# An Introduction to Beidleman Near Vector Spaces


Antsa Tantely Fandresena RAKOTONDRAFARA (antsa@aims.ac.za)
African Institute for Mathematical Sciences (AIMS)

Supervised by: Dr. Karin-Therese Howell
Stellenboch University, South Africa



14 May 2020

*Submitted in partial fulfillment of a structured masters degree at AIMS South Africa*

# Abstract

The concept of a near vector space was first introduced by Beidleman (1966). It is a generalization of a vector space where the right distributive law is omitted. The definition of a spanning set, basis and linear independence require the elements of the considered set to be in an irreducible submodule. A subset of a near vector space $M$ which is closed under addition and scalar multiplication is an $R$-subgroup but not necessarily a subspace of $M$. For a finite dimensional near vector space over a finite nearfield, an $R$-subgroup is a near vector space. The image of a near vector space under a linear mapping is an $R$-subgroup and not necessarily a subspace of the target near vector space. We construct a counterexample to illustrate this difference and then give a sufficient condition for a linear mapping to be normal. We also give a proof that a near vector space $R^n$ can be generated by $n - 1$ elements.

**Key words:** nearring, nearring module, $R$-subgroup, nearfield, Beidleman near vector space.

## Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

---

Antsa Tantely Fandresena RAKOTONDRAFARA 14 May 2020

# Contents

# 1. Introduction

Several applications for nearrings and nearfields have been discovered, for example in in agronomy, statistics, cryptography, automata, dynamical systems, graphs, homological and universal algebra, category theory, and so on. The discovery of nearfields by Dickson (1905) is the starting point. He began with a finite field and changed the multiplication in a genius way to have a structure which fulfills all the axioms of a field except one distributive law. These nearfields are now called "Dickson nearfields" in honour of him. Next, Zassenhaus (1935) characterized all finite nearfields. They are all Dickson nearfields with the exception of seven cases. Later, nearfield theory was applied in geometry to do coordinatization.

Furthermore, Blackett (1959) extensively investigated general nearrings. After this, many researchers worked in the area to transfer the results in ring theory to nearring theory and to find the important differences that can be applied in other areas. The main and natural example of a nearring is the nearring of mappings of an additive group. It is proved that every nearring can be embedded in a nearring of such a form.

The concept of a near vector space was first introduced in the PhD thesis of Beidleman (1966) using a right nearring module over a left nearfield. Later, André (1974) and Karzel and Kist (1984) introduced similar notions in a different way. After Beidleman's thesis, Djagba further investigated Beidleman's near vector spaces in (Djagba and Howell, 2019), (Djagba, 2019a) and (Djagba, 2019b).

Throughout the three chapters of this essay, we will introduce the basic theory of near vector spaces as defined by Beidleman (1966) and look at some similarities and differences between near vector spaces and traditional vector spaces. We start with the required background in the theory of nearrings, nearfields and nearring modules in Chapter 2. In Chapter 3, we will give the construction and define for a near vector space the concepts of basis, dimension and linear mappings which are fundamental in the theory of vector spaces. In Chapter 4, we will give analogous results to those from the theory of finite dimensional vector spaces and show some differences that arise from this lack of right distributivity, namely the notion of normal linear mappings and an important property of $R$-subgroups which hold only for proper near vector spaces.

# 2. Preliminary material

In this chapter, we will introduce some basic definitions, properties and theorems that are required to understand the concept of a Beidleman near vector space. We only include some selected proofs which are mainly from (Beidleman, 1966).

## 2.1 Nearrings

Originally the basic definitions stated are from (Blackett, 1959) and (Betsch, 1962) but we will state them as presented in (Beidleman, 1966).

**2.1.1 Definition.** (Beidleman, 1966) A **left nearring** is an algebraic structure consisting of a non-empty set $R$, together with two binary operations: addition "$+$" and multiplication "$\cdot$" with the following properties:

1. $(R, +)$ is a group (not necessarily abelian);
2. $(R, \cdot)$ is a semi-group;
3. $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b$ and $c$ in $R$ (the left distributive law).

**2.1.2 Remarks.**
- From now we will write the product $a \cdot b$ as $ab$ for all $a, b$ in $R$.
- A **right nearring** is obtained by replacing the left distributive law in the above definition with the right distributive law, $(a + b) \cdot c = a \cdot b + b \cdot c$ for all $a, b$ and $c$ in $R$.
- In this work, we will consider left nearrrings. All the properties that hold are valid for right nearrings.

If $R$ has an element $1$ which satisfies: $r1 = 1r = r$ for all elements $r$ in $R$, such an element $1$ is called the **identity** of $R$. In this work we assume that if $R$ has an identity, it is different from the neutral element $0$ of the addition.

From this definition, it is clear that every ring is automatically a nearring. As an example of a nearring which is not a ring, we consider the following example.

**2.1.3 Example.** The set of all mapping of an additive group $G$ into $G$ that map the additive identity $0$ to itself, denoted by $S(G)$, together with the addition $f + h$ where $g(f + h) = gf + gh$ for all $g$ in $G$ and composition $f \cdot h$ where $g(f \cdot h) = (gf)h$ for all $g$ in $G$ is the most natural example of a nearring and is called the **nearring associated with** $G$.

The following property, known in ring theory holds in nearring theory as well.

**2.1.4 Proposition.** (Beidleman, 1966) Let $R$ be a nearring and $r, s$ in $R$, we have: $r0 = 0$ and $r(-s) = -rs$.

*Proof.* We have that for $r$ in $R$, $r0 = r(0 + 0) = r0 + r0$, which implies $r0 = 0$. We also have for $r, s$ in $R$ that $rs + (-rs) = 0 = r(s + (-s)) = rs + r(-s)$ so $r(-s) = -rs$. □

Unlike in rings, we do not in general have that $0r = 0$ and $(-r)s = -rs$ for all $r, s$ in $R$. Nearrings that have this first property are called **zero symmetric nearrings** (Pilz, 2011). As many researchers in the area required this as an extra axiom in the definition; henceforth, we will consider only zero symmetric nearrings. Now we will look at a particular case of a nearring.

## 2.2 Nearfields

**2.2.1 Definition.** (Dickson, 1905)

Let $R$ be a nearring, if $R \neq \{0\}$ and $(R\backslash\{0\}, \cdot)$ is a group, then $R$ is called **division nearring** or **nearfield**.

From this, every field is automatically a nearfield.

We will give an example of proper nearfield which will be used in a further section.

**2.2.2 Example.** Let us consider the Dickson Nearfield $R = DN(3, 2)$ defined by taking the finite field $(GF(3^2), +, \cdot)$ and replacing the multiplication by an operation $\circ$ defined by:

$$a \circ b := \begin{cases} a \cdot b & \text{if } a \text{ is a square in } (GF(3^2), +, \cdot) \\ a \cdot b^3 & \text{otherwise} \end{cases}$$

Let $x$ be a root of the irreducible polynomial $x^2 + 1$. Then, the table of this new operation is the following:

Table 2.1: Table of the new operation $\circ$ for $(GF(3^2), +, \circ)$.

| $\circ$ | $0$ | $1$ | $2$ | $x$ | $1+x$ | $2+x$ | $2x$ | $1+2x$ | $2+2x$ |
|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $0$ | $1$ | $2$ | $x$ | $1+x$ | $2+x$ | $2x$ | $1+2x$ | $2+2x$ |
| $2$ | $0$ | $2$ | $1$ | $2x$ | $2+2x$ | $1+2x$ | $x$ | $2+x$ | $1+x$ |
| $x$ | $0$ | $x$ | $2x$ | $2$ | $2+x$ | $2+2x$ | $1$ | $1+x$ | $1+2x$ |
| $1+x$ | $0$ | $1+x$ | $2+2x$ | $1+2x$ | $2$ | $x$ | $2+x$ | $2x$ | $1$ |
| $2+x$ | $0$ | $2+x$ | $1+2x$ | $1+x$ | $2x$ | $2$ | $2+2x$ | $1$ | $x$ |
| $2x$ | $0$ | $2x$ | $x$ | $1$ | $1+2x$ | $1+x$ | $2$ | $2+2x$ | $2+x$ |
| $1+2x$ | $0$ | $1+2x$ | $2+x$ | $2+2x$ | $x$ | $1$ | $1+x$ | $2$ | $2x$ |
| $2+2x$ | $0$ | $2+2x$ | $1+x$ | $2+x$ | $1$ | $2x$ | $1+2x$ | $x$ | $2$ |

We can see from this table that this is a nearfield since $(GF(3^2), +)$ and $(GF(3^2)\backslash\{0\}, \circ)$ are groups and we have $a \circ (b + c) = a \circ b + a \circ c$ for all $a, b$ and $c$ in $GF(3^2)$. As we have for example, $x \circ (x + 1) \neq (x + 1) \circ x$, it is a proper nearfield.

From now, we will omit the symbol $\circ$ for the multiplication.

**2.2.3 Proposition.** (Zassenhaus, 1935) Let $R$ be a nearfield, then $(R, +)$ is abelian.

Since it is not the aim of this section and the proof requires more material than we stated here, we do not include it, but one can find the proof in (Pilz, 2011) .

## 2.3 Nearring modules

In this section, we want to generalise the concept of ring modules.

**2.3.1 Definition.** (Beidleman, 1966) A **(right) nearring module** $M$ over a nearring $R$ is an additive group $M$, together with a nearring $R$ and a mapping

$$\eta : M \times R \to M$$
$$(m, r) \mapsto m \cdot r$$

such that for all $r_1, r_2$ in $R$, $m$ in $M$,

$$m \cdot (r_1 + r_2) = m \cdot r_1 + m \cdot r_2,$$
$$m \cdot (r_1 r_2) = (m \cdot r_1) \cdot r_2.$$

If $M$ is a nearring module over $R$, we denote this by $M_R$ and $M$ is said to be an $R$-**module**. If $R$ contains an identity $1$ and $x \cdot 1 = x$ for all $x$ in $M$, we say that $M_R$ is **unitary** . The action of $\eta$ is called **scalar multiplication** as in the case of ring modules.

**2.3.2 Examples.** Any nearring is a nearring module over itself.

For a nearring $R$, the set $R^n$ for an integer $n$, where the elements are the tuples $(r_1, \ldots, r_n), r_i \in R$ for $i \in \{1, \ldots, n\}$, with addition defined for all $(r_1, \ldots, r_n), (r'_1, \ldots, r'_n)$ in $R^n$ by

$$(r_1, \ldots, r_n) + (r'_1, \ldots, r'_n) = (r_1 + r'_1, \ldots, r_n + r'_n)$$

and scalar multiplication defined for all $(r_1, \ldots, r_n)$ in $R^n$ and $r$ in $R$ by

$$(r_1, \ldots, r_n) \cdot r = (r_1 r, \ldots, r_n r)$$

is a nearring module over $R$.

