Mode Competition inside Laser Resonators

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Abstract

In this work the competition mode of the laser cavity resonator was achieved by finding the lowest loss mode. The results show that the largest eigenvalue of the propagator matrix represents the phase shift and the associated eigenvector corresponds to the lowest mode. It was also proven that the eigenvalues are always a monotonically decreasing series. Not that the fundamental mode (lowest loss) is changing as long as the optics parameters of the cavity change. In order to explore the problem of mode competition inside laser resonators, we studied and used the Li-Fox method and the matrix approach to model the cavity. We then used a power method to solve the eigenequations coming from the propagator matrix of the cavity. Finally, the simulation of the system in Matlab showed that there is an effect from the optics parameters on the output.

Keywords: Cavity resonator, lowest loss mode, phase shift, Li-Fox, and Matrix approch.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Nedjla Tazdait, 19 May 2020

Contents

1. Introduction

The cavity resonator is essential in the design of microwave oscillators as it is the frequency determining element. Their resonant frequency can be tuned by moving one of the walls of the cavity in or out, thereby changing its size (Pozar, 2009). Also the cavity resonator can be used for the measurement of microwave frequencies.

Light confined in the cavity reflects multiple times, producing standing waves for certain resonante frequencies (Paschotta et al., 2008). Optical resonators are used as laser resonators where the resonator losses are compensated by a gain medium to maintain or build up optical power and for filtering the frequency content of optical radiation. They are also used for filtering the transverse shape of optical radiation and for exploiting the resonant enhancement of interactivity power, e.g. in order to achieve efficient frequency doubling of light from a low-power single-frequency laser (Siegman, 1986).

In the design of a laser the most important calculation is the determination of the transverse mode of the cavity resonator. The resonator mode structure and spatial extent largely determines most of the important characteristics of a laser. Stable modes of oscillation are the variation in an electromagnetic field that can be kept in the resonator; those modes can be determined by the geometry of the resonator studied. Therefore, when the geometry is simple, an analytic method is used, but when it is complicated, numerical methods must be used to solve Maxwell's equations in the presence of the relevant boundary conditions. We use the numerical method of Fox and Li to obtain the operating mode of the device instead of solving the entire Maxwell's equations. Fox and Li used the Fresnel–Kirchhoff diffraction integral to mimic the physical process of wavefront propagation within the device, thus arriving at its stable mode of operation after several iterations (Kogelnik and Li, 1966a).

Numerous modes can be determined by this method, including the lowest-order mode in which the resonator operates under most conditions. This being the fundamental transverse mode of the laser resonator, and also has the same form as a Gaussian beam (Electriciantraining and Publishin, 2003).

Recall that in 1960, A G Fox and Li published their results concerning the lowest order transverse modes in Febry-Perot interferometers, i.e, the basic resonator in laser oscillation. Their calculations involved numerical competition where the simplest approach is the one dimensional system. However, the mode of two dimensional case is obtained by multiplying the mode in the orthogonal direction (Hall, 1990). Based on the matrix approach to find the eigenequation of the system, the power method was used to find these solutions.

1.1 Objectives

The objectives in this project are to model resonators using the matrix approach. By finding the eigenvalues and eigenvectors of this matrix, and the relationship between the eigenvectors of the matrix and the eigenmodes of the cavity. Also, if there is a relationship between them we can ask why are the number of eigenvectors dependent on the matrix size. If we find a set of eigenvectors, how do we know that the first is indeed the lowest loss. In other words, are the eigenvalues always a monotonically decreasing series? Finally, there is a comparison between the matrix approach and the Prony method of Siegman.

1.2 General Outlines

In this work Chapter 2 we will cover optical resonators, cavity modes, stability and the ABCD matrix, as well as the power method for eigenvalues and eigenvectors. chapter 3 explains how the resonator can be modelled as well as how to get the solutions of this model. In this chapter we are interested in finding the relationship between the eigenvalues and the eigenvectors of the propagator matrix, and how they can be used to find the lowest loss mode of this cavity. Finally, in Chapter 4, we conclude our work with a discussion of the results obtained.

2. Literature Review

2.1 Introduction

In this chapter some basic definitions and mathematical tools are presented. The definition of cavity modes and some properties concerning eigenvalues and eigenvectors are presented. In the last section an important method is presented that of the power method. This method is essential for our later calculations.

2.2 Optical Resonator

Lasers are optical resonators, as they are predominantly made up of three essential elements. These elements are the active medium, the pumping system and the resonating cavity. A simple cavity resonator consists of two mirrors M_1 and M_2 that are separated by a distance L. The point of interest in this study is to apply mathematical methods to model resonators. This is accomplished by calculating the modes of the cavity with the lowest loss. Figure 2.1 shows some of the different types of optical resonators we can consider.

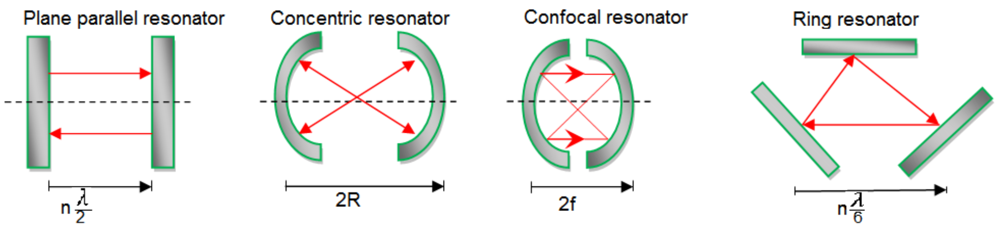


Figure 2.1: Cavity resonator types (QAo)

2.3 Cavity Modes

All radiation that comes from a laser's source are discrete optical frequencies that are usually separated by a non-similar frequency, and that are also they are associated with different modes. Each mode has variations in electromagnetic fields perpendicular to the axis of the resonator. Alternatively, this scenario may be viewed as the light reflected multiple times from mirrors inside the resonator which can interfere with itself. It follows that only a few wavelengths, and their associated waves, can be present in the cavity. These waves are called resonant modes, and they depend on the shape of the cavity (Koechner, 2013).

2.3.1 Longitudinal Modes.

From the quantum theory of light, the modes are the possible combinations of frequencies, polarisation and directions of electromagnetic field. However, not all modes in the cavity resonator are possible. The only modes possible are the modes that satisfy the condition

$$m\lambda = 2L\,, \tag{2.3.1}$$

where m is a particular mode number, λ the wave length of the laser inside the cavity, and L is the cavity length.

