Convergence of Two Point Flux Approximation Method for the Black-Scholes Option Pricing Model

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Abstract

In this essay, we analyze a numerical technique for the degenerate Black-Scholes partial differential equation. The analysis of the spatial discretization is based on the finite volume method with the two-point flux approximation method. We establish a rigorous proof for the convergence in space discretization. Theoretical results have been confirmed by numerical simulations.

Keywords: Black-Scholes PDE, Degenerate PDE, finite volume method, Two-Point Flux Approximation method.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Guy Franck Konan, 14 May 2020

Contents

1. Introduction

Financial markets are the markets in which long-term capital demand and supply are found. Such demand and supply are based on options trading. According to Hull (2003), an option is a contract that allows an holder to have the right to buy (call option) or to sell (put option) an underlying asset of a specified stock at a fixed price on (European option) or before (American option) a specified date (expiry date). It was shown by Black and Scholes (1973) that the value of an European option leads to a second order parabolic Partial Differential Equation (PDE) with respect to time and the price of the underlying asset. Moreover, it is well known that the Black-Scholes Equation (BSE) has an analytical solution only when the coefficients are constants for pricing European option. Therefore, numerical methods are used to solve the BSE. The first numerical algorithm was the lattice method in Cox et al. (1979) and Hull and White (1988). Many numerical methods have been developed subsequently such as the finite difference method Courtadon (1982), the fast Fourier transform approach Kwok et al. (2012), the finite elements method Topper (2005). But, when the price of the underlying asset is near to zero, the Black-Scholes equation becomes degenerated. In this case, classical numerical methods may not give a good accuracy in terms of approximation. To handle this degeneracy, Angermann and Wang (2007) proved the convergence of the fitted finite volume method, and the convergence of the novel Two-Point Flux Approximation (TPFA) coupled with fitted finite volume has been proved by Koffi and Tambue (2019). Their convergence proofs are based on the full discretization only for the theta Euler method. In order to adapt this proof to more efficient discretization methods in time such as the Rosenbrock methods, we will focus our study on the space discretization.

In this essay, we will provide a rigorous proof for a convergence in space discretization using TPFA method for a degenerated Black-Scholes PDE in one dimension. This essay is organised in five chapters. In chapter 2, we give the preliminaries and notations which will be used in this essay. In chapter 3, we will give a discretization of the Black-Scholes partial differential equation using TPFA method and its flux consistency. In chapter 4, we will provide a rigorous proof for the convergence in space which is given by the error estimate. In chapter 5, we will give a numerical experiment to support our theoretical results. Chapter 6 provides summary of our work and gives some future work.

2. Preliminaries

Here, we introduce some concepts to a better understanding of our work and afterwards, we provide a proof for the existence and the uniqueness of the solution of the continuous problem.

2.1 Definition of basic notions

2.1.1 Definition. Shreve (2004) Let Ω be a nonempty set, and let $\mathcal{F}$ be a collection of subsets of Ω. We say that $\mathcal{F}$ is a σ-algebra (called a σ-field by some authors) provided that:

(i) the empty set $\emptyset$ belongs to $\mathcal{F}$,

(ii) whenever a set A belongs to $\mathcal{F}$, its complement A^c also belongs to $\mathcal{F}$, and

(iii) whenever a sequence of sets $(A_n)_{n\in\mathbb{N}}$ belongs to $\mathcal{F}$, their union $\cup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{F}$.

2.1.2 Definition. Shreve (2004) Let Ω be a nonempty set, and let $\mathcal{F}$ be a σ-algebra of subsets of Ω. A probability measure $\mathbb{P}$ is a function that, to every set $A \in \mathcal{F}$, assigns a number in $[0,1]$, called the probability of A and written $\mathbb{P}(A)$. We require:

(i) $\mathbb{P}(\Omega) = 1$, and

(ii) (countable additivity) whenever $(A_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{F}$, then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n). \tag{2.1.1}$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

2.1.3 Definition. Shreve (2004) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable is a real valued function X defined on Ω with the property that for every Borel subset B of $\mathbb{R}$, the subset of Ω given by

$$\{X \in B\} = \{\omega \in \Omega; X(\omega) \in B\} \tag{2.1.2}$$

in the σ-algebra $\mathcal{F}$. We sometimes also permit a random variable to take the values ∞ and $-\infty$.

2.2 Notions in L^p-spaces

Let $(\Omega, \mathcal{M}, \mu)$ a measure space.

2.2.1 Definition. Brezis (2010)

Let $p \in \mathbb{R}$ with $1 < p < \infty$, we set

$$L^p(\Omega) = \Big\{ f : \Omega \to \mathbb{R}; f \text{ is mesurable and } |f|^p \in L^1(\Omega) \Big\}, \tag{2.2.1}$$

associated with the norm

$$\|f\|_{L^p} = \|f\|_p = \left[\int_{\Omega} |f(x)|^p \mathrm{d}\mu\right]^{\frac{1}{p}}. \tag{2.2.2}$$

2.2.2 Definition. Brezis (2010) We set

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,\middle|\, \begin{array}{c} f \text{ is mesurable and there is a constant } C \\ \text{such that } |f(x)| \le C \text{ a.e.(almost everywhere) on } \Omega \end{array} \right\}, \tag{2.2.3}$$

associated with the norm

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf\Big\{ C; |f(x)| \le C \text{ a.e. on } \Omega \Big\}. \tag{2.2.4}$$

2.3 Sobolev spaces and weighted Sobolev spaces

Let $N \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^N$ be an open set, and let $p \in \mathbb{R}$ with $1 \le p \le \infty$.

2.3.1 Definition. Brezis (2010) The Sobolev space $W^{1,p}$ is defined by

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \,\middle|\, \begin{array}{c} \exists g_1, g_2, \cdots, g_N \in L^p(\Omega) \text{ such that} \\ \int_\Omega u \frac{\partial \varphi}{\partial x_i} \mathrm{d}x = \int_\Omega g_i \varphi \mathrm{d}x, \ \forall \varphi \in C_c^\infty(\Omega), \quad \forall i = 1, 2, \cdots, N \end{array} \right\}. \tag{2.3.1}$$

We set

$$H^1(\Omega) = W^{1,2}(\Omega). \tag{2.3.2}$$

For $u \in W^{1,p}(\Omega)$, we define $\dfrac{\partial u}{\partial x_i} = g_i$, and we write

$$\nabla u = \text{grad} u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_N} \right). \tag{2.3.3}$$

The space is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|_p^p + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_p^p \right)^{\frac{1}{p}}, \text{ if } 1 \le p < \infty. \tag{2.3.4}$$

The space $H^1(\Omega)$ is equipped with the scalar product

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2(\Omega)} = \int_\Omega uv \mathrm{d}x + \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}. \tag{2.3.5}$$

The associated norm

$$\|u\|_{H^1(\Omega)} = \left(\|u\|_2^2 + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_2^2 \right)^{\frac{1}{2}}. \tag{2.3.6}$$

2.3.2 Definition. Brezis (2010) Let $1 \le p < \infty$; $W_0^{1,p}(\Omega)$ denotes the closure of $C_c^1(\Omega)$ in $W^{1,p}(\Omega)$. When $p = 2$,

$$H_0^1(\Omega) = W_0^{1,2}(\Omega). \tag{2.3.7}$$

The space $W_0^{1,p}(\Omega)$, equipped with the $W^{1,p}(\Omega)$ norm, is a separable Banach space. It is reflexive if $1 < p < \infty$. $H_0^1(\Omega)$, equipped with the $H^1(\Omega)$ scalar product, is a Hilbert space.

2.3.3 Standard notation.

Let $m \in \mathbb{N}$ and $I \subset \mathbb{R}$, and let us denote by T the expiry date of an option.
We define by $C^m(I)$ (respectively $C(\overline{I})$) the function set of which a function and its derivatives of up to order m are continuous on I (respectively on $\overline{I}$).
We set $H_0^m(I) = \Big\{u \in H^m(I) : v|_{\partial I} = 0\Big\}$.
For any Hilbert space $K(I)$ of classes of functions defined on I, we let $L^p(0, T, K(I))$ denote the space defined by

$$L^p(0,T;K(I)) = \Big\{u(\cdot,t) : u(\cdot,t) \in K(I) \text{ a.e. in } (0,T); \|v(\cdot,t)\|_K \in L^p((0,T))\Big\}, \tag{2.3.8}$$

where $1 \leq p \leq \infty$ and $\|\cdot\|_K$ denotes the natural norm in $K(I)$.
The norm on this space is denoted by

$$\|u\|_{L^p(0,T;K(I))} = \left(\int_0^T \|v(\cdot,t)\|_K^p \mathrm{d}t\right)^{\frac{1}{p}}. \tag{2.3.9}$$

Since the Black-Scholes equation is degenerated, then we define a weighted inner product on $L^2_\omega(I)$ by

$$(u,v)_\omega := \int_I x^2 uv \mathrm{d}x. \tag{2.3.10}$$

The corresponding weighted L^2-norm is defined by:

$$\|v\|_{0,w} := \left(\int_I x^2 v^2 \mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.3.11}$$

We define the space of all weighted square-integrable as follows:

$$L^2_w(I) := \Big\{u : \|u\|_{0,w} < \infty\Big\}. \tag{2.3.12}$$

It is easy to show that the pair $\Big(L^2_w(I), (\cdot,\cdot)_w\Big)$ is a Hilbert space.
Using $L^2(I)$ and $L^2_w(I)$, we define the weighted Sobolev space as follows:

$$H^1_{0,w}(I) := \Big\{u : u \in L^2(I), u' \in L^2_w(I) \text{ and } u|_{\partial I} = 0\Big\}. \tag{2.3.13}$$

Using also the inner product on $L^2(I)$ and $L^2_w(I)$, we define a weighted inner product on $H^1_{0,w}$ by $(\cdot,\cdot)_H = (\cdot,\cdot) + (\cdot,\cdot)_w$, where $(\cdot,\cdot)$ is the inner product on $L^2(I)$.
We can also prove easily that the pair $\Big(H^1_{0,w}(I), (\cdot,\cdot)_H\Big)$ is a Hilbert space with norm

$$\|u\|_{1,\omega} = \Big(\|u\|^2_{L^2(I)} + \|u'\|^2_{0,w}\Big)^{\frac{1}{2}} = \Big((u,u) + (x^2u', u')\Big)^{\frac{1}{2}}. \tag{2.3.14}$$

2.4 Derivation of the Black-Scholes PDE

Here, we follow the derivation method used in Hull (2003).
Let us define the following variables:

- $x =$ stock price
- $\mu =$ expected rate of return per unit of time from the stock price
- $t =$ time
- $\sigma =$ volatility
- $V = V(x,t) =$ option price
- Portfolio: 1 option, δ stocks

The assumptions we use to derive the Black-Scholes differential equation are as follows:

1. The stock price follows the geometric Brownian motion (GBM) with μ and σ constant.

$$dx = \mu x dt + \sigma x \mathrm{d}W, \tag{2.4.1}$$

   where W is a Brownian motion defined in Shreve (2004).

2. The short selling of securities with full use of proceeds is permitted.
3. There are no transactions costs or taxes. All securities are perfectly divisible.
4. There are no dividends during the life of the derivative.
5. There are no riskless arbitrage opportunities.
6. Security trading is continous.
7. The risk-free rate of interest r is constant and the same for all maturities.

