Pricing credit default swaps using finite volume methods

Karabo Vincent Mogotsi (karabo@aims.ac.za)
African Institute for Mathematical Sciences (AIMS)

Supervised by: Prof Phillip Mashele
North West University, South Africa

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African Institute for Mathematical Sciences
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Abstract

Credit Default Swaps (CDS) are financial derivatives that provide insurance against the risk of default by a particular company. They are known to be the most traded derivatives in the market whose main purpose include speculations and arbitrage. Credit Default Swaps valuation involve several models depending on the approach. The derivation of the CDS model lead to a Partial differential Equation (PDE) . The main objective of this study is to solve numerically The PDE for pricing CDS using finite volume methods. The Two Point Flux Approximation (**TPFA**) combined to the upwind method is applied for the space discretization and the $\theta-$Euler method for the time discretization. Numerical results will be provided to show the efficiency of the method.

Key words: Credit Default Swaps, Partial differential equation, Two Point Flux Approximation (**TPFA**), discretization, θ-Euler method

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

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1. Introduction

Credit default swap is a contract or an agreement between the buyer and the seller of CDS that allows the buyer to offset his/her credit risk with that of the CDS seller against default of the reference entity, thereby the CDS seller agrees to reimburse the CDS buyer in case the reference entity defaults. Moreover the buyer will make periodic premium payments either quarterly or semi-annually to the seller of CDS until the credit event occurs or the contract terminates. In a CDS contract credit events are denoted as failure to pay, bankruptcy and restructuring of the reference entity. CDS were initiated by Blythe Masters from JP Morgan & Co. in 1994 according to Zhao (2018) and are mostly used as a hedge or insurance against the default of a bond by a reference entity. The CDS are traded over-the-counter which makes them highly volatile. There exist two major pricing methods to evaluate CDS: the Structural approach and Reduced form method. Structural approach was built/formulated by Merton (1974), from the Black-Scholes option pricing theory, the model was further extended by Black and Cox (1976) and Longstaff and Schwartz, 1993, which assumes that the value of the firm's assets follows a stochastic process which implies that the probability of default of the reference entity is modelled indirectly because it use the firm's balance sheet data which is not reliable. Unlike The Structural approach, the reduced form method gathers the probability of default from exogenous data such as ratings, bonds or CDS spread, because the method focuses on real market data, therefore the probability of default is modelled directly. In general the credit default times of the reference entity is modelled as the first arrival times of Poisson Process with Stochastic intensities depending on interest rates (Jarrow and Turnbull, 1998). The method assumes the real market data as the short rate for instantaneous interest rate. Poisson process provide a convenient way to model the default arrival risk intensities based on the risk models according to (Cifuentes et al., 1996). These two approaches are widely viewed as competitive (Duffie, 1998). However both the model's goal is to calculate the probability of default of the reference entity. Furthermore these two models have mutual relationship.

Our main contribution is to solve efficiently the PDE arising from the CDS pricing model based on the previous work of He (2016), whereby he built a bridge between these two approaches by considering a non-linear PDE model which arises from CDS market. Attempting to solve this PDE leads to too many complexities whereby an analytical solution may sometimes not exist therefore we introduce the new methodology that is the Finite volume Methods (FVMs) to solve the PDE numerically. Moreover, other numerical techniques have been widely used to solve such complex CDS pricing problems in particular the Finite Difference method (FDM).

In this work we show how the Cox-Ingersoll-Ross (CIR) model is used to construct the stochastic interest rate, whereby some general assumptions are made under this model to ensure the model performs well. The main goal of this study is to introduce the new discretization techniques to aid in solving the PDE numerically.

1.1 Basic definitions in finance

1.1.1 Definition. Credit is a trust/arrangement in which a borrower receives a certain value now and agrees to repay the lender at a later date.

1.1.2 Definition. bond is a fixed income instrument that represents a loan given by lender to a borrower that includes the details of the loan and its payment.

1.1.3 Definition. Credit default Swap spread is the premium charged on CDS in basis points by the CDS seller.

1.1.4 Definition. basis point a unit of measure used i finance to describe the percentage change in the value of a financial asset. Example; oe basis point is equivalent to 0.01%(1/100th of a percent) or 0.0001 i dc form.

1.1.5 Definition. Over The Counter(OTC) refers to the process of how financial instruments are traded for companies that are not listed on a formal stock exchange such Johannesburg Stock Exchange (JSE)

1.1.6 Definition. Arbitrage refers to buying financial instruments in different financial markets in order to take advantage of the price difference for the same asset/security.

1.1.7 Definition. credit rating an assessment of the credit risk of a borrower.

1.1.8 Definition. Maturity the time when the financial transaction such as bond becomes ready to be paid.

1.1.9 Definition. Reference entity corporation, or government or other legal entity which issues bonds that underlies a credit event.

1.1.10 Definition. Term structure refers to the relationship between short-term and long-term interest rates.

1.1.11 Definition. Protection buyer an entity that is entitled to receive one or more payments under a CDS from its counterparty, protection seller

1.1.12 Definition. Protection seller an entity (usually a fund , or insurer) that is oblige to make periodic payments under CDS to its counterparty.

1.2 The Credit Default Swap

The credit risk modelling have been widely studied in literature. In brief, credit risk is the default on a bond due to the borrower's failure to meet his/her bond obligations. The importance of credit risk management/modelling is to better understand the credit risk exposures of an underlying entity in order to avoid losses due to restructuring or bankruptcies. The CDS market has played a major role in providing the insurance against this credit risk.

After the Global Financial crisis of 2008, there has been a rapid growth in the credit default swap market, where we saw the great development of models for CDS valuation. In the world of credit risk modelling there are two major pricing methods; the Structural approach model and the Reduced-form (intensity) model. In the Structural approach the value of the firm follows a diffusion process which makes it easy to predict the default times of the reference entity before it occurs according to Zhao (2018), because there is zero drop in the value of the firm. Unlike Structural approach, the Reduced form (intensity) models has no relation with any observable events. Furthermore the probability of default depends on the real market data because the model do not depend on the value of the firm. Moreover, the intensity model outperforms the Structural model because of its ability to capture unexpected credit events better. Even with the improvement made by Zho(1997) on structural approach the model is still relatively weak (Zhou, 1997).

Structural approach method has many disadvantages one being that they require vast amount of detailed information of the firm which can sometimes be unavailable. Furthermore, the model gives the relationship between the credit quality of a firm and its economic and financial conditions. It is thus quite challenging to observe the current make value of a firm.

The models are based on the specification of an exogenous process that governs the default event, moreover the default probabilities are generally generated directly from the market (Andrea, 2016).

The reduced form (intensity) model is fundamentally different from structural models in predicting the probability of default because they can easily capture sudden default events. Furthermore the model assumes that the default probabilities of the reference entity over any time is non-zero.

