationship between Galois and Pataki connections can be revealed by the following,

Theorem 5.4.9. *(Buglyó and Száz, 2008, Theorem 1.2) If $*$ and $\star$ establish an increasing Galois connection between $(X, \leq_X)$ and $(Y, \leq_Y)$ and $\diamond = * \circ \star$, then $*$ and $\diamond$ establish an increasing Pataki connection between X and Y.*

Corollary 5.4.10. *Define the maps R, S by*

$$R : \mathcal{Q}(X) \to \mathcal{M}_0(X); \quad R(d) = (d^s, \leq_d),$$

and

$$S : \mathcal{M}_0(X) \to \mathcal{Q}(X); \quad S((m, \leq)) = D_{(m,\leq)}.$$

*Then the maps R and $M = R \circ S$ establish an increasing **Pataki connection** between $\mathcal{Q}(X)$ and $\mathcal{M}_0(X)$.*

Remark 5.4.11. Buglyó and Száz (2008) defined the concepts of closure operator and interior operator using the idea of increasing Pataki connection. Thus, Pataki connections are more general objects than closure and interior operations.

6. Conclusion

In this last chapter, we conclude our investigation to the problem. We give the contribution in the study and give some open problems that one would look at for future work.

6.1 Rounding up

We give the conclusion to our investigations and propose some open problems. We have studied results for the quasi-metrization of a partially ordered metric space. In Chapter 2, we discussed the terminologies and presented some background from the theory of quasi-pseudometric spaces and orders. In Chapter 3, we presented the "interval condition", which is the case where $(X, m, \leq)$ is linearly ordered metric space we found that there exist a unique T_0-quasi-metric. In Chapter 4, we looked at a method that presented a resolution to the problem at hand, using a construction via a classical idea of paths. Finally, in Chapter 5, we presented the Galois connection in our context as well some immediate implications, like the existence of closure and interior operations and Pataki connection.

An interesting feature, which we could not cover is the linear context, where the underlying space X is a real vector space. There, the Galois connection can be re-interpreted as a connection between appropriate subsets of X.

Many interesting investigations were conducted during this research. Many questions, however, remain unanswered which can be addressed later. For future work we would like to look at the following:

Let $(X, m, \leq)$ be a partially ordered metric space. Suppose that there exists a proper subset E of X such that $(E, m, \leq)$ is induced by a quasi-pseudometric d. What topological or order property can we assume that E possesses in the aim to conclude that $(X, m, \leq)$ is induced?

It should be noted that this problem had already been stated by (Gaba, 2016, Problem 7.2.2) but the problem still remains open. Nevertheless, very recently Künzi et al. made a first attempt towards solving the problem (check Künzi and Yildiz (2017)).

6.2 Contribution

The problem that is addressed in this essay, has been investigated by Gaba, Künzi and Conradie. In Chapter 3, we provided an alternative proof of Proposition 3.1.1. In Chapter 4, in Lemma 4.1.1 we gave the detailed proof of the fact that the distance metric $D_{(m,\leq)}(x, y)$ is a quasi-metric and gave a detailed computation of $D_{(m,\leq)}(x, y)$. As we know that topologists are mainly interested in (X, m) metric space and together with the notion $(X, \leq)$ partial order, we were able to construct a first Galois connection in Proposition 5.2.1. The Remark 5.3.6 even though the proof is not rigorous we believe it shows that R and S are weak inverses. Although the idea of a the Galois connection came from the work by Conradie et al., we exploited the idea by linking it to the well known concept of the Pataki connection observed in Corollary 5.4.10.

Acknowledgements

Firstly, I would like to thank God. Also, I would like to express my sincere gratitude to my supervisor, Dr Yaé Gaba for his support, for sharing knowledge in his field and for the mathematical writing skills that he taught me. Also, I would like to thank my tutor, Prudence. A special thanks to Kenneth and Rock for all the advices they gave me. I would also like to thank AIMS for the scholarship, to the academic and non-academic staff for their support. I would also like to thank my friends William and Willie for making the AIMS experience a fun one and to the January intake it has been a pleasure being with you in this journey. To my parents, thank you for your love, support and for always pushing me to reach my fullest potential.

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