By applying Ito's formula we have:

$$dV = \left(\frac{\partial V}{\partial x}\mu x + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\sigma^2 x^2\right)dt + \frac{\partial V}{\partial x}\sigma x \mathrm{d}W. \tag{2.4.2}$$

Let Π be the value of the Portfolio. We know that the appropriate portfolio is as follows:

$$-1 : \text{derivative and } + \frac{\partial V}{\partial x} : \text{shares.}$$

The holder of this portfolio is short one derivative and long an amount $\frac{\partial V}{\partial x}$ of shares.
By definition,

$$\Pi = -V + \frac{\partial V}{\partial x}x. \tag{2.4.3}$$

The change $\delta\Pi$ in the value of the portfolio in the time interval δt is given by

$$\delta\Pi = -\delta V + \frac{\partial V}{\partial x}\delta x. \tag{2.4.4}$$

The discrete version of Equations (2.4.1) and (2.4.2) are

$$\delta x = \mu x \delta t + \sigma x \delta W, \tag{2.4.5}$$

and

$$\delta V = \left(\frac{\partial V}{\partial x} \mu x + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} \sigma^2 x^2 \right) \delta t + \frac{\partial V}{\partial x} \sigma x \delta W. \tag{2.4.6}$$

Substituting Equations (2.4.5) and (2.4.6) into Equation (2.4.4) yields

$$\delta \Pi = \left(- \frac{\partial V}{\partial t} - \frac{1}{2} \frac{\partial^2 V}{\partial x^2} \sigma^2 x^2 \right) \delta t. \tag{2.4.7}$$

Since the coefficient of δW in Equation (2.4.7) is 0, then the portfolio is riskless during time δt. Therefore

$$\delta \Pi = r \Pi \delta t \quad . \tag{2.4.8}$$

Substituting Equations (2.4.3) and (2.4.7) into equation (2.4.8), we obtain:

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} \sigma^2 x^2 \right) \delta t = r \left(V - \frac{\partial V}{\partial x} x \right) \delta t.$$

Hence, we obtain the Black-Scholes PDE:

$$LV(x,t) = 0. \tag{2.4.9}$$

with $LV(x,t) := -\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2(t) x^2 \frac{\partial^2 V}{\partial x^2} - r(t) x \frac{\partial V}{\partial x} + r(t) V = 0$,
where $(x,t) \in I_T = I \times [0,T)$, with $I = (0, x_{\text{max}})$, $\sigma > 0$ and $T > 0$ the expiry date; with the boundary and final (or payoff) conditions:

$$\left\{ \begin{array}{lcll} V(0,t) & = & g_1(t) & t \in [0,T), \\ V(x_{\text{max}},t) & = & g_2(t) & t \in [0,T), \\ V(x,T) & = & g_3(x) & x \in \overline{I}, \end{array} \right. \tag{2.4.10}$$

where g_1, g_2 and g_3 define the above boundary and final conditions and satisfy the following compatibility conditions:

$$g_3(0) = g_1(T) \text{ and } g_3(x_{\text{max}}) = g_2(T). \tag{2.4.11}$$

2.4.1 Assumption. We assume that the coefficients σ and r are sufficiently smooth and satisfy

$$\underline{\sigma} \le \sigma(t) \le \overline{\sigma}, \text{ and } \underline{r} \le r(t) \le \overline{r}, \tag{2.4.12}$$

and we define

$$\beta := \sup_{t \in (0,T)} \sigma^2(t). \tag{2.4.13}$$

for some positive constants $\underline{\sigma}$, $\overline{\sigma}$, $\underline{r}$, and $\overline{r}$.

2.4.2 Transformation of the Black-Scholes PDE.

In order to apply the finite volume method, we need to transform Equation (2.4.9) into its divergence form.

Let $u := e^{\beta t}(V - V_0)$, where

$$V_0(x,t) = g_1(t) + \frac{g_2(t) - g_1(t)}{x_{\text{max}}}x. \tag{2.4.14}$$

By multiplying Equation (2.4.9) with $e^{\beta t}$ and then adding $f(x,t) = -e^{\beta t}LV_0(x,t)$ on both sides, we obtain:

$$-\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(ax^2\frac{\partial u}{\partial x} + bxu\right) + cu = f(x,t), \tag{2.4.15}$$

where

$$a(t) := \frac{1}{2}\sigma^2(t),\ b(t) := r(t) - \sigma^2(t),\ \text{and}\ c(t) := 2r(t) + \beta - \sigma^2(t). \tag{2.4.16}$$

We transform (2.4.9) with the non-homogeneous Dirichlet boundary conditions into the homogenous boundary condition.

Then, from boundary conditions system (2.4.10), we obtain the homogeneous boundary conditions as follows:

$$\begin{cases} u(0,t) = 0 = u(x_{\text{max}},t) \quad t \in [0,T), \\ u(x,T) = e^{\beta T}\Big(g_3(x) - V_0(x,T)\Big), \quad x \in \overline{I}. \end{cases} \tag{2.4.17}$$

2.5 Variational formulation for the continuous problem

Assume that $\frac{\partial u}{\partial t} = 0$. The weak formulation equivalent to the equilibrium point form of Equations (2.4.15) and (2.4.17) is as follows:

Let $u, v \in H^1_{0,w}(I)$. By multiplying Equation (2.4.15) with v, and integrating by parts over I, we obtain:

$$-\left(a(t)x^2\frac{\partial u}{\partial x}\mathrm{d}x + b(t)xu\right)v\Big|_0^{x_{\text{max}}} + \int_0^{x_{\text{max}}}\left(a(t)x^2\frac{\partial u}{\partial x}\mathrm{d}x + b(t)xu\right)v'\mathrm{d}x + \int_0^{x_{\text{max}}} c(t)uv\mathrm{d}x = \int_0^{x_{\text{max}}} f(x,t)v\mathrm{d}x.$$

Since $v \in H^1_{0,w}(I)$, then $v(0) = v(x_{\text{max}}) = 0$. Therefore

$$\int_0^{x_{\text{max}}}\left(a(t)x^2\frac{\partial u}{\partial x}\mathrm{d}x + b(t)xu\right)v'\mathrm{d}x + \int_0^{x_{\text{max}}} c(t)uv\mathrm{d}x = \int_0^{x_{\text{max}}} f(x,t)v\mathrm{d}x. \tag{2.5.1}$$

From equation (2.5.1) we get:

$$\Big(a(t)x^2u' + b(t)xu, v'\Big) + \Big(c(t)u, v\Big) = (f, v). \tag{2.5.2}$$

Therefore, the weak formulation of (2.4.15) and (2.4.17) is:

$$\begin{cases} \text{Find } u \in H^1_{0,w}(I) \text{such that, for all } v \in H^1_{0,w}(I): \\ \Big(a(t)x^2u' + b(t)xu, v'\Big) + \Big(c(t)u, v\Big) = (f, v). \end{cases} \tag{2.5.3}$$

2.6 Existence and uniqueness of the solution for the continuous problem

Let us set $A(u,v) := \Big(a(t)x^2u' + b(t)xu, v'\Big) + \Big(c(t)u, v\Big)$ and $Tf := (f, v)$.
We want to show that the bilinear form A is coercive and continuous and the linear form Tf is continuous.

Now let us prove the continuity of A. We have

$$A(u,u) = \Big(a(t)x^2u', u'\Big) + \Big(b(t)xu, u'\Big) + \Big(c(t)u, u\Big).$$

Integrating (bxu, u') by part:

$$\Big(b(t)xu, u'\Big) = b(t)x\frac{u^2}{2}\Big|_0^{x_{\text{max}}} - \int_0^{x_{\text{max}}} b(t)\frac{u^2}{2}\mathrm{d}x.$$

Since $u \in H^1_{0,w}(I)$ then $u(0) = u(x_{\text{max}}) = 0$. Thus,

$$\Big(b(t)xu, u'\Big) = -\frac{1}{2}b(t)\int_0^{x_{\text{max}}} u^2\mathrm{d}x.$$

Therefore

$$\begin{aligned}\Big(b(t)xu, u'\Big) + \Big(c(t)u, u\Big) &= \int_0^{x_{\text{max}}} \Big(-\frac{1}{2}b(t) + c(t)\Big)u^2\mathrm{d}x \\ \Big(b(t)xu, u'\Big) + \Big(c(t)u, u\Big) &= \frac{1}{2}\Big(3r(t) - \sigma^2(t) + 2\beta\Big)\|u\|^2_{L^2(I)}.\end{aligned}$$

From Equation (2.4.14), we have:

$$2\beta - \sigma^2(t) > 0, \quad \forall t \in (0,T).$$

Therefore

$$\Big(b(t)xu, u'\Big) + \Big(c(t)u, u\Big) \geq \frac{3}{2}r(t)\|u\|^2_{L^2(I)}.$$

We then have:

$$\begin{aligned}A(u,u) \geq& \Big(a(t)x^2u', u'\Big) + \frac{3}{2}r(t)\|u\|^2_{L^2(I)} \\ \geq& \frac{1}{2}\sigma^2(t)(x^2u', u') + \frac{3}{2}r(t)\|u\|^2_{L^2(I)} \\ \geq& \frac{1}{2}\sigma^2(t)\|u'\|^2_{0,w} + \frac{3}{2}r(t)\|u\|^2_{L^2(I)} \\ \geq& \frac{1}{2}\underline{\sigma}^2\|u'\|^2_{0,w} + \frac{3}{2}\underline{r}\|u\|^2_{L^2(I)}.\end{aligned}$$

Hence

$$A(u,u) \geq \frac{1}{2}\min\Big(\underline{\sigma}^2, 3\underline{r}\Big)\Big(\|u'\|^2_{0,w} + \|u\|^2_{L^2(I)}\Big).$$

By setting $C = \frac{1}{2}\min\left(\underline{\sigma}^2, 3\underline{r}\right) > 0$, we have:

$$A(u,u) \geq C\|u'\|^2_{1,w}. \tag{2.6.1}$$

Then A is coercive.
Using the same transformation as before we prove that A and Tf are continuous. Then, by the Lax-Milgram theorem, the solution for the continuous problem (2.5.3) exists and is unique.

3. Discretization of the Black-Scholes PDE using the TPFA method

It is well known that Black-Scholes PDE with non-constant coefficients does not have an analytical solution. Then, we need numerical methods to get the option value associated to the problem (2.4.10). since the Black-Scholes PDE has diffusion and advection terms, we use the TPFA and up-wind method to approximate respectively the diffusion and the advection term. Finite volume method with TPFA is an efficient numerical method for solving Black-Scholes PDE. To approximate the Black-Scholes PDE with respect to space, we do the discretization in space by subdividing our study domain into grid points. Thereafter, we study the flux consistency of the TPFA.

3.1 Discretization in space by using TPFA

3.1.1 Subdivise our study domain into grids point.

Let $I = \bigcup_{i=0}^{N-1} I_i$.
Where

$$I_i = (x_i, x_{i+1}) \quad i = 0, \cdots, N-1, \tag{3.1.1}$$

with $0 = x_0 < x_1 < x_2 < \cdots < x_N = x_{\max}$. We define $h_i = x_{i+1} - x_i$ and $h = \max_{0 \le i \le N-1} h_i$. We also defined the following mid-points of the interval I_i by

$$x_{i-\frac{1}{2}} = \frac{x_{i-1} + x_i}{2} \text{ and } x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2} \qquad \text{for } i = 1, 2, \cdots, N-1,$$

with $x_{\frac{-1}{2}} = x_0$ and $x_{N+\frac{1}{2}} = x_{\max}$. With these mid-points, we form a second partition K_i of I such that

$$K_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}],$$

with $l_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \quad i = 0, \cdots, N.$

3.1.2 Two Point Flux Approximation method.

We want to approximate u in the second partition K_i of I.