There are limitations which are aligned to the intensity model one which was discovered by Duffie and Singleton (1999) and observed that there exist some complexities when it comes to explaining the term structure of the credit spread across firms with different credit risk.

Pricing CDS can lead to the resolution of a PDE, whereby one is required to apply numerical methods to solve the PDE.

Finite difference method is one of the common methods to solve differential equations. The application of the method in Financial mathematics and Computational Fluid Dynamics is sometimes not reliable for conservative differential equations and solutions bearing shocks because the method is tough to implement in complex geometry whereby the method needs complex mapping and mapping makes governing the equation even more tougher. It becomes quite difficult when comparing the two methods, because of the variations of both methods.

The Finite volume method or the Box model is a numerical scheme for solving PDE whereby the discretization schemes are based upon an integral form of the PDE which was firstly introduced in Financial Mathematics for pricing option by Zvan et al. (2001), because the model is more flexible than other numerical schemes in particular the finite difference method(FDM). One of the basic advantage of FVM over FDM is that it does not require the use of structured grids, thereby the effort to convert the given mesh in to structured numerical grid internally is completely avoided because the scheme normally determines the cell averaged values.

The finite volume methods that we shall consider are; Two Point Flux Approximation (**TPFA**), Mid-quadrature rule and the Upwind method.

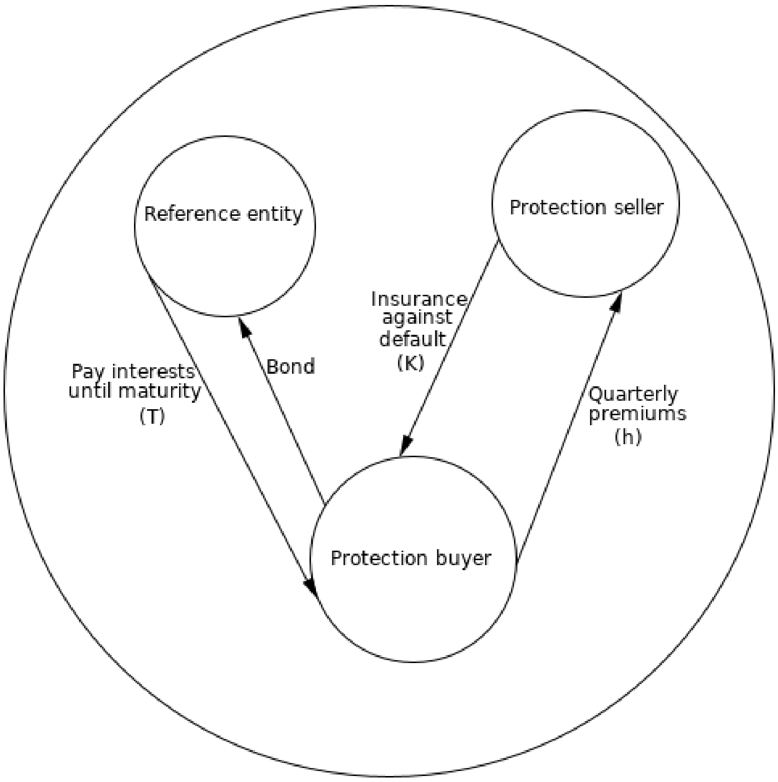

Figure 1.1: Transactions in a CDS contract between protection seller, protection buyer and the reference entity

Figure 1.1 above depicts a physical representation of a credit risk contract between the bond issuer(reference entity), the bond purchaser(protection buyer) and the bond insurer(protection seller).

Example

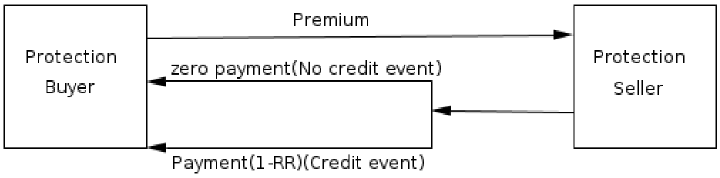

Figure 1.2: CDS contract between protection seller and protection buyer

Figure 1.2 above depicts the CDS contract between the bond purchaser and the bond insurer. The two parties has to agree on the terms of the contract i.e the protection seller will compensate the buyer if the reference entity (bond issuer) defaults; however, the protection buyer has to pay periodic premiums which can either be quarterly, semi-annually or annually to the seller until a credit event occurs or the bond terminates. For example lets suppose that a protection buyer make a purchase of a 5 year bond protection against default of a reference entity with the CDS spread of 100bps, where the notional amount of the bond is \$1 million. The protection buyer will make quarterly payments which equal to (\$1 million$\times$0.01$\times$0.25 = \$7500). Now lets assume that after some time the reference entity endures a credit event whereby the reference entity has a recovery rate of 40% of the notional value of the bond. The transactions of this contract are as follows

- The protection seller makes payment settlements to the protection buyer for the loss on the notional value of the bond i.e \$1 million$\times(100\% - 40\%) = \$600000$
- The protection buyer will also have to pay the interest accrued as the accrued premium from the previous payment date to time of default, i.e if the default occurs after a month the protection buyer will have to pay about \$1 million$\times(0.03 \times 1/12) = \$2500$

1.3 Problem statement

A CDS is a subset of the financial derivatives that guarantees lender against the bond risk. After the Global Financial crisis of 2008 there has been many challenges arising from pricing CDS due to the fluctuation of the interest rates. Our main objective is to solve numerically the PDE which arises from the CDS pricing model to help us determine the actual premiums that the protection buyer should pay to the protection seller. Therefore we construct a CDS pricing model considering the term structure of interest rates and credit events. Furthermore, when computing term structures of the interest rates and the credit events there are discontinuities which occurs in the pricing model, hence we chose to apply FVM for numerical approximation of the PDE instead of other methods because FVM uses the integral formula of conservation Laws as it does not make assumptions of smoothness.

This work is structured as follows: In chapter 2, we formulate the CDS pricing model, whereby we adopt the term structure modelling from interest rate, the CIR model and the credit events which is based on the work of (He, 2016). In chapter 3, we derive a PDE of the CDS intensity model for CDS valuation. In chapter 4, we apply Finite Volume Methods (FVMs) to solve the PDE numerically. In chapter 5 we perform numerical simulation to the CDS intensity model by FVMs. Chapter 6 is a conclusion.

2. Formulating the CDS model and the corresponding assumptions

In this section we formulate the CDS model based on the CDS contract and the corresponding assumption. Assuming that the CDS market is arbitrage free and complete, also the variables of model are defined as

2.0.1 Definition. K is the lump sum, h is the insurance premium and T is time to maturity of the bond.

2.0.2 Definition. τ is a stopping time to designate the occurrence of the underlying credit event.

The standard CDS Contract

This credit risk transaction binds two parties, the CDS seller and buyer, against credit and default event.

The contract terms are as follows.

