Evaluating Option prices using the Fast Fourier Transform

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19 May 2020

Submitted in partial fulfillment of a structured masters degree at AIMS South Africa

Abstract

When the characteristic function of the probability distribution of a stock is known, it is possible to efficiently calculate no-arbitrage option prices for a large number of strikes simultaneously using the Fast Fourier Transform. We outline the theory and introduce the Carr and Madan (1999) method for computing options. This method prices by introducing a damping factor, which will allow us to compute option prices numerically and we investigate its effect on the accuracy of the option values. We implement the valuation in Python and compare the analytical Black-Scholes formula with the Fast Fourier Transform results, then show the error convergence.

Keywords: Characteristic function, Option prices, Damping factor, Fast Fourier Transform.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Maluti Kgarose, 19 May 2020

Contents

1. Introduction

The formulation of the now famous Black-Scholes model in 1973, has revolutionised the way that financial markets price assets. This model seeks to mimic the volatile trajectories observed from market data by assuming that the price of the underlying assets follow a Geometric Brownian motion with constant drift and volatility. The model has been used extensively over the years to price European style call and put option prices generated by the price of an underlying asset.

Since the development of the Fourier series by French mathematician Joseph Fourier (March 1768 - May 1830), many advancements have been made to this area of mathematics. In particular, Fourier transform has been used in handling of vector data. The Fast Fourier Transform (FFT) is an algorithm used to compute discrete cases of Fourier Transform (DFT). First developed by Friedrich Gauss, the Cooley-Tukey algorithm is the most widely used in modern day and uses a divide-and-conquer algorithm to break down the DFT into smaller DFTs thus reducing the amount of computations needed to be done on a logarithmic scale.

In this essay, we aim to use the FFT to evaluate European option prices when the underlying asset are stock prices. To price options using this method we require the stock price and the strike price, that is, the price at which an option should be exercised at the expiration date. Given a range of strikes, to compute the option prices would be computationally expensive and we seek to show that we can use the FFT to accurately value option prices at a reduced time computationally using the FFT. In order ensure convergence of this method, we use the Carr and Madan (1999) approach which uses characteristic functions of a model, in our case the characteristic function of the Black-Scholes model, and Fourier transform to price options. Since the Carr and Madan (1999) method introduces a damping factor into the option price, this in turn has an impact on the accuracy of the results and we investigate this by comparing the results we obtain with the option prices computed using the Black-Scholes analytical formula.

The structure of the essay will be outlined as follows. The second chapter outlines the mathematical preliminaries of the Fourier transforms and characteristic functions, and the pricing formula for European options written as an expectation under risk-neutral measure presented in Wiersema (2008). In the third chapter, we use the Carr and Madan (1999) method and show that by introducing a damping factor, we can price option prices efficiently by using the FFT. We then implement the method and compare the results with the analytical solutions of the Black-Scholes.

2. Fourier Transform & Characteristic functions

2.1 Mathematical preliminaries

The definitions in the section follow from Hilpisch (2015).

2.1.1 Definition (Algebra)**.** A family $\mathcal{A}$ of sets is an algebra in Ω if:

1. $\Omega \in \mathcal{A}$
2. $M \in \mathcal{A} \implies M^c \in \mathcal{A}$
3. $M_1, M_2, \ldots, M_I \in M \implies \bigcup_{i=1}^{I} M_i \in \mathcal{A}$

M^c denotes the complement of the set M.

2.1.2 Definition (Filtration)**.** A filtration $\mathbb{F}$ is a non-decreasing family of algebras in Ω, i.e. $\mathbb{F} \equiv (\mathcal{A}_t)_{t\in[0,T]}$ where $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots \subseteq \mathcal{A}_{T-1} \subseteq \mathcal{A}_T$.

We call the collection $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ a filtered probability space.

2.1.3 Definition (Probability measure)**.** Let $\mathcal{A}$ be an algebra in Ω. A function $\mathbb{P} : \mathcal{A} \to [0, 1]$ is a probability measure if:

1. $\forall M \in \mathcal{A} : \mathbb{P}(M) \geq 0$
2. $\mathbb{P}\left(\bigcup_{i=1}^{I} M_i\right) = \sum_{i=1}^{I} \mathbb{P}(M_i)$ for disjoint sets $M_1, M_2, \ldots, M_I \in \mathcal{A}$
3. $\mathbb{P}(\Omega) = 1$

Two probability measures $\mathbb{P}$ and $\mathbb{Q}$, defined on an algebra $\mathcal{A}$, are equivalent if they agree on the same null-set $\mathbb{P}(M) = 0 \iff \mathbb{Q}(M) = 0$, where $M \in \mathcal{A}$. A collection $(\Omega, \mathcal{A}, \mathbb{P})$ of a state space Ω, a set of observable events $\mathcal{A}$, where $\mathcal{A}$ is an algebra, and a probability measure $\mathbb{P}$ defined on $\mathcal{A}$ is called a probability space.

2.1.4 Definition (Random variable)**.** Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a random variable X is a function

$$X : \Omega \to \mathbb{R}^+, s \mapsto X(s), \tag{2.1.1}$$

that is $\mathcal{A}$-measurable, i.e., for each $\mathbb{E} \in \{[a, b[\ : a, b \in \mathbb{R}, a < b\}$, we have

$$X^{-1}(M) \equiv \{s \in \Omega : X(s) \in M\} \in \mathcal{A}. \tag{2.1.2}$$

2.1.5 Definition (Expectation)**.** Let a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ be given where Ω is finite. The expectation $\mathbb{E}^{\mathbb{P}}[X]$ of a random variable X under a probability measure $\mathbb{P}$ is defined as

$$\mathbb{E}^{\mathbb{P}}[X] = \sum_{s\in\Omega} \mathbb{P}(s) \cdot X(s). \tag{2.1.3}$$

The expectation of a random variable is real-valued.