This condition means that the only modes that are permitted are those which create a standing wave. Longitudinal modes propagate along the axis of the laser cavity, which depends on the combination of directions. For a resonant mode to occur it must experience a phase shift equal to an integer multiple of 2 π over one closed loop path. Figure 2.2 represents some longitudinal modes

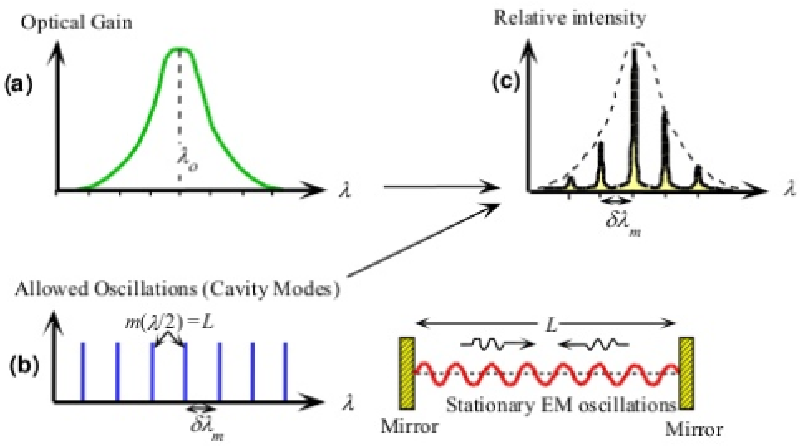


Figure 2.2: Longitudinal modes (Ackley, 1983)

2.3.2 Transverse Modes.

If we consider the electromagnetic field propagating axially, some modes perpendicular to the axis of the plan direction cavity are obtained as shown in Figure 2.3

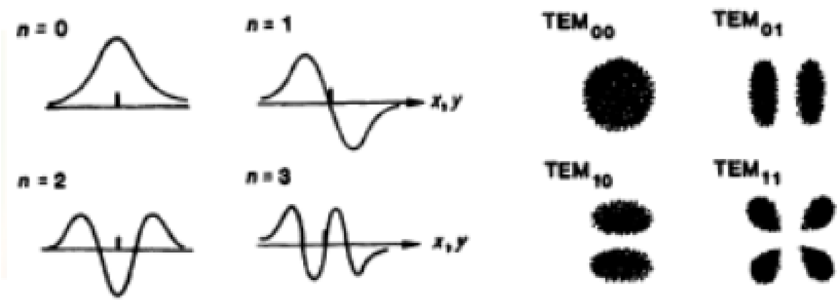


Figure 2.3: Transverse modes by curved mirror (Kogelnik and Li, 1966a)

By looking at the beam profile one can determine which kind of mode dominates its behave. The difference between the two types of cavity resonator modes are longitudinal and transverse. In the longitudinal mode the difference between the mode is the frequency of the oscillation. However, in the transverse mode, the difference between the modes is not just in the oscillation frequency, as for the first type, but also in their field distribution in a plane perpendicular to the direction of propagation.

In order to describe the modes of the wave structure inside the resonator we use the symbols TEM_{mnq} (rectangular coordinates) and TEM_{pql} (cylindrical coordinates). Where TEM are the Transverse Electromagnetic Waves, where the (mn) indices describe specific transverse modes and q describes the longitudinal mode (Paschotta et al., 2008).

2.4 Stability and the ABCD Matrix

2.4.1 Stable and unstable resonator.

Geometric optics used to study the stability of an optical resonator are based on a transfer matrix called the ABCD matrix. This makes it possible to determine the stability criteria of a resonator. Having geometrically stable or unstable optical resonators is related to the ray stability inside the cavity, where with each round-trip, the transverse profile is modified. Figure 2.4 shows the stable/unstable resonators.

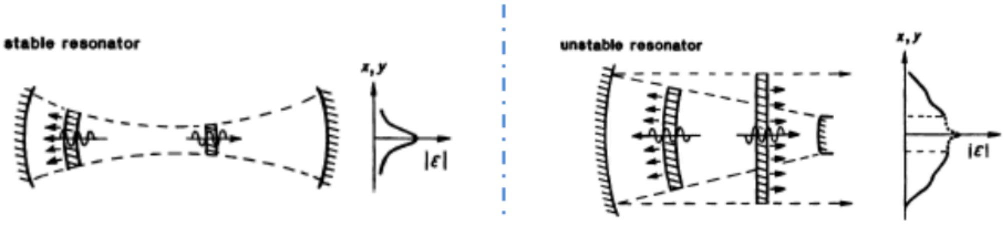


Figure 2.4: Stable/unstable resonators (Kogelnik and Li, 1966a)

The optical energy traveling with, is the segment of length z in the $+z$ direction of the cavity is given as

$$E(x, y, z) = Re\{\tilde{E}(x, y, z).e^{j(wt-kz)}\}\,, \tag{2.4.1}$$

where $\tilde{E}(x, y, z) = |\tilde{E}(x, y, z)|.e^{j\varphi(x,y,z)}$ and $e^{j(\omega t-kz)}$ is a plane wave.

For stable cavities the transverse modes depend on the type of the mirror in the cavity resonator. For curved mirrors the transverse modes are "Hermite-Gaussian functions" (cylindrical coordinates) and are written as

$$E_{mn}(x, y, z) = E_0 \frac{W_0}{W(Z)} H_n\left(\sqrt{2}\frac{X^2}{W(Z)^2}\right) H_m\left(\sqrt{2}\frac{y}{Z}\right) exp\left(-\frac{y^2}{W(Z)^2}\right)$$

$$\times exp\left(-i\left[kZ - (1+n+m)arctan\frac{Z}{Z_R} + \frac{k(X^2+y^2)}{2R(Z)}\right]\right)\,, \tag{2.4.2}$$

where E_0 is the field maximum, x and y are the axes that define a cross-section of the beam, Z is the axis of propagation, W_0 is the beam waist, $W(Z)$ is the beam radius at a given Z value, $H_n(x)$ and $H_m(x)$ are Hermite polynomials with non-negative integer indices n and m, k is the wavenumber (k=$2\pi/\lambda$), Z_R is the Rayleigh range, and $R(Z)$ is the radius of curvature of the wavefront. For the flat mirror the modes are Bessel functions (Kogelnik and Li, 1966a).

The diffraction loss is the amount of energy lost past the mirror edges, where determining the transverse mode properties of the cavity and now the profile in each rountrip is further modified by propagation, diffraction and bounces on end mirrors along with passes through lenses or apertures (Kogelnik and Li, 1966b)

2.4.2 The Principal of the ABCD Matrix Method.

The principle of this method is that each optic element (simple propagation in the medium, lens, mirror etc.) is associated to a specific 2×2 matrix. This can be determined by the characteristics related to the propagation with a simple multiplication of elementary matrices. Firstly, there are some considerations of the cavity resonators having two mirrors with radius R_1 and R_2 that are separated by a distance L. This system can be seen as an infinite succession of two lenses of focal distance $f_1 = \frac{R_2}{2}$ and $f_2 = \frac{R_2}{2}$.

To determine the stability of this criteria, a study of one trip for the ray inside the resonator should be observed. Therefore, using the ABCD matrices makes it possible to find some paraxial essential elements inside the cavity. More importantly, the ray of light is then given by the eigenvector $\vec{r_i}$ of the transfer matrix M_i corresponding to an eigenvalue γ_i, where we write

$$M_i \vec{r_i} = \gamma_i \vec{r_i} \,. \tag{2.4.3}$$

Writing for any arbitrary ray as a linear combination yields

$$\vec{r} = a_1 \vec{r_1} + a_2 \vec{r_2} \,. \tag{2.4.4}$$

As a consequence, a direct calculation of the ray vector after N round-trips inside the resonator gives

$$\vec{r_N} = a_1 \gamma_1^N \vec{r_1} + a_2 \gamma_2^N \vec{r_2} \,. \tag{2.4.5}$$

In the case of a stable resonator $|\gamma_1| = |\gamma_2| = 1$, which henceforth means that there is no ray that will leave the cavity. Otherwise the ray in equation (2.4.5) diverges in a radial position and the propagation angle converges to a bases vector or a zero vector. The condition for stability is then given by

$$0 \leqslant g_1 g_2 \leqslant 1 \,, \tag{2.4.6}$$

where $g_1 = 1 - \frac{L}{R_1}$ and $g_2 = 1 - \frac{L}{R_2}$.