The following proposition gives us some properties of nearring modules.

**2.3.3 Proposition.** (Beidleman, 1966) Let $M_R$ be a nearring module with an additive identity $0_M$. We have:

- $0_M \cdot 0 = 0_M$;
- $m \cdot 0 = 0_M$ for all $m \in M$;
- $0_M \cdot r = 0_M$ for all $r$ in $R$;
- $m \cdot (-r) = -m \cdot r$for all $r$ in $R$, $m$ in $M$.

*Proof.*

- $0_M \cdot 0 = 0_M \cdot (0 + 0) = 0_M \cdot 0 + 0_M \cdot 0$, so $0_M \cdot 0 = 0_M$.
- Let $m$ in $M$, then $m \cdot 0 = m \cdot (0 + 0) = m \cdot 0 + m \cdot 0$, so $m \cdot 0 = 0_M$.
- Let $r$ in $R$, then $0_M \cdot r = (0_M \cdot 0) \cdot r = 0_M \cdot (0r) = 0_M \cdot 0 = 0_M$.

- Let $m$ in $M$,$r$ in $R$, then $m \cdot r - m \cdot r = 0_M = m \cdot 0 = m \cdot (-r + r) = m \cdot (-r) + m \cdot r$ so $m \cdot (-r) = -m \cdot r$.

□

When there is no confusion, we will use just $0$ instead of $0_M$.

**2.3.4 Proposition.** (Beidleman, 1966) Let $R$ be a nearfield, $M_R$ an unitary $R$-module, $m \in M$ and $r \in R$. We have $m \cdot r = 0$ if, and only if, $m = 0$ or $r = 0$.

*Proof.* For $m = 0$ or $r = 0$, we always have that $m \cdot r = 0$.

Let $m \cdot r = 0$. If $r \neq 0$, then we have an element $r' \in R$ such that $rr' = 1$ and so $0 = m \cdot r = (m \cdot r) \cdot r' = m \cdot (rr') = m \cdot 1 = m$. □

**2.3.5 Definition.** (Beidleman, 1966) Let $G$ and $G'$ be additive groups and $T$ a mapping from $G$ to $G'$.

Let $A \subset G$ and $B \subset G'$ be non-empty. We call the set $AT = \{aT | a \in A\}$ the **image** of $A$ in $G'$, the set $\ker(T) = \{g \in G | gT = 0_{G'}\}$ the **kernel** of $T$ and the set $T^{-1}(B) = \{g \in G | gT \in B\}$ the **pre-image** of $B$ in $G$.

Now we will generalize the concept of a ring module homomorphism for the case of a nearring module.

**2.3.6 Definition.** (Beidleman, 1966) Let $M_R$ and $M'_R$ be two $R$-modules. A mapping $T : M \to M'$ is called an $R$-**homomorphism** if $(x + y)T = xT + yT$ and $(xT) \cdot r = (x \cdot r)T$ for all $x, y$ in $M, r$ in $R$.

If $T$ is bijective, we call it an $R$-**isomorphism** and we say that $M_R$ and $M'_R$ are $R$-isomorphic and we denote this by $M \underset{R}{\cong} M'$.

Let $T : M \to M'$ be an $R$-homomorphism. Our interest here is the structure of $MT$. We will need the following definition.

**2.3.7 Definition.** (Beidleman, 1966) Let $M_R$ a nearring module, a subset $A \subset M$ is called an $R$-**subgroup**, if and only if,

- $A$ is a subgroup of $(M, +)$;
- $A \cdot R = \{a \cdot r | a \in A, r \in R\} \subset A$.

We can now consider $MT$, where $M$ is an $R$-module and $T$ an $R$-homomorphism.

**2.3.8 Proposition.** (Beidleman, 1966) Let $R$ be a nearring, $M_R$ and $M'_R$ two $R$-modules. Let $T$ be an $R$-homomorphism from $M_R$ to $M'_R$. Then, $MT$ is a $R$-subgroup of $M'$.

*Proof.* As $T$ is a group homomorphism from $(M, +)$ to $(M', +)$, $(MT, +)$ is a subgroup of $(M', +)$. If $m \in M, r \in R$, then $(mT) \cdot r = (m \cdot r)T \in MT$. □

It is clear that $M_R$ and $\{0\}$ are $R$-subgroups.

An $R$-subgroup $A$ of $M_R$ is said to be **proper** if $A \neq M_R$ and $A \neq \{0\}$.

Now, let us define a submodule.

**2.3.9 Definition.** (Beidleman, 1966) Let $M_R$ be an $R$-module. A subset $A$ of $M_R$ is called a **submodule** if and only if:

- $A$ is a normal subgroup of the additive group $(M, +)$;
- $(m + a) \cdot r - m \cdot r \in A$ where $m \in M_R$, $a$ in $A$ and $r \in R$.

As in the case for a ring module, we have that:

**2.3.10 Proposition.** (Beidleman, 1966) The intersection of any set of submodules is a submodule.

*Proof.* Let $M_R$ be an $R$- module and $\{A_i, i \in \Omega\}$ be a set of submodules of $M_R$. From group theory we know that $\bigcap_{i\in\Omega} A_i$ is a normal subgroup of $(M, +)$. Then if $a$ is in $\bigcap_{i\in\Omega} A_i$, $m$ in $M_R$ and $r$ in $R$, $(m + a) \cdot r - m \cdot r$ is in $A_i$ for all $i$ in $\Omega$, so it is in $\bigcap_{i\in\Omega} A_i$. □

**2.3.11 Proposition.** (Beidleman, 1966) Every submodule is an $R$-subgroup of $M_R$.

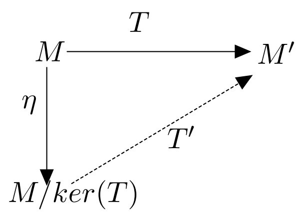
*Proof.* If $A$ is a submodule of $M_R$, then $A$ is a subgroup of $(M, +)$ and for all $a \in A, r \in R$, we have $a \cdot r = (0_M + a) \cdot r - 0_M \cdot r \in A$. □

Generally, not any $R$-subgroup is a submodule.

The definitions of a factor module and natural homomorphism from the module to the factor module derived similarly from group theory as for rings.

Next, we state the Fundamental Theorem of $R$-homomorphisms, which is similar to the one for ring homomorphisms.

**2.3.12 Theorem.** *(Beidleman, 1966) Let $M_R$ and $M'_R$ be $R$-modules and $T : M_R \to M'_R$ a surjective $R$-homomorphism. If $\eta$ is the natural $R$-homomorphism from $M_R$ onto $M/\ker(T)$, then $T$ induces an $R$-isomorphism $T' : M/\ker(T) \to M'_R$ such that $T = \eta \circ T'$.*




*Proof.* From group theory, we know that $T$ induces an additive isomorphism $T' : M/\ker(T) \to M'_R$ such that $T = \eta \circ T'$.

If $m \in M, r \in R$, then $(m + \ker(T))T' \cdot r = (mT) \cdot r = (m \cdot r)T = (m \cdot r + \ker(T))T' = ((m + \ker(T)) \cdot r)T'$. Thus, $T$ is a $R$-isomorphism. □

The following lemmas and theorem (First Isomorphism Theorem for Nearring Modules) follow similarly from the properties of group homomorphisms in group theory and will be stated without proof. The details can be found in (Beidleman, 1966), page 17.

**2.3.13 Theorem.** *(Beidleman, 1966) Let $T : M_R \to M'_R$ be a surjective $R$-homomorphism. We have:*

- *If $H$ is a submodule of $M_R$ that contains* $\ker(T)$*, then* $M/H \cong M'/HT$.
- *The mapping $f : H \to HT$ is a bijective mapping between the $R$-subgroup $H$ of $M_R$ that contains* $\ker(T)$ *and the $R$-subgroups of $M'_R$.*

**2.3.14 Lemma.** (Beidleman, 1966) Let $M_R$ and $M'_R$ be two $R$-modules and $T : M_R \to M'_R$ a surjective $R$-homomorphism. We have:

- The image of an $R$-subgroup $H$ of $M_R$ is an $R$-subgroup of $M'_R$.
- The pre-image of an $R$-subgroup $H'$ of $M'_R$ is an $R$-subgroup of $M_R$ containing $\ker(T)$.
- If $H$ is an $R$-subgroup such that $\ker(T) \subset H$, then $T^{-1}(HT) = H$.

**2.3.15 Lemma.** Beidleman (1966) Let $M_R$ and $M'_R$ be two $R$-modules and $T : M_R \to M'_R$ a surjective $R$-homomorphism. Then:

- The image of a submodule $A$ of $M_R$ is a submodule of $M'_R$,
- The pre-image of a submodule $A'$ of $M'_R$ is a submodule of $M_R$ containing $\ker(T)$,
- If $A$ is a submodule such that $\ker(T) \subset A$, then $T^{-1}(AT) = A$.

This result, the Jordan Hölder Theorem for Nearring Modules is again given without proof, all details can be found in (Beidleman, 1966), page 36. It will allow us to define the basis of a near vector space.

**2.3.16 Definition.** (Beidleman, 1966) A **Jordan Hölder series** of a nearring module $M_R$ is a sequence $\sum : M = M_0 \supset M_1 \supset \ldots \supset M_n = \{0\}$ such that for all $i$ in $\{1, \ldots, n-1\}$, $M_{i+1}$ is a proper submodule of the $R$-module $M_i$ and $M_i/M_{i+1}$ contains no proper submodules.

**2.3.17 Theorem.** *(Beidleman, 1966)*

*Let $\sum_1$ and $\sum_2$ be two Jordan Hölder series of an $R$-module $M_R$. Then there is bijective correspondence between the factors of the two series such that the paired factors are $R$-isomorphic.*

This theorem states that two Jordan Hölder series of an $R$-module $M_R$ have the same cardinality.

## 2.4 Direct sums of submodules

The definition of direct sum of submodules will be used in a later section to define a near vector space and to construct a basis for it.

**2.4.1 Definition.** (Beidleman, 1966) Let $M_R$ be a nearring module, $\{M_\lambda | \lambda \in \Omega\}$ a collection of submodules of $M_R$. We say that $M_R$ is a **direct sum of the submodules** $\{M_\lambda | \lambda \in \Omega\}$, if and only if, $(M, +)$ is a direct sum of the normal subgroups $\{M_\lambda | \lambda \in \Omega\}$. It is denoted by $M_R = \bigoplus_{\lambda \in \Omega} M_\lambda$.

As we use group theoretic properties for this definition, we can formalize in nearring modules some results from group theory.