2.1.6 Definition (Stochastic processes)**.** A stochastic process $(X_t)_{t\in[0,T]}$ is a date-ordered sequence of random variables X_t, $t \in [0,\ldots,T]$.

2.1.7 Definition (Adaptation)**.** A stochastic process $(X_t)_{t\in[0,T]}$ is said to be adapted to a filtration $\mathbb{F} = (\mathcal{A}_t)_{t\in[0,T]}$ if $\forall t : X_t$ is $\mathcal{A}_t$-measurable.

2.1.8 Definition (Algebra generation)**.** The algebra generated by a random variable X is denoted $\mathcal{A}(X)$ and is the smallest algebra to which X is measurable. The algebra generated by a stochastic process $(X_t)_{t\in[0,T]}$ up to date t is denoted $\mathcal{A}\left(X_i : i \in [0,t]\right)$ and is the smallest algebra with respect to which all random variables are X_i, $i \in [0,T]$ are measurable.

The stochastic process $(X_t)_{t\in[0,T]}$ generates the filtration $\mathbb{F}(\mathcal{A}_t)_{t\in[0,T]}$ where $\mathcal{A}_t \equiv \mathcal{A}(X_i : i \in [0,t])$. The stochastic process is adapted to the filtration it generates.

We now look at martingales. This is when a stock price process satisfying the condition that the expected discount price process at any future date equals its price today.

2.1.9 Definition (Brownian motion)**.** Let $(W_t)_{t\in[0,T]}$ be an $\mathbb{F}$-adapted process taking the values in $\mathbb{R}\left(\mathbb{R}^k, 1 < k < \infty\right)$. Then W is a k-dimensional standard Brownian motion if:

1. $W_0 = 0$ almost surely.
2. $W_t - W_s$ is independent of $\mathcal{A}_s$ for $0 \leq s < t \leq T$.
3. $W_t - W_s$ is a Gaussian random variable with mean zero and variance of $t - s$.

2.1.10 Definition (Conditional expectation)**.** Let $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ be given. The conditional expectation $\mathbb{E}_t^{\mathbb{P}}[X]$ of a random variable X given information $\mathcal{A}_t$ is the unique random variable that satisfies

1. $\mathbb{E}_t^{\mathbb{P}}[X]$ is $\mathcal{A}_t$-measurable.
2. $\forall M \in \mathcal{A}_t : \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}_t^{\mathbb{P}}[X] \cdot \mathbb{1}_M\right] = \mathbb{E}^{\mathbb{P}}\left[X \cdot \mathbb{1}_M\right]$

2.1.11 Definition (Martingale)**.** Let $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ be given. A $\mathbb{F}$-adapted stochastic process $(X_t)_{t\in[0,T]}$ is a martingale under the probability measure $\mathbb{Q}$ if

$$\forall t, s \geq 0, t + s \leq T \ : \mathbb{E}_t^{\mathbb{Q}}[X_{t+s}] = X_t. \tag{2.1.4}$$

A probability measure $\mathbb{Q}$ that makes a stochastic process defined on some filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$- a martingale is called a martingale measure. Whenever $\mathbb{Q}$ is $\mathbb{P}$ equivalent, it is called an equivalent martingale measure (EMM). We use the Radon-Nykodym derivative to change from one probability measure to an equivalent probability measure.

2.1.12 Definition (Radon-Nykodym derivative)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be given where Ω is finite. For a $\mathbb{P}$-equivalent probability measure $\mathbb{Q}$, the Radon-Nykodym derivative L, which is actually a random variable, is defined by

$$\forall s \in \Omega : L(s) \equiv \begin{cases} \frac{\mathbb{Q}(s)}{\mathbb{P}(s)} & \text{for} \quad \mathbb{P} \neq 0, \\ 0 & \text{for} \quad \mathbb{P} = 0. \end{cases} \tag{2.1.5}$$

2.1.13 Theorem (Itô's 1-dimensional formula)**.** *Let X_t be an Itô process, that is, a process given by*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{2.1.6}$$

Let $f(t, x) : \mathbb{R}^2 \to \mathbb{R}$, be a twice continuously differentiable function. Then

$$Y_t = f(t, X_t) \tag{2.1.7}$$

is again an Itô process, and

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2, \tag{2.1.8}$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is computed according to the rules

$$dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0 \quad and \quad dW_t \cdot dW_t = dt. \tag{2.1.9}$$

Proof. See Øksendal (2003). □

2.1.14 Definition. A stochastic process (X_t) is said to be a geometric Brownian motion if $\log X_t$ is a standard Brownian motion.

2.1.15 Definition. A portfolio is a pair of real-values (α_t, β_t) which are left continuous with right hand limits, called predictable processes. The value V_t of the portfolio is then given by

$$V_t = \alpha_t B(t) + \beta_t S(t). \tag{2.1.10}$$

The portfolio is said to be self-financing if

$$dV_t = \alpha_t dB(t) + \beta_t dS(t). \tag{2.1.11}$$

2.1.16 Proposition. If the drift of an SDE is zero, so that

$$dS_t = \sigma dW_t, \tag{2.1.12}$$

then the solution is a martingale Fusai and Roncoroni (2007).

2.2 Fourier transforms

We define the Fourier transform of an integrable function $f : \mathbb{R} \to \mathbb{R}$ as

$$\mathcal{F}[f(x)] = \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)dx. \tag{2.2.1}$$

The Inverse Fourier transform is then given by

$$\mathcal{F}^{-1}[\hat{f}(\omega)] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\hat{f}(\omega)dx, \tag{2.2.2}$$

and the integrals will converge if $f \in L^1$. That is

$$\int_{-\infty}^{\infty} |f(\omega)|d\omega < \infty$$

and will converge in mean-square if $f \in L^2$, so that

$$\int_{-\infty}^{\infty} |f(\omega)|^2 d\omega < \infty.$$

2.3 Characteristic functions

The following definitions follow from (Jondeau et al., 2007).