1. The contract expires after $\tau \wedge \tau_1 \wedge \tau_2 \wedge T$.
2. Before $\tau \wedge \tau_1 \wedge \tau_2 \wedge$ T, the buyer pays the seller insurance premium cash i.e cash flow of continuous rate h.
3. If $\tau \leq T \wedge \tau_1 \wedge \tau_2$, the seller pays the buyer at time τ the lump sum of K i.e insurance compensation.

The assumptions made under this contract are

1. Current time is t=0
2. T, K and h are positive constants
3. τ, τ_1, τ_2, are non-negative

2.1 Modelling the term structure for interest rate

In this section we shall use the Cox-Ingersoll-Ross interest rate model to estimate the risky-free short term interest rate which follows a random process, under the "risk neutral world" defined on a filtrated probability space (Ω,$\mathcal{F}$,$\mathbb{P}$), where Ω is the sample space, $\mathcal{F}$ is a σ-algebra and $\mathbb{P}$ is the probability measure.

2.1.1 The CIR Model. We use CIR model which was developed by Cox et al. (2005) to forecast interest rate, since the model follows a stochastic differential equation, thereby this model uses a square-root diffusion process to ensure that the calculated interest rates are always non-negative. Therefore the stochastic differential equation is given by

$$dr_t = (k - \beta r_t)dt + \sigma\sqrt{r_t}dW_t \tag{2.1.1}$$

where $\{W_t\}_{t\geq 0}$ is the standard Brownian motion defined on the probability space (Ω,$\mathcal{F}$,$\mathbb{P}$), whereby the diffusion is said to be time inhomogeneous since σ depends on time, and κ , β and σ. are positive constant. κ is the speed of mean reversion, σ is the variance rate of the diffusion process, r_t is the shot term interest rate. The interest rate will remain positive because of the stochastic term $\sigma\sqrt{r_t}dW$ which has a standard deviation factor that is proportion to the square root of the current rate. The standard deviation increase as the rate increase, as it decreases and goes to zero, $\sigma\sqrt{r_t}dW$ approaches zero. Therefore the interest rate moves towards $\frac{\kappa}{\beta}$. This implies that the interest rate will never tend to zero where $3\kappa \geq \sigma^2$ but there exist a positive probability that r_t will likely tend to zero when $0 < \kappa < \frac{\sigma^2}{2}$. Hence as the interest rate approaches zero the model then is refined so that the process continues.

2.1.2 Discount factor in CIR Model. In the CIR model, at given time $t \geq 0$ the discount factor of a zero-coupon bond is the rate of one unit cash of settlement at maturity T, which is given by

$$e^{-\int_0^t r_{\theta d\theta}}$$

This impies that the value of the T-bond maturing in T years under CIR is the discounted value that make the current value of the cash-flows equalling the value of the bond at time t, which is expressed as

$$B(r,T) = E[e^{-\int_0^t r_{\theta d\theta}}|_{r_0=r}]$$

Therefore we observe that there exist a close relation between the discount factor and price of the bond. so we evaluate the evolution of the short rate interest process r_t of the zero-coupon bonds through the maturity time T, furthermore the shape of the trajectory for the term structure yield can be flat, decreasing or increasing depending on the value of the interest rate r_t. We introduce the Black-Scholes operator $\mathcal{L}$ which is associated with the model

$$\mathcal{L}\phi(r,T) = \frac{\partial}{\partial T} - \frac{\sigma^2}{2}r\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r\phi(r,T) \tag{2.1.2}$$

2.2 Modelling Credit Events

Assuming that the default times follows the intensity based approach, so from the CDS contract mentioned above we can recall that τ, τ_1 and τ_2 are credit event and default times, furthermore we model τ, τ_1 and τ_2 as the first jump of a time inhomogeneous Poisson process, with intensities $\{\lambda_r\}_{t\geq 0}$, $\{\lambda_{1r}\}_{t\geq 0}$ and $\{\lambda_{2r}\}_{t\geq 0}$, respectively. Thus from previous work by He (2016), the default rate of the credit event which depends only on the short interest for both the seller and the buyer's side are expressed differently as follows

$$\lambda_t = \Lambda(r_t), \quad \lambda_{1t} = \Lambda_1(r_t), \quad \lambda_{2t} = \Lambda_2(r_t),$$

where $\Lambda(\cdot), \Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$ are non-negative real-valued known functions, and are defined as follows

$$\Lambda(r) = ar + b \quad \Lambda_1(r) = pH(r - B_2) \quad \Lambda_2(r) = qH(B_1 - r)$$

where a, b, B and B_2 are positive constant where $B_1 < B_2$ and q and p are non-negative constant.

Since it is assumed that τ, τ_1 and τ_2 are conditionally independent when $0 \leq s < t$, then the probability of occurrence of a credit event is given by

$$\mathbb{P}(\tau \in (t-dt,t), \tau_1 \wedge \tau_2 > t|\mathcal{F}_t, \tau \wedge \tau_1 \wedge \tau_2 > s) = \lambda_t e^{-\int_s^t(\lambda_\theta + \lambda_{1\theta} + \lambda_{2\theta})d\theta}dt,$$

2.3 Valuation of CDS Model

We shall discuss the valuation of a CDS from the buyer's point of view, where the current time (t=0) and Present value of all payment from the seller to the buyer is given by the notional value of the bond, discounting factor, and the default intensities. Let σ, κ and β be positive constant and $\mathcal{L}$ be the Cox-Ingersoll-Ross differential operator defined as

$$\mathcal{L}u = (-\frac{\sigma^2 r}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r))u \tag{2.3.1}$$

Given the default intensity

$$\Lambda(r) = ar - b, \quad \Lambda_1(r) = pH(r - B_2), \quad \Lambda_2 = qH(B_1 - r), \quad f(r) = K\Lambda(r) - h \tag{2.3.2}$$

Where the state space of the interest rate is defined on $\Omega = (0, \infty)$, if the current time is any $t \in [0, T)$ with $\tau \wedge \tau_1 \wedge \tau_2 \geq t$, from the buyer's point of view, the present value of all payments from the seller to the buyer is given by

$$\mathbf{p} = Ke^{-\int_0^t r\theta d\theta}\mathbf{1}_{\tau\wedge\tau_1\wedge\tau_2\wedge T} - h\int_0^T e^{-\int_0^t r\theta d\theta}dt.$$

then the price of the CDS $u(r, T)$ is the expected payoffs of the present value received by the protection buyer which is given by

$$u(r, T) := \mathbb{E}[\mathbf{p}|\Omega_0]$$
$$u(r, T) = \mathbb{E}[\int_0^T e^{-\int_0^s (r\theta + \lambda_\theta + \lambda_{1\theta} + \lambda_{2\theta})d\theta}\{K\lambda_s - h\}ds|\Omega_0],$$

where $\Omega_0 = \tau \wedge \tau_1 \wedge \tau_2 > 0$

Then by the Feynman-Kac formula, the value, $u(r, T)$ of the CDS is the solution of the second order PDE in 3.0.1 which is given in the next chapter.