**2.4.2 Proposition.** (Beidleman, 1966)

Let $\{M_\lambda | \lambda \in \Omega\}$ be a collection of submodules of $M_R$. We have:

- If $M_R = \bigoplus_{\lambda\in\Omega} M_\lambda$, then $M_R = \sum_{\lambda\in\Omega} M_\lambda$ and the elements of any two distinct submodules $M_\lambda$ permute;
- $M_R = \bigoplus_{\lambda\in\Omega} M_\lambda$, if and only if, $M_R = \sum_{\lambda\in\Omega} M_\lambda$ and $M_{\lambda_0} \cap \left( \sum_{\substack{\lambda\in\Omega \\ \lambda\neq\lambda_0}} M_\lambda \right) = \{0\}$ for each $\lambda_0 \in \Omega$;
- $M_R = \bigoplus_{\lambda\in\Omega} M_\lambda$, if and only if, $M_R = \sum_{\lambda\in\Omega} M_\lambda$ and every element of $M_R$ has a unique representation as a finite sum of elements chosen from the submodules $M_\lambda$.

Because of this uniqueness of representation we give the following definition.

**2.4.3 Definition.** (Beidleman, 1966) Let $M_R = \bigoplus_{\lambda\in\Omega} M_\lambda, M_\lambda$ is a submodule of $M_R$. An element $m \in M$ is of the form $m = \sum_{j=1}^{n} m_{\lambda_j}$ where $m_{\lambda_j} \in M_{\lambda_j}$, each $m_{\lambda_j}$ is called the $\lambda_j^{th}$ **component** of $m$.

The following lemma show that the elements of $R$ distribute over the components of an element.

**2.4.4 Lemma.** (Beidleman, 1966) Let $M_R = \bigoplus_{\lambda\in\Omega} M_\lambda, M_\lambda$ is a submodule of $M_R$. If $m = m_{\lambda_1} + \ldots + m_{\lambda_n}$ where $m_{\lambda_j} \in M_{\lambda_j}$ and $r \in R$, then $m \cdot r = \left(\sum_{j=1}^{n} m_{\lambda_j}\right) r = m_{\lambda_1} \cdot r + \ldots + m_{\lambda_n} \cdot r$.

*Proof.* We will use induction on $n$.

For only one non-zero component, it is obviously true. Assume that the lemma is true for all elements of $M$ with at most $\leq n-1$ non-zero components. Let $m$ be a non-zero element of $M$, $m = m_{\lambda_1} + \ldots + m_{\lambda_n}$ where $m_{\lambda_j} \in M_{\lambda_j}$.

If $r \in R$, then $(m_{\lambda_1} + \ldots + m_{\lambda_{n-1}}) \cdot r = m_{\lambda_1} \cdot r + \ldots + m_{\lambda_{n-1}} \cdot r$. Now as $M_{\lambda_n}$ and $\sum_{j=1}^{n-1} M_{\lambda_j}$ are submodules. We have $((m_{\lambda_1} + \ldots + m_{\lambda_{n-1}}) + m_{\lambda_n}) \cdot r - (m_{\lambda_1} + \ldots + m_{\lambda_{n-1}}) \cdot r$ in $M_{\lambda_n}$ and $(m_{\lambda_n} + (m_{\lambda_1} + \ldots + m_{\lambda_{n-1}})) \cdot r - m_{\lambda_n} \cdot r$ in $\sum_{j=1}^{n-1} M_{\lambda_j}$.

Using Proposition 2.4.2 and the property that two elements of different submodules in a direct sum commute, we have

$$\begin{aligned} m \cdot r - (m_{\lambda_1} + \ldots + m_{\lambda_{n-1}}) \cdot r - m_{\lambda_n} \cdot r &= ((m_{\lambda_1} + \ldots + m_{\lambda_{n-1}}) + m_{\lambda_n}) \cdot r \\ &\quad - (m_{\lambda_1} + \ldots + m_{\lambda_{n-1}}) \cdot r - m_{\lambda_n} \cdot r \\ &= (m_{\lambda_n} + (m_{\lambda_1} + \ldots + m_{\lambda_{n-1}})) \cdot r \\ &\quad - m_{\lambda_n} \cdot r - (m_{\lambda_1} + \ldots + m_{\lambda_{n-1}}) \cdot r \\ &\in M_{\lambda_n} \cap \left( \sum_{j=1}^{n-1} M_{\lambda_j} \right) = \{0\}. \end{aligned}$$

Hence, $m \cdot r = \sum_{j=1}^{n} m_{\lambda_j} \cdot r$. So the lemma is established. □

**2.4.5 Definition.** (Beidleman, 1966) Let $R$ be a nearring, $M_R$ an $R$-module. We say a submodule $A$ of $M_R$ is a **direct summand** if and only if, there is a submodule $B$ such that $M_R = A \oplus B$.

**2.4.6 Remarks.** We always have $\{0\}$ and $M_R$ as direct summands since $M_R \oplus \{0\} = \{0\} \oplus M_R = M_R$.

If $M_R$ does not have other direct summands, then $M_R$ is said to be **indecomposable**.

Below is a property of direct summands.

**2.4.7 Proposition.** (Beidleman, 1966) If $A$ is a direct summand of $M_R$, then each submodule of $A_R$ is a submodule of $M_R$.

*Proof.* As $A$ is a direct summand, we have a submodule $B$ satisfying $M_R = A \oplus B$.

We take a submodule $A'$ of $A_R$. Let $a' \in A', r \in R$, and $m = a + b$ where $a \in A, b \in B$. Proposition 2.4.2 and Lemma 2.4.4 give us:

- $-m + a' + m = -(a + b) + a' + (a + b) = -a + a' + a \in A'$,
- $(m + a') \cdot r - m \cdot r = (a + a') \cdot r + b \cdot r - a \cdot r - b \cdot r = (a + a') \cdot r - a \cdot r \in A'$.

So $A'$ is a submodule of $M_R$. □

## 2.5 Strictly semi-simple nearring modules

**2.5.1 Definition.** (Beidleman, 1966)

A nearring module $M_R$ is called **irreducible** if and only if, $M_R$ contains no proper $R$-subgroups.

The following theorem will allow us to give our first example of near vector space in a later section.

**2.5.2 Theorem.** *(Beidleman, 1966)*

*Let $R$ be a nearfield, then $R_R$ contains no proper $R$-subgroups.*

*Proof.* Let us consider a nearfield $R$. Let $B$ be a non-zero $R$-subgroup of $R_R$ and $b \in B \backslash \{0\}$. Since $R$ is a nearfield, $b$ has a multiplicative inverse $b' \in R$ satisfying $bb' = 1$ and so $1 \in B$. Therefore, $1 \cdot r = r \in B$ for all elements $r$ of $R$ and $B = R$. □

The following lemma will characterize the set $m \cdot R$.

**2.5.3 Lemma.** (Beidleman, 1966) If $M_R$ is an irreducible $R$-module, $m \in M$ and

$$m \cdot R = \{m \cdot r | r \in R\} \neq \{0\}$$

then, $M_R = m \cdot R$.

*Proof.* Let $m \in M_R$ with $m \cdot R \neq \{0\}$. Since $M_R$ is irreducible, the only nonzero $R$-subgroup is $M_R$, it means that we only need to prove that $m \cdot R$ is an $R$-subgroup of $M_R$. If $r_1, r_2 \in R$, then $m \cdot r_1 - m \cdot r_2 = m \cdot (r_1 - r_2) \in m \cdot R$ and $(m \cdot r_1) \cdot r_2 = m \cdot (r_1 r_2)$. It shows us that $m \cdot R$ is an $R$-subgroup of $M_R$, since it is non-zero, it is $M_R$ itself. □

From this, we give the following corollary which is used in the concept of basis of a near vector space.

**2.5.4 Corollary.** (Beidleman, 1966) For an unitary $R$-module $M_R$, $M_R$ is irreducible if and only if $m \cdot R = M_R$ for every non-zero element $m \in M$.

*Proof.* We have that $M_R$ is unitary, thus there exists an identity $1 \in R$ with $x \cdot 1 = x$ for all $x \in M_R$.

Let's assume now that $M_R$ is irreducible. If $m$ is any non-zero element of $M_R$, $m \cdot 1 = m \in m \cdot R$. Then Lemma 2.5.3 implies that $m \cdot R = M_R$.

Conversely, suppose that $m \cdot R = M_R$ for all non-zero elements $m \in M_R$. Let $A$ be any $R$-subgroup of $M_R$ with the property $A \neq \{0\}$. If $a \in A$ with $a \neq 0$, then $a \cdot R$ is an $R$-subgroup of $M_R$. $M_R = a \cdot R \subseteq A \subseteq M_R$ and $M_R = A$. □

We can now give the following definition.

**2.5.5 Definition.** (Beidleman, 1966) Let $M_R$ be a nearring module, we say that $M_R$ is **strictly semi-simple** if and only if, $M_R$ is a direct sum of irreducible submodules.

The following property will be given without proof, the reader can see the proof in (Beidleman, 1966), page 73.

**2.5.6 Theorem.** *(Beidleman, 1966) Let $R$ be a nearring, $M_R$ be an $R$-module. If $M_R = \sum_{\lambda \in \Omega} M_\lambda$, where $M_\lambda$ is an irreducible submodule, then*

- *$M_R$ is strictly semi-simple.*
- *Every submodule is a direct summand.*
- *Every submodule is strictly semi-simple.*

Now we are well equipped to understand the concept of a Beidleman near vector space.

# 3. Beidleman near vector spaces

The aim of this chapter is to define a Beidleman near vector space, to give some properties and to show similarities and differences between a vector space and a Beidleman near vector space.

## 3.1 Definition

We will start with the following.

**3.1.1 Definition.** (Beidleman, 1966) A **Beidleman near vector space** is a strictly semi-simple nearring module over a nearfield $R$.

From now, we use the term near vector space to refer to a Beidleman near vector space.
**3.1.2 Examples.**

- From this definition, every vector space over a field is a near vector space.
- Every irreducible $R$-module $M_R$ over a nearfield $R$ is a near vector space.
- By Theorem 2.5.2, the module $R_R$ is a near vector space for a nearfield $R$.
- For a nearfield $R$ and an integer $n$, the nearring module $R^n_R$ is a near vector space.

In this work, we will assume that our nearfield $R$ is not a field.

**3.1.3 Proposition.** (Beidleman, 1966) Let $R$ be a nearfield. Every irreducible $R$-module is $R$-isomorphic to $R_R$.

*Proof.* (Beidleman, 1966) Let $M_R$ be an irreducible $R$-module. If $m \in M$, $m \neq 0$, then the mapping $f : r \in R \mapsto m \cdot r \in M_R$ is a non-zero $R$-homomorphism of $R_R$ onto $M_R$.