2.3.1 Definition (Probability density function)**.** A real-valued random variable is said to be absolutely continuous if there exists a real-valued function p_X such that for any subset $\Omega \subset \mathbb{R}$

$$\mathbb{P}(X \in \Omega) = \int_\Omega p_X(x)dx. \tag{2.3.1}$$

Then p_X is called the probability density function (pdf) of the random variable X.

For any real numbers a and b, with $a < b$. If we have $\Omega = [a, b]$, equation

$$\mathbb{P}(a \leq X \leq b) = \int_a^b p_X(x)dx. \tag{2.3.2}$$

For p to be a valid pdf, it must be integrable on $\mathbb{R}$ and must satisfy the normalisation condition

$$\int_{-\infty}^{\infty} p_X(x)dx = 1. \tag{2.3.3}$$

2.3.2 Definition (Cumulative distribution function)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. The cumulative distribution function (cdf) of a real-valued random variable X is the function F_X given by

$$F_X(x) = \mathbb{P}(X \leq x), \qquad \text{for all} \quad x \in \mathbb{R}. \tag{2.3.4}$$

The three elements $(\Omega, \mathcal{A}, \mathbb{P})$ are known as the probability triplets where Ω is the sample space, $\mathcal{A}$ is the event space and $\mathbb{P}$ if the probability function.

2.3.3 Definition (Expected value)**.** Let X be a continuous real-valued random variable with pdf p_X. If $\int_{-\infty}^{\infty} xp_X(x)dx$ is absolutely convergent, then the mathematical expectation (or expected value) of X exists and is denoted by

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xp_X(x)dx. \tag{2.3.5}$$

2.3.4 Definition (Characteristic function)**.** Let a random variable X be distributed with pdf $p_X(x)$. The characteristic function of the random variable X is then defined as

$$\begin{aligned}
\phi_X(\omega) &= \mathbb{E}[e^{i\omega X}], \\
&= \int_{-\infty}^{\infty} e^{i\omega x} dF_X(x), \\
&= \int_{-\infty}^{\infty} e^{i\omega x} p_X(x)dx,
\end{aligned}$$

where $F_X(x)$ is the cumulative distribution function.

All characteristic functions are said to satisfy these basic properties:

1. $\phi_X(0) = 1$, since $\exp(0) = 1$.

2. $|\phi_X(\omega)| \leq 1$, for all real ω since

$$|\phi_X(\omega)| = \left|\int_{-\infty}^{\infty} e^{i\omega x} p_X(x)dx\right| \leq \int_{-\infty}^{\infty} |e^{i\omega x}| p_X(x)dx \leq 1. \tag{2.3.6}$$

3. $\overline{\phi_X(\omega)} = \phi_X(-\omega)$, since

$$\overline{\phi_X(\omega)} = \int_{-\infty}^{\infty} \overline{e^{i\omega x}} p_X(x)dx = \int_{-\infty}^{\infty} e^{-i\omega x} p_X(x)dx = \phi_X(-\omega). \tag{2.3.7}$$

4. $\phi_X(\omega)$ is continuous for a real ω. This follows directly from the fact that the exponential function is a continuous function.

5. $\phi_{aX+b}(\omega) = e^{i\omega b}\phi_X(a\omega)$. This follows immediately from $\phi_{aX+b}(\omega) = \mathbb{E}\left[e^{i\omega(aX+b)}\right]$.

The cumulative distribution function of a real-valued random variable X is given by

$$\begin{aligned} F_X(x) &= \mathbb{P}(X \leq x), \\ &= \mathbb{P}(X \in (-\infty, x)), \\ &= \int_{-\infty}^{x} p_X(s)ds. \end{aligned} \tag{2.3.8}$$

The following theorem proves that the distribution is uniquely determined by the characteristic function.

2.3.5 Theorem (Gil Pélaez Inversion theorem)**.** *Let $F_X(x)$ be a one-dimensional distribution of some real-valued random variable X with corresponding characteristic function*

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} dF_X(x), \tag{2.3.9}$$

then

$$F_X(x) = \frac{1}{2} + \frac{1}{2\pi}\int_0^{\infty} \frac{\phi_X(-\omega)e^{i\omega x} - \phi_X(\omega)e^{-i\omega x}}{i\omega}\, d\omega. \tag{2.3.10}$$

Proof. See See Gil-Pélaez (1951). □

From (2.3.10), we prove the following lemma.

2.3.6 Lemma. From theorem (2.3.5), we have the equation

$$\frac{\phi_X(-\omega)e^{i\omega x} - \phi_X(\omega)e^{-i\omega x}}{2\pi i\omega} = -\frac{1}{\pi}\mathcal{R}e\left(\frac{\phi_X(\omega)e^{-i\omega x}}{i\omega}\right), \tag{2.3.11}$$

thereby giving us the distribution

$$F_X(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \mathcal{R}e\left(\frac{\phi_X(\omega)e^{-i\omega x}}{i\omega}\right) d\omega. \tag{2.3.12}$$

Proof.

$$\phi_X(\omega)e^{-i\omega x} = a + bi, \quad \text{where} \quad a(\omega), b(\omega) \quad \text{are real valued.} \tag{2.3.13}$$

From property (3), we have $\phi_X(-\omega)e^{i\omega x} = \overline{\phi_X(\omega)e^{-i\omega x}} = a - bi.$

The LHS of equation (2.3.11) equation gives

$$\begin{aligned} \frac{\phi_X(-\omega)e^{i\omega x} - \phi_X(\omega)e^{-i\omega x}}{2\pi i\omega} &= \frac{a - bi - (a + bi)}{2\pi i\omega}, \\ &= -\frac{b}{\pi\omega}. \end{aligned}$$

And the RHS of equation (2.3.11) equation then gives

$$\begin{aligned} -\frac{1}{\pi}\mathcal{R}e\left(\frac{\phi_X(\omega)e^{-i\omega x}}{i\omega}\right) &= -\frac{1}{\pi}\mathcal{R}e\left(\frac{b}{\omega} - i\frac{a}{\omega}\right), \\ &= -\frac{b}{\pi\omega}. \end{aligned}$$

□

If X is a real valued random variable, then its cdf is

$$\begin{aligned} \mathbb{P}(X > x) &= 1 - \mathbb{P}(X \leq x), \\ &= 1 - F_X(x), \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathcal{R}e\left(\frac{\phi_X(\omega)e^{-i\omega x}}{i\omega}\right) d\omega. \end{aligned} \tag{2.3.14}$$

2.4 Option pricing under Risk-neutral valuation

In this section we present the formula for the pricing of European options as an expectation of the discounted option pay-off. We follow the theory outlined in Wiersema (2008).