3. Derivation of the PDE for CDS valuation

In this section. we derive a PDE for credit default swap model valuation based on the work of He (2016), this PDE arises from the Black-Scholes model. Lets consider a CDS from a buyer's perspective which depends on the interest rate, therefore our Black-Scholes operator can be expressed as follows Peng He expressed the PDE problem for u that arises from the CDS pricing model as follow

$$\begin{cases} (\frac{\partial}{\partial t} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2)u = f & \text{in} \quad (0, \infty)^2 \\ u(\cdot, 0) = 0 & \text{in} \quad (0, \infty) \end{cases} \tag{3.0.1}$$

where $\Lambda, \Lambda_1, \Lambda_2$ are defined in section 2 respective; $f = K\Lambda - h$ and $\sigma > 0, \kappa > 0, \beta > 0, a > 0, b > 0, 0 < B_1 < B_2, p \geq 0, q \geq 0, K > 0, h > 0$ So from 3.0.1 we have

$$\frac{\partial u}{\partial t} + \mathcal{L}u + (\Lambda + \Lambda_1 + \Lambda_2)u = f \tag{3.0.2}$$

substituting the CIR model and the intensities we get

$$\frac{\partial u}{\partial t} - \frac{\sigma^2}{2} r \frac{\partial^2 u}{\partial r^2} - (\kappa - \beta r)\frac{\partial u}{\partial r} +$$
$$(r + (ar + b) + pH(r - B_2) + qH(B_1 - r))u = K(ar + b) - h$$

where

$$(r + (ar + b) + pH(r - B_2) + qH(B_1 - r))u = C(r)u$$

therefore the PDE for CDS valuation can be expressed as follows

$$\frac{\partial u}{\partial t} - \frac{\sigma^2}{2} r \frac{\partial^2 u}{\partial r^2} - (k - \beta r)\frac{\partial u}{\partial r} + C(r)u + K(ar + b) - h \tag{3.0.3}$$

so now 3.0.3 this can be reduced further as follows.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial r}(a\frac{\partial u}{\partial r} + bu) + C(r)u + K(ar + b) - h \tag{3.0.4}$$

Now we solve 3.0.4 by differentiation by elimination and the right hand side becomes

$$= a\frac{\partial^2 u}{\partial r^2} + b\frac{\partial u}{\partial r} + \frac{\partial a}{\partial r}.\frac{\partial u}{\partial r} + \frac{\partial b}{\partial r} + du + K(ar + b) - h$$

which can be simplified to

$$= a\frac{\partial^2 u}{\partial r^2} + (b + \frac{\partial a}{\partial r})\frac{\partial u}{\partial r} + (\frac{\partial b}{\partial r} + d)u + K(ar + b) - h \tag{3.0.5}$$

Therefore the derived CDS model is given by

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial r}(\frac{\sigma^2}{2} r \frac{\partial u}{\partial r} + (k - \beta r - \frac{\sigma^2}{2})u) + (\beta + C(r))u + K(ar + b) - h \tag{3.0.6}$$

If we try to solve 3.0.6 we can see that it does not have analytical solution so in the next chapter we will use numerical method to solve the derived PDE.

4. Discretization of the PDE

In this chapter we introduce two discretization schemes, whereby the main objective is to completely discretize the PDE for pricing CDS.

4.1 Finite difference method

Finite difference method is a method which is widely used to approximate differential equations. The derivatives of the PDE are replaced by discrete approximations obtained by Taylor series, He (2016) proved the consistency, stability and convergence of the method. The PDE here is solved using the ODE45 function from matlab, hence there is no need to discretize the PDE.

There are disadvantages of using FDM over FVM one being that FDM does not yield satisfactory results whereby it yields non-physical oscillation which are caused by discontinuous derivatives which is discussed by (Ramírez-Espinoza and Ehrhardt, 2013). The method tends to loose efficiency for approximations made on an unstructured mesh when running large-scale simulations.

4.2 finite volume method

Finite volume method(FVM) is one of the numerical techniques derived from finite difference method (FDM) which is used widely for pricing financial derivatives where a PDE arise and an analytical solution cannot be attained therefore the PDE model is discretely approximated via Two Point Flux Approximation, Mid-quadrature method and Upwind-method, furthermore FVM is used as it defines a control volume where we approximate the average value of the unknown cells over the control volume.

Finite volume method has many advantages over finite difference method one being that it mimics the true solution. The other advantage is that the method can easily capture the shocks or discontinuities which are caused by large derivatives inside a domain. Furthermore, the method is easily formulated to allow unstructured meshes according to (Shukla et al., 2011).

In this chapter we shall apply finite volume methods to solve the the PDE numerically.

4.2.1 Discretization schemes. Firstly we truncate the space domain as follows, whereby the interval is defined as follows

$r_0 = r_{\frac{1}{2}} = 0 < r_1 < r_{\frac{3}{2}} < r_{\frac{5}{2}} < ... < r_N < r_{N+\frac{1}{2}} < r_{N+1} = 1$

and the space of integration which is divided into N+1 equally spaced points, thereby the length is given by h $= r_{i+1} - r_i$, where $i = 0, 1, 2, .., N$ so that $r_i = i \times h$, for $i = 0, 2, 3, ..., N + 1$. We also define the mid-points by further sub-dividing the space domain as follows

$$r_{i+\frac{1}{2}} = \frac{r_i + r_{i+\frac{1}{2}}}{2}, \quad i = 0, ..., N$$
$$r_{i-\frac{1}{2}} = \frac{r_i + r_{i-\frac{1}{2}}}{2}, \quad i = 1, ..., N + 1$$

Thereby these mid-points help us define what we call a control volume where we will integrate our PDE over, and it is defined by

$K_i = [r_{i-\frac{1}{2}}, r_{i+\frac{1}{2}}], \quad i = 0, 1, 2, .., N$

Now we do the space discretization as follows, let us consider a PDE in 3.0.6, then

Discretizing first derivative

We apply the mid-quadrature rule to obtain

$$\int_{K_i} \frac{\partial u}{\partial t} dr \approx (r_{i+\frac{1}{2}} - r_{i-\frac{1}{2}}).\frac{\partial u}{\partial t}|_{r=r_i}$$

we now substitute $r_{i+\frac{1}{2}}$ for $i = 1, ...N$ and $r_{i-\frac{1}{2}}$ for $i = 0, ...N+1$

$$\begin{aligned}
&= (\frac{r_i + r_{i+1}}{2} - \frac{r_i + r_{i-1}}{2})\frac{\partial u_i}{\partial t} \\
&= (\frac{r_i + r_{i+1} - r_i - r_{i-1}}{2})\frac{\partial u_i}{\partial t} \\
&= (\frac{r_{i+1} - r_{i-1}}{2})\frac{\partial u_i}{\partial t} \\
&= \frac{2h}{2}\frac{\partial u_i}{\partial t} \\
&= h\frac{\partial u_i}{\partial t}
\end{aligned}$$

