Crank-Nicolson method for real option valuation of risks related to climate change

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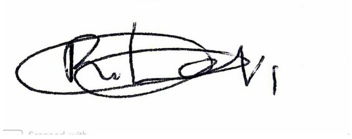
Abstract

With climate change becoming more and more of a global phenomenon, many industries around the globe experience climate-related damages each year. As an example, the risks associated with extreme temperature and sea-level rising is faced. We remain exposed to even more uncertainties with the changing climate, and as such, long term investment decisions in projects related to sea-level rise risks are important to consider. This essay builds an option model (real options), which considers sea level and temperature as the underlying, to estimate the risk associated with sea-level rise. We consider a finite difference method to find a semi-discrete solution to the resulting PDE in space. The Euler-θ-method (with $\theta = 0.5$) is used to find the solution of the semi-discrete problem in time. Then we give numerical simulations to show our results.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



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Introduction

In this essay, we develop a real option model with the temperature and the sea-level as our non-traded assets to evaluate risks that are caused by sea-level rise due to worldwide warming caused by the greenhouse gases in the atmosphere.

Since the nineteenth century the global temperature level has been increasing, consequently warming the earth. Its been found by scientists that the main cause of these rising temperatures is the increasing level of greenhouse gases in the atmosphere from human activities. Because of the high level of carbon dioxide in the atmosphere, it covers the ozone layer of the earth and does not allow the heat in the space to escape and consequently that heat is redirected down to the surface of the earth leading to the melting of the polar ice which causes an increasing sea-level Cli (2019).

An impact of global climate change on a number of the insurance firms affected is the issue of floods, droughts and therefore, the displacement of the population. Climate scientists say that global climate change is changing into an enormous issue, therefore insurance firms need to put into consideration the danger of climate change for things that they insure.

An option is a financial contract that gives right but not an obligation to purchase or sell any asset at an agreed time for a specific price. An example of such is a European and American which form a vanilla type of an option. Vanilla options deals with tradeable assets such as the stock of the option. Eschenbach et al. (2007).

Real options are more like Vanilla options since the two give the right, But the real option does not give an obligation to make business initiatives based on future conditions. Real options have non-tradeable assets and are linked with project decision making Shockley (2007).

For option valuation, we have a brief version from a finance textbook about the Black-Scholes-Merton (BSM) in the Seventies Nguyen (2016). Except for the real option valuation method, other methods can also be used for option pricing like the binomial method and Monte Carlo evaluation method Triantis (2003). Real option valuation and climate change are linked here because of the future uncertainties about the changes in the climate like rising sea level leading to floods, coastal erosion that can affect people residing in the coastal area. Other valuation methods like NVP do not include the benefits that the real option valuation method provides Such as an option of expanding, delaying, waiting or stopping the project in the future, In case certain conditions arise. It can be used in real estate projects development and research Linquiti and Vonortas (2011a).

We selected sea-level rise and the temperature because there's an uncertainty about the finance and of the rising sea-level condition in the future, Rick et al. (2011), surges of the storms are known as part through which the sea-level rise is the most dangerous and financially, all the damages caused by the rise of the sea need to be covered. This is what makes the model to sea-level rise and temperature suitable for the real options valuation method Linquiti and Vonortas (2011b).

The outline of the research: In Chapter 2 we use temperature and sea-level model to develop a real option pricing Black-Scholes PDE for a European option. Chapter 3 we use a finite difference method to approximate our two- factor Black-Scholes equation in space dimension and Euler-theta method(Crank-Nicolson method) in time t dimension. Chapter 4 we obtain numerical results and discussions.

Sea level rise as a result of Climate change

Rising sea-levels caused by worldwide warming already is causing problems for coastal cities around Cape Town. A place called Milnerton is being affected by coastal erosion. The town now is checking options they can take for the protection in Milnerton against the erosion, said by the climate experts in Cape Town on 15 August 2019. Apart from Milnerton several coastal regions in Cape Town, such as Glencairin, Hout Bay and Sea point has been affected by climate change earlier in August 2019. Climate experts discovered that the infrastructure and the facilities for the coastal cities have been underfunded and now they require proper funding for the coastal infrastructure and facilities for protection against sea-level rise.

As the climate changes because of the worldwide warming, the sand from the beaches may be eroded and we won't have proper beaches with enough sand in the coming decades as the sea-level continues to rise, said the climate scientist. The worldwide consensus at the time of the report suggested an average rise of the sea-level of 0.76m by 2100.

Figure 1: Rising sea level already is causing problems for Cape Town
photo by Mark Harley@ Groundup 2019

Even though this may appear to be small, it has a huge impact on the damage that is caused by the storms on the coastal cities. Climate scientist researcher and author at African center for cities have found a collection of research-based on the influences of the rising sea-level around the city of Capetown. He talked about the melting of the ice in Antarctica and Greenland, which is the main cause of the sea-level to rise. The IPCC report confirmed the global warming to be below 1.5-degrees Celcius to minimize chances of having a high rise of the sea-level, although the warming below 1.5 cannot be avoided depending on the future rates of emissions by human activities. Kretzmann (2019)

1. Basic notions in finance

1.1 Option

An option is a contract that gives the owner the right but not an obligation to buy or sell any asset for a known period(Maturity date or expiration date) at a specific price(strike price) Hull (2006).

1.2 Call option

A call option is a contract for an agreement between two parties to buy an asset for a specific price (strike price) at a certain period (expiration date). We take C(t, S) to be the price of a European call option where K is the strike price and T is the expiration date. Hence the pay-off for this call option is Hull (2006).

$$(S - K)^+ = max(S - K,\ 0)$$

The final condition for this call option is

$$C(T,\ S) = max(S - K,\ 0)$$

1.3 Put option

With a put option, the buyer of the "call" does not have an obligation to purchase the asset by the expiry date. but the other party should sell the asset to the buyer if the buyer exercises the option. We take P(t, S) to be the value of the Europian put option at expiration date T and strike price K, Therefore the pay-off for this option is Hull (2006).

$$(K - S)^+ = max(K - S,\ 0)$$

For this option the final condition for the option price is

$$P(T,\ S) = max(K - S,\ 0)$$

1.4 European option and American option

European option allows you to only exercise an option at the maturity date.

If there were 15 days to go and you wanted to exercise your option and buy the stock, you wait until the expiration date.

With the American option, you can exercise an option anytime on or before the expiration date.

For example:

From the previous example, if you have a call option and 15 days to maturity, you can still exercise that option and purchase the stock at the strike price Hull (2006).