Let us denote $u(x_i, t) \approx u_i$. Integrating Equation (2.4.15) over K_i, we have:

$$-\int_{K_i} \frac{\partial u}{\partial t} \mathrm{d}x - \int_{K_i} \frac{\partial}{\partial x}\left(a(t)x^2 \frac{\partial u}{\partial x}\right) \mathrm{d}x - \int_{K_i} \frac{\partial u}{\partial x}\left(b(t)xu\right) \mathrm{d}x + \int_{K_i} c(t)u \mathrm{d}x = \int_{K_i} f(x,t) \mathrm{d}x. \tag{3.1.2}$$

Applying the mid-quadrature rule on some integrals of Equation (3.1.2), we obtain:

$$\int_{K_i} \frac{\partial u}{\partial t} dx \approx l_i \frac{\mathrm{d}u_i}{\mathrm{d}t}, \; \int_{K_i} c(t)u \mathrm{d}x \approx l_i c(t) u_i \text{ and } \int_{K_i} f(x,t)\mathrm{d}x \approx l_i f_i \text{ with } f_i = f(x_i, t). \tag{3.1.3}$$

By applying the Green's theorem on the second term of equation Equation (3.1.2), we have:

$$\int_{K_i} \frac{\partial}{\partial x}\left(a(t)x^2 \frac{\partial u}{\partial x}\right) \mathrm{d}x = a(t)x^2 \frac{\partial u}{\partial x}\bigg|_{x_{i+\frac{1}{2}}} - a(t)x^2 \frac{\partial u}{\partial x}\bigg|_{x_{i-\frac{1}{2}}}. \tag{3.1.4}$$

Here, we apply the TPFA method to approximate P.
Let us define $P(x)$ as follows:

$$P(x) := k(x,t)\frac{\partial u}{\partial x}, \text{ with } k(x,t) = a(t)x^2. \tag{3.1.5}$$

Over an interval K_i, $k(x,t)$ will be replaced by its average value as follows:

$$k_i = \frac{1}{l_i}\int_{K_i} \frac{1}{2}\sigma^2(t)x^2 dx = \frac{1}{6}\sigma^2(t)\frac{x^3_{i+\frac{1}{2}} - x^3_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}. \tag{3.1.6}$$

According to Figure 3.1, $P_{i+\frac{1}{2}}$ can be evaluated at each side of $x_{i+\frac{1}{2}}$.

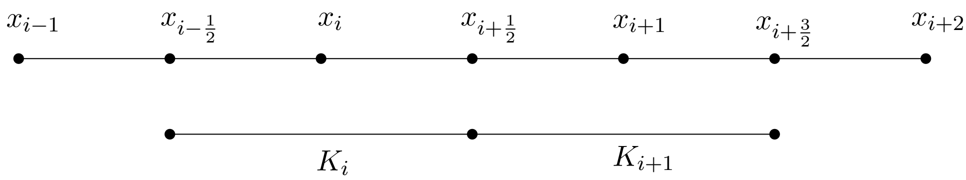

Figure 3.1: Subdivision

Then

$$P_{i+\frac{1}{2}} = 2k_i\frac{u_{i+\frac{1}{2}} - u_i}{l_i} \text{ and } P_{i+\frac{1}{2}} = 2k_{i+1}\frac{u_{i+1} - u_{i+\frac{1}{2}}}{l_{i+1}}. \tag{3.1.7}$$

Since the flux is continuous at $x_{i+\frac{1}{2}}$, then

$$u_{i+\frac{1}{2}} = \frac{\frac{k_i}{l_i}u_i + \frac{k_{i+1}}{l_{i+1}}u_{i+1}}{\frac{k_i}{l_i} + \frac{k_{i+1}}{l_{i+1}}}. \tag{3.1.8}$$

Setting $\eta_i = \frac{k_i}{l_i}$, we can rewrite $P_{i+\frac{1}{2}}$ in (3.1.7) as follows:

$$P_{i+\frac{1}{2}} = \lambda_{i+\frac{1}{2}}(u_{i+1} - u_i), \text{ with } \lambda_{i+\frac{1}{2}} = \frac{2\eta_i\eta_{i+1}}{\eta_i + \eta_{i+1}}. \tag{3.1.9}$$

3.1.3 Upwind method for the advection term.

Now let us approximate the advection term in Equation (3.1.2).
Using the upwind method, we have:

$$\int_{K_i} \frac{\partial}{\partial x}\Big(b(t)xu\Big)\mathrm{d}x \approx b(t)\Big(x_{i+\frac{1}{2}}u_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}u_{i-\frac{1}{2}}\Big), \tag{3.1.10}$$

with

$$u_{i+\frac{1}{2}} = \begin{cases} u_i & \text{if } b(t) > 0, \\ u_{i+1} & \text{if } b(t) < 0, \end{cases} \tag{3.1.11}$$

since $x_{i+\frac{1}{2}} > 0$.

Then

$$b(t)x_{i+\frac{1}{2}}u_{i+\frac{1}{2}} = x_{i+\frac{1}{2}}u_i \max\Big(b(t),0\Big) + x_{i+\frac{1}{2}}u_{i+1}\min\Big(b(t),0\Big). \tag{3.1.12}$$

We also have:

$$u_{i-\frac{1}{2}} = \begin{cases} u_{i-1} & \text{if } b(t) > 0, \\ u_i & \text{if } b(t) < 0, \end{cases} \tag{3.1.13}$$

because $x_{i-\frac{1}{2}} > 0$.
Then

$$b(t)x_{i-\frac{1}{2}}u_{i-\frac{1}{2}} = x_{i-\frac{1}{2}}u_{i-1} \max\Big(b(t),0\Big) + x_{i-\frac{1}{2}}u_{i}\min\Big(b(t),0\Big). \tag{3.1.14}$$

By using equations (3.1.3), (3.1.9), (3.1.12) and (3.1.14), the discrete formulation of (3.1.2) is then given by:

$$-\, l_i\frac{\mathrm{d}u_i}{\mathrm{d}t} + S_h(u_{i+\frac{1}{2}}) - S_h(u_{i-\frac{1}{2}}) + l_i c u_i = l_i f_i, \quad i = 1,\cdots,N-1 \tag{3.1.15}$$

where

$$S_h(u_{i+\frac{1}{2}}) = -\lambda_{i+\frac{1}{2}}(u_{i+1}-u_i) - x_{i+\frac{1}{2}}\left(u_i \max\Big(b(t),0\Big) + u_{i+1}\min\Big(b(t),0\Big)\right), \tag{3.1.16}$$

and

$$S_h(u_{i-\frac{1}{2}}) = -\lambda_{i-\frac{1}{2}}(u_{i}-u_{i-1}) - x_{i-\frac{1}{2}}\left(u_{i-1} \max\Big(b(t),0\Big) + u_{i}\min\Big(b(t),0\Big)\right). \tag{3.1.17}$$

3.2 Variational formulation of the discrete problem

Let $V_h \subset H^1_w(I)$ be the space of continuous functions that are piecewise continuous.
For an arbitrary $v \in C(\overline{I})$, we define the mass lumping operator
$L_h : C(\overline{I}) \to L^\infty(I)$ by

$$L_h v|_{K_i} := v(x_i) \quad i = 0,\cdots,N.$$

Also, if the function v satisfies homogeneous Dirichlet boundary conditions, we have:

$$L_h v|_{K_0} = L_h v|_{K_N} = 0.$$

Let us denote by $(\cdot,\cdot)_h$ the scalar product on $C(\overline{I}) \supset V_h$ by

$$(u,v)_h = (L_h u, L_h v) = \sum_{i=1}^{N-1} l_i u_i v_i \quad \forall u,v \in C(\overline{I}), \tag{3.2.1}$$

and its corresponding norm $\|\cdot\|_{0,h}$ by

$$\|v\|_{0,h} = \left(\sum_{i=1}^{N-1} l_i v_i^2\right)^{1/2}. \tag{3.2.2}$$

Multiplying Equation (3.1.15) with an arbitrary real number v_i and adding the results over all the intervals K_i of I, we obtain:

$$-\sum_{i=1}^{N-1} l_i \frac{du_i}{dt} v_i + \sum_{i=1}^{N-1} \Big(S_h(u_{i+\frac{1}{2}}) - S_h(u_{i-\frac{1}{2}}) \Big) v_i + c(t) \sum_{i=1}^{N-1} l_i u_i v_i = \sum_{i=1}^{N-1} l_i f_i v_i. \tag{3.2.3}$$

Equation (3.2.3) is equivalent to:

$$(\frac{du}{dt}, v)_h + a_{1,h}(u_h, v_h) + a_{2,h}(u_h, v_h) + c(t) \sum_{i=1}^{N-1} l_i u_i v_i = (f, v)_h \quad, \forall u_h, v_h \in V_h, \tag{3.2.4}$$

with

$$a_{1,h}(u_h, v_h) = \sum_{i=1}^{N-1} \Big(-\lambda_{i+\frac{1}{2}}(u_{i+1} - u_i) + \lambda_{i-\frac{1}{2}}(u_i - u_{i-1}) \Big) v_i, \tag{3.2.5}$$

and

$$a_{2,h}(u_h, v_h) = \sum_{i=1}^{N-1} \Big(-x_{i+\frac{1}{2}} \Big(u_i b_p + u_{i+1} b_n \Big) + x_{i-\frac{1}{2}} \Big(u_{i-1} b_p + u_i b_n \Big) \Big) v_i. \tag{3.2.6}$$

Where $b_p := \max\Big(b(t), 0\Big)$ and $b_n := \min\Big(b(t), 0\Big)$.
Note that Equation (3.2.4) can be written as follows:

$$A_h(u_h, v_h) = (f, v)_h, \tag{3.2.7}$$

with $A_h(u_h, v_h) := \sum_{i=1}^{N-1} \Big(S_h(u_{i+\frac{1}{2}}) - S_h(u_{i-\frac{1}{2}}) \Big) v_i + c(t) \sum_{i=1}^{N-1} l_i u_i v_i$, where S_h is the discrete flux given by Equations(3.1.14) and (3.1.15).

3.3 Existence and uniqueness of the discrete solution

Let us define the discrete $H_0^1(I)$ norm by

$$\|u_h\|_{0,w} = \Big(\sum_{i=1}^{N-1} \lambda_{i+\frac{1}{2}} (u_{i+1} - u_i)^2 \Big)^{1/2}, \tag{3.3.1}$$

and weighted discrete H_w^1-norm by

$$\|u_h\|_{w,d}^2 = \|u_h\|_{0,w}^2 + \|u_h\|_{0,h}^2. \tag{3.3.2}$$

It is easy to show that $\|\cdot\|_{0,w}$ is a semi-norm in V_h since $\lambda_{i+\frac{1}{2}} > 0$.

We know that the linear mapping $v \mapsto (f, v)_h$ given by (3.2.4) is continuous in V_h. Now, we want to prove the existence and the uniqueness of the discrete solution u_h by proving the following theorem.

3.3.1 Theorem. Let Assumption 2.4.1 be fulfilled. If h is sufficiently small, then, for all $u_h \in V_h$, we have

$$A(u_h, u_h) \geq \gamma \|u_h\|_{w,d}^2, \tag{3.3.3}$$

where γ denotes a positive constant, independent of h and u_h.