Therefore the mid-quadrature rule estimate for the first derivative is thus given by

$$\int_{K_i} \frac{\partial u}{\partial t} dr \approx h\frac{\partial u_i}{\partial t}$$

Discretizing second derivative

Since there are second order terms in the PDE we shall apply the **Two Point Flux Approximation** method where approximations will be done by tracking backward $(r_{i+\frac{1}{2}})$ and forward $(r_{i-\frac{1}{2}})$ as depicted in Figure 4.1 below

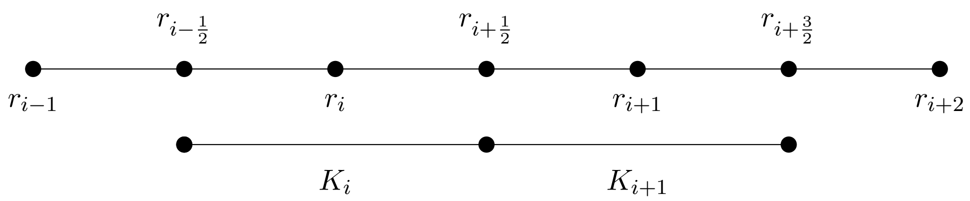

Figure 4.1: Interval

the higher order terms are then approximated as follows

$$\int_{K_i} \frac{\partial}{\partial r}(\frac{\sigma^2}{2} r\frac{\partial u}{\partial r} + (k - \beta r - \frac{\sigma^2}{2})u)dr \approx [\frac{\sigma^2}{2} r\frac{\partial u}{\partial r} + (k - \beta r - \frac{\sigma^2}{2})u]\Big|_{r_i-\frac{1}{2}}^{r_i+\frac{1}{2}}$$

so it follow that

$$\frac{\sigma^2}{2} r \frac{\partial u}{\partial r}\bigg|_{r=r_{i+\frac{1}{2}}} - \frac{\sigma^2}{2} r \frac{\partial u}{\partial r}\bigg|_{r=r_{i-\frac{1}{2}}} + (k - \beta r - \frac{\sigma^2}{2}) u\bigg|_{r=r_{i+\frac{1}{2}}} - (k - \beta r - \frac{\sigma^2}{2}) u\bigg|_{r=r_{i-\frac{1}{2}}}$$

now we determine the approximation of the continuous flux at the mid-points by tracking backward. Let

$$\begin{aligned} H_{i+\frac{1}{2}} &= \frac{\sigma^2}{2} r \frac{\partial u}{\partial r}\bigg|_{r=r_{i+\frac{1}{2}}} \\ H_{i+\frac{1}{2}} &= \frac{\sigma^2}{2} r_i \frac{u_{i+\frac{1}{2}} - u_i}{\frac{h}{2}} \\ &= \sigma^2 r_i \frac{u_{i+\frac{1}{2}} - u_i}{h} \end{aligned}$$

and

$$\begin{aligned} H_{i+\frac{1}{2}} &= \frac{\sigma^2}{2} r_i \frac{u_{i+1} - u_{i+\frac{1}{2}}}{\frac{h}{2}} \\ &= \sigma^2 r_{i+1} \frac{u_{i+1} - u_{i+\frac{1}{2}}}{h} \end{aligned}$$

so now for

$$H_{i+\frac{1}{2}} = H_{i+\frac{1}{2}}$$

we have

$$\begin{aligned} \sigma^2 r_i \frac{u_{i+\frac{1}{2}} - u_i}{h} &= \sigma^2 r_{i+1} \frac{u_{i+1} - u_{i+\frac{1}{2}}}{h} \\ r_i(u_{i+\frac{1}{2}} - u_i) &= r_{i+1}(u_{i+1} - u_{i+\frac{1}{2}}) \\ r_i u_{i+\frac{1}{2}} + r_{i+1} u_{i+\frac{1}{2}} &= r_i u_i + r_{i+1} u_{i+1} \\ \implies u_{i+\frac{1}{2}}(r_i + r_{i+1}) &= r_i u_i + r_{i+1} u_{i+1} \end{aligned}$$

therefore

$$u_{i+\frac{1}{2}} = \frac{r_i u_i + r_{i+1} u_{i+1}}{(r_i + r_{i+1})}$$

so substituting $u_{i+\frac{1}{2}}$, we get

$$H_{i+\frac{1}{2}} = \sigma^2 r_{i+1} \frac{u_{i+1} - \frac{r_i u_i + r_{i+1} u_{i+1}}{(r_i + r_{i+1})}}{h}$$

which reduces to

$$H_{i+\frac{1}{2}} = \frac{h\sigma^2 r_i r_{i+1}}{r_i + r_{i+1}} (u_{i+1} - u_i) \tag{4.2.1}$$

Similarly the forward tracking is giving as follows, let

$$H_{i-\frac{1}{2}} = \frac{\sigma^2}{2} r \frac{\partial u}{\partial r}\bigg|_{r=r_{i-\frac{1}{2}}}$$

$$H_{i-\frac{1}{2}} = \frac{\sigma^2}{2} r_i \frac{u_i - u_{i-\frac{1}{2}}}{\frac{h}{2}}$$

$$= \sigma^2 r_i \frac{u_i - u_{i-\frac{1}{2}}}{h}$$

and

$$H_{i-\frac{1}{2}} = \frac{\sigma^2}{2} r_{i-1} \frac{u_{i-\frac{1}{2}} - u_{i-1}}{\frac{h}{2}}$$

$$= \sigma^2 r_{i-1} \frac{u_{i-\frac{1}{2}} - u_{i-1}}{h}$$

so now for

$$H_{i-\frac{1}{2}} = H_{i-\frac{1}{2}}$$

we have

$$\sigma^2 r_i \frac{u_i - u_{i-\frac{1}{2}}}{h} = \sigma^2 r_{i-1} \frac{u_{i-\frac{1}{2}} - u_{i-1}}{h}$$

this simplifies to

$$r_i(u_i - u_{i-\frac{1}{2}}) = r_{i-1}(u_{i-\frac{1}{2}} - u_{i-1})$$

$$r_i u_i - r_i u_{i-\frac{1}{2}} = r_{i-1} u_{i-\frac{1}{2}} - r_{i-1} u_{i-1}$$

$$\implies u_{i-\frac{1}{2}}(r_i + r_{i-1}) = r_i u_i + r_{i-1} u_{i-1}$$

therefore

$$u_{i-\frac{1}{2}} = \frac{r_i u_i + r_{i-1} u_{i-1}}{(r_i + r_{i-1})}$$

so substituting $u_{i-\frac{1}{2}}$, we get

$$H_{i-\frac{1}{2}} = \sigma^2 r_{i-1} \frac{\frac{r_i u_i + r_{i-1} u_{i-1}}{(r_i + r_{i-1})} - u_{i-1}}{h}$$

which reduces to

$$H_{i-\frac{1}{2}} = \frac{h\sigma^2 r_i r_{i-1}}{r_i + r_{i-1}}(u_i - u_{i-1}) \tag{4.2.2}$$