2.4.1 Definition (European option)**.** The purchase of a European $call$ option gives the owner the right to buy a stock S, at a given date T, for a predetermined strike price K. The purchase of a European put option gives the owner the right to $sell$ the stock S at a given date T for a predetermined strike price K. The date the option is exercised is called the maturity date.

The call option's pay-off at exercise date T is given by

$$\max(S_T - K, 0) := \begin{cases} S_T - K & \text{if} \quad S_T > K, \\ 0 & \text{if} \quad S_T \leq K. \end{cases}$$

We consider the Stochastic differential equation (SDE) for the stock price given by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \tag{2.4.1}$$

We then introduce the discounted stock price as

$$S^* = e^{-rt}S_t, \tag{2.4.2}$$

where r is the riskless growth rate of the savings account. This then gives us the stock price in terms of the savings account as a numeraire and using Ito's formula we get the (SDE)

$$\frac{dS_t^*}{S_t^*} = (\mu - r)dt + \sigma dW_t. \tag{2.4.3}$$

To achieve this, we rewrite it as follows

$$\begin{aligned} \frac{dS_t^*}{S_t^*} &= \sigma\left(\frac{\mu - r}{\sigma}dt + dW_t\right), \\ &= \sigma\left(\varphi dt + dW_t\right), \end{aligned} \tag{2.4.4}$$

where $\varphi = \frac{\mu - r}{\sigma}$. Let $y = \varphi t + x$. At $W_t = x$, we then have the probability density of W_t being given by

$$\begin{aligned} \frac{1}{\sqrt{t}\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{t}{\sqrt{t}}\right)^2\right] &= \frac{1}{\sqrt{t}\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\varphi t + x}{\sqrt{t}}\right)^2\right]\exp\left[\frac{1}{2}\varphi^2 t + \varphi x\right], \\ &= \frac{1}{\sqrt{t}\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{y}{\sqrt{t}}\right)\right]. \end{aligned} \tag{2.4.5}$$

Equation (2.4.5) is the probability density of a new Brownian motion W_t^* which defines $W_t^* = \varphi t + W_t$. Its differential is

$$dW_t^* = \varphi dt + dW_t. \tag{2.4.6}$$

Substituting (2.4.6) into the SDE (2.4.3) gives

$$\frac{dS_t^*}{S_t^*} = \sigma dW_t^* \quad \text{and} \quad \frac{dS_t}{S_t} = rdt + \sigma dW_t^*. \tag{2.4.7}$$

From proposition (2.1.16), under the probability distribution of Brownian motion W_t^*, we note that S_t^* is a Martingale and we denote this probability by $\mathbb{Q}$. Let V_t be the value of the replicating portfolio at time t consisting of a quantity α_t of stock S_t and a risk-free borrowing amount β_t. The evolution of β_t is modelled by the following differential equation

$$d\beta_t = \beta_t r dt. \tag{2.4.8}$$

The solution of which is given by

$$\beta_t = e^{rt} \quad \text{with} \quad \beta(0) = 1. \tag{2.4.9}$$

The replicating portfolio must be self-financing so that the change in the value of the portfolio must only come from the change in the value of the stock and the change in the borrowing. This condition is represented by

$$dV_t = \alpha_t dS_t + d\beta_t. \tag{2.4.10}$$

The discounted value of the portfolio is

$$V_t^* = e^{-rt}V_t, \tag{2.4.11}$$

where V_t^* is a function of V and t.

From Ito's formula we have

$$dV^* = \frac{\partial V^*}{\partial t}dt + \frac{\partial V^*}{\partial V}dV + \frac{1}{2}\frac{\partial^2 V^*}{\partial^2}(dV)^2 + \frac{\partial^2 V^*}{\partial t \partial V}dtdV. \tag{2.4.12}$$

Differentiating equation (2.4.11) and substituting into (2.4.12) gives us

$$dV^* = -rV^*dt + e^{-rt}dV. \tag{2.4.13}$$

Substituting (2.4.10) and (2.4.11) into (2.4.13) then gives us

$$\begin{aligned} dV^* &= -r\left(e^{-rt}V\right)dt + e^{-rt}\left(\alpha_t dS + d\beta_t\right), \\ &= \alpha_t e^{-rt}\sigma S_t dW^*, \\ &= \alpha_t dS_t^*. \end{aligned}$$

From proposition (2.1.16), the solution will be a Martingale under $\mathbb{Q}$ since there is no drift term and thus the expected value at future time T is equal to its value at time $t = 0$. We write this as

$$\mathbb{E}^{\mathbb{Q}}\left[V_T^*|V_0^*\right] = V_0^* = V_0. \tag{2.4.14}$$

This the gives us the following initial value

$$\begin{aligned} V_0 &= \mathbb{E}^{\mathbb{Q}}\left[V_T^*|S_0\right], \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}V_T|S_0\right]. \end{aligned} \tag{2.4.15}$$

Equation (2.4.15) will be used in the valuation of European options where V can either be a call or a put option. We then have the valuation formula for a European call option with strike price K, time to expiration T under the risk-neutral measure $\mathbb{Q}$ given as

$$C_{T-t}(K) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\left(S_T - K\right)^+|\mathcal{A}_t\right]. \tag{2.4.16}$$

3. Fourier-based Option pricing

In this chapter we outline the Carr and Madan (1999) method used in the pricing of options. We also show the limitations of the Heston (1993) approach to pricing options in section (3.1). We introduce a damping factor α in the valuation of the call price to ensure that our solution converges to zero. We also look at the effect that α has on the accuracy of the results. We show that an analytical solution for the characteristic equation exists and as a result, using Fourier transforms, we are able to compute the option price. Since this method ensures integrability, we use Fast Fourier Transform to price option for a large number of strike.