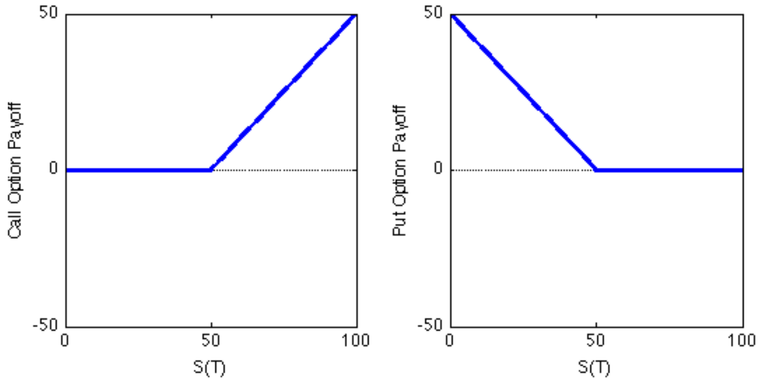

Figure 1.1: Pay-offs for a European call option and European put option both with the exercise price (K) of 50. Knight (2014)

1.5 Black-Scholes equation

Black Scholes equation is a model which explains the dynamics of the option prices:

The financial Black-Scholes PDE is

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} - rF = 0$$

With F as the price of the option and a function of the underlying's price S at time t, r is the risk-free interest rate, and σ is the underlying's constant volatility of the A.Dobrushkin (2017).

1.6 Stochastic Process

The stochastic process deals with the randomness on the interest rate in the future, its a random process. It is a variable whose value unusually changes overtime.

It is a family of random variables X(t) depending on a set tϵT. For the random X(t) is defined on a sample space Ω, where we have the field $\mathcal{A}$ if subsets of Ω and a probability measure P on $\mathcal{A}$ Prabhu (2007).

1.7 Brownian motion

A continuous stochatsic process $W_{t|t\leq 0}$ is a Brownian motion process with volatility σ if

1. W_0= 0
2. The process W_t has mean 0 and variance $\sigma^2 t$, hence is normally distributed.
3. W_t has stationary increments, that is, for $s < t$, the increment $W_t - W_s$ depends only on the value $(t - s)$. Hence it satisfies (2)
4. W_t has independent increments. Hence, for any form $t_2 \leq t_2 \leq ... \leq t_n$, the increments

$$W_2 - W_1, W_3 - W_2, ..., W_{t_n} - W_{t_{n-1}}$$

are independent random variables Roman (2004).

1.8 Standard Brownian motion

The process $W_{t|t\geq 0}$ with $\mu = 0$ and $\sigma = 1$ is called a standard Brownian motion, where W_t has mean 0 and variance t.

If $W_{t|\geq 0}$ is a Brownian motion with variance σ^2 and with drift μ then we can write

$W_t = \mu t + \sigma Z_t$

where $W_{t|t\geq 0}$ is a standard Brownian motion Roman (2004).

Brownian motion with drift

Brownian motion with drift is a stochastic process of the form $\mu + W_{t|t\geq 0}$ where we have a constant μ and a Brownian motion W_t.

A continuous stochastic process $W_{t|t\geq 0}$ is a Brownian motion process with volatility σ and flow μ if Roman (2004)

1. (1) W_0 =0
2. W_t has a normal distribution with μt and variance $\sigma^2 t$.
3. W_t has stationary increments. Thus, $W_t - W_s$ is commonly distributed with the mean μt(t-s) and variance $\sigma^2 t$(t-s).
4. W_t has independent increments.

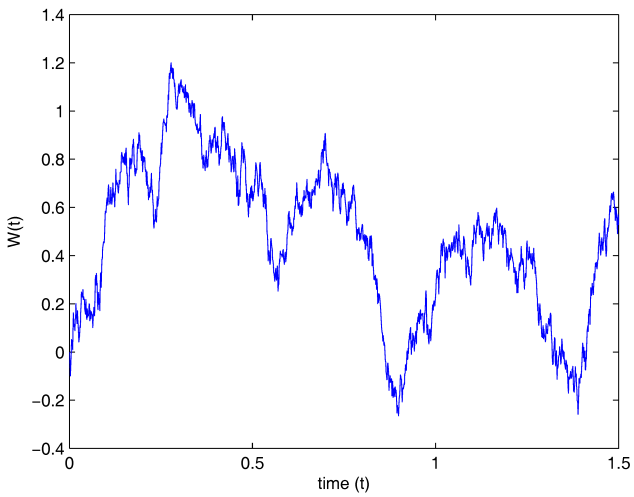

Figure 1.2: Brownian motion(Wiener process)
Bro (2019)

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2. Partial differencial equation(PDE) for the real option.

In this chapter, we apply Ito's formula on the two-dimensional function $F(t,X,Y)$ with the sea-level X and temperature Y as non-traded assets to find a PDE for the real option model depending on the sea level X and the temperature Y. We require a final condition, which is the payoff of the particular type of option we need to price, depending on whether the option is a put or a call and to the option's price.

Theorem:(Two- dimensional It$\hat{o}$ formula)

Suppose a function F(t, X, Y) has continuous partial derivatives $F_t,\ F_X,\ F_Y,\ F_{XX},\ F_{YY},\ F_{XY}$. Where $X(t)$ and $Y(t)$ are the stochastic process. Therefore a two-dimensional It$\hat{o}$ formula is in the form Shreve (2004)

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{\partial F}{\partial Y}dY + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}dX^2 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2}dY^2 + \frac{\partial^2 F}{\partial X \partial Y}dXdY \tag{2.0.1}$$

2.1 Mathematical model

Assuming that globally the mean temperature $(Y_t)_{t\geq 0}$ is a one dimensional Markov process with dynamics under the historical measure $\mathbb{P}$ and sea level process $(X_t)_{t\geq 0}$ is a function of the temperature under $\mathbb{P}$ measure given by

$$dY_t = \theta(\bar{Y}_1(t) - Y_t)dt + \sigma_Y d\hat{W}_Y \tag{2.1.1}$$

$$dX_t = \mu(Y_t, t)dt + \sigma_X(X_t, t)d\hat{W}_X \tag{2.1.2}$$

By applying Ito's formular on the two-dimensional function $F(t,X,Y)$ with

$$X = X_t,\ \ Y = Y_t, \tag{2.1.3}$$

We substitute Equation (2.1.1), (2.1.2) and (2.1.3) into Equation (2.0.1) to get,

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}(\mu(Y_t,t)dt + \sigma_X(X_t,t)d\hat{W}_X) + \frac{\partial F}{\partial Y}(\theta(\bar{Y}_1(t) - Y_t)dt + \sigma_Y d\hat{W}_Y)$$

$$+\frac{1}{2}\frac{\partial^2 F}{\partial X^2}(\mu(Y_t,t)dt + \sigma_X(X_t,t)d\hat{W}_X)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2}(\theta(\bar{Y}_1(t) - Y_t)dt +$$

$$\sigma_Y d\hat{W}_Y)^2 + \frac{\partial^2 F}{\partial X \partial Y}(\mu(Y_t,t)dt + \sigma_X(X_t,t)d\hat{W}_X)(\theta(\bar{Y}_1(t) - Y_t)dt + \sigma_Y d\hat{W}_Y)$$