So now we solve $(k - \beta r - \frac{\sigma^2}{2})u\Big|_{r=r_{i+\frac{1}{2}}} - (k - \beta r - \frac{\sigma^2}{2})u\Big|_{r=r_{i-\frac{1}{2}}}$ using the **Upwind method** as follows, the backward and forward difference of $u_{i+\frac{1}{2}}$ is given by

$$u_{i+\frac{1}{2}} = \begin{cases} u_{i+1} & \text{if } b(r_{i+1}) < 0. \\ u_i & \text{if } b(r_{i+1}) > 0 \end{cases} \tag{4.2.3}$$

so

$$b(r_{i+\frac{1}{2}})u_{i+\frac{1}{2}} = max(b(r_{i+1}), 0)u_{i+1} + min(b(r_{i+1}), 0)u_i$$

and also the backward and forward difference of $u_{i-\frac{1}{2}}$ is thus given by

$$u_{i-\frac{1}{2}} = \begin{cases} u_{i-1} & \text{if } b(r_{i-1}) > 0. \\ u_i & \text{if } b(r_{i-1}) < 0 \end{cases} \tag{4.2.4}$$

so

$$b(r_{i-\frac{1}{2}})u_{i-\frac{1}{2}} = max(b(r_{i-1}), 0)u_{i-1} + min(b(r_{i-1}), 0)u_i$$

Discretizing the third part

Applying the **mid-quadrature rule**, to the third part of our PDE we have

$$\int_{K_i} (\beta + C(r))udr \approx h(\beta + C(r))u|_{r=r_i}$$

therefore the mid-quadrature rule estimate is given by

$$\approx h(\beta + C(r_i))u_i$$

Discretizing the fourth part

we now use the **mid-quadrature rule** to approximate the last part of our PDE as follows

$$\int_{K_i} (K(ar + b) - h)dr \approx h(K(ar + b) - h)\Big|_{r=r_i}$$

therefore the mid-quadrature rule estimate is given by

$$\approx h(K(ar_i + b) - h)$$

so now our PDE sums up to

$$h\frac{\partial u_i}{\partial t} = H_{i+\frac{1}{2}} - H_{i-\frac{1}{2}} + (k - \beta r_{i+\frac{1}{2}} - \frac{\sigma^2}{2})u_{i+\frac{1}{2}} - (k - \beta r_{i-\frac{1}{2}} - \frac{\sigma^2}{2})u_{i-\frac{1}{2}} + h(C(r_i) + \beta)u$$
$$+ h(K(ar_i + b) - h)$$

so we set

$$g(r_i) = h(K(ar_i + b) - h)$$

and substituting both $H_{i+\frac{1}{2}}$ and $H_{i-\frac{1}{2}}$ to get

$$h\frac{\partial u_i}{\partial t} = \frac{h\sigma^2 r_i r_{i+1}}{r_i + r_{i+1}}(u_{i+1} - u_i) - \frac{h\sigma^2 r_i r_{i-1}}{r_i + r_{i-1}}(u_i - u_{i-1}) + max(b(r_{i+1}), 0)u_{i+1} + min(b(r_{i+1}), 0)u_i$$
$$- max(b(r_{i-1}), 0)u_{i-1} - min(b(r_{i-1}), 0)u_i + h(C(r_i) + \beta)u_i + g(r_i)$$

so now for simplicity we let

$$\alpha_{i+1} = \frac{h\sigma^2 r_i r_{i+1}}{r_i + r_{i+1}}, \quad \alpha_{i-1} = \frac{h\sigma^2 r_i r_{i-1}}{r_i + r_{i-1}}, \quad \gamma_{r+1} = max(b(r_{i+1}), 0), \quad \gamma_{r-1} = max(b(r_{i-1}), 0)$$

and

$$\zeta_{r+1} = min(b(r_{i+1}), 0), \quad \zeta_{r-1} = min(b(r_{i-1}), 0)$$

$$h\frac{\partial u_i}{\partial t} = \alpha_{i+1}(u_{i+1} - u_i) - \alpha_{i-1}(u_i - u_{i-1}) + \gamma_{r+1}u_{i+1} - \gamma_{r-1}u_{i-1} + \zeta_{r+1}u_i - \zeta_{r-1}u_i + h(C(r_i) + \beta)u + g(r_i)$$

it further simplifies to

$$h\frac{\partial u_i}{\partial t} = (\alpha_{r_{i+1}} + \gamma_{r_{i+1}})u_{i+1} + (\zeta_{r_{i+1}} + \alpha_{r_{i+1}} - \zeta_{r_{i-1}} - \alpha_{r_{i-1}} + h(C(r_i) + \beta))u_i - (\gamma_{r_{i-1}} - \alpha_{r_{i-1}})u_{i-1} + g(r_i)$$

setting

$p_i = (\alpha_{r_{i+1}} + \gamma_{r_{i+1}})$, $q_i = -(\zeta_{r_{i+1}} + \alpha_{r_{i+1}} - \zeta_{r_{i-1}} - \alpha_{r_{i-1}} + h(C(r_i) + \beta))$ and $s_i = (\gamma_{r_{i-1}} - \alpha_{r_{i-1}})$

so now the PDE can be represented algebraically as follow

Case 1: i = 1

$$h\frac{\partial u_i}{\partial t} = p_i u_{i+1} + q_i u_i + s_i u_{i-1} + g(r_i)$$

for $i = 1$

$$h\frac{\partial U_1}{\partial t} = p_1 u_2 + q_1 u_1 + s_1 u_0 + g(r_1)$$

since $u_{1-1} = u_0$, but we want to start iterating from u_1 so

$$h\frac{\partial U_1}{\partial t} = p_1 u_2 + q_1 u_1 + s_1 u_1 + g(r_1)$$

Case 2: i = 2

$$h\frac{\partial U_2}{\partial t} = p_2 u_3 + q_2 u_2 + s_2 u_1 + g(r_2)$$

Case 3: i = 3

$$h\frac{\partial U_3}{\partial t} = p_3 u_4 + q_3 u_3 + s_3 u_2 + g(r_3)$$

Case 4: i = N-1

$$h\frac{\partial U_{N-1}}{\partial t} = p_{N-1}u_N + q_{N-1}u_{N-1} + s_{N-1}u_{N-2} + g(r_{N-1})$$