3.1 Delta Probability Decomposition

We begin by letting x be logarithm of the terminal stock price S_T for a European call such that $\log S_T = x$, with strike price K. The corresponding *pdf* is denoted by $q(x)$ and $C_T(k)$ is the call price of the option given in terms of the logarithm of the strike $k = \log K$, under the risk-neutral measure $\mathbb{Q}$. We have the following characteristic function

$$\phi(\omega) = \int_{-\infty}^{\infty} e^{i\omega x}\, q(x)dx, \tag{3.1.1}$$

where $q(x)$ is the density of the random variable x and following from (2.4.16), the initial call price, that is, when $t = 0$ is given by

$$\begin{aligned} C_T(K) &= e^{-rT}\mathbb{E}_t^{\mathbb{Q}}[(S_T - K)^+], \\ &= e^{-rT}\mathbb{E}_t^{\mathbb{Q}}[(e^x - e^k)^+], \\ &= e^{-rT}\int_{-\infty}^{\infty}[(e^x - e^k)^+]q(x)\, dx, \\ &= e^{-rT}\left(\int_k^{\infty} e^x q(x)\, dx - e^k\int_k^{\infty} q(x)dx\right). \end{aligned} \tag{3.1.2}$$

3.1.1 Proposition. Assuming no dividends and risk-free interest rate, the initial option value is

$$C_T(K) = S_0\Pi_1 - Ke^{-rT}\Pi_2, \tag{3.1.3}$$

with probabilities

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathcal{R}e\left(\frac{e^{-i\omega x}\tilde{\phi}(\omega)}{i\omega}\right) d\omega \tag{3.1.4}$$

and

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathcal{R}e\left(\frac{e^{-i\omega x}\phi(\omega)}{i\omega}\right) d\omega. \tag{3.1.5}$$

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Proof. Equation (3.1.2) is composed of two integrals. We define the second integral to be

$$\Pi_2 = \int_k^{\infty} q(x)dx.$$

This is the probability (2.3.14). We then have

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathcal{R}e\left(\frac{\phi(\omega)e^{-i\omega x}}{i\omega}\right)\, d\omega. \tag{3.1.6}$$

From the first integral in (3.1.2) we have

$$\mathbb{E}[S_T] \quad = \quad \int_{-\infty}^{\infty} e^x q(x)dx.$$

By manipulating the equation we have

$$\int_{-\infty}^{\infty} \frac{e^x}{\mathbb{E}[S_T]} q(x)dx = 1.$$

From (2.3.3), we note that $\frac{e^x}{\mathbb{E}[S_T]}q(x)$ is a density function since it integrates to one and is positive and from (2.3.6) the characteristic function is

$$\begin{aligned}
\tilde{\phi}(\omega) &= \int_{-\infty}^{\infty} e^{i\omega x} \frac{e^x}{\mathbb{E}[S_T]} q(x)dx, \\
&= \frac{1}{\mathbb{E}[S_T]} \int_{-\infty}^{\infty} e^{ix(\omega - i)} q(x)dx, \\
&= \frac{1}{\mathbb{E}[S_T]} \phi(\omega - i).
\end{aligned}$$

□

As in the case of Chourdakis (2008), the value of the call option is

$$\begin{aligned}
C_T(K) &= e^{-rT}\left[\mathbb{E}[S_T]\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathcal{R}e\left(\frac{e^{-i\omega x}\tilde{\phi}(\omega)}{i\omega}\right) d\omega\right)\right. \\
&\quad \left. + K\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathcal{R}e\left(\frac{e^{-i\omega x}\phi(\omega)}{i\omega}\right) d\omega\right)\right].
\end{aligned} \tag{3.1.7}$$

Due to the expression $1/\omega$ in the integrals, this pricing approach causes difficulties in the numerical computation. This problem is addressed by Carr and Madan (1999). We then follow their approach in the next subsection (3.2).

3.2 Carr and Madan method

We again have that $x = \log S_T$ is the terminal price of the stock for a European call option with strike price K and corresponding pdf denoted by $q(x)$. Then $C_T(k)$ is the price for the call option as a function of the log-strike $k = \log K$, under the risk-neutral measure $\mathbb{Q}$. We calculate the call price as follows

$$\begin{aligned}
C_T(k) &= e^{-rT}\mathbb{E}_t^{\mathbb{Q}}[(S_T - K)^+], \\
&= e^{-rT}\mathbb{E}_t^{\mathbb{Q}}[(e^x - e^k)^+], \\
&= e^{-rT}\int_{-\infty}^{\infty} [(e^x - e^k)^+] q(x)\, dx, \\
&= e^{-rT}\int_k^{\infty} (e^x - e^k) q(x)\, dx.
\end{aligned}$$

Taking the limit as $k \to -\infty$ of the call price, we have

$$\begin{aligned} \lim_{k \to -\infty} C_T(k) &= e^{-rT}\mathbb{E}[e^x], \\ &= S_0. \end{aligned} \tag{3.2.1}$$

The call price does not converge to zero as $k \to -\infty$ and thus $C_T(k)$ is not in L^1 and as a result the Fourier transform will not exist.