Figure 2.5 shows an example of an ABCD matrix which best describes different optics system

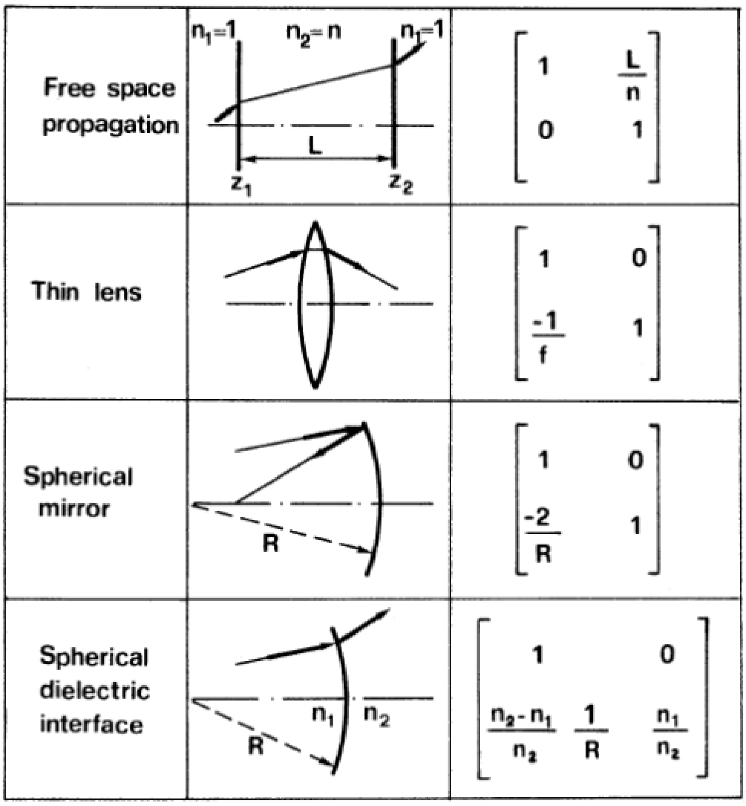


Figure 2.5: ABCD matrix examples (Svelto and Hanna, 2010)

2.5 Eigenvectors and Eigenvalues

Matrices are a powerful mathematical tool to be used in different applied fields of engineering and science. In this section we will present important properties of matrices, eigenvalues and eigenvectors to answer some interesting questions.

To begin we recall that linear transformation, T, of a vector, v, from a vector space, V, over a field, F, into itself where v is a non-zero vector in V, is defined as

$$T(v) = \lambda v.$$

v is the eigenvector of T if $T(v)$ is a scalar multiple of v. The eigenvalue, or characteristic root, corresponds to the eigenvector v. If we have a finite-dimensional vector space, then the linear transformation T can be represented as a square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}. \tag{2.5.1}$$

Let A be an $n \times n$ matrix. An eigenvalue λ, and its corresponding eigenevector v, are a scalar and a non-zero vector respectively, that satisfy

$$Av = \lambda v\,. \tag{2.5.2}$$

That is

$$(A - \lambda I)v = 0\,. \tag{2.5.3}$$

If v is an eigenvector associated with the value λ, then there is an αv for every non-zero scalar α, such that

$$A(\alpha v) = \alpha(Av) = \alpha(\lambda v) = \lambda(\alpha v)\,. \tag{2.5.4}$$

So the subspace of eigenvectors associated with λ has repeated eigenvalues.

An eigenvalue is a root of the characteristic polynomial given by the determinant of the singular matrix $A - \lambda I$.

2.6 Proprieties of Eigenvalues

Consider the equation $as^2 + bs + c = 0$, where when we divide by a, we get

$$s^2 + \frac{b}{a}s + \frac{c}{a} = 0\,,$$

where $\frac{c}{a}$ represent the product of the roots, and $\frac{-b}{a}$ is the sum of the roots.

We have

$$\lambda^2 - (a+d)\lambda + (ab - bc) = 0\,, \tag{2.6.1}$$

where $\lambda_1 + \lambda_2 = a + d = Tr(A)$, and the product $\lambda_1\lambda_2 = ad - bc = |A|$

2.6.1 Remark. (a) It is important to note that the total summation of the eigenvalues, coming from the matrix A, is the sum of its diagonal elements. It is also known as the trace of A, which is by definition given by (Apostol, 2007),

$$\sum \lambda = Tr(A)\,,$$

$$\prod \lambda = |A|\,. \tag{2.6.2}$$

(b) If $A = 0$ the eigenvalue is zero.

(c) If a matrix A has three eigenvalues $\lambda_1, \lambda_2, \lambda_3$, then the eigenvalues of the transpose matrix A^T has the same eigenvalues.

Give the eigenvalues as the square eigenvalues of A, that is λ_1^2, λ_2^2, λ_3^2.

So for

$$A^n \to \lambda_1^n, \lambda_2^n, \lambda_3^n\,. \tag{2.6.3}$$

$$A^{-2} \to \frac{1}{\lambda_1^2}, \frac{1}{\lambda_2^2}, \frac{1}{\lambda_3^2}\,, \tag{2.6.4}$$

2.6.2 Vector and matrix sequences.
An infinite sequence of elements $x_0, x_1 ...$ of a set X will be denoted $(x_k)_k >$ 0 "or simply (x_k). In a vector space V with a standard norm $|| \bullet ||$. We say that a sequence (v_k) of elements of V converges to an element $v \epsilon V$, or that v is the limit of the sequence (v_k), if

$$\lim_{k \to \infty} ||v_k - v|| = 0\,, \tag{2.6.5}$$

and we write

$$v = \lim_{k \to \infty} v_k. \tag{2.6.6}$$

If the space is of finite dimension, the equivalence of the standards shows that the convergence of a series is independent of the chosen standard. The particular choice of standard norm $|| \bullet ||_\infty$ shows that the convergence of a sequence of vectors is equivalent to the convergence of the n sequences (n = dimension of the space) of scalars formed by the components of the vectors.

2.6.3 Theorem. *Let A be a square matrix. The following conditions are equivalent:*

$$\lim_{k \to \infty} A^k = 0\,. \tag{2.6.7}$$

and for all of vectors v

$$\lim_{k \to \infty} A^k v = 0\,. \tag{2.6.8}$$

$$\varrho(A) < 1\,, \tag{2.6.9}$$

where ϱ is the "radius" of the matrix A, which represents the largest eigenvalue. For at least one subordinate matrix standard $|| \bullet ||$ (Ciarlet).

$$||A|| < 1\,. \tag{2.6.10}$$

2.7 Similarity Transformation

A matrix SAS^{-1} has the same eigenvalues as A for any non-singular matrix S, in which case the matrices are said to be similar. We write

$$Av_j = \lambda_j v_j \tag{2.7.1}$$

for $j = 1, 2..., n$.

$$AV = VD\,, \tag{2.7.2}$$

where

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \lambda_n \end{bmatrix}.$$

the diagonal matrix of eigenvalues λ_n, and where V=$[v_1, v_2, .., v_n]$.

We can write the left hand side (LHS) of equation (2.7.2) as

$$Av = A[v_1, v_2, ...v_n] = [Av_1, Av_2, ..., Av_n] \tag{2.7.3}$$

and the right hand side (RHS) of (2.7.2) as

$$[v_1, ..., v_n]D = [\lambda_1 v_1, \lambda_2 v_2, ..., \lambda_n v_n]\,. \tag{2.7.4}$$

By comparing the two column vectors (2.7.2) and (2.7.3) with the diagonal matrix of eigenvalues λ, where if the vectors $v_1, .., v_n$ are linearly independent, then V is non-singular and $A = VDV^{-1}$.