Case 5: i = N

$$h\frac{\partial U_N}{\partial t} = p_N u_{N+1} + q_N u_N + s_N u_{N-1} + g(r_N)$$

since u_{N+1} is the last point in our domain, our ending point will be set to N, hence we set $u_N = u_{N+1}$ therefore

$$h\frac{\partial U_N}{\partial t} = p_N u_N + q_N u_N + s_N u_{N-1} + g(r_N)$$

Now to get the semi-discretized equation from the algebraic equation we need to construct a tri-diagonal matrix from the coefficients as follows

$$A = \begin{pmatrix} s1+q_1 & p_1 & 0 & \dots & \dots & 0 \\ s_2 & q_2 & p_2 & \dots & \dots & 0 \\ 0 & s_3 & q_3 & p_3 & \dots & 0 \\ 0 & . & . & \dots & \dots & 0 \\ 0 & \dots & \dots & s_{N-1} & q_{N-1} & p_{N-1} \\ 0 & \dots & \dots & .. & s_N & p_N + q_N \end{pmatrix}$$

Matrix B consists of the values of $g(r_i)$

$$B = \begin{pmatrix} g(r_1) \\ g(r_2) \\ g(r_3) \\ \dots \\ \dots \\ \dots \\ g(r_{N-1}) \\ g(r_N) \end{pmatrix}$$

so now from matrix A and B we have a semi-discretized equation which can be expressed as

$$h\frac{\partial U_i}{\partial t} = AU + B \tag{4.2.5}$$

To get the fully discretized model we need to fully discretize the semi-discreatized equation in time, this will be done in the following section.

4.3 Time discretization

We use θ-Euler method to completely discretize 4.2.5. We prefer to discretize time t, where t$\in [0, T]$ in N+1 intervals j grid points such that $0 = t_1, < t_2 < t_3 < ... < t_j = T$, $dt = \frac{T}{n}$, so from 4.2.5 using θ-Euler method where $\theta = 0.5$, we have

$$h\frac{U^{j+1} - U^j}{dt} = \theta[AU^{j+1} + B^{j+1}] + (1-\theta)[AU^j + B^j]$$
$$hu^{j+1} + U^j = \theta dtAU^{j+1} + \theta dtB^{j+1} + dtAU^j - \theta dtAU^j + dtB^j - \theta dtB^j$$
$$hu^{j+1} - \theta dtAU^{j+1} = U^j + \theta dtB^{j+1} + dtAU^j - \theta dtAU^j + dtB^j - \theta dtB^j,$$ where I is the identitty matrix
$$U^{j+1}[Ih - \theta dtA] = [I + dt(1-\theta)A]U^j + dtB^j$$

therefore we have a fully discretized model which is given as,

$$U^{j+1} = ([I - \theta dtA]^{-1}[I + dt(1-\theta)A]\frac{1}{h})U^j + dtB^j$$

Some special cases of θ are given by:

$$\begin{aligned} \theta &= 0, \quad \text{implicit Euler scheme} \\ \theta &= 1, \quad \text{explicit Euler scheme} \\ \theta &= \frac{1}{2}, \quad \text{Crank-Nicolson scheme} \end{aligned}$$

In this study we prefer to chose $\theta = \frac{1}{2}$

5. Numerical experiments

In this section we perform numerical approxmation on the 1-d model whereby we compare results from FVMs with FDM which was obtained by (He, 2016). The codes are written in Matlab. The domain of computation is (r,T)$\in \Omega = (0,1) \times [0,T)$. We shall compare the results of this two models using Monte-Carlo method as a benchmark, to determine which method is better at approximating the derived PDE for pricing credit default swaps.

And this will be done as follows.

5.1 Using finite difference method

To test finite difference scheme, we shall adopt the results of the model from the previous work of He (2016), where he used the ODE45 function in Matlab to solve the PDE, moreover time space(T) remained fixed and the interest rate space(r) is divided into m subintervals from 0 to 1. The simulation were performed until N = 1500 since the computation cost and memory were considered. However, the ODE solver tends to save results of the matrix in each time step thereby the memory gets exhausted very fast and the cost of computation is high.

5.2 Using finite volume method

To test finite volume schemes, we shall adopt the calibrated parameters in CIR model from Table 5.1, which appeared in the PDE problem. To match the results of the previous work of (He, 2016), the simulation were performed with N = 100 to N = 1500, and the computational cost and memory were considered.

5.3 Results and Analysis

We use similar parameters adopted from the work of He (2016) which can be found in Table 5.1 below and also varying N and σ. The results of simulations are shown in Figure 5.1, 5.2 and 5.3.

Table 5.1: Setting up parameter

Constants	Value
T	10(years)
θ	0.5
κ	6%
β	0.780(/year)
σ	12.2%(/year)
a	0.5($)
b	0.1(/year)
B_1	4%(/year)
B_2	6%(/year)
p	0.5(/year)
q	0.3(/year)
K	1($)
h	0.125($/year)
n	360
N	1500(dim)

From the work by He (2016), Figure 5.2 below we can see that the finite volume scheme when $r = 0.06$, $N = 750$ the graph converges to maturity, but the finite difference method slightly diverges away.