To ensure convergence to zero Carr and Madan (1999) multiply the call price by an exponential and integrable damping factor $e^{\alpha k}$ with $\alpha \in \mathbb{R}^+$, so that

$$C_T^d(k) = e^{\alpha k} C_T(k). \tag{3.2.2}$$

Hence the damped call price is given by

$$C_T^d(k) = e^{-rT} \int_k^\infty e^{\alpha k} \left(e^x - e^k \right) q(x)\ dx. \tag{3.2.3}$$

Taking the Fourier transform of the damped call price gives

$$\begin{aligned} \hat{C}(\omega) &= \mathcal{F}[C_T^d(k)], \\ &= \int_{-\infty}^{\infty} e^{i\omega k}\ C_T^d(k) dk, \\ &= \int_{-\infty}^{\infty} e^{i\omega k} \left(e^{-rT} \int_k^\infty e^{\alpha k} \left(e^x - e^k \right) q(x)\ dx \right) dk, \\ &= \int_{-\infty}^{\infty} e^{-rT} q(x) \left(\int_{-\infty}^{x} e^{i\omega k} e^{\alpha k} \left(e^x - e^k \right)\ dk \right) dx, \qquad \text{Fubini's theorem.} \end{aligned} \tag{3.2.4}$$

Isolating the inner integral, we observe that

$$\begin{aligned} \int_{-\infty}^{x} e^{(\alpha+i\omega)k} \left(e^x - e^k \right)\ dk &= \int_{-\infty}^{x} e^{(\alpha+i\omega)k} e^x\ dk - \int_{-\infty}^{x} e^{(1+\alpha+i\omega)k}\ dk, \\ &= e^x \left(\frac{e^{(\alpha+i\omega)k}}{\alpha+i\omega} \right)\Bigg|_{-\infty}^{x} - \left(\frac{e^{(1+\alpha+i\omega)k}}{1+\alpha+i\omega} \right)\Bigg|_{-\infty}^{x}, \\ &= \frac{e^{(1+\alpha+i\omega)x}}{\alpha+i\omega} - \frac{e^{(1+\alpha+i\omega)x}}{1+\alpha+i\omega}, \\ &= \frac{e^{(1+\alpha+i\omega)x}}{\alpha^2+\alpha+(2\alpha+1)i\omega-\omega^2}. \end{aligned}$$

From the second equation we notice that

$$\lim_{k \to -\infty} e^{(\alpha+i\omega)k} = 0,$$

hence giving us the inner integral as

$$\int_{-\infty}^{x} e^{(\alpha+i\omega)k} \left(e^x - e^k \right)\ dk = \frac{e^{(1+\alpha+i\omega)x}}{\alpha^2+\alpha+(2\alpha+1)i\omega-\omega^2}. \tag{3.2.5}$$

Going back to equation (3.2.4) and substituting the integral (3.2.5), we get

$$\begin{aligned}\hat{C}(\omega) &= e^{-rT}\int_{-\infty}^{\infty} q(x)\left(\frac{e^{(1+\alpha+i\omega)x}}{\alpha^2+\alpha+(2\alpha+1)i\omega-\omega^2}\right)\,dx,\\ &= e^{-rT}\int_{-\infty}^{\infty} q(x)\left(\frac{e^{i(\omega-(\alpha+1)i)x}}{\alpha^2+\alpha+(2\alpha+1)i\omega-\omega^2}\right)\,dx.\end{aligned}$$

From the definition of characteristic functions, we note that

$$\begin{aligned}\phi(\omega-(\alpha+1)i) &= \mathbb{E}\left[e^{i(\omega-(\alpha+1)i)x}\right],\\ &= \int_{-\infty}^{\infty} e^{i(\omega-(\alpha+1)i)x}q(x)\,dx. \qquad (3.2.6)\end{aligned}$$

Therefore, solution to the Fourier transform of the damped call price is obtained as

$$\hat{C}(\omega) = \frac{e^{-rT}\ \phi(\omega-(\alpha+1)i)}{\alpha^2+\alpha+(2\alpha+1)i\omega-\omega^2}. \qquad (3.2.7)$$

To recover the damped call price we use the Inverse Fourier transform, thereby giving us

$$\begin{aligned}C_T^d(k) &= \mathcal{F}^{-1}[\hat{C}(\omega)],\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega k}\ \hat{C}(\omega)d\omega,\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega k}\ \frac{e^{-rT}\ \phi(\omega-(\alpha+1)i)}{\alpha^2+\alpha+(2\alpha+1)i\omega-\omega^2}\ d\omega. \qquad (3.2.8)\end{aligned}$$

From equations (3.2.2) and (3.2.8), the call price is given by

$$\begin{aligned}C_T(k) &= e^{-\alpha k}C_T^d(k), \qquad (3.2.9)\\ &= \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega k}\ \mathcal{F}[C^d](\omega)\ d\omega. \qquad (3.2.10)\end{aligned}$$

where

$$\mathcal{F}[C^d](\omega) = \frac{e^{-r\tau}\ \phi(\omega-(\alpha+1)i)}{\alpha^2+\alpha+(2\alpha+1)i\omega-\omega^2}. \qquad (3.2.11)$$

3.3 Numerical evaluation using the FFT

In this section we approximate the call price

$$C_T(k) = \frac{e^{-\alpha k}}{\pi}\int_{0}^{\infty} e^{-i\omega k}\ \mathcal{F}[C^d](\omega)\ d\omega, \qquad (3.3.1)$$

where

$$\mathcal{F}[C^d](\omega) = \frac{e^{-rT}\ \phi(\omega-(\alpha+1)i)}{\alpha^2+\alpha+(2\alpha+1)i\omega-\omega^2}, \qquad (3.3.2)$$

is the Fourier Transform of the damped call price written in terms of the characteristic function. Let M be a good point to truncate the integral (3.3.1), thereby giving us

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi}\int_{0}^{M} e^{-i\omega k}\ \mathcal{F}[C^d](\omega)\ d\omega. \qquad (3.3.3)$$