Proof. According to Equation (3.2.5), we have:

$$\begin{aligned} a_{1,h}(u_h,u_h) &= \sum_{i=1}^{N-1} -\lambda_{i+\frac{1}{2}}(u_{i+1}-u_i)u_i + \sum_{i=1}^{N-1} \lambda_{i-\frac{1}{2}}(u_i-u_{i-1})u_i \\ &\geq \sum_{i=1}^{N-1} \lambda_{i+\frac{1}{2}}(u_{i+1}-u_i)^2. \end{aligned}$$

Then using Equation (3.3.1), we get:

$$a_{1,h}(u_h,u_h) \geq \|u_h\|_{0,w}^2 \tag{3.3.4}$$

Also, from Equation (3.2.6), we have:

$$\begin{aligned} a_{2,h}(u_h,u_h) &= \sum_{i=1}^{N-1} -x_{i+\frac{1}{2}}\Big(u_i b_p + u_{i+1} b_n\Big)u_i + \sum_{i=1}^{N-1} x_{i-\frac{1}{2}}\Big(u_{i-1} b_p + u_i b_n\Big)u_i \\ &= -\, b_n\left(\sum_{j=0}^{N-1} l_i u_i^2\right). \end{aligned}$$

Since $b_n \leq 0$, we get:

$$a_{2,h}(u_h,u_h) \geq 0. \tag{3.3.5}$$

Putting Inequalities (3.3.4) and (3.3.5) together, we have:

$$a(u_h,u_h) \geq \|u_h\|_{0,w}^2 + c(t)\sum_{i=1}^{N-1} l_i u_i^2. \tag{3.3.6}$$

So

$$a(u_h,u_h) \geq \|u_h\|_{0,w}^2 + c(t)\|u\|_{0,h}^2. \tag{3.3.7}$$

Since $c = 2r(t) + \beta - \sigma^2(t) > 0$, we have:

$$a(u_h,u_h) \geq \gamma\Big(\|u_h\|_{0,w}^2 + \|u\|_{0,h}^2\Big), \tag{3.3.8}$$

where $\gamma = \min\Big(1, c(t)\Big) > 0$.
Therefore,

$$a(u_h,u_h) \geq \gamma\|u_h\|_{w,d}^2. \tag{3.3.9}$$

□

Then, the discrete solution u_h exists and it is unique.

3.4 Flux consistency of the TPFA method

Let us set $\omega' := \frac{\partial \omega}{\partial x}$.

3.4.1 Lemma (Flux consistency)**.** Let I_h be the interpolation operator defines as follows:

$$
\begin{aligned}
I_h \colon C(\overline{I}) &\longrightarrow \quad V_h \\
v &\longmapsto I_h v \colon \overline{I} \longrightarrow V_h \\
&\qquad\qquad x \longmapsto \sum_{k=1}^{N-1} v(x_i)\psi_{x_i}(x),
\end{aligned}
$$

where $\{\psi_{x_i}\}_{i=1}^{N-1}$, with $\psi_{x_i}(x_j) = \delta_{ij}$, is the nodal basis to $\{x_i\}_{i=1}^{N-1}$, where $x_i \in K_i$.
Let S be the continuous flux function $\omega \in C(\overline{I})$ as

$$
S(\omega(x_{i+\frac{1}{2}})) := -k(x_{i+\frac{1}{2}})\omega'(x_{i+\frac{1}{2}}) - bx_{i+\frac{1}{2}}\omega(x_{i+\frac{1}{2}}), \quad x_{i+\frac{1}{2}} \in K_i. \tag{3.4.1}
$$

When the TPFA method is applied for the spatial discretization, that means that the discrete flux is given by S_h defined in Equations (3.1.14) and (3.1.15) for $\omega \in H^2_{0,\omega}(I)$, there exists a positive constant $C_{1,2}$ such that

$$
\left| S(w(x_{i+\frac{1}{2}})) - S_h(I_h w(x_{i+\frac{1}{2}})) \right| \le C_{1,2} \int_{x_i}^{x_{i+1}} \Big(|S'(\omega(x))| + |\omega'(x)| + |\omega(x)| \Big) \mathrm{d}x \qquad i = 0, \cdots, N. \tag{3.4.2}
$$

Proof. We have

$$
\begin{aligned}
\left| S(w(x_{i+\frac{1}{2}})) - S_h(I_h w(x_{i+\frac{1}{2}})) \right| =& \Big| -k(x_{i+\frac{1}{2}})\omega'(x_{i+\frac{1}{2}}) - bx_{i+\frac{1}{2}}\omega(x_{i+\frac{1}{2}}) + \lambda_{i+\frac{1}{2}}\Big(\omega(x_{i+1}) - \omega(x_i)\Big) \\
&+ x_{i+\frac{1}{2}}\Big(\omega(x_i)b_p + \omega(x_{i+1})b_n\Big)\Big| \\
=& \Big| -k(x_{i+\frac{1}{2}})\Big[\omega'(x_{i+\frac{1}{2}}) - \frac{\omega(x_{i+1}) - \omega(x_i)}{h_i}\Big] \\
&+ x_{i+\frac{1}{2}}\Big[-b\omega(x_{i+\frac{1}{2}}) + \Big(\omega(x_i)b_p + \omega(x_{i+1})b_n\Big)\Big] \\
&+ \Big(\frac{k(x_{i+\frac{1}{2}})}{h_i} - \lambda_{i+\frac{1}{2}}\Big)\Big(\omega(x_{i+1}) - \omega(x_i)\Big)\Big|
\end{aligned}
$$

$$
\begin{aligned}
\left| S(w_{x_{i+\frac{1}{2}}}) - S_h(I_h w_{x_{i+\frac{1}{2}}}) \right| \le& \left| k(x_{i+\frac{1}{2}})\Big[\frac{\omega(x_{i+1}) - \omega(x_i)}{h_i} - \omega'(x_{i+\frac{1}{2}})\Big]\right| \\
&+ \left| x_{i+\frac{1}{2}}\Big[b\omega(x_{i+\frac{1}{2}}) - \Big(\omega(x_i)b_p + \omega(x_{i+1})b_n\Big)\Big]\right| \\
&+ \left| \Big(\frac{k(x_{i+\frac{1}{2}})}{h_i} - \lambda_{i+\frac{1}{2}}\Big)\Big(\omega(x_{i+1}) - \omega(x_i)\Big)\right|.
\end{aligned}
$$

Let us define

$$\begin{aligned} T_1 :=& k(x_{i+\frac{1}{2}})\Big[\frac{\omega(x_{i+1})-\omega(x_i)}{h_i} - \omega'(x_{i+\frac{1}{2}})\Big],\\ T_2 :=& x_{i+\frac{1}{2}}\Big[b\omega(x_{i+\frac{1}{2}}) - \Big(\omega(x_i)b_p + \omega(x_{i+1})b_n\Big)\Big],\\ T_3 :=& \Big(\frac{k(x_{i+\frac{1}{2}})}{h_i} - \lambda_{i+\frac{1}{2}}\Big)\Big(\omega(x_{i+1}) - \omega(x_i)\Big). \end{aligned}$$

Let us find the upper bound of $|T_1|$.
According to the Corollary 9.15 in Brezis (2010), $H^2(I) \hookrightarrow C^1(\overline{I})$. Then, using Taylor expansion with integral remainder, we have:

$$\begin{aligned} \omega(x_{i+1}) =& \omega(x_{i+1} - x_{i+\frac{1}{2}} + x_{i+\frac{1}{2}})\\ =& \omega(x_{i+\frac{1}{2}}) + (x_{i+1} - x_{i+\frac{1}{2}})\omega'(x_{i+\frac{1}{2}}) + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \omega''(x)(x_{i+1}-x)\mathrm{d}x. \end{aligned}$$

Then

$$\omega(x_{i+1}) = \omega(x_{i+\frac{1}{2}}) + \frac{h_i}{2}\omega'(x_{i+\frac{1}{2}}) + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \omega''(x)(x_{i+1}-x)\mathrm{d}x. \tag{3.4.3}$$

Using the same rule, we obtain:

$$\omega(x_i) = \omega(x_{i+\frac{1}{2}}) - \frac{h_i}{2}\omega'(x_{i+\frac{1}{2}}) + \int_{x_{i+\frac{1}{2}}}^{x_i} \omega''(x)(x_i-x)\mathrm{d}x. \tag{3.4.4}$$

Using Equations (3.4.3) and (3.4.4), we have:

$$\frac{\omega(x_{i+1})-\omega(x_i)}{h_i} - \omega'(x_{i+\frac{1}{2}}) = \frac{1}{h_i}\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \omega''(x)(x_{i+1}-x)\mathrm{d}x + \frac{1}{h_i}\int_{x_i}^{x_{i+\frac{1}{2}}} \omega''(x)(x_i-x)\mathrm{d}x. \tag{3.4.5}$$

Since $\left|\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \omega''(x)(x_{i+1}-x)\mathrm{d}x\right| \leq \frac{h_i}{2}\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} |\omega''(x)|\mathrm{d}x$,

and $\left|\int_{x_i}^{x_{i+\frac{1}{2}}} \omega''(x)(x_i-x)\mathrm{d}x\right| \leq \frac{h_i}{2}\int_{x_i}^{x_{i+\frac{1}{2}}} |\omega''(x)|\mathrm{d}x$,

then

$$\begin{aligned} \left|\frac{\omega(x_{i+1})-\omega(x_i)}{h_i} - \omega'(x_{i+\frac{1}{2}})\right| \leq& \frac{1}{2}\Big(\int_{x_i}^{x_{i+\frac{1}{2}}} |\omega''(x)|\mathrm{d}x + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} |\omega''(x)|\mathrm{d}x\Big)\\ \leq& \frac{1}{2}\int_{x_i}^{x_{i+1}} |\omega''(x)|\mathrm{d}x. \end{aligned}$$

Moreover, we have $k(x_{i+\frac{1}{2}}) = \frac{1}{2}\sigma^2 x_{i+\frac{1}{2}}^2$.

Then

$$|T_1| \leq \frac{1}{4}\sigma^2 x_{i+\frac{1}{2}}^2 \int_{x_i}^{x_{i+1}} |\omega''(x)| \mathrm{d}x.$$

Hence

$$|T_1| \leq \frac{1}{4}\sigma^2 \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+\frac{1}{2}}}{x}\right)^2 |x^2\omega''(x)| \mathrm{d}x, \tag{3.4.6}$$

since

$$x_i \leq x \leq x_{i+1} \Leftrightarrow \frac{x_{i+\frac{1}{2}}}{x_{i+1}} \leq \frac{x_{i+\frac{1}{2}}}{x} \leq \frac{x_{i+\frac{1}{2}}}{x_i}.$$

We also have:

$$\begin{aligned}\frac{x_{i+\frac{1}{2}}}{x_i} &= \frac{x_i + x_{i+1}}{2x_i}\\ &= 1 + \frac{h_i}{2x_i}\\ &= \frac{1}{2}\left(2 + \frac{h_i}{h_{i-1} + x_{i-1}}\right).\end{aligned}$$

Since

$$\frac{h_i}{h_{i-1} + x_{i-1}} \leq \frac{h_i}{h_{i-1}}, \tag{3.4.7}$$

and

$$\frac{h_{i+1}}{c} \leq h_i \leq ch_{i+1} \Rightarrow \frac{h_i}{c} \leq h_{i-1} \leq ch_i \Rightarrow \frac{1}{c} \leq \frac{h_i}{h_{i-1}} \leq c, \qquad c > 0, \tag{3.4.8}$$

we obtain:

$$\frac{x_{i+\frac{1}{2}}}{x_i} \leq \frac{1}{2}(2 + c). \tag{3.4.9}$$

Therefore, using Inequalities (3.4.6) and (3.4.9), we have:

$$|T_1| \leq \frac{\sigma^2}{4}\left(1 + \frac{c}{2}\right)^2 \int_{x_i}^{x_{i+1}} |x^2\omega''(x)| \mathrm{d}x. \tag{3.4.10}$$