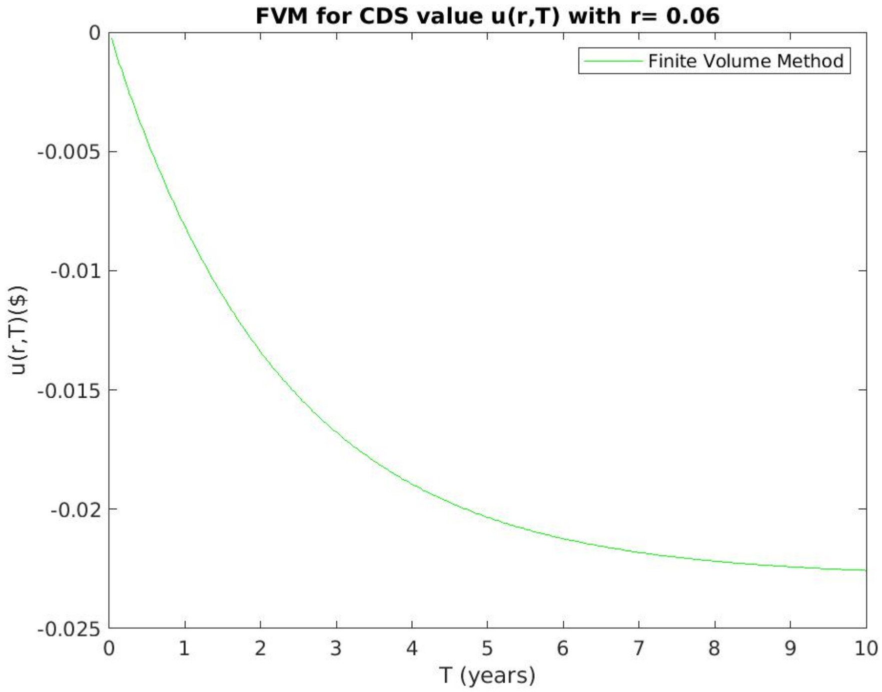

Figure 5.1: Finite Volume Method: $\frac{2k}{\sigma^2} < 1$, with N=500

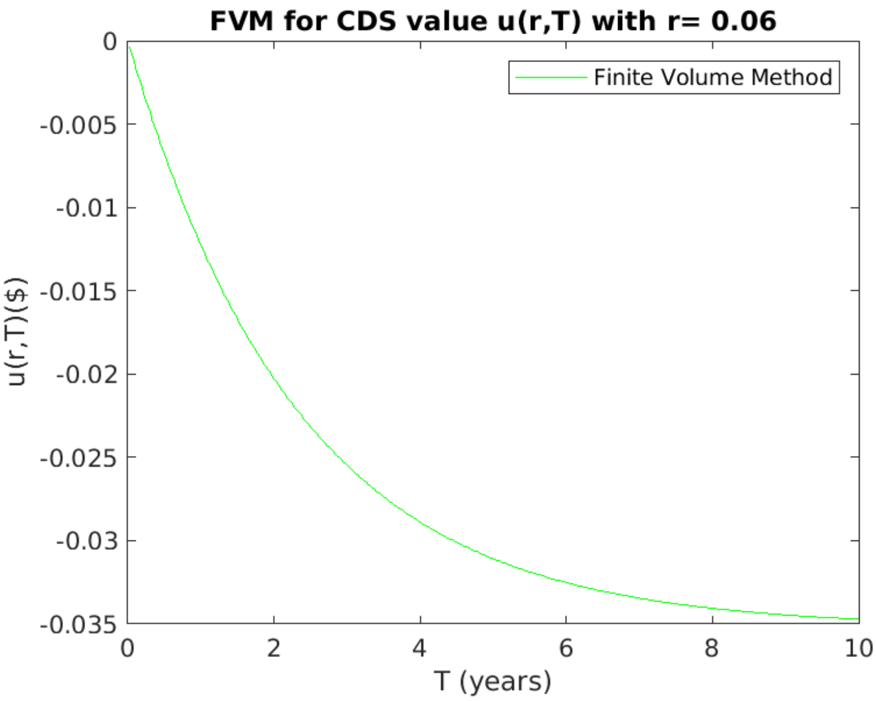

Figure 5.2: Finite Volume Method: $\frac{2k}{\sigma^2} < 1$, with N=750

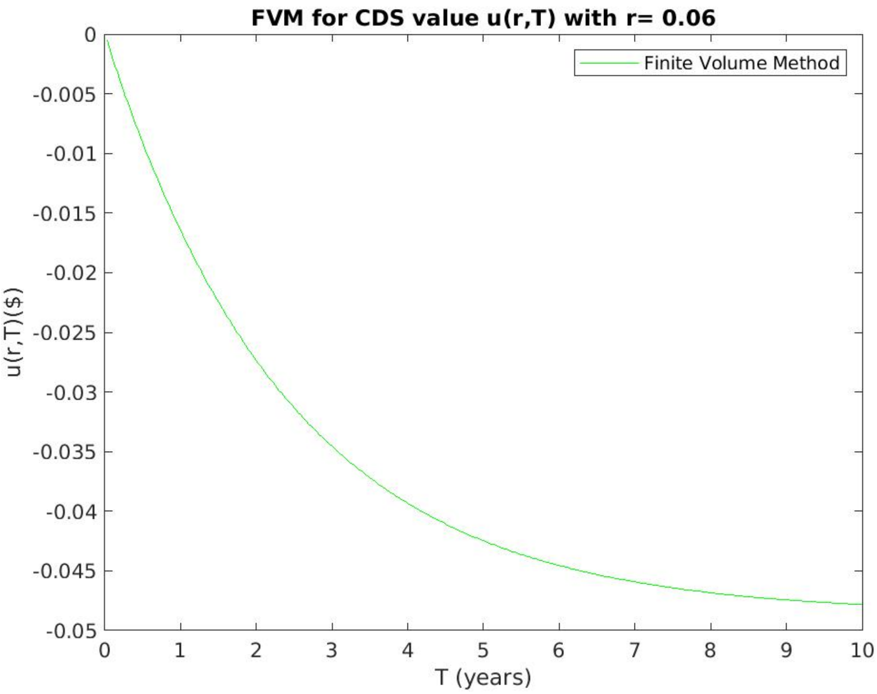

Figure 5.3: Finite Volume Method: $\frac{2k}{\sigma^2} < 1$, with N=1500

Table 5.2: Relative errors

	Numerical Methods(relative errors(%))	
	Finite difference method	Finite volume method
Space		
N = 100	0.5122	0.4472
N = 250	0.3912	0.3732
N = 500	0.2403	0.2202
N = 750	0.1038	0.0971

To determine which method is more accurate than the other in terms of solving the PDE we calculate the relative errors of both methods using Monte Carlo method as a benchmark. From the results obtained in Table 5.2 we can see that finite FVM outperforms FDM. The method of calculating relative errors in different space were defined as:

$$Relativeerror = \frac{|P_{approximated} - P_{analytical}|}{|P_{analytical}|},$$

where $P_{approximated}$ is the solution given by the numerical method and $P_{analytical}$ is the solution given by the Monte Carlo Method.

5.4 Conclusion on numerical results

After analysing the results above, we see that from Figure 5.2 and 5.3, the simulated solution of the PDE problem almost matched.From Figure 5.2 and 5.3, one can see that the first two years, the contract favours the CDS buyer because the price $u(0.06, T) \geq 0$ for $0 \leq T \leq 2$ and $u(0, 06, T) \leq 0$ for $2 \leq T \leq 10$. Varying the number of steps produces different numerical solutions, so from Figure 5.1 increasing the variance (σ) also causes the model to perform badly, therefore the numerical solution diverges. From Figure 5.2 we can see that when the number of steps N = 750, the method performs well because the value of the CDS converges to maturity T. After calculating the relative errors of the two methods, we find that from Table 5.2, finite volume method has less relative errors compared to finite difference method. Therefore finite volume methods performs better than the finite difference method, thereby the method is more accurate and the method is better in approximating the price of CDS than finite difference method.

6. Conclusion

In this end, this paper has introduced the new framework in the world of credit derivatives for approximating non-linear PDE which arises from the credit default market, we studied the pricing of CDS under the work of He (2016) focusing on both the structural and reduced form approach. The CIR model (Cox, Ingersoll Jr, and Ross, 2005) is employed to model the risk-free interest rates. Furthermore, the intensity based model was used to model the credit events which may arise during the term of the bond or before the bond terminates. A one dimensional PDE model from the risk-neutral and real world default probabilities with different maturities were derived. Finite volume methods (**TPFA**, upwind method and mid-quadrature rule) were applied in approximating the solution of the PDE. Moreover, the PDE approach is more computationally efficient. The **TPFA** method is more accurate than the finite difference method.

For future work, we can study the stability and rate of convergence and extend the approach to a high-dimensional problem.

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