Since $M$ is irreducible and $Rf$ is an $R$-subgroup of $M_R$, $Rf$ is either $\{0\}$ or $M$. Our function $f$ is a non-zero $R$-homomorphism so $Rf = M$. From Theorem 2.5.2, $R_R$ contains no proper $R$-subgroups. But $\ker(f)$ is an $R$-subgroup of $R_R$ since for $r_1, r_2$ in $\ker(f)$, $r$ in $R$, $m \cdot (r_1 - r_2) = m \cdot r_1 + m \cdot r_2 = 0$ and $m \cdot (r_1 r) = (m \cdot r_1) \cdot r = 0$. This implies that $\ker(f) = \{0\}$ since $f$ is non-zero.

From the Fundamental Theorem of $R$-homomorphisms for nearring modules we have $R = R/\ker(f) \underset{(R)}{\cong} Rf = M$. □

From this, we can conclude that:

**3.1.4 Corollary.** (Beidleman, 1966) If $m \neq 0$ is an element of an irreducible $R$-module $M_R$, then $m \cdot R$ is an irreducible $R$-module.

*Proof.* Since $(R, +)$ is abelian, $(M, +)$ is also abelian. Hence, the set $m \cdot R$ is a submodule of $M_R$, because $a + m \cdot r - a = m \cdot r$ is in $m \cdot R$ for all $a$ in $M_R$ and $(m \cdot R) \cdot R = m \cdot R$. □

The following corollary is also a consequence of Proposition 3.1.3.

**3.1.5 Corollary.** If $M_R$ is near vector space over $R$ then $(M, +)$ is abelian.

*Proof.* Since $M = \bigoplus_{\lambda\in\Omega} M_\lambda$, $m = \sum_{\lambda\in\Omega} m_\lambda$ and $m' = \sum_{\lambda\in\Omega} m'_\lambda$, where $m_{\lambda i}$ are in $M_\lambda$ for $i$ in $1, 2$.

$$\begin{aligned} m + m' - (m' + m) &= m + m' - m - m' \\ &= \sum_{\lambda\in\Omega} m_\lambda + \sum_{\lambda\in\Omega} m'_\lambda - \sum_{\lambda\in\Omega} m_\lambda - \sum_{\lambda\in\Omega} m'_\lambda \end{aligned}$$

By Proposition 2.4.2 every element in different submodules commute.

$$m + m' - (m' + m) = \sum_{\lambda\in\Omega} (m_\lambda + m'_\lambda - m_\lambda - m'_\lambda)$$

From Proposition 3.1.3, $M_\lambda$ is $R$-isomorphic to $R_R$ so it is abelian.

$$m + m' - (m' + m) = \sum_{\lambda\in\Omega} (m_\lambda - m_\lambda + m'_\lambda - m'_\lambda) = 0.$$

□

One of the most important features of a vector space is the notion of basis and dimension. We also have this for a near vector space as we will see in the following section.

## 3.2 Basis and Dimension

As in the case of a vector space, the definition of a basis depends on the notion of linear combination and spanning set.

**3.2.1 Definition.** (Beidleman, 1966) Let $R$ be a nearfield and $M_R$ a near vector space over $R$ and $\{m_1, \ldots, m_n\}$ a finite set of elements from $M_R$. We say that $m \in M_R$ is a **linear combination** of $\{m_1, \ldots, m_n\}$ if, and only if, we have elements $\{r_1, \ldots, r_n\}$ of $R$ such that $m = \sum_{i=1}^{n} m_i \cdot r_i$.

The definition of spanning set in near vector space theory differs from the one in vector space theory.

**3.2.2 Definition.** (Beidleman, 1966) Let $M_R$ be a near vector space, $X \subseteq M_R$, we say that $X$ is a **spanning set** for $M_R$ if, and only if,

- every element of $X$ is contained in an irreducible submodule,
- every element of $M_R$ can be written as a linear combination of a finite set of elements from $X$.

This first condition makes the notion of a spanning set different to what a spanning set is for a vector space.

**3.2.3 Definition.** (Beidleman, 1966) Let $M_R$ be a near vector space. A **basis** of $M_R$ is a spanning set $X$ for $M_R$ such that each element of $M_R$ can be uniquely represented as linear combination of elements from $X$.

As in the case of vector space, every near vector space has a basis. The following theorem shows that.

**3.2.4 Theorem.** *(Beidleman, 1966) A near vector space $M_R$ has a basis.*

*Proof.* Suppose $M_R$ is a near vector space. Then $M_R = \bigoplus_{\lambda \in \Omega} M_\lambda$ where $M_\lambda$ is a non-zero irreducible submodule. For each $\lambda \in \Omega$, we take $m_\lambda \neq 0$ and so we have $m_\lambda \cdot R = M_\lambda$ by Corollary 2.5.4. This gives us that $M_R = \bigoplus_{\lambda \in \Omega} m_\lambda \cdot R$ and it proves that $\{m_\lambda | \lambda \in \Omega\}$ is a basis for $M_R$. □

**3.2.5 Remark.** Particularly, if $M_R$ is irreducible, then for all non-zero element $m$ of $M$, $M_R = m \cdot R$ and $\{m\}$ is a basis for $M_R$.

Now, we will generalise the concept of linear independence we know from vector spaces to near vector spaces.

**3.2.6 Definition.** (Beidleman, 1966) Let $R$ be a nearfield, $M_R$ a near vector space over $R$. We take a finite set $X = \{m_1, \ldots m_n\}$. $X$ is **linearly independent** if and only if,

- every element of $X$ is contained in an irreducible submodule,
- the equality $m_1 \cdot r_1 + \cdots + m_n \cdot r_n$, with $r_1, \ldots r_n \in R$ implies $r_1 = \cdots = r_n = 0$.

**3.2.7 Remark.** If $X$ is infinite, we say that $X$ is linearly independent if each finite subset of $X$ is linearly independent.

If $X$ is not linearly independent, we say that $X$ is **linearly dependent**.

**3.2.8 Proposition.** (Beidleman, 1966) Let $X$ be a subset of $M$. If $0$ is in $X$, then $X$ is linearly dependent.

*Proof.* From Proposition 2.3.4 $0 \cdot r = 0$ for all $r$ in $R$. Let $\{ x_1, \ldots, x_n\}$ be a finite subset of $X$. Then $0 \cdot r + x_1 \cdot 0 + \cdots + x_n \cdot 0 = 0$, hence $\{ x_1, \ldots, x_n, 0\}$ is linearly dependent so $X$ must be linearly dependent too. □

As in the case of vector space, we have the following.

**3.2.9 Lemma.** (Beidleman, 1966)

Let $R$ be a nearfield, $M_R$ a near vector space over $R$ and $X$ a linearly independent set of $M_R$. We have $\sum_{x \in X} x \cdot R = \bigoplus_{x \in X} x \cdot R$.

*Proof.* As $X$ is linearly independent, each $x \in X$ is an element of a non-zero irreducible submodule $M_x$. From Proposition 3.2.8, $x \neq 0$ so $x \cdot R = M_x$. Therefore, $\sum_{x \in X} x \cdot R$ is a submodule of $M_R$.

Let's take $m \in x \cdot R \cap \left( \sum_{\substack{y \in X \\ y \neq x}} y \cdot R \right)$, then there exists elements $y_1, \ldots, y_n$ from $X$ and elements $r, r_1, \ldots, r_n$ from $R$ such that $m = x \cdot r = \sum_{i=1}^{n} y_i \cdot r_i$.

It means that $y_1 \cdot r_1 + \ldots + y_n \cdot r_n + x \cdot (-r) = 0$ and so $r_1 = r_2 = \ldots = r_n = r = 0$.

We have $r = 0$ and hence $m = 0$. Using Proposition 2.4.2, $R \cap \left( \sum_{\substack{y \in X \\ y \neq x}} y \cdot R \right) = \{0\}$ implies that $\sum_{x \in X} x \cdot R = \bigoplus_{x \in X} x \cdot R$. □

Now, we will show that as in vector space, we can extend a linearly independent set in a near vector space to a basis.

**3.2.10 Theorem.** *(Beidleman, 1966) A linearly independent set $X$ of a near vector space $M_R$ can be extended to a basis for $M_R$.*

*Proof.* From Lemma 3.2.9, $\sum_{x \in X} x \cdot R = \bigoplus_{x \in X} x \cdot R$ is a submodule of $M_R$. Since $M_R$ is strictly semi-simple, it follows from Theorem 2.5.6 that there is a submodule $M'_R$ such that $M_R = (\bigoplus_{x \in X} x \cdot R) \oplus M'$ and $M'_R$ is a near vector space over $R$. According to Theorem 3.2.4, $M'_R$ has a basis $Y$. Then, $X' = X \cup Y$ is a basis for $M_R$. Hence $X$ can be extended to a basis.

□

From this, we will prove the followings properties that is similar to these of vector spaces.

**3.2.11 Theorem.** *(Beidleman, 1966) Let $M_R$ be a near vector space and $X$ a non-empty subset of $M_R$. Then the following statements are equivalent:*

- *$X$ is a basis for $M_R$,*
- *$X$ is a spanning linearly independent set,*
- *$X$ is a maximal linearly independent set,*
- *$X$ is a minimal spanning set.*

In order to prove this theorem, we will split it in four lemmas stated below.

**3.2.12 Lemma.** (Beidleman, 1966)

Let $X$ be a basis for $M_R$. Then $X$ is a linearly independent spanning set.

*Proof.* Since $X$ is a basis, it is a spanning set by definition. Let us assume that there are elements $\{m_1, \ldots, m_n\} \subseteq X$ and $r_1, \ldots, r_n \in R$ such that $m_1 r_1 + \cdots + m_n \cdot r_n = 0 = m_1 \cdot 0 + \cdots + m_n \cdot 0$. Since $X$ is a basis, the representation is unique. Thus $r_1 = \cdots = r_n = 0 \in R$. This show that $X$ is linearly independent. □

As in the case of vector space, we have also the following.

**3.2.13 Lemma.** (Beidleman, 1966)

Let $X$ be a linearly independent set that spans $M_R$. Then $X$ is a maximal linearly independent set.

*Proof.* Let us assume that $X'$ is a linearly independent subset of $M_R$ and $X$ a proper subset of $X'$.

Let $x' \in X'\backslash X$. Since $X$ is a spanning set for $M_R$, there exists $x_1, \ldots, x_n \in X$ and $r_1, \ldots, r_n \in R$ such that $x' = \sum_{i=1}^{n} x_i \cdot r_i$. This mean that $x_1 \cdot r_1 + \cdots + x_n \cdot r_n + x' \cdot (-1) = 0$, so $X'$ is linearly dependent. Hence $X$ is a maximal linearly independent set. □

This next lemma show a well known property in vector space.