Using the definition of S given by Equation (3.4.1) and the fact that S is differentiable, we obtain:

$$S'(\omega(x)) = -a(t)x^2\omega''(x) - \Big(2a(t) + b(t)\Big)x\omega'(x) - b(t)\omega(x). \tag{3.4.11}$$

It follows that

$$-a(t)x^2\omega''(x) = S'(\omega(x)) + \Big(2a(t) + b(t)\Big)x\omega'(x) + b(t)\omega(x). \tag{3.4.12}$$

Then, from Equation (3.4.12), we have:

$$\int_{x_i}^{x_{i+1}} |x^2\omega''(x)| \mathrm{d}x = \frac{2}{\sigma^2(t)} \int_{x_i}^{x_{i+1}} \left| S'(\omega(x)) + \Big(2a(t) + b(t)\Big)x\omega'(x) + b(t)\omega(x) \right| \mathrm{d}x \tag{3.4.13}$$

Since

$$\int_{x_i}^{x_{i+1}} \left|\Big(2a(t)+b(t)\Big)x\omega'(x)\right|\mathrm{d}x = r(t)\int_{x_i}^{x_{i+1}} |x\omega'(x)|\mathrm{d}x \\ \leq \overline{r}x_{\mathsf{max}}\int_{x_i}^{x_{i+1}} |\omega'(x)|\mathrm{d}x,$$

and

$$\int_{x_i}^{x_{i+1}} \Big|b(t)\omega(x)\Big|\mathrm{d}x = \Big|r(t)-\sigma^2(t)\Big|\int_{x_i}^{x_{i+1}} \Big|\omega(x)\Big|\mathrm{d}x \\ \leq (\overline{r}+\beta)\int_{x_i}^{x_{i+1}} \Big|\omega(x)\Big|\mathrm{d}x.$$

Therefore, using Inequality (3.4.10), we have:

$$|T_1| \leq C_1 \int_{x_i}^{x_{i+1}} \Big(|S'(\omega(x))| + |\omega'(x)| + |\omega(x)|\Big)\mathrm{d}x, \tag{3.4.14}$$

where $C_1 = \frac{1}{2}\Big(1+\frac{c}{2}\Big)^2 \max\Big(1, \overline{r}x_{\mathsf{max}}, \overline{r}+\beta\Big)$.

Let us estimate $|T_2|$.

1. Suppose that $b(t)<0$, then

$$T_2 = b(t)x_{i+\frac{1}{2}}\Big(\omega(x_{i+\frac{1}{2}}) - \omega(x_{i+1})\Big).$$

   By applying Taylor expansion with integral remainder,

$$\omega(x_{i+1}) = \omega(x_{i+\frac{1}{2}}) + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \omega'(x)\mathrm{d}x. \tag{3.4.15}$$

   So

$$\left|bx_{i+\frac{1}{2}}\Big(\omega(x_{i+\frac{1}{2}}) - \omega(x_{i+1})\Big)\right| = |b(t)|x_{i+\frac{1}{2}}\left|\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \omega'(x)\mathrm{d}x\right|. \tag{3.4.16}$$

   Since

$$x_{i+\frac{1}{2}} \leq x_{\mathsf{max}} \text{ and } \left|\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \omega'(x)\mathrm{d}x\right| \leq \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} |\omega'(x)|\mathrm{d}x, \tag{3.4.17}$$

   then

$$|T_2| \leq |b(t)|x_{\mathsf{max}}\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} |\omega'(x)|\mathrm{d}x. \tag{3.4.18}$$

2. Suppose that $b(t)>0$, then

$$T_2 = b(t)x_{i+\frac{1}{2}}\Big(\omega(x_{i+\frac{1}{2}}) - \omega(x_i)\Big).$$

By applying Taylor expansion with integral remainder,

$$\omega(x_{i+\frac{1}{2}}) = \omega(x_i) + \int_{x_i}^{x_{i+\frac{1}{2}}} \omega'(x)\mathrm{d}x. \tag{3.4.19}$$

So

$$\left|b(t)x_{i+\frac{1}{2}}\Big(\omega(x_{i+\frac{1}{2}}) - \omega(x_i)\Big)\right| = |b(t)|x_{i+\frac{1}{2}}\left|\int_{x_i}^{x_{i+\frac{1}{2}}} \omega'(x)\mathrm{d}x\right|. \tag{3.4.20}$$

Since

$$x_{i+\frac{1}{2}} \leq x_{\text{max}} \;\; \text{and} \;\; \left|\int_{x_i}^{x_{i+\frac{1}{2}}} \omega'(x)\mathrm{d}x\right| \leq \int_{x_i}^{x_{i+\frac{1}{2}}} |\omega'(x)|\mathrm{d}x, \tag{3.4.21}$$

then

$$|T_2| \leq |b(t)|x_{\text{max}} \int_{x_i}^{x_{i+\frac{1}{2}}} |\omega'(x)|\mathrm{d}x. \tag{3.4.22}$$

Using Inequalities (3.4.18) and (3.4.22), we have:

$$|T_2| \leq \frac{1}{2}\Big(r(t) + \sigma^2(t)\Big)x_{\text{max}} \int_{x_i}^{x_{i+1}} |\omega'(x)|\mathrm{d}x. \tag{3.4.23}$$

therefore

$$|T_2| \leq C_2 \int_{x_i}^{x_{i+1}} |\omega'(x)|\mathrm{d}x, \tag{3.4.24}$$

where $C_2 = \frac{1}{2}(\overline{r} + \beta)x_{\text{max}}$.

Finally, let us estimate T_3.
By applying Taylor expansion with integral remainder, we obtain:

$$\omega(x_{i+1}) = \omega(x_i) + \int_{x_i}^{x_{i+1}} \omega'(x)\mathrm{d}x. \tag{3.4.25}$$

It follows that

$$|\omega(x_{i+1}) - \omega(x_i)| \leq \int_{x_i}^{x_{i+1}} |\omega'(x)|\mathrm{d}x. \tag{3.4.26}$$

Now, Let us check the upper bound of $\left|\frac{k(x_{i+\frac{1}{2}})}{h_i} - \lambda_{i+\frac{1}{2}}\right|$.

We have:

$$\left|\frac{k(x_{i+\frac{1}{2}})}{h_i} - \lambda_{i+\frac{1}{2}}\right| \leq \left|\frac{k(x_{i+\frac{1}{2}})}{h_i}\right| + \left|\lambda_{i+\frac{1}{2}}\right|. \tag{3.4.27}$$

Besides, we have:

$$\begin{aligned}\left|\frac{k(x_{i+\frac{1}{2}})}{h_i}\right| &= \frac{\sigma^2(t)x^2_{i+\frac{1}{2}}}{2h_i} \\ &= \frac{\sigma^2(t)}{8}x_{i+1}\frac{\left(1 + 2\frac{x_i}{x_{i+1}} + (\frac{x_i}{x_{i+1}})^2\right)}{\left(1 - \frac{x_i}{x_{i+1}}\right)}.\end{aligned}$$

Setting $\xi_i = \frac{x_i}{x_{i+1}}$, with $0 \leq \xi_i < 1$, we obtain:

$$\begin{aligned}\left|\frac{k(x_{i+\frac{1}{2}})}{h_i}\right| &= \frac{\sigma^2(t)}{8} x_{i+1} \frac{1+2\xi_i+\xi_i^2}{1-\xi_i} \\ &\leq \frac{\sigma^2(t)}{8} x_{\text{max}} \frac{1+2\xi_i+\xi_i^2}{1-\xi_i}. \end{aligned} \tag{3.4.28}$$

Also, using Taylor expansion, we have:

$$\frac{1}{1-\xi_i} - (1+\xi_i) = \mathcal{O}(\xi_i^2),$$

Then there exists $k > 0$ such that

$$\left|\frac{1}{1-\xi_i} - (1+\xi_i)\right| \leq k\xi_i^2. \tag{3.4.29}$$

Since $\frac{1}{1-\xi_i} - (1+\xi_i) > 0$, thus

$$\frac{1}{1-\xi_i} - (1+\xi_i) \leq k\xi_i^2. \tag{3.4.30}$$

Then, we have:

$$\frac{1}{1-\xi_i} \leq 1+\xi_i+k\xi_i^2. \tag{3.4.31}$$

Hence

$$\frac{1}{1-\xi_i} \leq 2+k. \tag{3.4.32}$$

Also, using $0 \leq \xi_i < 1$, we have

$$0 < 1+2\xi_i+\xi_i^2 < 4. \tag{3.4.33}$$

Then, using Inequalities (3.4.28), (3.4.32) and (3.4.33), we obtain:

$$\left|\frac{k(x_{i+\frac{1}{2}})}{h_i}\right| \leq C_3, \tag{3.4.34}$$

where $C_3 = \frac{\beta}{2}(2+k)x_{\text{max}}$.

Moreover, we have

$$\begin{aligned}\left|\lambda_{i+\frac{1}{2}}\right| &= \frac{2\eta_i\eta_{i+1}}{\eta_i+\eta_{i+1}} \\ &= \frac{\sigma^2(t)}{3} \frac{\left(x^3_{i+\frac{1}{2}} - x^3_{i-\frac{1}{2}}\right)\left(x^3_{i+\frac{3}{2}} - x^3_{i+\frac{1}{2}}\right)}{l^2_{i+1}\left(x^3_{i+\frac{1}{2}} - x^3_{i-\frac{1}{2}}\right) + l_i^2\left(x^3_{i+\frac{3}{2}} - x^3_{i+\frac{1}{2}}\right)} \\ &\leq \frac{\sigma^2(t)}{3} \times \frac{x_{i+\frac{1}{2}}\left(1+\frac{x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}}} + \left(\frac{x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}}}\right)^2\right)}{1-\frac{x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}}}}. \end{aligned}$$

By setting $a_i = \frac{x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}}}$, where $0 \leq a_i < 1$, we have:

$$\left|\lambda_{i+\frac{1}{2}}\right| \leq \frac{\sigma^2(t)}{3} \frac{x_{i+\frac{1}{2}}\left(1 + a_i + a_i^2\right)}{1 - a_i}. \tag{3.4.35}$$

It follows that

$$\left|\lambda_{i+\frac{1}{2}}\right| \leq \frac{\sigma^2(t)}{3} \frac{x_{\max}\left(1 + a_i + a_i^2\right)}{1 - a_i}. \tag{3.4.36}$$

Using Taylor expansion, we obtain:

$$\frac{1}{1 - a_i} = 1 + a_i + \mathcal{O}(a_i^2).$$

Then, using the same rules in Inequalities (3.4.31), (3.4.32) and (3.4.33), we have:

$$\left|\lambda_{i+\frac{1}{2}}\right| \leq C_4, \tag{3.4.37}$$

where $C_4 = \beta x_{\max}(2 + k_1), \quad k_1 > 0.$

Then, using Inequalities (3.4.27),(3.4.34) and (3.4.37), we obtain:

$$\left|\frac{k(x_{i+\frac{1}{2}})}{h_i} - \lambda_{i+\frac{1}{2}}\right| \leq C_5, \tag{3.4.38}$$

where $C_5 = \beta x_{\max}\left(3 + \frac{1}{2}k + k_1\right)$.