Hence,

$$dF = \frac{\partial F}{\partial t}dt + \mu(Y_t,t)\frac{\partial F}{\partial X}dt + \sigma_X(X_t,t)\frac{\partial F}{\partial X}d\hat{W}_X + \theta(\bar{Y}_1(t) - Y_t)\frac{\partial F}{\partial Y}dt + \sigma_Y\frac{\partial F}{\partial Y}d\hat{W}_Y + \tag{2.1.4}$$

$$\frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma_X^2(X_t,t)(d\hat{W}_X)^2 +$$

$$\frac{1}{2}\frac{\partial^2 F}{\partial Y^2}\sigma_Y^2(Y_t,t)(d\hat{W}_Y)^2 + \frac{\partial^2 F}{\partial X \partial Y}(\sigma_X(X_t,t)d\hat{W}_X)(\sigma_Y d\hat{W}_Y)$$

Where,

$$d\hat{W_X}^2 = d\hat{W_Y}^2 = dt, \; d\hat{W}_X dt = d\hat{W}_Y dt = 0, \tag{2.1.5}$$

At the same time, assuming the two underlying assets are correlated:

$$d\hat{W}_X d\hat{W}_Y = \rho_{XY} dt \tag{2.1.6}$$

where ρ_{XY} is the correlation between the two stochastic process $\hat{W}_X$ and $\hat{W}_Y$.

Substitute Equation (2.1.5) and (2.1.6) into Equation (2.1) to get

$$dF = \frac{\partial F}{\partial t}dt + \mu(Y_t,t)\frac{\partial F}{\partial X}dt + (\theta(\bar{Y}_1(t) - Y_t))\frac{\partial F}{\partial Y}dt + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2}\sigma_X^2(X_t,t)dt + \frac{1}{2}\frac{\partial^2}{\partial Y^2}\sigma_Y^2 dt + \tag{2.1.7}$$

$$\frac{\partial^2 F}{\partial X \partial Y}\sigma_Y \sigma_X X \rho_{XY} dt + \sigma_X X \frac{\partial F}{\partial X}d\hat{W}_X + \sigma_Y \frac{\partial F}{\partial Y}d\hat{W}_Y$$

From Equation (2.1.2) we assume that

$$\mu(Y_t,t) = \eta(Y_t - \bar{Y}_0) \; and \; \sigma_X(X_t,t) = \sigma_X X_t \tag{2.1.8}$$

Hence,

$$dF = \left(\frac{\partial F}{\partial t} + \eta(Y - \bar{Y}_0)\frac{\partial F}{\partial X} + \theta(\bar{Y}_1(t) - Y_t)\frac{\partial F}{\partial Y} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma_X^2 X^2 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2}\sigma_Y^2 + \right. \tag{2.1.9}$$

$$\left. \rho_{XY}\sigma_X\sigma_Y X \frac{\partial^2 F}{\partial X \partial Y}\right)dt + \sigma_X X \frac{\partial F}{\partial X}d\hat{W}_X + \sigma_Y \frac{\partial F}{\partial Y}d\hat{W}_Y$$

Dividing both sides by F, we get the relative change equation:

$$\frac{dF}{F} = \left(\frac{\partial F}{\partial t}dt + \eta(Y - \bar{Y}_0)\frac{\partial F}{\partial X} + \theta(\bar{Y}_1(t) - Y_t)\frac{\partial F}{\partial Y} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma_X^2 X^2 + \right. \tag{2.1.10}$$

$$\left. \frac{1}{2}\frac{\partial^2 F}{\partial Y^2}\sigma_Y^2 + \rho_{XY}\sigma_X\sigma_Y X \frac{\partial^2 F}{\partial X \partial Y}dt + \sigma_X X \frac{\partial F}{\partial X}d\hat{W}_X + \sigma_Y \frac{\partial F}{\partial Y}d\hat{W}_Y\right) \Big/ F$$

We define s_1, s_2 and k for simplicity

where

$$k = \left(\frac{\partial F}{\partial t}dt + \eta(Y - \bar{Y}_0)\frac{\partial F}{\partial X} + \theta(\bar{Y}_1(t) - Y_t)\frac{\partial F}{\partial Y} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma_X^2 X^2 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2}\sigma_Y^2 + \rho_{XY}\sigma_X\sigma_Y X\frac{\partial^2 F}{\partial X \partial Y}\right)\Big/ F \tag{2.1.11}$$

$$s_1 = \left(\sigma_X X\frac{\partial F}{\partial X}\right)\Big/ F, \quad s_2 = \left(\sigma_Y\frac{\partial F}{\partial Y}\right)\Big/ F \tag{2.1.12}$$

We invest the amounts $x_2,\ x_2,\ x_3$ in bonds of maturity $\lambda_1,\ \lambda_2,\ \lambda_3$ For equal relationship between expected returns on bonds of varying maturities to form a portfolio P. The portfolio return rate is

$$\frac{dP}{P} = [x_1 k + x_2 k + x_3 k]dt + [x_1 s_1 + x_2 s_1 + x_3 s_1]d\hat{W}_X + [x_1 s_2 + x_2 s_2 + x_3 s_2]d\hat{W}_Y \tag{2.1.13}$$

The portfolio return rate is non-stochastic if the coefficients of the Brownian motions $d\hat{W}_X$ and $d\hat{W}_Y$ in Equation (2.1.13) are zeros:

$$[x_1 s_1 + x_2 s_1 + x_3 s_1] = 0, \quad [x_1 s_2 + x_2 s_2 + x_3 s_2] = 0 \tag{2.1.14}$$

To avoid arbitrage profits, return rate on this portfolio and the risk free rate of interest r must be equal.

Portfolio risk premium

To find portfolio risk premium we use market risk premium,

Market risk premium = expected returns - riskfree rate

For the return on a portfolio we have,

$$\frac{dF}{F} = kdt + s_1 d\hat{W}_X + s_2 d\hat{W}_Y \tag{2.1.15}$$

Taking the expectation both sides of the Equation (2.1.15) we get,

$$\mathbb{E}\left[\frac{dF}{F}\right] = \mathbb{E}[kdt] + s_1\mathbb{E}[d\hat{W}_X] + s_2\mathbb{E}[d\hat{W}_Y]$$

where t denotes time, $d\hat{W}_X$ and $d\hat{W}_Y$ are Wiener processes with E[$d\hat{W}_X$]=E[$d\hat{W}_Y$]=0

Then,

$$\mathbb{E}\left[\frac{dF}{F}\right] = \mathbb{E}[kdt]$$

$$= kdt$$

With the arbitrage condition for the portfolio we have,

$$r_p = \mathbb{E}[r_m] - r$$

where the expected return is more than risk-free interest rate For non-arbitrage condition we have,

$$\mathbb{E}[r_m] = r$$
$$\Rightarrow r_P = r - r = 0$$

where the expected return on the market is equal to the risk-free rate of interest.

$$r_P = x_1(k(\lambda_1) - r) + x_2(k(\lambda_2) - r) + x_3(k(\lambda_3) - r) = 0 \tag{2.1.16}$$

The zero risk conditions Equation (2.1.14) and no-arbitrage condition Equation (2.1.16) have a solution if and only if

$$k - r = \lambda_X s_1 + \lambda_Y s_2 \tag{2.1.17}$$

The above equation contains the relative risk premia on bonds of varying maturities. It expresses the instantaneous risk premium on a discount bond of any maturity as the sum of two elements,

Where λ_X and λ_Y are market prices for the sea level and temperature.