**3.2.14 Lemma.** (Beidleman, 1966)

Let $X$ be a maximal linearly independent subset of $M_R$. Then $X$ is a minimal spanning set.

*Proof.* From Theorem 3.2.10 $X$ can be extended to a basis. As every basis is linearly independent according to Lemma 3.2.12, $X$ is already a basis. So $X$ is a spanning set as well. Let us prove that $X$ is a minimal spanning set.

For that let us assume there is a proper subset $Y \subsetneq X$ that also spans $M_R$. If $x \in X\backslash Y$, then $x = \sum_{i=1}^{n} y_i \cdot r_i$ for some $y_1, \ldots, y_n \in Y$ and $r_1, \ldots, r_n \in R$. Then, $\{y_1, \ldots, y_n, x\} \subset X$ is a linearly dependent set and so we have a contradiction. So $X$ is a minimal spanning set. □

We will finish the proof with the following.

**3.2.15 Lemma.** (Beidleman, 1966) If $X$ is a minimal spanning set for $M_R$, then $X$ is a basis.

*Proof.* Since $X$ is a spanning set, $x \cdot R$ is an irreducible submodule for each $x \in X$ and $M_R = \sum_{x \in X} x \cdot R$. From Theorem 2.5.6 there is a subset $X' \subseteq X$ such that $M_R = \sum_{x \in X'} x \cdot R$. From this $X'$ is a basis for $M_R$, and since $X$ is minimal, we conclude that $X = X'$. □

We now establish the notion of dimension in the case of a near vector space.

**3.2.16 Lemma.** (Beidleman, 1966) Let $M_R$ be a near vector space, Let $M_R = \bigoplus_{i=1}^{n} M_i$ where $M_i$ is a non-zero irreducible submodule. Then any other decomposition of $M_R$ into non-zero irreducible submodules has exactly $n$-summands.

*Proof.* Let $\sum : M = M_1 \oplus \cdots \oplus M_n \supset M_2 \oplus \cdots \oplus M_n \supset \cdots \supset M_n \supset \{0\}$ and note that $\sum$ is a Jordan-Hölder series, thus it follows that such a decomposition has exactly $n$-summands by the Jordan-Hölder Theorem. □

From the previous Lemma, we have that if $M_R$ has a finite basis, then any two bases have the same number of elements.

Let $M_R$ be a near vector space. We denote by $|X|$ the cardinal number of a non-empty subset $X \subset M_R$.

**3.2.17 Theorem.** *(Beidleman, 1966) Let $M_R$ be a near vector space, $X_1$ and $X_2$ bases for $M_R$. Then $|X_1| = |X_2|$.*

*Proof.* We only have left the case where $|X_1|$ is infinite. For each $x \in X_1$, let

$$X_2(x) = \{y_1, \ldots, y_n \in X_2 | x = \sum_{i=1}^{n} y_i \cdot r_i,\ r_i \in R \text{ and } r_i \neq 0\}.$$

Since $X_2$ is a basis $X_2(x)$ is well defined. If $X = \bigcup_{x \in X_1} X_2(x)$, then note that $X$ is a spanning set for $M_R$ and $X \subseteq X_2$. From Theorem 3.2.11, $X_2$ is a minimal spanning set so $X = X_2$. Since $X_1$ is infinite and each $X_2(x)$ is finite, we have $|X_2| \leq |X_1|$. Similarly, $|X_1| \leq |X_2|$ and so $|X_1| = |X_2|$. □

From this we can give the following definition:

**3.2.18 Definition.** (Beidleman, 1966) Let $M_R$ be a near vector space over $R$. The **dimension** of $M_R$ is the cardinality of any basis for $M_R$. We denote it as $\dim M$.

If $\dim M$ is finite, $M$ is said to be a **finite dimensional near vector space**, else $M$ is an **infinite dimensional near vector space**. Since the submodule $\{0\}$ of a near vector space does not have a basis, we will agree that $\dim\{0\} = 0$.

## 3.3 Subspaces

As in the case of vector space, we will define the notion of subspace. The following is the original definition.

**3.3.1 Definition.** (Beidleman, 1966) Let $R$ be a nearfield, $M_R$ a near vector space over $R$. Every submodule of $M_R$ is called a **subspace** of $M_R$.

An equivalent definition can be given knowing that according to our Corollary 3.1.5, the additive group of a near vector space is always abelian so every subgroup is normal.

**3.3.2 Definition.** (Djagba, 2019b) Let $R$ be a nearfield, $M_R$ a near vector space over $R$. A non-empty subset $M'$ of $M$ is a subspace if and only if,

- $(A, +)$ is a subgroup of $(M, +)$,
- $(m + a) \cdot r - m \cdot r$ is in $M'$ for all $m$ in $M$, $a$ in $M'$ and $r$ in $R$.

From Theorem 2.5.6, every subspace of $M_R$ is a near vector space. It is clear from Proposition 2.3.10 that the intersection of a set of subspaces is a subspace.

Let us now give a property known in vector spaces that also hold for near vector spaces.

**3.3.3 Proposition.** (Djagba, 2019b) Let $R$ be a nearfield, $M_R$ a near vector space over $R$, $M_1$ and $M_2$ subspaces of $M_R$. Then $M_1 + M_2 = \{m_1 + m_2 | m_1 \in M_1, m_2 \in M_2\}$ is also a subspace of $M$.

*Proof.* Let $a, a'$ be two elements of $M_1 + M_2$ such that $a = m_1 + m_2$ and $a' = m_1' + m_2'$ for $m_1, m_1'$ in $M_1$ and $m_2, m_2'$ in $M_2$. We have $a - a' = (m_1 + m_2) - (m_1' + m_2') = (m_1 - m_1') + (m_2 - m_2')$ in $M_1 + M_2$ so $(M_1 + M_2, +)$ is a subgroup of $(M, +)$.

Let $m$ be in $M$, $a$ in $M_1 + M_2$ and $r$ in $R$. We have $a = m_1 + m_2$ for $m_1$ in $M_1$ and $m_2$ in $M_2$. We have

$$\begin{aligned}(m+a)\cdot r - m\cdot r &= (m+m_1+m_2)\cdot r - m\cdot r\\ &= ((m+m_1)+m_2)\cdot r - m\cdot r\\ &= ((m+m_1)+m_2)\cdot r - (m+m_1)\cdot r + (m+m_1)\cdot r - m\cdot r\\ &= (m+m_1)\cdot r - m\cdot r + ((m+m_1)+m_2)\cdot r - (m+m_1)\cdot r\end{aligned}$$

because $(M,+)$ is abelian.

As $M_1$ and $M_2$ are subspaces, $((m+m_1)+m_2)\cdot r-(m+m_1)\cdot r$ is in $M_2$ and $(m+m_1)\cdot r-m\cdot r$ is in $M_1$. It shows that $(m+a)\cdot r-m\cdot r$ is in $M_1+M_2$. Thus $M_1+M_2$ is a subspace of $M$. □

We will now define the complementary of a subspace.

**3.3.4 Definition.** (Beidleman, 1966) Let $M'$ be a subspace of $M_R$. We say that $M'$ is **complemented** by a subspace $M''$ if and only if, $M_R = M' \oplus M''$. The subspace $M''$ is called the **complementary** of $M'$.

The following theorem shows the uniqueness of the complementary.

**3.3.5 Theorem.** *(Beidleman, 1966)*

*Every subspace of a near vector space is complemented uniquely up to $R$-isomorphism.*

*Proof.* Let $M'$ be a subspace of a near vector space $M_R$. Theorem 2.5.6 tells us that $M'$ is a direct summand. Thus there is a subspace $M_1$ such that $M = M' \oplus M_1$ and $M_1$ is a complementary of $M'$.

Now let $M_2$ be another complementary of $M'$. So each element $m$ of $M$ can be written uniquely in the form $m = a_1 + m_1 = a_2 + m_2$ where $a_1, a_2 \in M', m_1 \in M_1, m_2 \in M_2$.

As every subspace is a near vector space, we will consider the mapping $T$ from $M_1$ to $M_2$ defined by $m_1T = m_2$ such that $m_1 = a_2 + m_2$ for some $a_2 \in M'$ and $m_2 \in M_2$. It is well defined since every element of $M_1$ is an element of $M$ and $M = M' \oplus M_2$ so the representation $m_1 = a_2 + m_2$ is unique for each $m_1 \in M_1$.

It is an $R$-homomorphism since if we have two elements $m_1, m_1' \in M_1$ and $r \in R$, $m_1 = a_2 + m_2$ and $m_1' = a_2' + m_2'$, then $m_1 + m_1' = a_2 + m_2 + a_2' + m_2' = a_2 + a_2' + m_2 + m_2'$ since $(M,+)$ is abelian by Corollary 3.1.5.

Thus, $m_1T + m_1'T = m_2 + m_2' = (m_1 + m_1')T$. Also, $(m_1\cdot r)T = (a_2\cdot r + m_2\cdot r)T = m_2\cdot r = m_1T\cdot r$.

Since $M = M' \oplus M_1$, then for each element $m_2 \in M_2$, there are elements $a_1 \in M', m_1 \in M_1$ such that $m_2 = a_1 + m_1$ so $m_1 = m_2 - a_1$ and $m_1T = m_2$. Thus $T$ is surjective.

We need to prove that $T$ is injective.

Suppose there are two elements $m_1$ and $m_1'$ such that $m_1T = m_1'T = m_2$. So $m_1 = m_2 + a_2$ and $m_1' = m_2 + a_2'$ and $m_1 - m_1' = a_2 - a_2' \in M'$. But from the definition of direct summands, $M' \cap M_1 = \{0\}$ so $m_1 - m_1' = 0$. Thus, $M_1 \underset{R}{\cong} M_2$. □

The following theorem states the relation between complementarity and dimension.

**3.3.6 Theorem.** *(Beidleman, 1966) Let $R$ be a nearfield and $M_R$ be a near vector space over $R$. Assume there are proper subspaces $M'$ and $M''$ of $M_R$ such that $M_R = M' \oplus M''$. Then $\dim M = \dim M' + \dim M''$.*

*Proof.* Let $X'$ be a basis for $M'$ and $X''$ be a basis for $M''$. So $X = X' \cup X''$ is a basis for $M_R$. By Theorem 3.2.11 $X'$ is a linearly independent subset of $M'$ and so Proposition 3.2.8 gives us $0 \notin X'$. From this $\dim M = |X| = |X'| + |X''| = \dim M' + \dim M''$. □

As a consequence of this, we have the following.