If V is nonsingular and the inverse exists we can write: $A = VDV^{-1}$. This is called eigenvalue decomposition, and we say A is diagonalisable. It is worth noting that not all square matrices are diagonalisable.

2.8 The Power Method for Eigenvalues and Eigenvectors

Generally we can not find the roots of the characteristic polynomial by inspection, so numerical methods must be used. When the degree of the characteristic equation is greater than three, then there is a need to in general this is the case solve it numerically, where finding the eigevectors is more difficult than finding eigenvalues. In particular, when the eigenvectors corresponding to eigenvalues are complex, are repeated, or are close together. For an arbitrary $n \times n$ matrix there are many methods to compute eigenvalues and eigevectors. The power method shall be used in this project.

To develop the power method we begin by noting that the eigenvalue with the largest absolute value is called the *dominant eigenvalue*, and the eigenvector associated to this eigenvalue is the *dominant eigenvector*. The power method is iterative and will determine the dominant eigenvalue, and the eigenvector associated whit it. It concludes when the N^{th} and $(N+1)^{th}$ iterations give no change in m^{th} decimal place (accuracy) of the dominant eigenvalue and the elements of the dominant eigenvector. The method is established by use of the fact that given an $n \times n$ diagnolisable matrix A with arbitrary elements, an arbitrary n column vector v can always be expressed in the form

$$v = c_1 v_1 + c_2 v_2 + ... + c_n v_n \,, \tag{2.8.1}$$

where, $v_1, v_2, ..., v_n$ are the n eigenvectors of A, and $c_1, c_2, .., c_n$ are suitable constants. This method results in eigenvalues readily ordered by their absolute value with $|\lambda_1| > |\lambda_2| > |\lambda_3| > .... > |\lambda_n|$, so we can write as in equation (2.7.1), for $i = 1, 2, 3..., n$, which gives

$$Av = c_1 Av_1 + c_2 Av_2 + ... + c_n Av_n = \lambda_1 \left(c_1 v_1 + c_2 \frac{\lambda_2}{\lambda_1} v_2 + ... + c_n \frac{\lambda_n}{\lambda_1} v_n \right) \,, \tag{2.8.2}$$

When iterating this result r times we get

$$A^r v = \lambda_1^r \left\{ c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^r v_2 + ... + c_n (\frac{\lambda_n}{\lambda_1})^r v_n \right\} . \tag{2.8.3}$$

As we can see here, the ordering of the magnitude of the eigenvalues means that $|\lambda_r/\lambda_1| < 1$ for $r = 2, 3...$ So when r becomes large, all terms on the RHS of (2.8.3) become small, except the first term $\lambda_1^r c_1 v_1$. If $|\lambda_1| > 1$, the scaling factor multiplying v_1 is growing when r increases, which is a problem as we will find a large eigenvector corresponding to the dominant eigenvalue. As well we need to use the other condition that $|\lambda < 1|$ which means that the factor will vanish rapidly as r increases. To overcome this difficulty, there is a need to normalise the successive eigenvector approximation for v_1 of each stage of iteration by scaling successive iterations in such away that the first element of the approximate vector is 1. As the eigenvector is an unknown, we need to use unit column vector to initiate the iterative ... such as,

$$v_1^{(0)} = [1, 1, 1, ..., 1]^T \,, \tag{2.8.4}$$

though any vector can be used as, the result $Av^{(0)} = u_1^{(1)}$ is computed already, where

$$u_1^{(1)} = [u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, ..., u_n^{(n)}] \,.$$

So the vector $\mathbf{u_1}^{(1)}$ is normalised by dividing each of its elements by $\beta_1 = u_1{}^{(1)}$ to reach the next approximation

$$v_1^{(1)} = [1, u_2^{(1)}, ..., u_n^{(1)}] \,. \tag{2.8.5}$$

By doing the same operation many times, the sequence of numbers $\beta_1, \beta_2, ...$ will converge to the dominant eigenvalue λ_1 and the series of the vectors $v_1^{(0)}, v_1^{(1)}, ...$ will converge to the dominant eigenvector v_1. Note that there is a method for calculating specific eigenvalues called the Shift Inverse power method. For this work, we are interested in only finding the largest eigenvalue and its associated eigenvector.

2.8.1 Convergence of the Power Method.

If A is an $n \times n$ diagnolisable matrix with a dominant eigenvalue, then there exists a nonzero vector such that the sequence of vectors given by

$$Ax_0, A^2x_0, A^3x_0, ..., A^kx_0\,, \tag{2.8.6}$$

approaches a multiple of the dominant eigenvector of A.

The power method will converge rapidly if $|\lambda_2|/|\lambda_1|$ is small, and slowly if $|\lambda_2|/|\lambda_1|$ is close to 1 (Jeffrey, 2010).

3. Mathematical Modelling

In this chapter, the calculation of the modes within the cavity resonator shall be studied using the Fox-Li method, where a matrix approach is used to model it. The mode with the lowest loss of light intensity is then found by computing the eigenvalues.

A simple system for a cavity resonator contains two mirrors separated along the optical axis by a distance L. A schematic diagram of this resonator is shown in figure 3.1,

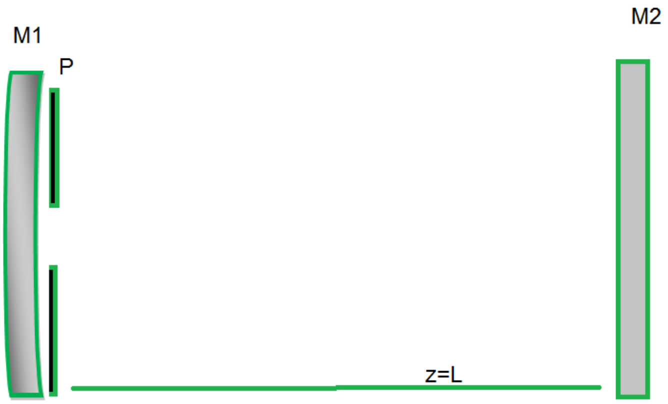


Figure 3.1: Simple cavity resonator system

where M_1 is the concave mirror, M_2 is a flat mirror, and P is the aperture. We will treat the concave mirror as a lens with focal length f, which means we will study the system represented in the figure 3.2,

Figure 3.2: cavity resonator

where the mirrors M_1, M_2 have radius of curvature X_1 and X_2 respectively.

We will describe and use the Fox-Li method, with a matrix approach, to find the matrix of each round-trip in this system.

3.1 Fox-Li Method

There is a need for a fast and precise method for computing the resonant field. Fox-Li was first applied on Fabry-Pero interferometr to compute the fundamental mode is containing amplitude and phase diffractive optics. The method was based on the fact that the fundamental mode required to go back and forth before settling down in the cavity. In this method we model the laser cavity as a sequence of lenses, where a distribution of the initial field propagates several times until a stable state solution is reached. (Cagniot et al., 2011)

The idea of Fox-Li is that starting from an initial random field we will use diffraction theory to compute the electromagnetic field at the second mirror. An integral of this field is then used to find the field at the first mirror.

Several normal modes are possible, so it depends on the choice of the initial field distribution, which influences the convergence of the method. The order of the lowest loss mode is the mode with the greatest intensity at the center of the mirror and with low intensities at the edges (Fox and Li, 1961).

Note that in this method we consider the special case where we ignore the diffraction of the laser field through the gain medium.

Given an initial field $E(x_0, z_0)$, we want to find the field $E(x_1, z_1)$ using a random system, where the optical propagator method uses the propagation kernel, $K(x_1, x_0)$ (a Greens function) to find the final field.