Then, using equations (3.4.26) and (3.4.38), we have:

$$|T_3| \leq C_5 \int_{x_i}^{x_{i+1}} |\omega'(x)| \mathrm{d}x. \tag{3.4.39}$$

Therefore, according to (3.4.14), (3.4.24) and (3.4.39), we get:

$$\left|S(w(x_{i+\frac{1}{2}})) - S_h(I_h w(x_{i+\frac{1}{2}}))\right| \leq C_{1,2} \int_{x_i}^{x_{i+1}} \left(|S'(\omega(x))| + |\omega'(x)| + |\omega(x)|\right) \mathrm{d}x, \tag{3.4.40}$$

where $C_{1,2} = C_1 + C_2 + C_5 > 0$.

$\square$

4. Error estimate

In this chapter, we analyze the error which makes the TPFA method in the approximation of the solution by looking at the difference between the solution of the continuous problem (2.5.3) and that of the discrete problem (3.1.15).

4.0.1 Theorem. *Let $\theta \in [1/2, 1]$. Then, if u of Equation (2.5.2) is such that $u \in H^1(0, T, H^1(I)) \cap H^2(0, T, H^1(I))$ and $S(u) \in C(0, T, H^1(I))$, then*

$$\|u - u_h\|_{w,d} \leq C_{a,6} h \bigg(|S'(u)|_1 + |u'|_1 + |u|_1 \bigg), \tag{4.0.1}$$

where $C_{a,6}$ is a positive constant, independent of h, u and N.

Proof. Let $u \in H^1(I)$. According to the Theorem 3.25 in Knaber and Angermann (2003), there exists a constant $\gamma_0 > 0$ such that:

$$|u - I_h u|_0 = \|u - I_h u\|_{0,h} \leq \gamma_0 h |u|_1, \tag{4.0.2}$$

where $|\cdot|_1$ is the semi-norm of $H^1(I)$, and $|\cdot|_0$ is the semi-norm of $H^1_\omega(I)$.
Since $l_i > 0$, then $|\cdot|_0$ is a norm of $H^1_\omega(I)$.
Also, since $H^1_\omega(I)$ and V_h are finite dimensional, then they are both isomorph to $\mathbb{R}^N$.
Therefore the norm of $H^1_\omega(I)$ and the norm of V_h are equivalents.
Then there exits $\gamma_2 > 0$ such that

$$\gamma_2 \|u - I_h u\|_{\omega,d} \leq |u - I_h u|_0. \tag{4.0.3}$$

From Inequalities (4.0.2) and (4.0.3), we obtain:

$$\|u - I_h u\|_{\omega,d} \leq \gamma_3 h |u|_1, \tag{4.0.4}$$

where $\gamma_3 = \frac{\gamma_0}{\gamma_2}$.

Now, let us estimate $\|I_h u - u_h\|_{w,d}$. We have:

$$\begin{aligned}
A_h(I_h u - u_h, v_h) =& A_h(I_h u, v_h) - A_h(u_h, v_h) \\
=& \sum_{i=1}^{N-1} \Big[S_h(I_h u_{i+\frac{1}{2}}) - S_h(I_h u_{i-\frac{1}{2}}) \Big] v_i + (c I_h u, v_h)_h - \sum_{i=1}^{N-1} \Big[S_h(u_{i+\frac{1}{2}}) - S_h(u_{i-\frac{1}{2}}) \Big] v_i \\
& - (cu, L_h v_h) \\
=& \sum_{i=1}^{N-1} \Big[S_h(I_h u_{i+\frac{1}{2}}) - S_h(u_{i+\frac{1}{2}}) \Big] v_i - \sum_{i=1}^{N-1} \Big[S_h(I_h u_{i-\frac{1}{2}}) - S_h(u_{i-\frac{1}{2}}) \Big] v_i \\
& + c \Big[(L_h I_h u, L_h v_h) - (u, L_h v_h) \Big] \\
=& \sum_{i=1}^{N-1} \Big[S_h(I_h u_{i+\frac{1}{2}}) - S_h(u_{i+\frac{1}{2}}) \Big] v_i - \sum_{j=0}^{N-2} \Big[S_h(I_h u_{j+\frac{1}{2}}) - S_h(u_{j+\frac{1}{2}}) \Big] v_{j+1} \\
& + c \Big[(L_h I_h u, L_h v_h) - (u, L_h v_h) \Big]
\end{aligned}$$

$$\begin{aligned}
A_h(I_hu-u_h,v_h) = &\sum_{i=1}^{N-2}\Big[S_h(I_hu_{i+\frac{1}{2}})-S_h(u_{i+\frac{1}{2}})\Big]v_i+\Big(S_h(I_hu_{N-\frac{1}{2}})-S_h(u_{N-\frac{1}{2}})\Big)v_{N-1}\\
&-\Big(S_h(I_hu_{\frac{1}{2}})-S_h(u_{\frac{1}{2}})\Big)v_1-\sum_{j=1}^{N-2}\Big[S_h(I_hu_{j+\frac{1}{2}})-S_h(u_{j+\frac{1}{2}})\Big]v_{j+1}\\
&+c\Big[(L_hI_hu,L_hv_h)-(u,L_hv_h)\Big]\\
=&\sum_{i=1}^{N-2}\Big[S_h(I_hu_{i+\frac{1}{2}})-S_h(u_{i+\frac{1}{2}})\Big]v_i+\Big(S_h(I_hu_{N-\frac{1}{2}})-S_h(u_{N-\frac{1}{2}})\Big)(v_{N-1}-v_N)\\
&+\Big(S_h(I_hu_{\frac{1}{2}})-S_h(u_{\frac{1}{2}})\Big)(v_0-v_1)-\sum_{j=1}^{N-2}\Big[S_h(I_hu_{j+\frac{1}{2}})-S_h(u_{j+\frac{1}{2}})\Big]v_{j+1}\\
&+c\Big[(L_hI_hu,L_hv_h)-(u,L_hv_h)\Big]\\
=&\sum_{i=0}^{N-1}\Big[S_h(I_hu_{i+\frac{1}{2}})-S_h(u_{i+\frac{1}{2}})\Big](v_i-v_{i+1})+c\Big[(L_hu,L_hv_h)-(u,L_hv_h)\Big].
\end{aligned}$$

Since

$$L_hI_hu=L_h(I_hu)=L_hI_hu\big|_{K_j}=I_hu(x_j)=\sum_{i=1}^{N-1}u(x_i)\phi_{x_i}(x_j)=u(x_j)=L_hu.$$

Then

$$A_h(I_hu-u_h,v_h)=\sum_{i=0}^{N-1}\Big[S_h(I_hu_{i+\frac{1}{2}})-S_h(u_{i+\frac{1}{2}})\Big](v_i-v_{i+1})+c\Big[\Big((L_h-I)u,L_hv_h\Big)\Big]. \quad (4.0.5)$$

Let us estimate $P_1=\sum_{i=0}^{N-1}\Big[S_h(I_hu_{i+\frac{1}{2}})-S_h(u_{i+\frac{1}{2}})\Big](v_i-v_{i+1})$, and $P_2=c\Big((L_h-I)u,L_hv_h\Big)$.

We have:

$$\begin{aligned}
\left|\Big[S_h(I_hu_{\frac{1}{2}})-S_h(u_{\frac{1}{2}})\Big](v_0-v_1)\right| &=\left|\frac{1}{\lambda_{\frac{1}{2}}}\Big[S_h(I_hu_{\frac{1}{2}})-S_h(u_{\frac{1}{2}})\Big]\lambda_{\frac{1}{2}}(v_0-v_1)\right|\\
&=\frac{1}{\lambda_{\frac{1}{2}}}\left|S_h(I_hu_{\frac{1}{2}})-S_h(u_{\frac{1}{2}})\right|\lambda_{\frac{1}{2}}|v_1|.
\end{aligned}$$

Let us estimate $\frac{1}{\lambda_{i+\frac{1}{2}}}$.

$$\begin{aligned}
\left|\frac{1}{\lambda_{i+\frac{1}{2}}}\right| &=\frac{l_{i+1}k_i+l_ik_{i+1}}{2k_ik_{i+1}}\\
&=\frac{3}{\sigma^2(t)}\frac{l_{i+1}^2(x_{i+\frac{1}{2}}^3-x_{i-\frac{1}{2}}^3)+l_i^2(x_{i+\frac{3}{2}}^3-x_{i+\frac{1}{2}}^3)}{(x_{i+\frac{1}{2}}^3-x_{i-\frac{1}{2}}^3)(x_{i+\frac{3}{2}}^3-x_{i+\frac{1}{2}}^3)}
\end{aligned}$$

$$\left|\frac{1}{\lambda_{i+\frac{1}{2}}}\right| = \frac{3}{\sigma^2(t)}\left[\frac{l_{i+1}^2}{x_{i+\frac{3}{2}}^3 - x_{i+\frac{1}{2}}^3} + \frac{l_i^2}{x_{i+\frac{1}{2}}^3 - x_{i-\frac{1}{2}}^3}\right]$$
$$= \frac{3}{\sigma^2(t)}\left[\frac{l_{i+1}^2}{x_{i+\frac{3}{2}}^3}\frac{1}{1-\left(\frac{x_{i+\frac{1}{2}}}{x_{i+\frac{3}{2}}}\right)^3} + \frac{l_i^2}{x_{i+\frac{1}{2}}^3}\frac{1}{1-\left(\frac{x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}}}\right)^3}\right]$$
$$= \frac{3}{\sigma^2(t)}\left[\frac{1}{x_{i+\frac{3}{2}}^2}\frac{l_{i+1}^2}{x_{i+\frac{3}{2}}}\frac{1}{1-\left(\frac{x_{i+\frac{1}{2}}}{x_{i+\frac{3}{2}}}\right)^3} + \frac{1}{x_{i+\frac{1}{2}}^2}\frac{l_i^2}{x_{i+\frac{1}{2}}}\frac{1}{1-\left(\frac{x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}}}\right)^3}\right].$$

Since

$$x_{i+\frac{1}{2}} < x_{i+\frac{3}{2}},$$

then

$$\frac{1}{x_{i+\frac{3}{2}}^2} < \frac{1}{x_{i+\frac{1}{2}}^2}.$$

Therefore

$$\left|\frac{1}{\lambda_{i+\frac{1}{2}}}\right| \leq \frac{3}{\sigma^2(t)}\left[\frac{1}{x_{i+\frac{1}{2}}^2}\frac{l_{i+1}^2}{x_{i+\frac{3}{2}}}\frac{1}{1-\left(\frac{x_{i+\frac{1}{2}}}{x_{i+\frac{3}{2}}}\right)^3} + \frac{1}{x_{i+\frac{1}{2}}^2}\frac{l_i^2}{x_{i+\frac{1}{2}}}\frac{1}{1-\left(\frac{x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}}}\right)^3}\right]$$
$$\leq \frac{3}{\sigma^2(t)x_{\mathsf{max}}^2}\frac{1}{\left(\frac{x_{i+\frac{1}{2}}}{x_{\mathsf{max}}}\right)^2}\left[\frac{l_{i+1}^2}{x_{i+\frac{3}{2}}}\frac{1}{1-\left(\frac{x_{i+\frac{1}{2}}}{x_{i+\frac{3}{2}}}\right)^3} + \frac{l_i^2}{x_{i+\frac{1}{2}}}\frac{1}{1-\left(\frac{x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}}}\right)^3}\right].$$

Since

$$l_i \leq x_{i+\frac{1}{2}} \text{ and } l_{i+1} \leq x_{i+\frac{3}{2}},$$

then

$$\frac{1}{x_{i+\frac{1}{2}}} \leq \frac{1}{l_i} \text{ and } \frac{1}{x_{i+\frac{3}{2}}} \leq \frac{1}{l_{i+1}}.$$