**3.3.7 Corollary.** (Beidleman, 1966)

Let $R$ be a nearfield, $M_R$ an $n$- dimensional near vector space and $M'$ a subspace of $M_R$. We have $M' = M$ if, and only if, $\dim M = \dim M'$.

*Proof.* Let us assume that $\dim M' = \dim M$ and $M'$ is a proper subspace of $M_R$. From Theorem 3.3.5 there is a non-zero subspace $M''$ such that $M_R = M' \oplus M''$. So $\dim M = \dim M' + \dim M'' > \dim M$ which is impossible, so $M' = M$.

Conversely, if we have $M' = M$, we have always $\dim M = \dim M'$. □

In a vector space, every subset that is closed under addition and scalar multiplication is a subspace. For us, we only have the implication:

**3.3.8 Lemma.** (Djagba, 2019b) Let $R$ be a nearfield, $M_R$ a near vector space over $R$. If $M'$ is a subspace of $M$ then $M'$ is closed under addition and scalar multiplication.

*Proof.* If $M'$ is a subspace of $M_R$ then $(M', +)$ is a subgroup of $(M, +)$ so it is closed under addition. Then, $(m + a) \cdot r - m \cdot r$ is in $M'$ for all $m$ in $M$, $a$ in $M'$ and $r$ in $R$ so also for $m = 0$, $a \cdot r$ is in $M'$. □

From our Definition 2.3.7, a subset closed under addition and scalar multiplication is an $R$-subgroup. We will show an example of an $R$-subgroup which is not a subspace.

**3.3.9 Example.** (Djagba, 2019b) We will use the Dickson nearfield $R = DN(3, 2)$ stated in Example 2.2.2.

We will take the near vector space $R_R^2$. Let us consider the set

$$\begin{aligned} H &= \{(1, x) \cdot r | r \in R\} \\ &= \{(0, 0), (1, x), (2, 2x), (x, 2), (x + 1, x + 2), (x + 2, 2x + 2), (2x + 1, x + 1), (2x + 2, 2x + 1)\}. \end{aligned}$$

Since $(1, x) \cdot r_1 - (1, x) \cdot r_2 = (1, x) \cdot (r_1 - r_2)$ and $((1, x) \cdot r_1) \cdot r = (1, x) \cdot (r_1 r)$, $H$ is an $R$-subgroup of $R^2$. It is not a subspace since if we take $(1, x + 1)$ in $R^2$, $(1, x)$ in $H$ and $x$ in $R$, $((1, x + 1) + (1, x)) \cdot x - (1, x + 1) \cdot x = (x, 1)$ is not in $H$.

## 3.4 Linear mappings

Linear mappings are one of the fundamental components in the theory of vector spaces, we will have a corresponding notion in near vector spaces.

**3.4.1 Definition.** (Beidleman, 1966)

Let $R$ be a nearfield, $M_R$ and $A_R$ two near vector spaces over $R$. An $R$-homomorphism $T$ of a near vector space $M_R$ into $A_R$ is called a **linear mapping**.

If a linear mapping $T$ is an $R$-isomorphism, then we say that $T$ is a linear isomorphism.

In the next lemma, we will see that as it is expected to be, the properties of a linear mapping $T$ is determined by any basis of $M_R$.

**3.4.2 Lemma.** (Beidleman, 1966) Let $R$ be a nearfield, $M_R$ and $A_R$ be two near vector spaces over $R$ and $T$ a linear mapping of $M_R$ into $A_R$. Then $T$ is uniquely determined on any basis for $M_R$.

*Proof.* Let $X$ be a basis for $M_R$ and $T'$ a linear mapping of $M_R$ into $A_R$ such that $xT = xT'$ for all $x \in X$. If $m$ in $M_R$, then $m = \sum_{i=1}^{n} x_i \cdot r_i$ where $x_i \in X$ and $r_i \in R$. Hence

$$\begin{aligned} mT &= \left(\sum_{i=1}^{n} x_i \cdot r_i\right) T \\ &= \left(\sum_{i=1}^{n} (x_iT) \cdot r_i\right) \\ &= \left(\sum_{i=1}^{n} (x_iT') \cdot r_i\right) \\ &= \left(\sum_{i=1}^{n} x_i \cdot r_i\right) T' \end{aligned}$$

and so $T = T'$ on $M_R$. □

The following theorems show the well known fact in the theory of vector spaces, namely that dimension characterizes isomorphism.

**3.4.3 Theorem.** *(Beidleman, 1966) Let $R$ be a nearfield, $M_R$ and $A_R$ be two near vector spaces over $R$. If $M_R$ and $A_R$ are $R$-isomorphic, then $\dim M = \dim A$.*

*Proof.* Let $T$ be a linear isomorphism of $M_R$ into $A_R$. Since $T$ is an onto mapping and $T$ maps irreducible subspaces of $M_R$ onto irreducible subspaces of $A_R$ by Lemma 2.3.15, it follows that $\dim M = \dim A$. □

Let us prove the converse of this theorem.

**3.4.4 Theorem.** *(Beidleman, 1966) Let $R$ be a nearfield, $M_R$ and $A_R$ be two near vector spaces over $R$. If $\dim M = \dim A$, then $M_R$ and $A_R$ are $R$-isomorphic.*

*Proof.* Let $X$ be a basis for $A_R$ and $Y$ a basis for $M_R$. Then $|X| = |Y|$ and so there is a bijective mapping $T$ of $X$ onto $Y$. Let $a \in A, a = \sum_{i=1}^{n} x_i \cdot r_i$ where $x_i$ in $X$ and $r_i$ in $R$. Let $T'$ the mapping from $A$ into $M$ defined by $aT' = \sum_{i=1}^{n} (x_iT) \cdot r_i$. Since a basis representation is unique and $xT \in Y$ for all $x$ in $X$, we see that $T'$ is a single-valued onto mapping.

If $a' = x'_1 \cdot r'_1 + \cdots + x'_k \cdot r'_k$ where $x'_j \in X, r'_j \in R$, then

$$\begin{aligned}(a+a')T &= \left(\sum_{i=1}^{n} x_i \cdot r_i + \sum_{j=1}^{k} x'_j \cdot r'_j\right) T' \\ &= \sum_{i=1}^{n} (x_i T) \cdot r_i + \sum_{j=1}^{k} (x'_j T) \cdot r'_j\end{aligned}$$

so $T'$ is a group homomorphism of $(A,+)$ onto $(M,+)$.

Now let us prove that $(aT') \cdot r = (a \cdot r)T'$ for $a \in A, r \in R$.

Since $Y$ is a basis for $M_R$, $xT \neq 0$ for all $x \in X$. Moreover, $xT$ is contained in a non-zero irreducible submodule of $M_R$. From Lemma 2.4.4, we have

$$\begin{aligned}(aT') \cdot r &= \left[\sum_{i=1}^{n} (x_i T) \cdot r_i\right] \cdot r \\ &= \sum_{i=1}^{n} (x_i T) \cdot r_i r \\ &= \left[\sum_{i=1}^{n} x_i \cdot (r_i r)\right] T' \\ &= (a \cdot r)T'.\end{aligned}$$

We now assume $a \neq 0$ where $a$ is the element given above. Hence, there is at least one index $i = 1, 2, \ldots, n$ such that $x_i \cdot r_i \neq 0$. If $aT' = 0$, then $aT' = \sum_{i=1}^{n} (x_i T) \cdot r_i = 0$.

Since $Y$ is a basis, $r_1 = \cdots = r_n = 0$ by Theorem 3.2.11. This shows $T'$ is a one-to-one mapping and so $A \underset{(R)}{\cong} M$. □

# 4. Finite dimensional near vector spaces

In this section we will focus on finite dimensional near vector spaces.

## 4.1 Similarities with finite dimensional vector space

Firstly, we will give a characterisation.

**4.1.1 Theorem.** *(Djagba, 2019b) Let $R$ be a nearfield. An $R$-module $M_R$ is a finite dimensional near vector space if and only if there exists an integer $n = \dim M_R$ such that $M_R \underset{R}{\cong} R^n_R$.*

*Proof.* If $M_R \underset{R}{\cong} R^n_R$, then it is clear that $M_R$ is a finite dimensional near vector space.

Let us assume that $M_R$ is a finite dimensional near vector space, so $M_R = \bigoplus_{i=1}^{n} M_i$, where $M_i$ are non-zero irreducible submodules of $M_R$ for $i \in \{1, \ldots, n\}$. From Corollary 2.5.4 there exists an non-zero element $m_i$ of $M_i$ such that $m_i \cdot R = M_i$ for $i \in \{1, \ldots, n\}$.

So, we have $M_R = \bigoplus_{i=1}^{n} m_i \cdot R$. The set $B = \{m_1, \ldots, m_n\}$ is a basis of $M_R$. Let $T$ be the map from $R^n$ to $M$ defined by $(r_1, \ldots, r_n)T = \sum_{i=1}^{n} m_i \cdot r_i$.

We will show that it is an $R$-isomorphism.

Let $(r_1, \ldots, r_n), (r'_1, \ldots, r'_n)$ in $R^n$. We have:

$$\begin{aligned}
((r_1, \ldots, r_n) + (r'_1, \ldots, r'_n))\, T &= \sum_{i=1}^{n} m_i \cdot (r_i + r'_i) \\
&= \sum_{i=1}^{n} (m_i \cdot r_i + m_i \cdot r'_i) \\
&= \sum_{i=1}^{n} m_i \cdot r_i + \sum_{i=1}^{n} m_i \cdot r'_i \\
&= (r_1, \ldots, r_n)T + (r'_1, \ldots, r'_n)T.
\end{aligned}$$

Let $r \in R$ and $(r_1, \ldots, r_n) \in R^n$. Using Lemma 2.4.4 we obtain:

$((r_1, \ldots, r_n) \cdot r)\, T = \sum_{i=1}^{n} (m_i \cdot r_i) \cdot r = \left( \sum_{i=1}^{n} m_i \cdot r_i \right) \cdot r = ((r_1, \ldots, r_n)\, T) \cdot r.$

Since $B$ is a basis of $M_R$,

$$\begin{aligned}
(r_1, \ldots, r_n)\, T = 0 &\Rightarrow \sum_{i=1}^{n} m_i \cdot r_i = 0 \\
&\Rightarrow r_1 = \cdots = r_n = 0.
\end{aligned}$$

Hence $\ker(T) = \{(0, \ldots, 0)\}$ and $T$ is injective.

Let $m$ in $M_R$. Since $B$ is a basis of $M_R$, there exists $r_1, \ldots, r_n \in R$ such that $m = \sum_{i=1}^{n} m_i \cdot r_i = (r_1, \ldots, r_n)T$.