The field $E_n(x_0)$ at a specific reference plane is mapped to the field $E_{n+1}(x_1)$ in one transverse dimension using

$$E_{n+1}(x_1) = \int_{-\infty}^{+\infty} K(x_1, x_0) E_n(x_0) dx_0 \tag{3.1.1}$$

where Collin's Kernel (Collins, 1970) related the propagation through a purely ABCD matrix system, which depends on the reference plane:

$$K_c(x_1, z_1; x_0, z_0) = \sqrt{-\frac{ik_0}{2\pi B(z_1, z_0)}} exp\left\{\frac{ik}{2B(z_1, z_0)} D(z_1, z_0)x_1^2 + A(z_1, z_0)x_0^2 - 2x_1x_0\right\}, \tag{3.1.2}$$

where A,B and D are the elements of the round trip ABCD matrix which also depends on the location reference plane.

By using ray-optics, the matrix be determined by ABCD we will get the paraxial resonance equation, such that certain cavity geometries result in a closed ray path after N round trips. Solving this equation using the Li-Fox method gives the transverse mode. Siegman here referred to such modes as a multipass transverse mode (Maes, 2003), where our goal is to find the transverse modes of the cavity resonator. From the general transformation between the amplitude of the field after one trip in the plane $z{=}z_0$ to the field amplitude in the same plane for the previous round-trip, we get the propagation kernel. The propagation integral is then given by

$$\tilde{E}(x, y) = e^{-jkp} \int \tilde{K}(x, y, x_0, y_0) \tilde{E}_0(x_0, y_0) dx_0 dy_0\,, \tag{3.1.3}$$

where p is the length of one trip, K the propagator or propagation kernel which is related to the reference plane and contains the optical elements of the cavity resonator. The RHS of equation (3.1.3) represents the input plane.

3.2 Matrix Approach

Coherent light propagation links diffraction theory to ray optics, where the ray matrix describes the complete lens system. The kernel of diffraction integral is written in terms of this matrix, where the matrix approach provides a simple method for writing the diffraction integral. To implement Li-Fox, there is a need for a fast calculation so we replace in each round-trip the diffraction integral by a matrix, which contains the optic parameters. (Collins, 1970)

3.2.1 Formulation of the problem.

The electromagnetic field u_1 propagates and reflects on mirror M_2 as u_2, where we will study this propagation using the matrix approach. The propagation integral in 1 dimension only, which is not a "real" case, is given by (Litvin and Forbes, 2009)

$$u_0(x_1, L) = \sqrt{\frac{i}{\lambda L}} \int_{-\infty}^{+\infty} u_2(x_2) exp\left[-\frac{i\pi}{\lambda L}(x_1 - x_2)^2\right] dx_2\,, \tag{3.2.1}$$

where L is the distance separating the two mirrors, λ is the wave length, x_0 is the first point of contact of the light ray with the mirror M_1, and x_2 is the first point of contact of the light ray with mirror M_2. The field u_0 represents the initial field between the mirror M_1 and f is the focal length.

We there considerate the field passing through the lens as u_0, which is described by the complex transmission function

$$t_1 = A(x_1)e^{(i\phi(x_1))}\,. \tag{3.2.2}$$

The new field resulting on the right of the mirror M_1, which is the product of the transmission function with the initial field, is given as

$$u_1(x_1, L) = \sqrt{\frac{i}{\lambda L}} \int_{-\infty}^{+\infty} u_2(x_2) exp\left[-\frac{i\pi}{\lambda L}(x_1 - x_2)^2\right] A(x_1)exp(i\phi(x_1))dx_2 \tag{3.2.3}$$

$$= \sqrt{\frac{i}{\lambda L}} \int_{-\infty}^{+\infty} u_2(x_2)A(x_1)exp\left[-\frac{i\pi}{\lambda L}(x_1 - x_2)^2 + i\phi(x_1)\right] dx_2\,. \tag{3.2.4}$$

The propagation integral the becomes

$$u_1(x_1, L) = \sqrt{\frac{i}{\lambda L}} \int_{-X_2}^{X_2} u_2(x_2)A(x_1)exp\left[-\frac{i\pi}{\lambda L}(x_1^2 - 2x_1x_2 + x_2^2) + i\phi(x_1)\right] dx_2\,. \tag{3.2.5}$$

When the field u_1 reflects on the mirror M_2, the resultant field is the product of u_1 with a new transmission function t_2, which is an element of the second mirror. Note that we know that the mirror M_2 does not have effect on the field u_1, so in this case

$$t_2 = A(x_2)e^{(i\phi(x_2))} = 1\,. \tag{3.2.6}$$

Now we assume that we observe the field u_2 on the right of mirror M_1, which will be given as

$$u_2(x_2, L) = u_1(x_1, L) = \sqrt{\frac{i}{\lambda L}} \int_{-X_2}^{X_2} u_2(x_2)A(x_1)exp\left[-\frac{i\pi}{\lambda L}(x_1^2 - 2x_1x_2 + x_2^2) + i\phi(x_1)\right] dx_2\,. \tag{3.2.7}$$

We now need to find the relationship between the two fields in one pass, where we know that we will implement the matrix approach method so we will write the diffraction integral as a summation. We will subdivide the two mirrors into N parts, each part with size $\quad x = 2X_2/N$ for M_2, and $\quad x = 2X_1/N$ for M_1. where X_1 is the radius of the mirror M_1, and X_2 is the radius of M_2.

3.2.2 The discretisation.

We know that the size of each segment of the two mirrors is defined as

$$x_1 = \frac{2X_1}{N}, \quad x_2 = \frac{2X_2}{N}\,. \tag{3.2.8}$$

So we can write, in general, that

$$x_i = X_1 - i \quad x_1\,, \tag{3.2.9}$$

$$x_j = X_2 - j \quad x_2\,. \tag{3.2.10}$$

We can write the diffraction integral as a summation, where the electromagnetic field $u_1(x_1)$ is defined as

$$u_1(x_1, L) = \sqrt{\frac{i}{\lambda L}} \int_{-X_2}^{X_2} u_2(x_2)A(x_1)exp\left[-\frac{i\pi}{\lambda L}(x_1^2 - 2x_1x_2 + x_2^2) + i\phi(x_1)\right] dx_2 \tag{3.2.11}$$

$$= \sum_{i=0}^{N} \sqrt{\frac{i}{\lambda L}} \int_{-X2-i \quad x}^{X_2-(i+1) \quad x} u_2(X_2 - i \quad x)A(x_1)exp\left[-\frac{i\pi}{\lambda L}(x_1^2 - 2x_1x_2 + x_2^2) + i\phi(x_1)\right] dx_2\,. \tag{3.2.12}$$

By substituting from equation (3.2.9) x_1 into (3.2.11) we get

$$u_1(X_1 - i \quad x_1, L) = \sum_{j=0}^{N-1} u_2(X_2 - j \quad x_2)\sqrt{\frac{i}{\lambda L}} \int_{X_2-(j+1) \quad x_2}^{X_2-j \quad x_2} A(X_1 - i \quad x_1)$$

$$exp\left[-\frac{i\pi}{\lambda L}(X_1 - i \quad x_1 - x_2)^2 + i\phi(X_1 - i \quad x_1)\right] dx_2\,. \tag{3.2.13}$$

By replacing $X_1 - i \quad x_1 = X_i$ and $X_2 - j \quad x_2 = X_j$ in equation (3.2.13) we get

$$u_1(X_i, L) = \sum_{j=0}^{N-1} u_2(X_j) \sqrt{\frac{i}{\lambda L}} \int_{X_j - \quad x_2}^{X_j} A(X_i) exp \left[-\frac{i\pi}{\lambda L} (X_i - x_2)^2 + i\phi(X_i) \right] dx_2 \, . \quad (3.2.14)$$