We show that,

$$\lambda_X s_1 + \lambda_Y s_2 = k - r \tag{2.1.18}$$

From Equation (2.1.12)

$$s_1 = (\sigma_X X \frac{\partial F}{\partial X})/F, \quad s_2 = (\sigma_Y \frac{\partial F}{\partial Y})/F$$

$$\Rightarrow \lambda_X(\sigma_X X \frac{\partial F}{\partial X}/F) + \lambda_Y(\sigma_Y \frac{\partial F}{\partial Y}/F) = k - r \tag{2.1.19}$$

Now we assume that k = r and solve for X(sea level)

$$\lambda_X(\sigma_X X \frac{\partial F}{\partial X}/F) + \lambda_Y(\sigma_Y \frac{\partial F}{\partial Y}/F) = 0 \tag{2.1.20}$$

$$\lambda_X(\sigma_X X \frac{\partial F}{\partial X}/F) = -\lambda_Y(\sigma_Y \frac{\partial F}{\partial Y}/F)$$

$$\lambda_X(\sigma_X X \frac{\partial F}{\partial X}) = -\lambda_Y(\sigma_Y \frac{\partial F}{\partial Y})$$

$$\lambda_X \sigma_X X = -\lambda_Y \frac{\partial X}{\partial Y}$$

$$X = -\frac{\lambda_Y \sigma_Y}{\lambda_X \sigma_X} \frac{\partial X}{\partial Y} \tag{2.1.21}$$

Substitute Equation (2.1.21) into Equation (2.1.20)

$$-\lambda_Y (\frac{\sigma_Y \partial X}{\partial Y} \frac{\partial F}{\partial X})/F + \lambda_Y (\sigma_Y \frac{\partial F}{\partial Y})/F = 0$$

$$\Rightarrow -\lambda_Y (\sigma_Y \frac{\partial F}{\partial Y})/F + \lambda_Y (\sigma_Y \frac{\partial F}{\partial Y})/F = 0$$

therefore,

$$k - r = \lambda_X S_1 + \lambda_Y S_2 \tag{2.1.22}$$

substitute (2.1.11) and (2.1.12) into (2.1.22)

$$\left(\frac{\partial F}{\partial t} dt + \eta(Y - \bar{Y}_0) \frac{\partial F}{\partial X} + (\theta(\bar{Y}_1(t) - Y_t)) \frac{\partial F}{\partial Y} + \frac{1}{2} \frac{\partial^2 F}{\partial Y^2} \sigma_X^2 X^2 + \frac{1}{2} \frac{\partial^2 F}{\partial Y^2} \sigma_Y^2 + \rho_{XY} \sigma_X \sigma_Y X \frac{\partial^2 F}{\partial X \partial Y}\right) \Big/ F - r$$
$$= \lambda_X \left(\sigma_X X \frac{\partial F}{\partial X}\right) \Big/ F + \lambda_Y \left(\sigma_Y \frac{\partial F}{\partial Y}\right) \Big/ F$$

Multiplying both sides by F and rearranging the above equation where $F(x,\ y,\ t)$ is the unknown function to be solved for x, y are the coordinates in space, and t is time:

$$\frac{\partial F}{\partial t} + (\eta(Y - \bar{Y}_0) - \lambda_X \sigma_X X) \frac{\partial F}{\partial X} + (\theta(\bar{Y}_1(t) - Y_t) - \lambda_Y \sigma_Y) \frac{\partial F}{\partial Y} + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} \sigma_X^2 X^2 + \frac{1}{2} \frac{\partial^2 F}{\partial Y^2} \sigma_Y^2 +$$
$$\rho_{XY} \sigma_X \sigma_Y X \frac{\partial^2 F}{\partial X \partial Y} - rF = 0 \tag{2.1.23}$$

Finally, we get the PDE for the real options model for the sea level X and the temperature of Y.

Then we need to approximate the above PDE using the finite difference method in space and Crank-Nicolson method in time.

We remark here that Equation (2.1.23) is the European real option PDE. Generally, Equation (2.1.23) has no analytical solution. We, therefore, resort to numerical methods. In this essay, we consider the finite difference method in space and the Crank-Nicolson method for the time discretization.

2.2 Boundary and final conditions

In this section, we discuss the payoffs, whether to continue with the investment or not due to the case where the condition for sea-level rise is high and the case where the sea-level rise is low. Whether is appropriate or not appropriate to build protection even if we have fewer damages or more damages than the cost of adjusting the defense to a specific height.

The Final Condition

Case 1:

$$V(X_T) < K \Rightarrow V(X_T) - K < 0$$

$F = \max\{V(X_T) - K,\ 0\} = 0$

If the avoided damages are much less than the cost of increasing the defense to a particular height, as shown above, then the cost of the real option is equal to zero, and it might not be essential for us to exercise an option due to the fact this may lead to wasted resources.

For example, building walls for the sea today because of the expected storm surges at some unknown date in the future may lead to the loss of resources. Instead, the resources should have been helpful on the other way for the community.

Case 2:

But if avoided damages are more than the cost of increasing defense to a specific height, then we exercise an option because insufficient protection may lead to floods in case of the sea level rises.

But if avoided damages are more than the cost of increasing defense to a particular height, then we exercise an option because inadequate protection may lead to floods in case of the sea level rises.

Similarly leaving adaptation measures too late can additionally incur costs due to pain and damage inflicted on people and property.

$$V(X_T) > K \Rightarrow V(X_T) - K > 0$$

F $=\max\{V(X_T) - K,\ 0\} = V(X_T) - K$

Where $V(X)$ denote the function of avoided damages caused by the sea level rise via increasing the defense which is an increasing function of X. K denote the cost of increasing a defense to a specific height.

2.3 Free boundary condition for an American option

Considering the sea level X. The valuation problem consists of figuring out the most reliable exercising policy, like the exercising time that maximizes the option value. To resolve the equation, we use the follwing two boundary conditions:

$$F(X^*(t),\ Y,\ t) = V(X^*(t) - K) \tag{2.3.1}$$

$$\frac{\partial F(X^*(t),\ Y,\ t)}{\partial X} = V^{'}(X^*(t) - K) \tag{2.3.2}$$

where X_t is the crucial value point on which the investment is triggered. If $X(t) \geq X^*(t)$, we have to commit the investment two immediately. If $X(t) < X(t)$, it is beneficial to delay the choice for investment. The boundary condition (2.3.1) says that if at $X(t)$ the investment is optimal, then the option price $F(X,\ Y,\ t)$ must be equal to the price of the termination condition at time t.