Therefore, $T$ is surjective. So $T$ is bijective and a linear isomorphism. $\square$

The following properties are basic results in vector spaces and also hold in near vector spaces. They do not appear in the literature and are new.

**4.1.2 Proposition.** Let $R$ be a nearfield, $M_R$ be a finite dimensional near vector space over $R$. If $M'_R$ is a subspace of $M_R$, then $\dim M' \leq \dim M$.

*Proof.* It follows directly from Theorem 3.3.6. $\square$

We also have the following.

**4.1.3 Proposition.** Let $R$ be a nearfield, $M_R$ be a near vector space of dimension $n$. Then every linear independent set of elements of cardinality $n$ is a basis.

*Proof.* From Theorem 3.2.10 a linear independent set can be extended to a basis. Theorem 3.2.17 states that this basis needs to be of cardinality $n$. Thus the extension is the trivial one. $\square$

Similarly, the following holds.

**4.1.4 Proposition.** Let $R$ be a nearfield, $M_R$ a finite dimensional near vector space of dimension $n$. Then every spanning sets of cardinality $n$ is a basis.

*Proof.* A spanning set contains a basis from Theorem 3.2.11. As this basis needs to be of cardinality $n$ according to Theorem 3.2.17, a spanning set of cardinality $n$ is minimal and so is a basis. $\square$

We saw in Proposition 3.3.3 that $M_1 + M_2$ is a subspace, so here we will consider its dimension.

**4.1.5 Proposition.** Let $R$ be a nearfield, $M$ a near vector space over $R$, $M_1$ and $M_2$ two subspaces of $M$. Then we have $\dim(M_1 + M_2) = \dim M_1 + \dim M_2 - \dim(M_1 \cap M_2)$.

*Proof.* Let $\{m_1, \ldots, m_l\}$ be a basis of $M_1 \cap M_2$ where $l = \dim(M_1 \cap M_2)$. This set is at the same time linearly independent in $M_1$ and $M_2$ so can be extended to be a basis in both subspaces. Let $\{m_1, \ldots, m_l, v_1, \ldots, v_k\}$ be the resultant basis of $M_1$ and $\{m_1, \ldots, m_l, w_1, \ldots, w_j\}$ the one for $M_2$. Let us show now that the set $\{m_1, \ldots, m_l, v_1, \ldots, v_k, w_1, \ldots, w_j\}$ of cardinality $l + k + j = \dim M_1 + \dim M_2 - \dim(M_1 \cap M_2)$ is a basis for $M_1 + M_2$.

By definition, the subspace spanned by $\{m_1, \ldots, m_l, v_1, \ldots, v_k, w_1, \ldots, w_j\}$ contains all linear combinations of each element so it spans $M_1 + M_2$. Next, we need to prove that it is linearly independent.

For this, let us consider a linear combination of its elements which is equal to $0$.

We have $m_1 \cdot r_1 + \ldots + m_l \cdot r_l + v_1 \cdot r'_1 + \ldots + v_k \cdot r'_k + w_1 \cdot r''_1 + \ldots + w_j \cdot r''_j = 0$.

This implies that $w_1 \cdot r''_1 + \ldots + w_j \cdot r''_j = -m_1 \cdot r_1 + \ldots - m_l \cdot r_l - v_1 \cdot r'_1 - \ldots - v_k \cdot r'_k$ is in $M_1$. As it is a linear combination of elements of the basis of $M_2$, it is an element of $M_2$ and then an element of $M_1 \cap M_2$. Writing it as linear combination of $\{m_1, \ldots, m_l\}$, we have, $w_1 \cdot r''_1 + \ldots + w_j \cdot r''_j = m_1 \cdot \lambda_1 + \ldots + m_l \cdot \lambda_l$. So back to our equation, we have that $m_1 \cdot r_1 + \ldots + m_l \cdot r_l + v_1 \cdot r'_1 + \ldots + v_k \cdot r'_k + m_1 \cdot \lambda_1 + \ldots + m_l \cdot \lambda_l = 0$.

This linear combination involves only elements of the basis of $M_1$ so all the coefficients are $0$. Thus, the set $\{m_1, \ldots, m_l, v_1, \ldots, v_k, w_1, \ldots, w_j\}$ is linearly independent and a basis of $M_1 + M_2$. $\square$

Now, we will prove the rank-nullity theorem for near vector spaces.

**4.1.6 Theorem.** *(Beidleman, 1966)* ***(Rank-nullity theorem)*** *Let $R$ be a nearfield, $A_R$ and $M_R$ two finite dimentional near vector spaces over $R$. Let $T$ be a linear mapping of $M_R$ into $A_R$. Then $MT$ is a near vector space over $R$ and $\dim M = \dim \ker(T) + \dim MT$.*

*Proof.* From the Fundamental theorem of $R$-homomorphisms, we have $M/\ker(T) \underset{(R)}{\cong} MT$ and so $MT$ is a near vector space over $R$. If $\ker(T) = \{0\}$, then $\dim M = \dim MT$ by Theorem 3.4.3. Let $\ker(T) \neq \{0\}$. Since $\ker(T)$ is a subspace of $M_R$, it follows from Theorem 3.3.5 that there is a subspace $M'$ of $M_R$ such that $M' \oplus \ker(T) = M_R$.

We take a basis $R_B$ of $M'$ and a basis $N_B$ of $\ker(T)$. Then $R_B \cup N_B$ is a basis of $M$.

Now let us prove that $R_B T$ is a basis of $MT$.

For this, let $a \in MT$. So there exists $m \in M$ such that $a = mT$.

Writing $m$ as linear combination of the basis $R_B \cup N_B$, we have $m = \sum_{r_i \in R_B} r_i \cdot \lambda_i + \sum_{n_i \in N_B} n_i \cdot \gamma_i$, where $\lambda_i, \gamma_i$ in $R$.

Now,

$$\begin{aligned}
a &= mT \\
&= \left( \sum_{r_i \in R_B} r_i \cdot \lambda_i + \sum_{n_i \in N_B} n_i \cdot \gamma_i \right) T \\
&= \sum_{r_i \in R_B} (r_i \cdot \lambda_i) T + \sum_{n_i \in N_B} (n_i \cdot \gamma_i) T \\
&= \sum_{r_i \in R_B} r_i T \cdot \lambda_i + \sum_{n_i \in N_B} n_i T \cdot \gamma_i \\
&= \sum_{r_i \in R_B} r_i T \cdot \lambda_i \text{ since } n_i T = 0 \, \forall n_i \in N_B, \\
&= \sum_{r_i \in R_B} (r_i T) \cdot \lambda_i.
\end{aligned}$$

So $R_B T$ is a spanning set of $MT$.

Let us now prove that it is linearly independent.

Let us take a linear combination of elements in $R_B T$ which equal $0$. We have

$$\begin{aligned}
\sum_{r_i \in R_B} (r_i T) \cdot \lambda_i = 0 &\Rightarrow \sum_{r_i \in R_B} (r_i \cdot \lambda_i) T = 0 \\
&\Rightarrow \left( \sum_{r_i \in R_B} (r_i \cdot \lambda_i) \right) T = 0
\end{aligned}$$

$$\Rightarrow \sum_{r_i \in R_B} (r_i \cdot \lambda_i) \in \ker(T)$$
$$\Rightarrow \sum_{r_i \in R_B} (r_i \cdot \lambda_i) = \sum_{n_i \in N_B} n_i \cdot \gamma_i \text{ for some } \gamma_i \in R$$
$$\Rightarrow \sum_{r_i \in R_B} (r_i \cdot \lambda_i) - \sum_{n_i \in N_B} n_i \cdot \gamma_i = 0$$
$$\Rightarrow \lambda_i = \gamma_i = 0 \text{ for all } i \text{ since it is a linear combination of element in a basis of } M.$$

So $R_B T$ is a basis of $MT$ and since $|R_B| = |R_B T|$, we have $\dim M' = \dim MT$.

Thus, $\dim M = \dim M' + \dim \ker(T) = \dim MT + \dim \ker(T)$. □

**4.1.7 Corollary.** (Beidleman, 1966) Let $R$ be a nearfield. $M_R$ and $A_R$ two near vector spaces over $R$. Let $T$ be a surjective linear mapping of $M_R$ into $A_R$. We have that $T$ is bijective if and only if, $\dim M = \dim A$.

*Proof.* If $T$ is bijective, then $T$ is a linear isomorphism of $M_R$ onto $A_R$. From Theorem 3.4.3, $\dim M = \dim A$.

Assume $\dim M = \dim A$ and $\ker(T) \neq \{0\}$. From Theorem 4.1.6, $\dim M = \dim A + \dim \ker(T)$ and this is a contradiction. This implies that $T$ is injective. Therefore, $T$ is a bijective linear mapping. □

## 4.2 Normal linear mappings

One of the differences between vector space and near vector space is stated below.

**4.2.1 Theorem.** *(Beidleman, 1966)*

*Let $R$ be a nearfield. $M_R$ and $A_R$ two near vector spaces over $R$. Let $T$ be a linear mapping of $M_R$ onto $A_R$. In general, $MT$ is not a subspace of $A_R$.*

*Proof.* In the following, we will construct a counterexample of this. For this we will use the the nearfield $DN(3,2)$ with the near vector space $R^2$ and the $R$ -subgroup $H$ stated in Example 3.3.9.

We will construct a map from $R^2$ to $R^2$ such that the image is $H$.

Let $T$ be the map defined by:

$$\begin{aligned} T : R^2 &\to R^2 \\ (a,b) &\mapsto (1,x) \cdot a. \end{aligned}$$

We see that $((a,b)+(c,d))T = (a+c,b+d)T = (1,x)\cdot(a+c) = (1,x)\cdot a+(1,x)\cdot c = (a,b)T+(c,d)T$ and $((a,b)T)\cdot r = ((1,x)\cdot a)\cdot r = (1,x)\cdot(ar) = ((a,b)\cdot r)T$. So $T$ is a linear mapping. By definition of $T$, $(a,b)T = (1,x)\cdot a$ is in $H$ for all $(a,b)$ in $R^2$ so $R^2 T = H$. But as we saw below, $H$ is not a subspace of $R^2$. □

From this counterexample, we want to have a sufficient condition for a linear mapping to have its image be a subspace of a near vector space.

**4.2.2 Definition.** (Beidleman, 1966) Let $R$ be a nearfield and $M_R$ a near vector space over $R$. We say that a linear mapping $T$ from $M_R$ to itself is **normal** if and only if, $MT$ is a subspace of $M_R$.