We know that

$$x_i = X_1 - i \quad x_1 \, , \quad (3.2.15)$$

and

$$x_j = X_2 - j \quad x_2 \, . \quad (3.2.16)$$

So in order to generalise, for the electromagnetic field u_1 we have

$$u_1^i(L) = u_1(x_i, L) = u_1(X_1 - i \quad x_1, L) \, , \quad (3.2.17)$$

and for the field u_2 we have

$$u_2^j(L) = u_1(x_j, L) = u_2(X_2 - i \quad x_2, L) \, . \quad (3.2.18)$$

We there replace X_i in the equation (3.2.14) by (3.2.18) to get

$$u_1(x_i, L) = \sum_{j=0}^{N-1} u_2(X_j) \sqrt{\frac{i}{\lambda L}} \int_{X_j - \quad x_2}^{x_j} A(x_i) exp \left[-\frac{i\pi}{\lambda L} (x_i^2 - 2x_i x_j + x_j^2) + i\phi(x_i) \right] dx_2 \, . \quad (3.2.19)$$

So after the descritisation we end up with equation (3.2.19). Since it does not change with changing fields we can write the fields as vectors, so that equation (3.2.19) in matrix form is

$$\vec{\vec{u}}_1 = T \vec{\vec{u}}_2 \, . \quad (3.2.20)$$

From that the matrix T_{ij} is

$$T_{ij} = \sqrt{\frac{i}{\lambda L}} \int_{X_2 - (j+1) \quad x_2}^{X_2 - j \quad x_2} A(x_i) exp \left[-\frac{i\pi}{\lambda L} (x_2^2 - 2x_j x_2 + x_j^2) + i\phi(x_i) \right] dx_2 \, . \quad (3.2.21)$$

The matrix in equation (3.2.20) represents the relationship between the field at the first mirror M_1 and the field at the mirror M_2 after one pass. When

$$x_2 \to 0 \, , \quad (3.2.22)$$

We end up with

$$T_{ij} \approx T_{ij} = \sqrt{\frac{i}{\lambda L}} A(x_i) exp \left[-\frac{i\pi}{\lambda L} (x_i^2 - 2x_i x_j + x_j^2) + i\phi(x_i) \right] \quad x \, . \quad (3.2.23)$$

We know that

$$u_1^i(L) = u_1(x_i, L) = u_1(X_1 - i \quad x_1, L) \, ,$$

and

$$u_2^j(L) = u_1(x_j, L) = u_2(X_2 - j \quad x_2, L)\,. \tag{3.2.24}$$

We can then write equation (3.2.20) in the general form:

$$u_1^i(L) = [T_{ij}(L)]\, u_2^j\,. \tag{3.2.25}$$

Recall that we need to find the matrix that represents the relationship between the two fields after one trip, so we first need to find the matrix when the field reflects at the second mirror M_2, back to the first mirror M_1 (second pass).

We know that after one trip the mirror M_2 does not affect the field u_1, so u_2 is given by

$$u_2(x_2, f) = \sqrt{\frac{i}{\lambda L}} \int_{-X_2}^{X_2} u_2(x_2) A(x_1) exp\left[-\frac{i\pi}{\lambda L}(x_1^2 - 2x_1x_2 + x_2^2) + i\phi(x_1)\right] dx_2\,. \tag{3.2.26}$$

Then we can write the general formula as

$$u_2^j(L) = I u_1^i\,, \tag{3.2.27}$$

where I is the identity matrix. This matrix represents the relationship between the field at the mirror M_2 and the field arriving at the mirror M_1 (second pass of the field). As such we notice that in this case there is just one matrix which represents one trip, therefore, the field does not change. The matrix representing one trip is the multiplication of the two matrices when the field performs the first pass and when the field returns to same point as

$$M = I.T_{ij} = \sqrt{\frac{i}{\lambda L}} \int_{X_2-(j+1) \quad x_2}^{X_2-j \quad x_2} A(x_i) exp\left[-\frac{i\pi}{\lambda L}(x_2^2 - 2x_2x_j + x_j^2) + i\phi(x_i)\right] dx_2\,. \tag{3.2.28}$$

This matrix contains the optic parameters of the cavity resonator.

3.2.3 How does the method work.

The idea for the implementation of the method starts by considering an arbitrary field u_1.

Let u_1 be a random field

$$u_1(1^{st}\text{pass}) = \text{random field} \tag{3.2.29}$$

For the first pass

$$u_2 = M_1 u_1(1^{st}\text{pass})\,. \tag{3.2.30}$$

For the second pass,.

$$u_1(2^{nd}\text{pass}) = M_2 u_2\,. \tag{3.2.31}$$

So for one trip

$$u_1(2^{nd}\text{pass}) = M_2 M_1 u_1(1^{st}\text{pass})\,. \tag{3.2.32}$$

Let $M = M_2 M_1$, and replace in equation (3.2.32) to get

$$u_1(2^{nd}\text{pass}) = Mu_1(1^{st}\text{pass})\,. \tag{3.2.33}$$

Generalise to the n-trip,

$$u_1(n^{th}\text{pass}) = M^n u_1(1^{st}\text{pass})\,, \tag{3.2.34}$$

and when

$$n \to \infty,\ u_1(n^{th}\text{pass}) \to \text{mode we decide to calculate}\,.$$

Instead of using integrals we have simplified the calculations by using of matrices.

3.2.4 Eigen-equation.

Since we are ignoring the presence of the active medium and the effect of the mirror, the relationship with the initial and final field after one round trip is the propagator matrix M. For exploring the resonator modes behaviour, an alternative to the scalar-diffraction iteration algorithm is the matrix eigenequation. We can find the cavity resonator modes by solving numerically this equation discretised for the eigenvectors U_n and eigenvalues γ_n, which satisfy the matrix eigenequation

$$MU = \gamma U\,, \tag{3.2.35}$$

where M is the complex matrix derived from diffraction theory (by discretisation of the Kirchhoff integral equation), and the U_n are the complex eigenvectors that coincide with the optical resonator model.

The iterative algorithm applied to equation (3.2.5) is summarised in following paragraph.

An arbitrary initial vector field

$$U = V_0 = \sum^{n} k_n U_n\,, \tag{3.2.36}$$

where k_n is a multimode summation, is frequently multiplied by the matrix M to obtain a sequence of vectors V_1, V_2, V_3 ect. After sufficiently many iterations, the higher loss mode components drop out of the sequence of vectors leaving the lowest-loss resonator mode U_1 and the corresponding eigenvalue λ_1. From linear algebra a conclusion on the convergence of $\sum^n A^n$ is written as in chapter 2 equation (2.6.10). This condition means our model A exists. The necessary condition for the convergence of $\sum A^n$ can be interpreted as $||A^n|| \longrightarrow 0$ as $n \longrightarrow \infty$.

4. Results and Observations

In this chapter was have applied the ideas of linear algebra presented in chapter 3 to the study of lasers. We start by revealing the relationship between the eigenvectors of the matrix propagator and the modes of the cavity resonator, which we have modeled in the previous chapter. Thus eigenvalues found as a series, will have their behaviours, studied to determine whether they are and why the lowest loss mode with the largest eigenvalue. We want to also plot the fundamental mode of this cavity for large matrix elements, and want to see what the intensity of the modes look like, and the effect of some optics parameters on the shape and intensity of the output. The numerical results are presented using Matlab.