Condition as X→ X_{max} for a European option

Boundary condition at $X = X_{max}$

$$F(X = X_{max},\ Y,\ t) = F(X = X_{max},\ Y,\ T) = V(X = X_{max}) - K,$$

Due to an increase of the sea-level we make an investment.

As X→ X_{min}

Dirichlet boundary condition is

$$F(X \to X_{max},\ Y,\ t) = 0,$$

As the sea-level is low then we will have a low investment value.

As $Y \to Y_{min}$ and $Y \to Y_{max}$

Using the third and fifth terms from Equation (2.1.23) equating to zeroes for $Y \to Y_{min}$ and $Y \to Y_{max}$ using the boundary and the final conditions as described we get the boundary conditions $Y \to Y_{min}$ and $Y \to Y_{max}$

Re-writing our PDE model for the real options:

$$\frac{\partial F}{\partial t} + (\eta(Y - \bar{Y}_0) - \lambda_X \sigma_X X)\frac{\partial F}{\partial X} + (\theta(\bar{Y}_1(t) - Y_t) - \lambda_Y \sigma_Y)\frac{\partial F}{\partial Y} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma_X^2 X^2 +$$

$$\frac{1}{2}\frac{\partial^2 F}{\partial Y^2}\sigma_Y^2 + \rho_{XY}\sigma_X\sigma_Y X\frac{\partial^2 F}{\partial X \partial Y} - rF = 0 \tag{2.3.3}$$

with the boundary conditions

$$F(X_{max},\ Y,\ t) = V(X_{max}) - K$$

For a European real option and

$$F(X \to X_{min},\ Y,\ t) = 0,\ F(X \to X_{min},\ Y,\ t) = g_1(X,\ t),\ F(X \to X_{max},\ Y,\ t) = g_2(X,\ t)$$

with final condition

$$F(X,\ Y,\ t) = max(V(X_T) - K,\ 0) \tag{2.3.4}$$

For European option, where g_1 and g_2 are determined by solving related one-dimensional European option problems.

2.4 Real option for American:

We solve a parabolic partial differential equation in order to determine the boundary condition for $g_1(X,\ t)$ for the boundary $Y = Y_{min}$.

$$\begin{cases} -\dfrac{\partial F}{\partial t} - \frac{1}{2}\sigma_X^2 X^2 \dfrac{\partial^2 F}{\partial X^2} - h(X,\ Y_{min},\ t) + rf - \lambda[F^* - F]_+^k = 0 \\ F(X_{min},\ t) = 0,\ F(X_{min},\ Y,\ t) = V(X_{max}) - K, \\ F(X,\ t) = max(V(X) - K,\ 0). \end{cases} \tag{2.4.1}$$

We solve the initial boundary problem in order to determine boundary condition for a function $g_2(X, t)$

for the boundary $Y \to Y_{max}$

$$\begin{cases} -\frac{\partial F}{\partial t} - \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 F}{\partial X^2} - h(X,\ Y_{min},\ t) + rf - \lambda[F^* - F]_+^k = 0 \\ F(X_{min},\ t) = 0,\ F(X_{min},\ Y,\ t) = V(X_{max}) - K, \\ F(X,\ t) = max(V(X) - K,\ 0). \end{cases} \tag{2.4.2}$$

Equation (2.4.1) and Equation (2.4.2) are the same to that one of a European option when $\lambda = 0$.

2.5 The power penalty approach

For the formulation of a problem into a complementary problem, Let $F(X,\ Y,\ T)$ represent the value of the real option with the expiry date T.

We solve American option problems for two assets by making use of a penalty approach We add a non-linear penalty term to the Black–Scholes equation. This way it gives a constant solution, eliminating the free and moving boundary.

Same application for the one-asset American two option, American option price for the two-asset is mostly greater than the pay-off. Therefore, the two-asset American option pricing problem can be written in the form of a linear complementary problem as

$$LF = -\frac{\partial F}{\partial t} - \frac{1}{2}\Big(\sigma_X^2 X^2 \frac{\partial^2 F}{\partial X^2} + \sigma_Y^2 \frac{\partial^2 F}{\partial Y^2} + 2\rho_{XY}\sigma_X\sigma_Y X \frac{\partial^2 F}{\partial X \partial Y}\Big) - (h(X,\ Y,\ t)\frac{\partial F}{\partial X} +$$

$$g(Y,\ t)\frac{\partial F}{\partial Y}) + rF, \qquad (2.5.1)$$

Where,

$$h(X,\ Y,\ t) = \eta(Y - \bar{Y}_0) - \lambda_x \sigma_X X$$

and

$$g(Y,\ t) = \theta(\bar{Y}_1(t) - Y) - \lambda_Y \sigma_Y.$$

The free X^t divides the

$$\Omega = (X_{min},\ X_{max}) \times (Y_{min},\ Y_{max})$$

into the continuation of region $\sum_1$ and the stopping region $\sum_2$.

In the continuation region $\sum_1$,

$$F > F(X,Y,T), LF = 0;$$

In the stopping region $\sum_2$,

$$V(X) > K,\ F = V(X) - K,$$

Note that,

A real option with flexibility for each year has two values:

1. Termination Value (when an option is implemented)
2. Continuation Value (when an option is deferred)

Then $LF > 0$. In other words the real option value F satisfies the follwing partial differential complementarity problem:

$$\begin{cases} LF \geq 0, \\ F - F^* \geq 0, \\ LF.(F - F^*) = 0, \end{cases} \qquad (2.5.2)$$

For

$$(X,\ Y,\ t)\epsilon\Omega \times [0,\ T)$$

with the boundary conditions

$$\begin{cases} F(X_m, Y, t) = 0, F(X_m, Y, t) = V(X_{max} - K), \\ F(X, Y_{min}, t) = g_1(X_t), F(X, Y_{max}, t) = g_2(X_t), \\ \end{cases} \tag{2.5.3}$$

and the terminal condition

$$F(X, Y, T) = F^*(X, Y), \tag{2.5.4}$$

Where

$$F^*(X, Y) = \max(V(X) - K, 0)$$

is the payoff function.

3. Numerical Methods

In Chapter 2, we have shown that European option pricing for two-underlying assets temperature and sea-level results in a two-dimensional Black-Scholes PDE which is parabolic with variables time t and the two underlying assets for sea level X and temperature Y. Different two-asset options are defined by different pay-off functions. Numerical methods are used to approximate differential equations. Finite difference methods (FDMs) are numerical methods that we used for solving differential equations. These methods are widely used in financial mathematics for the options valuation since 1977 Schwartz (1977). The idea of FDMs is to discretize the domain with several grid points and use finite differences to approximate the derivatives on these grid points. After the application of the finite difference discretization along space and time dimensions to the Black-Scholes, A large sparse linear system is solved. In this chapter, we also introduce this discretization method for the two-asset Black-Scholes problem.

3.1 The finite difference method

In this method, we replace the differential notation with some expressions for the first derivative order in time t and second-order derivatives in the spacial dimension. Mostly several finite difference approximations to ODEs are derived by using Taylor series expansion.