We then discretise $[0, M]$ as follows. Let the mesh points ω_j be defined as

$$0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{N-1} = M,$$

with $\omega_j = j\Delta\omega, \quad j = 0, \ldots, N-1$ and

$$\Delta\omega = \frac{M}{N-1}.$$

We now apply the Trapezoidal rule on the definite integral in equation (3.3.3), giving us

$$\begin{aligned} \int_0^M e^{-i\omega k}\, \mathcal{F}[C^d](\omega)\, d\omega &\approx \frac{\Delta\omega}{2}\Big(e^{-i\omega_0 k}\, \mathcal{F}[C^d](\omega_0) + e^{-i\omega_N k}\, \mathcal{F}[C^d](\omega_N) \\ &\quad +2\sum_{j=1}^{N-1} e^{-i\omega_j k}\, \mathcal{F}[C^d](\omega_j)\Big), \\ &= \frac{\Delta\omega}{2}\Big(\mathcal{F}[C^d](0) + e^{-ij\Delta\omega k}\, \mathcal{F}[C^d](N\Delta\omega) \\ &\quad +2\sum_{j=1}^{N-1} e^{-ij\Delta\omega k}\, \mathcal{F}[C^d](j\Delta\omega)\Big), \\ &= \sum_{j=0}^{N-1} e^{-ij\Delta\omega k}\, \mathcal{F}[C^d](j\Delta\omega)\Delta\omega\, v_j, \end{aligned} \tag{3.3.4}$$

where $v_j = 1$ for $j = 1, \ldots, N-2$ and $v_0 = v_{N-1} = \frac{1}{2}$ are the weights. Now, substituting equation (3.3.4) into (3.3.3), we get

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi}\sum_{j=0}^{N-1} e^{-ij\Delta\omega k}\, \mathcal{F}[C^d](j\Delta\omega)\Delta\omega\, v_j. \tag{3.3.5}$$

Next, we then introduce another N size equidistant grid for the logarithmic strike prices $k_0, \ldots, k_{N-1}$ with

$$k_n = k_0 + n\Delta k,$$

and $k_{N-1} = -k_0 = \frac{N\Delta k}{2}$. Substituting (3.3) into (3.3.5) gives us

$$\begin{aligned} C_T(k_n) &\approx \frac{e^{-\alpha k_n}}{\pi}\sum_{j=0}^{N-1} e^{-i\Delta\omega j(k_0+n\Delta k)}\mathcal{F}[C^d](j\Delta\omega)v_j\Delta\omega, \\ &= \frac{e^{-\alpha k_n}}{\pi}\sum_{j=0}^{N-1} e^{-i\Delta\omega j k_0}e^{-in\Delta k\Delta\omega j}\mathcal{F}[C^d](j\Delta k)v_j\Delta\omega, \end{aligned} \tag{3.3.6}$$

for $n = 0, \ldots, N-1$. Setting $\Delta\omega\Delta k \equiv \frac{2\pi}{N}$, we then apply the FFT. We choose

$$\Delta\omega\Delta k = \frac{2\pi}{N},$$

we then have that $\Delta\omega k_0 = \Delta\omega\Delta k\frac{N}{2} = -\pi$. Setting these in equation (3.3.6) then gives us

$$C_T(k) \approx \frac{e^{-\alpha k_n}}{\pi}\sum_{j=0}^{N-1} e^{-ijn\frac{2\pi}{N}} h_j, \tag{3.3.7}$$

where

$$h_j = e^{ij\pi}\mathcal{F}[C^d](j\Delta\omega)\Delta\omega\ v_j. \tag{3.3.8}$$

We follow the pseudo-code below in the implementation of the Carr and Madan (1999) FFT.

- We input $\Delta\omega, \Delta k, N$ and α.
- We then construct vectors $\omega = \{(j-1)\Delta\omega : j = 0, \dots, N-1\}$ and $\mathbf{k} = \frac{N}{2}\Delta k + (k-1)\Delta k : k = 0, \dots, N-1\}$.
- Construct the Fourier transform of the modified call price

  $$\mathbf{c}^d = \exp(-rT)\phi(T, \omega - i(\alpha + i)) \oslash [(i\omega + \alpha) \odot (i\omega + \alpha + 1)]$$

  where $\odot$ is an element-by-element vector multiplication, while $\oslash$ is element-by-element vector division.
- We compute the vector $\mathbf{c} = \exp(-ix_1\omega) \odot \mathbf{c}^d$. $c_1 = -\frac{c_1}{2}$ and $c_N = -\frac{c_N}{2}$ for the trapezoidal rule. Next, we run the `fft` on $\mathbf{c}$, giving us $\hat{\mathbf{c}} = FFT(\mathbf{c})$
- Finally, we compute the call option values $\mathbf{c} = \frac{1}{\pi}\exp(-\alpha\mathbf{x}) \odot [\hat{\mathbf{c}}]$. The output is the log-strike price vector $\mathbf{k}$ and corresponding call prices $\mathbf{c}$.

3.4 Results

The formula for the Black-Scholes of a European call price at time of maturity T, strike price K, risk-free interest rate r, stock price S and present time t with volatility σ is given as

$$C(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \tag{3.4.1}$$

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma(\sqrt{T-t})} \tag{3.4.2}$$

and

$$d_2 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma(\sqrt{T-t})}, \tag{3.4.3}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution .

We use this formula to compute analytical values for the European call price. The characteristic function for the Black-Scholes model is

$$\phi(\omega) = \exp\left(\left(\left(r - \frac{1}{2}\sigma^2\right)i\omega - \frac{1}{2}\frac{\sigma^2}{\omega^2}\right)T\right), \tag{3.4.4}$$

and we use it in the FFT option pricing Hilpisch (2015) using the equation (3.3.2).