So

$$\frac{l_i^2}{x_{i+\frac{1}{2}}} \leq l_i \text{ and } \frac{l_{i+1}^2}{x_{i+\frac{3}{2}}} \leq l_{i+1}.$$

Then, we have:

$$\frac{1}{\lambda_{i+\frac{1}{2}}} \leq \frac{3}{\sigma^2(t)x_{\mathsf{max}}^2}\frac{1}{\left(\frac{x_{i+\frac{1}{2}}}{x_{\mathsf{max}}}\right)^2}\left[l_{i+1}\frac{1}{1-\left(\frac{x_{i+\frac{1}{2}}}{x_{i+\frac{3}{2}}}\right)^3} + l_i\frac{1}{1-\left(\frac{x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}}}\right)^3}\right] \tag{4.0.6}$$

Setting

$$b_i = \frac{x_{i+\frac{1}{2}}}{x_{\mathsf{max}}}, \quad c_i = \frac{x_{i+\frac{1}{2}}}{x_{i+\frac{3}{2}}} \quad \text{and} \quad d_i = \frac{x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}}}, \tag{4.0.7}$$

with
$$0 < b_i < 1, \quad 0 < c_i < 1 \quad \text{and} \quad 0 < d_i < 1 \quad \text{for } i = 0, \cdots, N, \tag{4.0.8}$$

Then, Inequality (4.0.6) becomes
$$\frac{1}{\lambda_{i+\frac{1}{2}}} \leq \frac{3}{\sigma^2(t)x^2_{\max}} \frac{1}{b_i^2}\left[\frac{c}{1-c_i^3} + \frac{1}{1-d_i^3}\right] l_i. \tag{4.0.9}$$

Using Taylor expansion, we have:
$$\begin{aligned} \frac{1}{b_i^2} =& \frac{1}{1-(1-b_i^2)} = 1 + 1 - b_i^2 + \mathcal{O}(b_i^4), \\ \frac{1}{1-c_i^3} =& 1 + c_i^3 + \mathcal{O}(c_i^6), \\ \text{and } \frac{1}{1-d_i^3} =& 1 + d_i^3 + \mathcal{O}(d_i^6). \end{aligned}$$

Then, there exists $k_b, k_c, k_d > 0$ such that
$$\begin{aligned} \frac{1}{b_i^2} \leq& 1 + 1 - b_i^2 + k_b b_i^4 \leq 2 + k_b, \\ \frac{1}{1-c_i^3} \leq& 1 + c_i^3 + k_c c_i^6 \leq 2 + k_c, \\ \text{and } \frac{1}{1-d_i^3} \leq& 1 + d_i^3 + k_d d_i^6 \leq 2 + k_d. \end{aligned}$$

Hence,
$$\frac{1}{\lambda_{i+\frac{1}{2}}} \leq \frac{3}{\sigma^2 x^2_{\max}}(2+k_b)\Big(2(c+1) + ck_c + k_d\Big) l_i \tag{4.0.10}$$

Since
$$l_i \leq \frac{1}{2}(1+c)h_i, \quad \text{for } i = 0, \cdots, N,$$

then
$$\frac{1}{\lambda_{i+\frac{1}{2}}} \leq C_6 h_i, \tag{4.0.11}$$

where $C_6 = \frac{3(1+c)}{2\sigma^2 x^2_{\max}}\Big(2 + k_b\Big)\Big(2(c+1) + ck_c + k_d\Big)$.
Therefore, by using Inequality (4.0.11), we have:
$$\begin{aligned} \left|\Big[S_h(I_h u_{\frac{1}{2}}) - S_h(u_{\frac{1}{2}})\Big](v_0 - v_1)\right| &\leq C_6 h_0 \Big|S_h(I_h u_{\frac{1}{2}}) - S_h(u_{\frac{1}{2}})\Big| C_4 |v_1| \\ &\leq C_a h_0 \left[\int_{x_0}^{x_1} \Big(|S'(u(x))| + |u'(x)| + |u(x)|\Big) \mathrm{d}x\right] |v_1|, \end{aligned}$$

where $C_a = C_4 C_6 C_{1,2} > 0$.
Then, using Cauchy-Schwartz inequality, we have:

$$\left|\Big[S_h(I_h u_{\frac{1}{2}}) - S_h(u_{\frac{1}{2}})\Big](v_0 - v_1)\right| \le C_a h_0 \left[\int_{x_0}^{x_1} \mathsf{d}x\right]^{\frac{1}{2}} \left[\int_{x_0}^{x_1} \Big(|S'(u(x))| + |u'(x)| + |u(x)|\Big)^2 \mathsf{d}x\right]^{\frac{1}{2}} |v_1|,$$
$$\le C_a h_0 h_0^{\frac{1}{2}} \int_{x_0}^{x_1} \Big(|S'(u(x))| + |u'(x)| + |u(x)|\Big)^2 \mathsf{d}x\Big]^{\frac{1}{2}} |v_1|.$$

Therefore

$$\left|\Big[S_h(I_h u_{\frac{1}{2}}) - S_h(u_{\frac{1}{2}})\Big](v_0 - v_1)\right| \le \ C_a h_0 \int_{x_0}^{x_1} \Big(|S'(u(x))| + |u'(x)| + |u(x)|\Big)^2 \mathsf{d}x\Big]^{\frac{1}{2}} \sqrt{h_0 v_1^2}. \quad (4.0.12)$$

Also, we have:

$$\left|\sum_{i=1}^{N-1} \Big[S_h(I_h u_{i+\frac{1}{2}}) - S_h(u_{i+\frac{1}{2}})\Big](v_i - v_{i+1})\right| = \Bigg|\sum_{i=1}^{N-1} \frac{1}{\sqrt{\lambda_{i+\frac{1}{2}}}} \Big[S_h(I_h u_{i+\frac{1}{2}}) - S_h(u_{i+\frac{1}{2}})\Big]$$
$$\sqrt{\lambda_{i+\frac{1}{2}}}(v_i - v_{i+1})\Bigg|$$
$$\le C_{1,2}\sqrt{C_4}$$
$$\sum_{i=1}^{N-1} \sqrt{h_i}\left[\int_{x_i}^{x_{i+1}} \Big(|S'(u(x))| + |u'(x)| + |u(x)|\Big)\mathsf{d}x\right]$$
$$|v_i - v_{i+1}|\sqrt{\lambda_{i+\frac{1}{2}}}$$
$$\le C_{1,2}\sqrt{C_4}$$
$$\sum_{i=1}^{N-1} h_i\left[\int_{x_i}^{x_{i+1}} \Big(|S'(u(x))| + |u'(x)| + |u(x)|\Big)^2\mathsf{d}x\right]^{\frac{1}{2}}$$
$$|v_i - v_{i+1}|\sqrt{\lambda_{i+\frac{1}{2}}}$$

$$\left|\sum_{i=1}^{N-1}\left[S_h(I_h u_{i+\frac{1}{2}})-S_h(u_{i+\frac{1}{2}})\right](v_i-v_{i+1})\right| \leq hC_{1,2}\sqrt{C_4}$$
$$\sum_{i=1}^{N-1}\left[\int_{x_i}^{x_{i+1}}\Big(|S'(u(x))|+|u'(x)|+|u(x)|\Big)^2 \mathrm{d}x\right]^{\frac{1}{2}}$$
$$|v_i-v_{i+1}|\sqrt{\lambda_{i+\frac{1}{2}}}$$
$$\leq hC_{1,2}\sqrt{C_4}$$
$$\left[\sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}}\Big(|S'(u(x))|+|u'(x)|+|u(x)|\Big)^2 \mathrm{d}x\right]^{\frac{1}{2}}$$
$$\left[\sum_{i=1}^{N-1}|v_i-v_{i+1}|^2\lambda_{i+\frac{1}{2}}\right]^{\frac{1}{2}}.$$

It follows that

$$\left|\sum_{i=1}^{N-1}\left[S_h(I_h u_{i+\frac{1}{2}})-S_h(u_{i+\frac{1}{2}})\right](v_i-v_{i+1})\right| \leq C_b h\left[\int_{x_1}^{x_N}\Big(|S'(u(x))|+|u'(x)|+|u(x)|\Big)^2 \mathrm{d}x\right]^{\frac{1}{2}}\|v_h\|_{0,w}. \tag{4.0.13}$$

Where $C_b = C_{1,2}\sqrt{C_4}$.
Then, using Equations (4.0.12) and (4.0.13) we get:

$$|P_1| \leq C_{ab}\left[h_0\left(\int_{x_0}^{x_1}\Big(|S'(u(x))|+|u'(x)|+|u(x)|\Big)^2 \mathrm{d}x\right)^{\frac{1}{2}}\sqrt{h_0 v_1^2} + \left(\int_{x_1}^{x_N}\Big(|S'(u(x))|+|u'(x)|+|u(x)|\Big)^2 \mathrm{d}x\right)^{\frac{1}{2}} h\|v_h\|_{0,w}\right],$$

where $C_{ab} = \max(C_a, C_b)$.
Since,

$$h_0\int_{x_0}^{x_1}\Big(|S'(u(x))|+|u'(x)|+|u(x)|\Big)^2 \mathrm{d}x\Bigg]^{\frac{1}{2}}\sqrt{h_0 v_1^2} \leq h\sqrt{c}\int_{x_0}^{x_1}\Big(|S'(u(x))|+|u'(x)|+|u(x)|\Big)^2 \mathrm{d}x\Bigg]^{\frac{1}{2}}\sqrt{h_0 v_1^2}, \tag{4.0.14}$$

Using Equation (4.0.14), we obtain

$$|P_1| \leq C_{ab}C_{ab,1}h\left[\left(\int_{x_0}^{x_1}\Big(|S'(u(x))|+|u'(x)|+|u(x)|\Big)^2 \mathrm{d}x\right)^{\frac{1}{2}}\sqrt{l_1 v_1^2} + \left(\int_{x_1}^{x_N}\Big(|S'(u(x))|+|u'(x)|+|u(x)|\Big)^2 \mathrm{d}x\right)^{\frac{1}{2}}\|v_h\|_{0,w}\right],$$

where $C_{ab,1} = \max(\sqrt{c}, 1) > 0$. Then

$$|P_1| \leq C_{a,1} \cdot h\left(\int_{x_0}^{x_N} \left(|S'(u(x))| + |u'(x)| + |u(x)|\right)^2 \mathsf{d}x\right)^{\frac{1}{2}} \left(\sqrt{l_1 v_1^2} + \|v_h\|_{0,w}\right), \tag{4.0.15}$$

where $C_{a,1} = C_{ab}C_{ab,1} > 0$.
So,

$$|P_1| \leq C_{a,2}h\left(\int_{x_0}^{x_N} \left(|S'(u(x))| + |u'(x)| + |u(x)|\right)^2 \mathsf{d}x\right)^{\frac{1}{2}} \left(l_1 v_1^2 + \|v_h\|_{0,\omega}^2\right)^{\frac{1}{2}}, \tag{4.0.16}$$

where $C_{a,2} = C_{a,1}\gamma_1, \quad \gamma_1 > 1.$
Then,

$$|P_1| \leq C_{a,2}\left(\int_{\Omega} \left(|S'(u(x))| + |u'(x)| + |u(x)|\right)^2 \mathsf{d}x\right)^{\frac{1}{2}} h\|v_h\|_{\omega,d}. \tag{4.0.17}$$