3.2 Discretization

Consider a domain D: 0$\leq x \leq a$, 0$\leq y \leq b$

for $i = 1, 2, 3, \ ..., \ n-1$ and for $j = 1, 2, 3, \ ..., \ m-1$ where Δx and Δy are the steplengths obtained by Δx=$\frac{a}{n}$ and Δy=$\frac{b}{m}$.

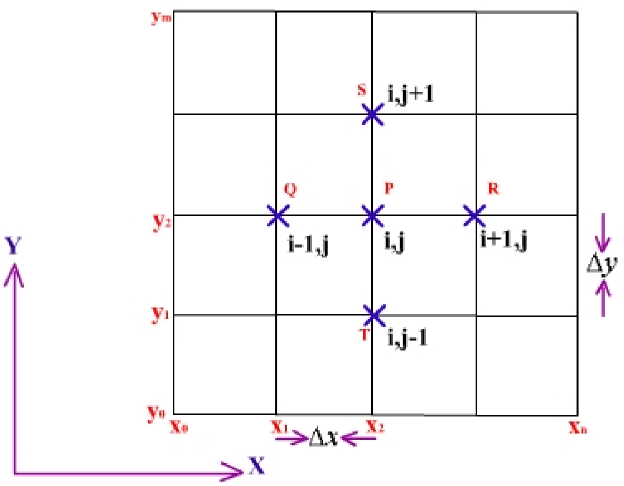

Figure 3.1: Mesh grid

Let $Q_{i,j} = Q_{x_i,y_j}$ be any point in D, x_i and y_j are as follows

$$x_i = x_0 + i\Delta x \tag{3.2.1}$$
$$y_j = y_0 + j\Delta y \tag{3.2.2}$$

where x_0, y_0 is the coordinate for the point $(0, 0)$. If $F(x, y)$ is any continuous function with all necessary derivatives exisit in D then

$$F_{i\pm1,j} = F(x \pm \Delta x, y) = F_{i,j} \pm \Delta x \frac{\partial F_{i,j}}{\partial x} + ... \tag{3.2.3}$$

$$F_{i,j\pm1} = F(x, y \pm \Delta y) = F_{i,j} \pm \Delta y \frac{\partial F_{i,j}}{\partial y} + ... \tag{3.2.4}$$

From Equation (3.2.3) and (3.2.4) above partial derivatives can be approximated as

Central difference

$$\frac{\partial F_{i,j}}{\partial x} = \frac{F_{i+1,j - F_{i-1,j}}}{2\Delta x} + ...$$

Forward difference

$$\frac{\partial F_{i,j}}{\partial x} = \frac{F_{i+1,j} - F_{i,j}}{\Delta x} + ...$$

Backward difference

$$\frac{\partial F_{i,j}}{\partial x} = \frac{F_{i,j} - F_{i-1,j}}{\Delta x} + ...$$

Now we re-write our partial differential equation from chapter 2 as follows:

$$\frac{\partial F}{\partial t} + (\eta(Y - \bar{Y}_0) - \lambda_X \sigma_X X)\frac{\partial F}{\partial t} + (\theta(\bar{Y}_1(t) - Y_t) - (\lambda_Y \sigma_Y))\frac{\partial F}{\partial Y} + \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 F}{\partial X^2} + \frac{1}{2}\sigma_Y^2 \frac{\partial^2 F}{\partial Y^2} +$$

$$\rho_{XY}\sigma_X\sigma_Y X \frac{\partial^2 F}{\partial X \partial Y} - rF = 0 \tag{3.2.5}$$

We focus on Equation (3.2.5) as the Black- Scholes equation, firstly we replace the differential notations with finite difference expressions for second-order in space. Numerically the space dimensions and the time dimension for parabolic PDEs such as Black-Scholes PDE are solved using FDMs differently. First, we discretize the PDE in (3.2.5).

where,

$$\Delta X = \frac{X_{max} - X_{min}}{N_X}, \quad \Delta Y = \frac{Y_{max} - Y_{min}}{N_Y}, \quad t\epsilon[0,\ T] \tag{3.2.6}$$

$$\frac{dF_{i,j}}{dt} + \left[\left(\frac{(\eta(Y_j - \bar{Y}_0) - \lambda_X \sigma_X X_i)}{2\Delta X}\right) + \frac{1}{2}\left(\frac{\sigma_X^2 X_i^2}{\Delta X^2}\right)\right] F_{i+1,j} + \left(\frac{\rho_{XY}\sigma_X\sigma_Y X_i}{4\Delta X \Delta Y}\right) F_{i+1,j+1} + \tag{3.2.7}$$

$$\left[\frac{((\theta(\bar{Y}_1(t)-Y_j)-\lambda_Y\sigma_Y))}{2\Delta Y}+\left(\frac{\frac{1}{2}\sigma_Y^2}{\Delta Y^2}\right)\right]F_{i,j+1}+\left(-\frac{\sigma_X^2 X_i^2}{\Delta X^2}-\frac{\sigma_Y^2}{\Delta Y^2}-r\right)F_{i,j}-$$

$$\left[(\eta(Y_j-\bar{Y}_0)-\lambda_X\sigma_X X_i)-\frac{1}{2}\left(\frac{\sigma_X^2 X_i^2}{\Delta X^2}\right)\right]F_{i-1,j}-\left[(\theta(\bar{Y}_1(t)-Y_j)-\lambda_Y\sigma_Y)-\left(\frac{\sigma_Y^2}{2\Delta Y^2}\right)\right]F_{i,j-1}+$$

$$\left(-\frac{\rho_{XY}\sigma_X\sigma_Y X_i}{4\Delta X\Delta Y}\right)F_{i+1,j-1}+\left(-\frac{\rho_{XY}\sigma_X\sigma_Y X_i}{4\Delta X\Delta Y}\right)F_{i-1,j+1}+\left(\frac{\rho_{XY}\sigma_X\sigma_Y X_i}{4\Delta X\Delta Y}\right)F_{i-1,j-1}=0$$

Hence we obtain the following Equation from Equation (3.2)

$$\frac{\partial F}{\partial t}+a_{ij}F_{i+1,j}+b_{ij}F_{i+1,j+1}+c_{i,j}F_{i,j+1}+d_{i,j}F_{i,j}+e_{i,j}F_{i-1,j}+f_{i,j}F_{i,j-1}+g_{i,j}F_{i+1,j-1}+h_{i,j}F_{i-1,j+1}+$$

$$k_{i,j}F_{i-1,j-1}=0 \tag{3.2.8}$$

for $i=1,2,...,N_X-1$ and $j=1,2,...,N_Y-1$, these form an $(N_X-1)^2$ X $(N_X-1)^2$ linear system of equations for

$F=(F_{1,1},\ ...,\ F_{1,N_Y-1},\ F_{2,1},\ ...,\ F_{2,N_Y-1},\ ...,\ F_{N_X-1,1},\ ...,\ F_{N_X-1,2},\ ...,\ F_{N_X-1},\ F_{N_Y-1})^T.$

with

$F_0,\ j(t),\ F_i,\ 0(t),\ F_0,\ N_Y(t),\ F_{N_X},\ 0(t).$

Discretization of the space leads to a semi-discrete ordinary differential equation which is re-written as:

$$\frac{dF_h(t)}{dt}+A_hF_h(t)+B_h=0 \tag{3.2.9}$$

Where,

F_h is the unknown, B_h is the known vector which has the values of F_h at nth time level and $A_h$is the square matrix of order $(N-1)X(N-1)$ with a block of tri-diagonal structure.