4.1 Eigenequation and Eigenmodes

Using the equation (3.2.35), the initial field shape propagates to the mirror M_1 as a Gaussian with distribution intensity. The field propagates along the optical axis, and is there reflecting back to the mirror M_1, where it should have the same shape but with different amplitude distribution intensity. This is because there is a diffraction on the edges of the mirror and the aperture. When this happens a phase shift is created, as show in the figure 4.1. To explain this we firstly need to explain the relation-ship

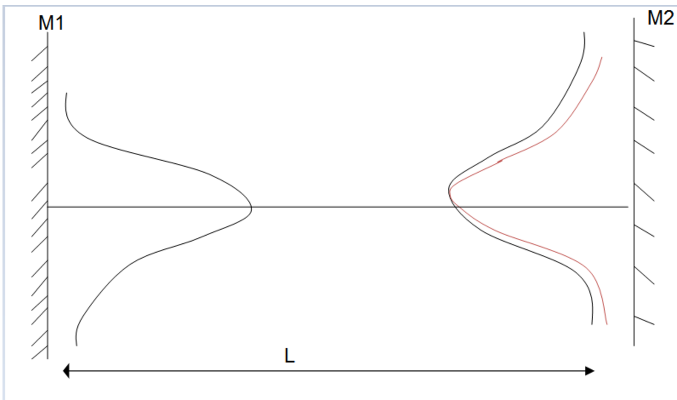


Figure 4.1: Relative intensity

between the largest eigenvalue and the eigenmode (eigenfunction) of the cavity resonator. Note that it is important to know that if the largest eigenvalues of the propagator "M" are too close to one $\lambda = 1$ there is no loss, so there is a laser gain. However, we know that the condition for λ, the equation (2.6.9), the eigenvalue to be associated with an eigenvector which is the largest eigenvector, represents the lowest loss mode of the cavity.

On the other hand, if the spectral radius is less than one, then there is a need to find the lowest loss mode of the cavity, which in turn has the largest eigenvalue. Therefore we will be using of the spectral radius of A in order to explain the relationship between the eigenvectors of this matrix and the eigenmode of the cavity.

We know that the dominant mode is growing after some trips, so the fractional round trip power loss of a given resonator mode is $|1 - \gamma_{nm}^2|$, or $|1 - \gamma_{nm}|$ for strip resonators. This means no laser gain in the cavity, where λ represents the phase shift.

4.2 Eigenvalues Series

To find the solution of the eigenequation which comes from the matrix modeling of the resonator, we use

$$\tilde{\gamma}_{nm}\tilde{E}_{mn}(x,y) = \int \tilde{K}(x,y,x_0,y_0)\tilde{E}_{mn}(x_0,y_0)dx_0dy_0\,. \tag{4.2.1}$$

and we know that the solution is a mode of the cavity

$$\tilde{E^1}_{mn}(x,y) = \tilde{\gamma}_{nm}\tilde{E^0}_{mn}(x,y)e^{-jkp}\,. \tag{4.2.2}$$

From the equation (4.2.2) we can make a conjecture for the k^{th} order as

$$\frac{\tilde{E}^k_{nm}(x,y)}{\tilde{E}^0_{nm}(x,y)} = \tilde{\gamma}^k_{nm}\,, \tag{4.2.3}$$

where we know that $|\tilde{\gamma}_{nm}| < 1 \rightarrow |\tilde{\gamma}^k_{nm}| < 1$, which means that

$$|\tilde{E}^k_{nm}(x,y)| < |\tilde{E}^0_{nm}(x,y)| \tag{4.2.4}$$

The eigenvalues are found as a series, where we will prove that if it is a monotonically increasing/decreasing series with recurrence. We then have

$$\tilde{E}^k_{nm}(x,y) = \tilde{\gamma}^k_{nm}\tilde{E^0}_{mn}(x,y)e^{-jkp}\,, \tag{4.2.5}$$

which for k+1 is

$$\tilde{E}^{k+1}_{mn}(x,y) = \tilde{\gamma}^{k+1}_{nm}\tilde{E^0}_{mn}(x,y)e^{-jkp}\,. \tag{4.2.6}$$

Dividing equation (4.2.5) by equation (4.2.6), we get

$$\frac{\tilde{E}^k_{nm}(x,y)}{\tilde{E}^{k+1}_{nm}(x,y)} = \frac{\tilde{\gamma}^k_{nm}}{\tilde{\gamma}^{k+1}_{nm}}\,, \tag{4.2.7}$$

which is equal to

$$|\frac{\tilde{E}^k_{nm}(x,y)}{\tilde{E}^{k+1}_{nm}(x,y)}| = |\frac{1}{\tilde{\gamma}^1_{nm}}| > 1\,, \tag{4.2.8}$$

this means that

$$|\tilde{\gamma}^k_{nm}| > |\tilde{\gamma}^{k+1}_{nm}|\,. \tag{4.2.9}$$

From the results (4.2.9), (4.2.8) and (4.2.4) for any $k = 1...N$ we always have the first eigenvalue as the largest one. We can write $|{\gamma_{nm}}^0| > |{\gamma_{nm}}^1| > |{\gamma_{nm}}^2|.... > |{\gamma_{nm}}^k|$, so $\lambda_0 > \lambda_1 > .... > \lambda_k$.

We experimented with this is our Matlab code using a large order matrix element with closely spaced dominant eigenvalues, where convergence was extremely slow (Fox-Li gives us a problem slow of convergence). The Prony method is used to take the advantage of the orthogonality properties of optical resonator eigenmodes to convert the discretised matrix M into a complex symmetric non-hermitian matrix. The eigen-value problem then makes the set of N coupled linear simultaneous equations solve by "standard matrix symmetrisation techniques"Siegman (2000).

4.3 Results

The implementation of the model (see equation (3.2.25)) shows that the result below represents the fundamental mode.

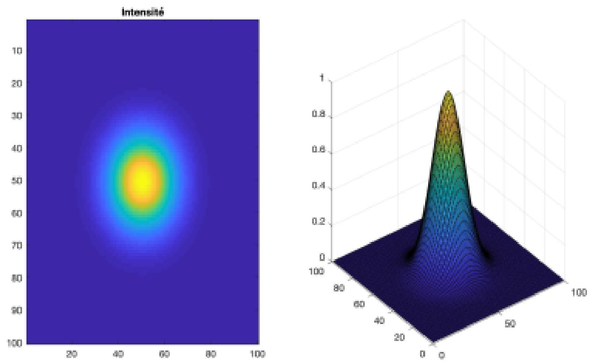


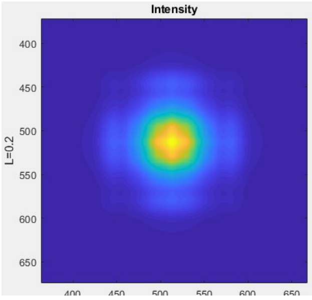
Figure 4.2: *The fundamental mode of Gaussian beam for L=0.7m, P<26.25×10^{-7}m*

This figure represent the solution of the matrix propagator implemented in the Matlab code, so its the approximation of the fundamental mode which is represent the lowest loss mode of the cavity resonator. The Table 4.1 shows some eigenvectors from the matrix propagator,