So, by using Minkowski inequality, we obtain:

$$|P_1| \leq C_{a,2}\left[\left(\int_{\Omega} |S'(u(x))|^2 \mathsf{d}x\right)^{\frac{1}{2}} + \left(\int_{\Omega} |u'(x)|^2 \mathsf{d}x\right)^{\frac{1}{2}} + \left(\int_{\Omega} |u(x)|^2 \mathsf{d}x\right)^{\frac{1}{2}} \mathsf{d}x\right] h\|v_h\|_{\omega,d}.$$

therefore,

$$|P_1| \leq C_{a,2}\left(|S'(u)|_1 + |u'|_1 + |u|_1\right) h\|v_h\|_{\omega,d}. \tag{4.0.18}$$

Similarly, there exists $C_{a,3} > 0$ such that

$$|P_2| = \left|c\left((L_h - I)u, L_h v_h\right)\right| \leq C_{a,3} h|u|_1\|v_h\|_{0,h}. \tag{4.0.19}$$

Then,

$$|P_2| \leq C_{a,3} h|u|_1\|v_h\|_{\omega,d}. \tag{4.0.20}$$

Therefore,

$$\left|A_h(I_h u - u_h, v_h)\right| \leq C_{a,2}\left(|S'(u)|_1 + |u'|_1 + |u|_1\right) h\|v_h\|_{\omega,d} + C_{a,3} h|u|_1\|v_h\|_{\omega,d}. \tag{4.0.21}$$

Then

$$\left|A_h(I_h u - u_h, v_h)\right| \leq C_{a,4}\left(|S'(u)|_1 + |u'|_1 + |u|_1\right) h\|v_h\|_{\omega,d}, \tag{4.0.22}$$

where $C_{a,4} = \max\left(C_{a,2}, C_{a,3} + 1\right)$.
Since A_h is coercive, then

$$\gamma\|I_h u - u_h\|_{\omega,d}^2 \leq \left|A_h(I_h u - u_h, I_h u - u_h)\right| \leq C_{a,4}\left(|S'(u)|_1 + |u'|_1 + |u|_1\right) h\|I_h u - u_h\|_{\omega,d}. \tag{4.0.23}$$

Since $I_h u \neq u_h$, then, from equation (4.0.23) we have:

$$\|I_h u - u_h\|_{\omega,d} \leq \frac{C_{a,4}}{\gamma}\Bigg(|S'(u)|_1 + |u'|_1 + |u|_1\Bigg)h. \tag{4.0.24}$$

So, using (4.0.24), we have:

$$\|I_h u - u_h\|_{\omega,d} \leq C_{a,5} h\Bigg(|S'(u)|_1 + |u'|_1 + |u|_1\Bigg) \tag{4.0.25}$$

where $C_{a,5} = \frac{C_{a,4}}{\gamma}, \quad$ with $\gamma > 0$.
Since

$$\|u - u_h\|_{\omega,d} \leq \|u - I_h u\|_{\omega,d} + \|I_h u - u_h\|_{\omega,d}, \tag{4.0.26}$$

then, from equations (4.0.4) and (4.0.25), we obtain:

$$\|u - u_h\|_{\omega,d} \leq \gamma_3 h |u|_1 + C_{a,5} h\Bigg(|S'(u)|_1 + |u'|_1 + |u|_1\Bigg). \tag{4.0.27}$$

Therefore

$$\|u - u_h\|_{\omega,d} \leq C_{a,6} \cdot h\Bigg(|S'(u)|_1 + |u'|_1 + |u|_1\Bigg), \tag{4.0.28}$$

where $C_{a,6} = C_{a,5} + \gamma_3 > 0$.

□

4.0.2 Remark. From Theorem 4.0.1, we deduce that the spatial discretization has $\mathcal{O}(h)$ as order error bound.

5. Numerical simulation of the TPFA method

In this chapter, we aim to confirm the theoretical results. To do this, we first use the Euler method to approximate the partial derivative with respect to time. Afterward, we analyze numerically the errors by using L^2 error formula, increasing the space step, and fixing the time step. We finalize this chapter, by giving the order of convergence in space of the TPFA.

Subdivise $[0, T)$ into subintervals.
We have:

$$0 = t_0 < t_1 < \cdots < t_{M-1} = T,$$

with $0 \leq m \leq M-1, M \in \mathbb{N}, \Delta\tau = \max_{0 \leq m \leq M-1} \Delta t_m$ and $\Delta t_m = t_m - t_{m-1}$.
Let us set $\tau := T - t$. By applying the TPFA to Equation (2.4.9), it will behave like (3.1.14). We get then:

$$L\frac{\mathrm{d}V}{\mathrm{d}\tau} = AV + G, \tag{5.0.1}$$

where $L = \begin{pmatrix} l_1 & 0 & \dots & \dots & 0 \\ 0 & l_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & l_{n-1} \end{pmatrix}$, $A = \begin{pmatrix} \alpha_1 & e_1 & 0 & \dots & 0 \\ a_1 & \alpha_2 & e_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & e_{n-1} \\ 0 & \dots & \dots & a_{n-1} & \alpha_{n-1} \end{pmatrix}$,

and $G = \begin{pmatrix} \Big(\lambda_{1/2} + x_{1/2}b_p\Big)V(x_0,\tau) \\ 0 \\ \vdots \\ 0 \\ \Big(\lambda_{N-1/2} + x_{N+1/2}\Big)V(x_N,\tau) \end{pmatrix}$, $V = \begin{pmatrix} V(x_1,\tau) \\ V(x_2,\tau) \\ \vdots \\ V(x_{N-2},\tau) \\ V(x_{N-1},\tau) \end{pmatrix}$.

with $\alpha_i = \lambda_{i+\frac{1}{2}} - x_{i+\frac{1}{2}}b_p + \lambda_{i-\frac{1}{2}} + x_{i-\frac{1}{2}}b_n + l_i c,\ a_i = -\lambda_{i+\frac{1}{2}} + x_{i+\frac{1}{2}}b_p$ and $e_i = -\lambda_{i+\frac{1}{2}} - x_{i+\frac{1}{2}}b_n$.

Then, we have:

$$\frac{\mathrm{d}V}{\mathrm{d}\tau} = BV + H, \tag{5.0.2}$$

where $B = L^{-1}A,\ H = L^{-1}G$.
By applying Euler method on Equation (5.0.2), we have:

$$\frac{V^{m+1} - V^m}{\Delta\tau} = \theta(BV^{m+1} + H^{m+1}) + (1-\theta)(BV^m + H^m). \tag{5.0.3}$$

From Equation (5.0.3), we deduce:

$$V^{m+1} = \Big(I_{N-1} - \theta\Delta\tau B\Big)^{-1}\Big[\Big(I_{N-1} + (1-\theta)\Delta\tau B\Big)V^m + \theta\Delta\tau H^{m+1} + \Delta\tau(1-\theta)H^m\Big], \tag{5.0.4}$$

where I_N is an identity matrix of size N.
We know from Wilmott et al. (1993) that when r and σ are constants, the analytical solution for the

European call is

$$C(x, \tau) = xN(d_1) - Ke^{-r\tau}N(d_2), \tag{5.0.5}$$

where $N(\cdot)$ is the cumulative distribution function for a standardised normal random variable given by,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{x} e^{-\frac{1}{2}y^2} \mathrm{d}y, \tag{5.0.6}$$

where

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \tag{5.0.7}$$

and

$$d_1 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \tag{5.0.8}$$

With the time τ.

Using the domain $I = (0, x_{\text{max}}) \times (0, T)$, with $x_{\text{max}} = 300$, T the maturity time, the strike price $K = 300$, the risk free interest rate $r = 0.1$, the volatility $\sigma = 0.7$ and $\theta = 0.5$, we obtain the following graphs:

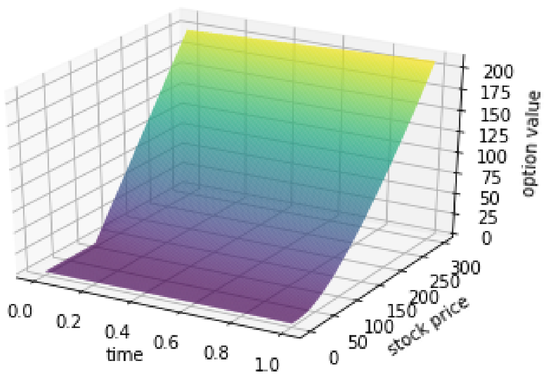

Figure 5.1: Analytical solution

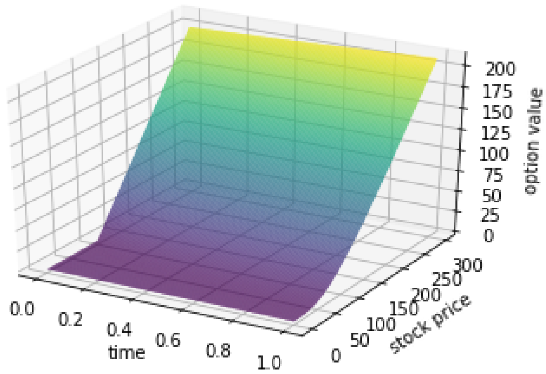

Figure 5.2: Numerical solution

Also using L^2-errors formula we get the following table of space errors by choosing $\Delta\tau = \frac{1}{100}$:

Number of grid points	Errors
50	0.15990361532342243
100	0.13738831721189304
150	0.11558600582767335
200	0.09011863193895389
250	0.061529826335805934
300	0.0328905797134449
350	0.02373368993757551

Table 5.1: Table of errors space

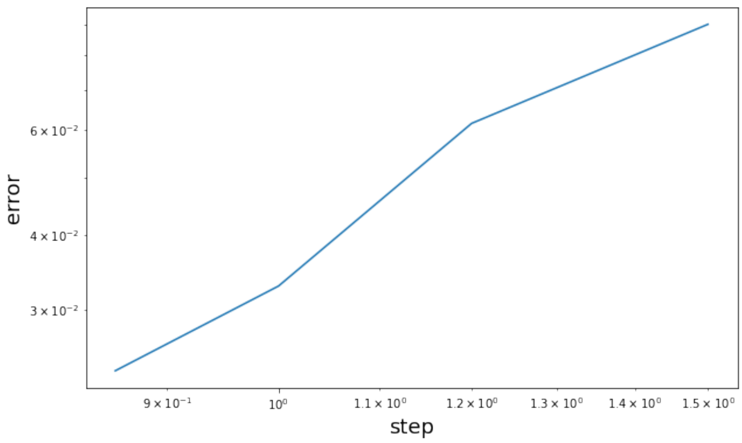

Figure 5.3: Log Log scale convergence plot

Figure 5.3 show us that the order of the convergence in space of the TPFA is $\mathcal{O}(h)$, which confirms the theoretical result on the error estimate.

6. Conclusion

In this essay, we presented and analyzed a novel Two Point Flux Approximation method for the Black-Scholes equation governing European option pricing. The method is based on the finite volume method coupled with the TPFA techique. It has been shown that the spatial discretization has $\mathcal{O}(h)$ as order error bound. Numerical results has been presented to confirm the theoretical rate of convergence.

In future work, we want to see the behavior of the option value when the stock price is not continuous on time so that we will consider the different perturbations in market prices due to some events. To do this, we need to extend this study by adding a non-local integration term on the Black-Scholes equation which will lead us to a jump diffusion model. Afterward, we will analyze the convergence in full discretization of the TPFA method for pricing options under a jump diffusion processes, by using the Rosenbrock methods for the time discretization.

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