Then we replace continuous-time derivative by divided differences and focus on two-level schemes,

We then approximate $\frac{dF}{dt}$ using the Crank-Nicolson Method for time discretization:

$$\frac{dF}{dt}\approxeq\frac{F_h^{n+1}-F^n}{\Delta t} \tag{3.2.10}$$

For $i=1,2,...,N_X, j=1,2,...,N_Y$ in Equation (3.2.8) being equal to the given boundary conditions.

We have,

$$a_{ij} = \left[\frac{(\eta(Y_j - \bar{Y}_0) - \lambda_X \sigma_X X_i)}{2\Delta X} + \frac{1}{2} \frac{\sigma_X^2 X_i^2}{\Delta X^2} \right] \tag{3.2.11}$$

$$b_{ij} = \left(\frac{\rho_{XY} \sigma_X \sigma_Y X_i}{4\Delta X \Delta Y} \right) \tag{3.2.12}$$

$$c_{i,j} = \left[\frac{((\theta(\bar{Y}_1(t) - Y_j) - \lambda_Y \sigma_Y))}{2\Delta Y} + \frac{\frac{1}{2}\sigma_Y^2}{\Delta Y^2} \right] \tag{3.2.13}$$

$$d_{i,j} = \left(- \frac{\sigma_X^2 X_i^2}{\Delta X^2} - \frac{\sigma_Y^2}{\Delta Y^2} - r \right) \tag{3.2.14}$$

$$e_{i,j} = \left[(\eta(Y_j - \bar{Y}_0) - \lambda_X \sigma_X X_i) - \frac{1}{2} \frac{\sigma_X^2 X_i^2}{\Delta X^2} \right] \tag{3.2.15}$$

$$f_{i,j} = \left[(\theta(\bar{Y}_1(t) - Y_j) - \lambda_Y \sigma_Y) - \left(\frac{\sigma_Y^2}{2\Delta Y^2} \right) \right] \tag{3.2.16}$$

$$g_{i,j} = \left(- \frac{\rho_{XY} \sigma_X \sigma_Y X_i}{4\Delta X \Delta Y} \right) \tag{3.2.17}$$

$$h_{i,j} = \left(- \frac{\rho_{XY} \sigma_X \sigma_Y X_i}{4\Delta X \Delta Y} \right) \tag{3.2.18}$$

$$k_{i,j} = \left(\frac{\rho_{XY} \sigma_X \sigma_Y X_i}{4\Delta X \Delta Y} \right) \tag{3.2.19}$$

for $i = 1, ..., N_X - 1$, $\quad j = 1, ..., N_Y - 1$

Where N =3, we get

$$A_h = \begin{pmatrix} d_{11} & c_{11} & 0 & a_{11} & b_{11} & 0 & 0 & 0 & 0 \\ f_{12} & d_{12} & c_{12} & g_{12} & a_{12} & b_{12} & 0 & 0 & 0 \\ 0 & f_{13} & d_{13} & 0 & g_{13} & a_{13} & 0 & 0 & 0 \\ e_{21} & h_{21} & 0 & d_{21} & c_{21} & 0 & a_{21} & b_{21} & 0 \\ k_{22} & e_{22} & h_{22} & f_{22} & d_{22} & c_{22} & g_{22} & a_{22} & b_{22} \\ 0 & k_{23} & e_{23} & 0 & f_{23} & d_{23} & 0 & g_{23} & a_{23} \\ 0 & 0 & 0 & e_{31} & h_{31} & 0 & d_{31} & c_{31} & 0 \\ 0 & 0 & 0 & k_{32} & e_{32} & h_{32} & f_{32} & d_{32} & c_{32} \\ 0 & 0 & 0 & 0 & k_{23} & e_{33} & 0 & f_{33} & d_{33} \end{pmatrix}$$

$$F_h = \begin{pmatrix} F_{11} & F_{12} & F_{13} & F_{21} & F_{22} & F_{23} & F_{31} & F_{32} & F_{33} & \end{pmatrix}^T$$

$$B_h = \begin{pmatrix} e_{11}F_{01} + f_{11}F_{10} + g_{11}F_{20} + h_{11}F_{02} + k_{11}F_{00} \\ e_{12}F_{02} + h_{12}F_{03} + k_{12}F_{01} \\ b_{13}F_{24} + c_{13}F_{14} + e_{13}F_{03} + h_{13}F_{04} + k_{13}F_{02} \\ f_{21}F_{20} + g_{21}F_{30} + k_{21}F_{10} + \\ 0 \\ b_{23}F_{34} + c_{23}F_{24} + h_{23}F_{14} \\ a_{31}F_{41} + b_{31}F_{42} + f_{31}F_{30} + g_{31}F_{40} + k_{31}F_{20} \\ a_{32}F_{42} + b_{32}F_{43} + g_{32}F_{41} \\ a_{33}F_{43} + b_{33}F_{44} + c_{33}F_{34} + h_{33}F_{24} \end{pmatrix}$$

3.3 Euler-θ Method

Inbmoco (https://math.stackexchange.com/users/493857/Inbmoco)

In this section, we use the Euler-θ-method to solve an Ordinary differential equation at the given boundary conditions and find the convergence of the solution.

Dividing the interval $[0, T]$ into N sub-intervals, we get

$$0 = t_0 < t_1 < t_2 ... < t_N = T$$

We then approximate $\frac{dF}{dt}$ by replacing it with the finite difference expression for time discretization:

$$\frac{dF}{dt} \approxeq \frac{F_h^{n+1} - F^n}{\Delta t} \tag{3.3.1}$$

We now define a discrete system as,

$$\frac{F_h^{n+1} - F_h^n}{\Delta t} + \theta A_h F_h^{n+1} + (1-\theta) A_h F_h^n + B_h = 0$$

$$F_h^{n+1} - F^n + \Delta t \theta A_h F^{n+1} + \Delta t(1-\theta) A_h F^n - h + \Delta t B_h = 0$$

$$(1 + \Delta t \theta A_h) F^{n+1} = -\Delta t(1-\theta) A_h F_h^n + F^n - \Delta t B_h$$

For Crank-Nicolson

$$(I + \Delta t \theta A_h)^{-1} [(I - (1-\theta)\Delta t A_h) F^n - \Delta t B_h]$$

For $\theta = 0$

$$F^{n+1} = (I - \Delta t A_h) F^n - \Delta t B_h$$

For $\theta = 1$

$$F^{n+1} = (I + \Delta t)^{-1} (F^n - \Delta t B_h)$$

Where θ ϵ[0,1]

4. Computational results and discussion

This table shows the premium paid by an investor for the real American option. We see that when t = 0.49.