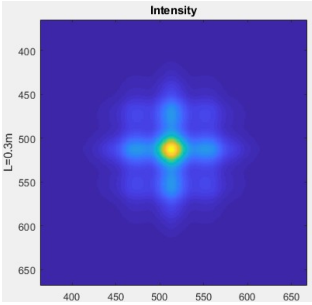
Some of eigenvecors			
-0.0084-0.0012i	-0.0168+0.0216i	-0.0340+0.0040i	-0.0339-0.0108i
-0.0356-0.0041i	0.0155-0.0034i	-0.0123-0.0108i	0.0109-0.0115i
-0.0159-0.0308i	0.0201-0.0304i	0.0124-0.0140i	-0.0059+0.0115i
-0.0307+0.0008i	0.0141+0.0099i	0.0124-0.0140i	-0.0059+0.0333i
0.0232+0.0075i	0.0200-0.0102i	-0.0030+0.0276i	-0.0059+0.0219i
0.0313+0.0029i	-0.0510+0.0152i	-0.0189+0.0041i	-0.0011+0.01651i
0.0365+0.0029i	-0.0018+0.0152i	0.0392-0.0092i	-0.0183+0.0051i
-0.0019-0.0017i	-0.0187-0.0078i	0.0161+0.0168i	0.0161-000045i
0.0247-0.0133i	0.0173+0.0439i	0.0395+0.0048i	-0.0151-0.0190i

Table 4.1: *The eigenvectors (Approximation of the fundamental mode of Gaussian beam) for L=0.7m, P<26.25×10^{-7}m*

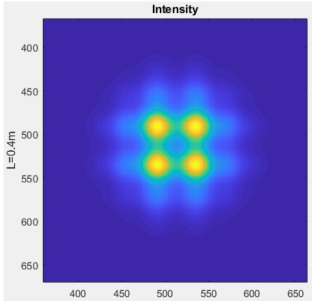
carefully changed the parameters of the model, such as the distance separating the two mirrors. Secondly, the radius of the aperture is changed, where by fixing the aperture-radius P<26.25×10^{-7}m, we got the results of Figure 4.4:



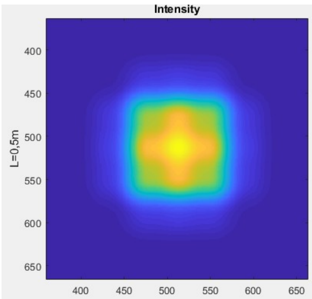
(a) P<26.25×10^{-7}m,L=0.2m



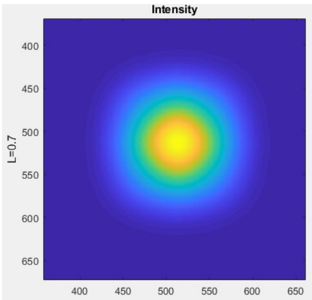
(b) P<26.25×10^{-7}m,L=0.3m



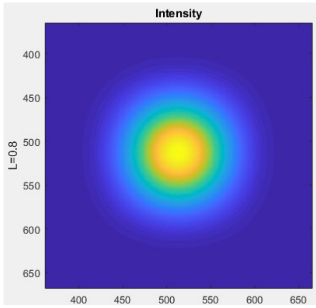
(c) P<26.25×10^{-7}m,L=0.4m



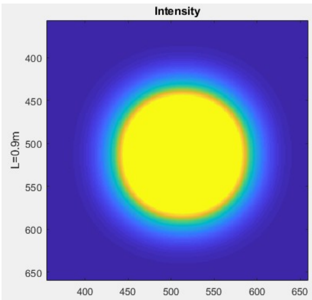
(a) P<26.25×10^{-7}m,L=0.5m



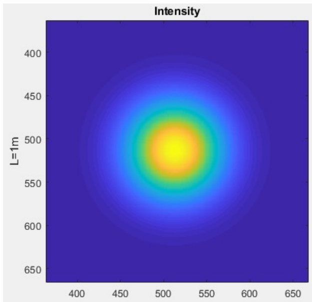
(b) P<26.25×10^{-7}m,L=0.7m



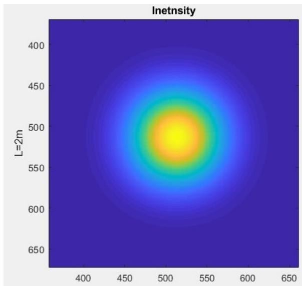
(c) P<26.25×10^{-7}m,L=0.8m



(d) P<26.25×10^{-7}m,L=0.9m



(e) P<26.25×$10^{-7}m, L = 1m$



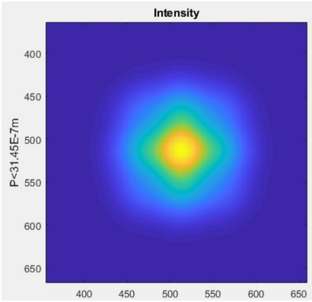
(f) P<26.25×10^{-7}m,L=2m

Figure 4.4: Visualisation of the fundamental modes of the cavity resonator for different values of L with different the radius of the aperture

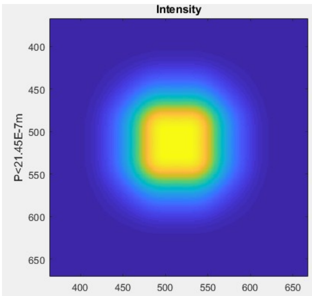
There is a relationship between the optical parameters of the cavity and its modes. By defining a Fresnel number F, which is a combination of the parameters L, X_1 and X_2, as defined in Section 3 where a is the radius of the aperture, the Fresnel number is given as

$$N = \frac{a^2}{L\lambda}. \tag{4.3.1}$$

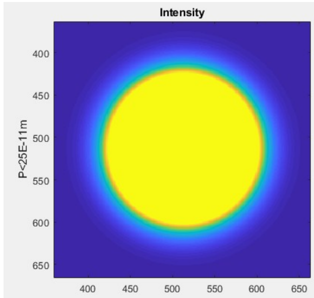
In the presented series of results we notice that there is a change in the form and the intensity of the beam, which means that the variation of L affects the field. The intensity in the centre of the mode represents the intensity in the center of the mirror, and the distribution of the intensity around represents the intensity distribution on the edges of the mirror. The results show that there are some specific values of parameters which can generate a flat topped beam (see Figure 4.5; (c) and (d)).



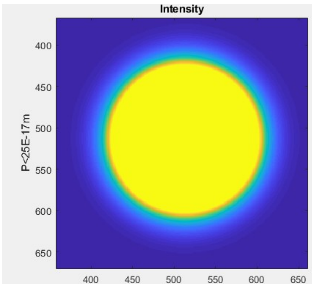
(a) P<31.45$\times 10^{-7}$m



(b) P<21.45$\times 10^{-7}$m



(c) P<25$\times 10^{-17}$m



(d) P<25$\times 10^{-17}$m

Figure 4.5: Visualisation of the fundamental modes of the cavity resonator for different values of aperture radius with a fixed cavity length L

We conclude by changing longitudinal mode, where we have to change the distance between the mirrors, and for the transverse mode, which we do by changing the aperture size.

5. Conclusion and Future Work

5.1 Conclusion

The study of cavity resonators is essential for any laser construction. In this work we have addressed how to apply the Fox-Li method with the matrix approach to model a resonator cavity, where, we use the power method to obtain the eigenvalues of the eigenequation. we have also explained that these are related to the modes of the cavity resonator, finding that the largest eigenvalue is associated with the eigenvector which is represented by the lowest loss mode of that cavity. It was observed that the eigenvalues were monotonically increasing which intuitively implies that the lowest loss mode is always the fundamental mode. Finally it was observed that the variation of the distance between the mirrors and the radius of the aperture gives various loss intensities.

5.2 Future Work

In the future work of this project, we are going to make the method better, and to apply it in complex system because the one we using here is simple.

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