The figures below were generated from the following parameter values

$$\begin{aligned} S &= 100 \ \text{ stock price}, \\ T &= 1 \ \text{ expiration time}, \\ r &= 0.05 \ \text{ risk-free rate}, \\ \sigma &= 0.2 \ \text{ volatility}, \end{aligned}$$

and the same grid of strike prices. The damping parameter is α, and we plot for different values to investigate its effect on the accuracy of the option prices. For Figure (3.4), $\alpha = 2$.

Figure 3.1: Comparison of the Black-Scholes analytical solution vs FFT for European call options.

Figure (3.4) shows a plot of the call prices computed using the FFT in comparison with the analytical solution of the Black-Scholes. Figure (3.4) shows the effect of α on the accuracy of the option prices. For the values $\alpha = 1$ and $\alpha = 2$ the results deviate from the analytical solution. For $\alpha = 2$,$\alpha = 5$ and $\alpha = 6$, the stock prices become more accurate.

Figure 3.2: Effect of α on the call price.

By using a negative value for α, we get the put option prices Fusai and Roncoroni (2007). Figure (3.4) shows the European put option prices and from Figure (3.4) we observe the effect that the value of α has on the accuracy of the results. The put option prices are less accurate for $\alpha = -1.5$ and $\alpha = -2$ and more accurate for $\alpha = -3$, $\alpha = -5$ and $\alpha = -6$.

Figure 3.3: Comparison of the Black-Scholes analytical solution vs FFT for European put options.

Figure 3.4: Effect of α on the put option price.

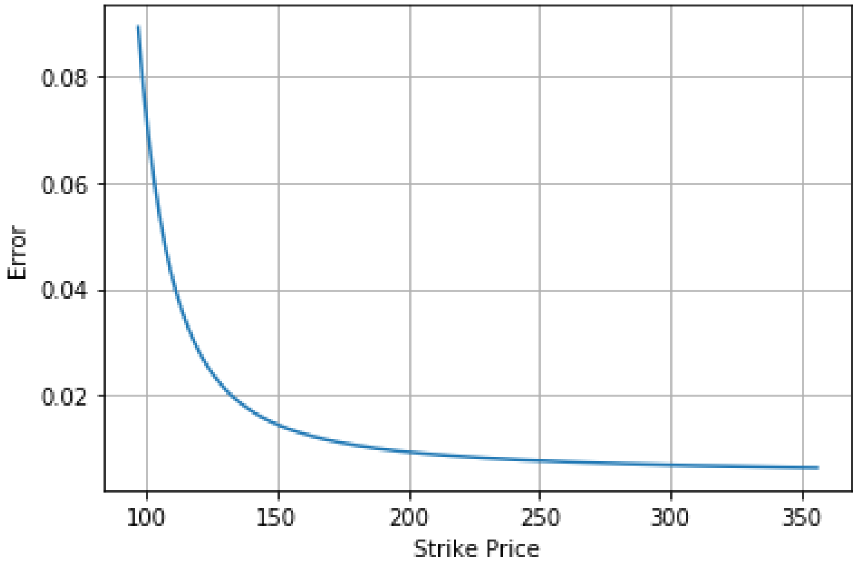

Figure 3.5: Error of the $BSFFT$ in comparison to the BS analytical values.

Figure (3.5) shows the plot of the `true relative error` for the European call price computed using the FFT displayed in figure (3.4), compared with the values computed using the Black-Scholes analytical formula. We used the following formula

$$true\ relative\ error = \frac{|BS - BSFFT|}{|BS|}, \tag{3.4.5}$$

in computing the error where BS is the Black-Scholes analytical solution and $BSFFT$ is the option prices computed using the FFT where the characteristic function (3.4.4) used was that of the Black-Scholes model. The error decreases as the strike prices move away from the stock price $S = 100$. The following table shows a sample of strike prices with corresponding error values from (3.4.5).

Strike Prices	Error values
97.044553	0.089026
98.019867	0.083161
99.004983	0.077804
100.0000	0.072905
101.005017	0.068418
102.020134	0.064303
103.045453	0.060526
104.081077	0.057053

We also computed option prices for a range of strike prices for an array of size $2^8 = 256$ using the FFT and it takes `3.9999940781854e-07s` to compute. This is the same time it takes to compute a single option price using the Black-Scholes formula in Python.

4. Conclusion

In this essay we used the Fast Fourier Transform to valuate European option prices, given a range of strike prices. We derived the option price under risk-neutral measure to avoid arbitrage opportunities. We then introduced the damping factor into the option pricing formula which allowed us to Fourier Transform the call price, which we obtained in terms of the characteristic function. In the implementation of the FFT, we used the characteristic function for the Black-Scholes model. This allowed us to recover the call price by Fourier Inversion. We then implemented the FFT and numerically computed the option prices for a range strike prices. The same range of strike prices were used to generate option values computed from the Black-Scholes analytical formula. We showed the effect of the damping factor on the accuracy of the results when comparing the option values obtained numerically by the FFT and those obtained from the Black-Scholes analytical formula. We also showed that the FFT is an efficient method since we can compute option prices for a range of strike prices in the same time it takes to compute a single option price using the analytical Black-Scholes formula.

Acknowledgements

I wish to express my gratitude and appreciation to my supervisor Prof. Ronnie Becker for his patience and guidance during this essay period. I am very grateful to my tutor Rock Stephane Koffi for his motivation and support throughout the essay. I would also like to thank Alice Nanyanzi for her support during the course phase. And also a special thank you to all the tutors, the academic director Simukai Utete and IT staff, under the management of Jan Groenewald for their support during the academic year. The role played by the facilities manager, Igsaan Kamalie and his team during this difficult time is admirable and I would like to acknowledge all their efforts.

I would like to thank Dr ZI Ali, a friend and advisor, without whom, my time at AIMS might not have been possible. I would also like to acknowledge Willie Mohlongo and Shadrack Madonsela for their advices in writing the codes. I would also like to thank my family for the support they have given me over the years.

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