Table 4.1: American real option values for some points

(X, Y)	t			
	t=0	t=0.19	t=0.49	t=1
(76, 2.4)	0	0.0052	0.1813	1.019
(106.4, 2)	6.40	8.214	10.66	13.72
(155.8, 2)	55.8	56.00	56.29	56.83

American options with parameters:

$X_{min} = 20cm,\ X_{max} = 190cm,\ Y_{min} = 0.9,\ Y_{max} = 3.1,\ \eta = 0.3,\ \theta = 0.01,\ T = 1,\ r = 0.1,\ \hat{Y_0} = 0.5,$

$Y_1 = 1.5 + \hat{6}.57X10^{-5}t + 10.4sin\left(\frac{2\pi}{365}t - 2.01\right),\ \rho_{XY} = 0.9,\ \sigma_X = 0.2,\ \lambda_X = -0.2,\ \sigma_Y = 0.3,\ \lambda_Y = -0.3,$

$$K = 100,\ \lambda = 10,\ k = 4$$

We use the above parameters by choosing $V(X) = X$ and dividing (X_{min}, X_{max}), (Y_{min}, Y_{max}), and $(0, T)$ uniformly into 50, 50, and 50 sub-intervals. To solve the American real option. We use the final and the boundary condition from Equations, (2.3.4), (2.4.2), (2.5.3), (2.5.4), and (2.3.4). Numerical values of these conditions are determined by the one-dimensional initial-value problems which are sketched below. We see that the valuation of the real option results is more reasonable because the value of the real option increase with an increase in the sea level.

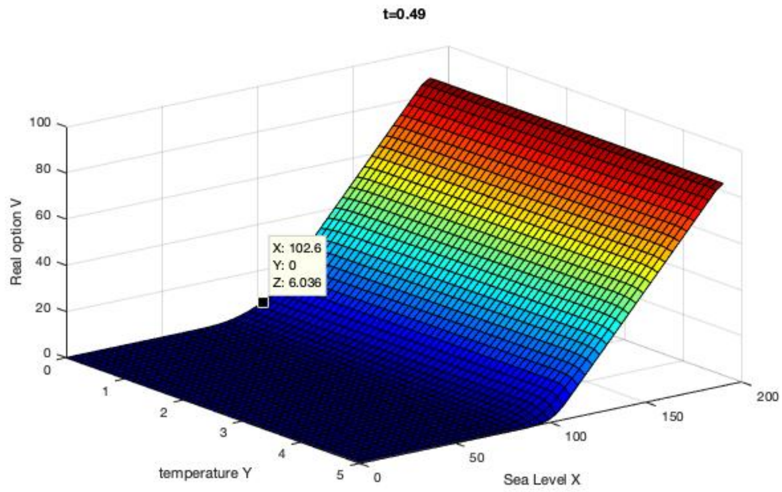

Figure 4.1: real option valuation for American at t = 0.49

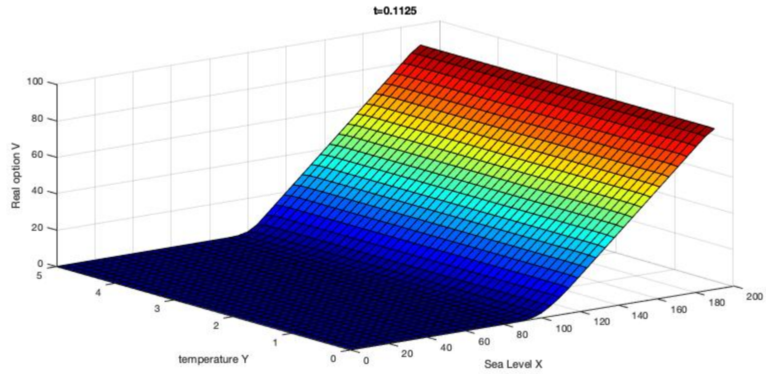

Figure 4.2: real option valuation for American at t=0.1125

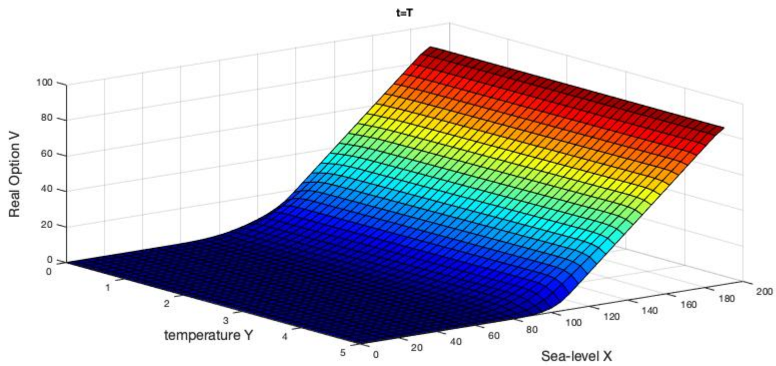

Figure 4.3: real option valuation for American at $t = T$

From table 4.1. The first temperature at 2.4 shows a lower sea level of 76. This shows a very low option value of 0 starting from $t = 0$ until the option value of 1.019 to $t = 1$. The investor here will pay a low premium since we won't be having risks of the sea level, the temperature is not high. The sea level starts to rise by 106.4 with the temperature at 2. This has an option value of 6.40 from $t = 0$ to $t = 1$ with the final option value of 13.72. The sea level has risen by 155.2 but the temperature is constant. This has an increasing option value of 55.8 at $t = 0$ until 56.83 at $t = 1$. The temperature rise shows an increase on the sea level and this makes an option value to be high because of high risks that can affect the asset. But low temperature shows a normal sea level hence low risk and premium. Therefore the value of the real option gets higher as option approaches maturity. This increases the value of the real option.

5. Conclusion and futurework

In this thesis we developed a real option model with the sea-level and tempature as our underlying assets to investigate the management of seal-level rise risk. We discretized the Black-Scholes PDE for the real option by using the finite difference method in space discretization on a mesh grid and the Crank-Nicolson method with $\theta = 0.5$ in time t. We have turned our PDE into algebraic equations, also often called discrete equations. The key property of the equations is that they are algebraic, which makes it easy to solve. We obtained our results using numerical experiments , for varying American real option values and at the varying temperatures and sea-level points to show the efficiency of this real option model. Moreover, we observed that an increase in temperature increases the sea-level and the sea-level rise risks, therefore the value for the real option also increases. As the sea-level is below the critical point(k $= 100$) the option value is almost zero. But if the sea-level is above critical point the option value increases.

In this thesis we were not able to make use higher dimensions, we used only two-dimension in our spatial discretization hence for in the future, we could extend current work and make use of numerical methods for higher dimensions.

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