Let $R$ be a nearring and $M_R$ a near vector space over $R$. For each pair $(m, r)$ in $M \times R$, let $s_{m \cdot r}$ be the mapping defined from $M_R$ to itself by $s_{m \cdot r}(x) = (m + x) \cdot r - m \cdot r$ for every $x$ in $M_R$. Define

$$S(M, R) = \{s_{m \cdot r} | m \in M, r \in R\}.$$

We have a sufficient condition for $T$ to be normal.

**4.2.3 Theorem.** *(Beidleman, 1966) (Beidleman, 1966) Let $R$ be a nearfield and $M_R$ a near vector space over $R$. Let $T$ be a linear mapping from $M$ to itself. If for all elements $s_{m \cdot r}$ in $S(M, R)$ we have $T \circ s_{m \cdot r} = s_{m \cdot r} \circ T$, then $T$ is normal.*

*Proof.* Let us assume that for all elements $s_{m \cdot r}$ in $S(M, R)$ we have $T \circ s_{m \cdot r} = s_{m \cdot r} \circ T$. In particular, if we fix $r$ to be $1$ and since $M_R$ is unitary, we have for all $m$ and $a$ in $M$, $aT \circ s_{m \cdot 1} = as_{m \cdot 1} \circ T$.

We have $as_{m \cdot 1} = (m + a) \cdot 1 - m \cdot 1 = m + a - m$. It means $m + aT - m = (m + a - m)T$ in $MT$. It shows that $(MT, +)$ is a normal subgroup of $(M, +)$.

Now using the fact that $T \circ s_{m \cdot r} = s_{m \cdot r} \circ T$, we have for any element $m$ in $M$, $r$ in $R$ and $aT$ in $MT$, $(m + xT) \cdot r - m \cdot r = [(m + x) \cdot r - m \cdot r]T$ which in $MT$ and so $MT$ is a subspace of $M_R$. Thus, $T$ is normal. □

This next result show that every subspace can be associated with a normal linear mapping.

**4.2.4 Theorem.** *(Beidleman, 1966)*

*Let $R$ be a neafield, $M_R$ be a near vector space over $R$ and $A$ an arbitrary subspace of $M$. There is a linear mapping $T$ such that $MT = A$ and for all elements $m$ in $M$ and $r$ in $R$, we have $T \circ s_{m \cdot r} = s_{m \cdot r} \circ T$.*

*Proof.* From Theorem 3.3.5, there is a subspace $B$ of $M_R$ such that $M_R = A \oplus B$. If $m = a + b$, then the mapping $T$ from $M$ into itself defined by $mT = a$ is a surjective linear mapping of $M_R$ into $A$. Since for each $a$ in $A$ we have $aT = a$, $((a + b) + (a' + b'))T = a + a' = (a + b)T + (a' + b')T$ and $((a + b)T) \cdot r = a \cdot r = ((a + b) \cdot r)T$ for all $a, a'$ in $A$, $b, b'$ in $B$ and $r$ in $R$.

Let $m = a + b$ and $m' = a' + b'$, and $r$ in $R$ where $a, a'$ in $A$, $b, b'$ in $B$.

We will show now that $s_{m \cdot r} \circ T = T \circ s_{m \cdot r}$. From Proposition 2.4.2 and Lemma 2.4.4, we have

$$\begin{aligned} m'(s_{m \cdot r} \circ T) &= [(m + m') \cdot r - m \cdot r]T \\ &= [(a + a') \cdot r + (b + b') \cdot r - a \cdot r - b \cdot r]T \\ &= (a + a') \cdot r - a \cdot r \end{aligned}$$

and

$$\begin{aligned} (m'T)s_{m \cdot r} &= (m + m'T) \cdot r - m \cdot r \\ &= (a + b + a') \cdot r - (a + b) \cdot r \\ &= (a + a') \cdot r + b \cdot r - a \cdot r - b \cdot r \\ &= (a + a') \cdot r - a \cdot r. \end{aligned}$$

□

Thus, $T$ commutes with the elements of $S(M, R)$.

Here we will have a characterisation of the subspaces of a near vector space.

**4.2.5 Corollary.** (Beidleman, 1966) Let $R$ be a nearfield, $M_R$ be a near vector space over $R$ and $A$ a subset of $M$. We have $A$ is a subspace of $M_R$ if and only if, there is a linear mapping $T$ of $M_R$ into itself such that for all elements $m$ in $M$ and $r$ in $R$, we have $T \circ s_{m \cdot r} = s_{m \cdot r} \circ T$ and $MT = A$.

*Proof.* By the definition of a normal linear map, the image is always a subspace. From Theorem 4.2.3 and Theorem 4.2.4 we prove that if $A$ is a subspace then we have a normal linear map of $M_R$ such that $T \circ s_{m \cdot r} = s_{m \cdot r} \circ T$ and $MT = A$. □

For the following, we will consider the analogue of a projection map in vector spaces.

**4.2.6 Theorem.** *(Beidleman, 1966) Let $R$ be a nearfield, $M_R$ a near vector space over $R$, $T$ a non-zero normal linear mapping such that $T^2 = T$. Then we have that $M_R = MT \oplus \ker(T)$.*

*Proof.* Since $T$ is a normal linear mapping, $MT$ is a subspace of $M_R$. Let $x$ be an element of $MT \cap \ker(T)$, then there exists $m$ in $M$ such that $mT = x$ and $0 = xT = mT^2 = mT = x$. We will show that $M_R = MT + \ker(T)$. Let $m$ be an element of $M_R$. Then $m = mT - mT + m$ and $(-mT + m)T = (-mT)T + mT = -mT^2 + mT = 0$. Thus, $M_R = MT + \ker(T)$ and so $M_R = MT \oplus \ker(T)$. □

## 4.3 On the properties of $R$-subgroups

One other difference between vector space and near vector space will be developed in this section. Let $R$ be a finite nearfield and $M_R$ a finite dimensional near vector space over $R$. In the case where $R$ is a finite field, $M$ is a vector space and the concept of an $R$-subgroup coincides with a subspace. Thus, we are particularly interested in the case where $R$ is a proper nearfield, this means there are some elements $\lambda, \alpha$ and $\beta$ in $R$ such that $(\alpha + \beta)\lambda \neq \alpha\lambda + \beta\lambda$ because of the lack of right distributivity. As all finite dimensional near vector spaces over $R$ are $R$-isomorphic to $R^n$ for some integer $n$, we will consider only the near vector spaces $R^n$ and all results will hold for other finite dimensional near vector spaces by isomorphism.

**4.3.1 Definition.** (Djagba, 2019b) Let $\{v_1, v_2, \dots v_k\}$ be a finite set of elements of $R^n$. The smallest $R$-subgroup containing $\{v_1, v_2, \dots v_k\}$ is called the **gen** of $\{v_1, v_2, \dots v_k\}$.

We want to describe $gen(\{v_1, v_2, \dots v_k\})$. For this, we will introduce some notation.

Let $LC_0(\{v_1, v_2, \dots v_k\}) := \{v_1, v_2, \dots v_k\}$ and for $n \geq 0$, let $LC_{n+1}(\{v_1, v_2, \dots v_k\})$ be the set of all linear combinations of elements in $LC_n(\{v_1, v_2, \dots v_k\})$.

It means $LC_{n+1}(\{v_1, v_2, \dots v_k\}) = \left\{ \sum\limits_{w \in LC_n} w \cdot \lambda_w : \lambda_w \in R \, \forall w \in LC_n(\{v_1, v_2, \dots v_k\}) \right\}$. If there will be no ambiguity with regard to the initial set of elements we will write $LC_n$ instead of $LC_n(\{v_1, v_2, \dots v_k\})$.

**4.3.2 Theorem.** *(Djagba, 2019b) Let $R$ be a finite nearfield and $v_1, v_2, \dots v_k$ elements of $R^n$. Then, $gen(\{v_1, v_2, \dots v_k\}) = \bigcup\limits_{i=0}^{\infty} LC_i$.*

*Proof.* For this, we will prove that $\bigcup_{i=0}^{\infty} LC_i$ is an $R$-subgroup of $R^n$ and contained in every $R$-subgroup containing $\{v_1, v_2, \ldots, v_k\}$.

Since $0_{R^n} = v_1 \cdot 0$, $0_{R^n}$ is in $LC_1$. Thus $\bigcup_{i=0}^{\infty} LC_i$ is non-empty. Let $x, y$ be two elements of $\bigcup_{i=0}^{\infty} LC_i$. So for some indices $i$ and $j$, $x$ is $LC_i$ and $y$ is in $LC_j$. Let $l = max\{i, j\}$. So $LC_i \subseteq LC_l$ and $LC_j \subseteq LC_l$. Then, $x - y$ is in $LC_{l+1}$.

This shows that $\left(\bigcup_{i=0}^{\infty} LC_i, +\right)$ is a subgroup of $(R^n, +)$. Now, let $x \in \bigcup_{i=0}^{\infty} LC_i$. So $x$ is in $LC_l$ for some $l$. So $x \cdot r$ is in $LC_{l+1}$ as a linear combination of $x$.

Let us now show that $\bigcup_{i=0}^{\infty} LC_i$ is contained in every $R$-subgroup $S$ of $R^n$ containing the set $\{v_1, \ldots, v_k\}$. We will prove that for all $i$, $LC_i$ is a subset of $S$, for this we will use induction on $i$. For $i = 0$ we have $LC_0 \subseteq S$. Let us assume that $LC_i \subseteq S$ for $i$ in $\mathbb{N}$. Then let $x$ be an element in $LC_{i+1}$. We have $x = \sum_{w \in LC_i} w \cdot \lambda_w$, where $\lambda_w$ is in $R$. But $w$ is in $LC_i \subseteq S$, it implies that $w \cdot \lambda_w$ is also in $S$ since $S$ is an $R$-subgroup. Then the linear combination $x = \sum_{w \in LC_i} w \cdot \lambda_w$ is also in $S$. Hence $\bigcup_{i=0}^{\infty} LC_i \subseteq S$. □

**4.3.3 Example.** Let us take the Dickson nearfield $R = DN(3, 2)$. We see that for $\alpha = 1+x, \beta = 2+x$, and $\gamma = x$, we have $(\alpha+\beta)\gamma = (2x)x = 1$ and $\alpha\gamma+\beta\gamma = 1+2x+1+x = 2$. Let us consider the near vector space $R^3$ with the elements $v_1 = (1, 1, 0)$ and $v_2 = (1, 0, 1)$. We have: $LC_0(v_1, v_2) = \{v_1, v_2\}$. The element

$$\begin{aligned} v_3 &= (v_1\alpha + v_2\beta) \cdot \gamma - v_1 \cdot (\alpha\gamma) - v_2 \cdot (\beta\gamma) \\ &= ((1, 1, 0) \cdot (1 + x) + (1, 0, 1) \cdot (2 + x)) \cdot x - (1, 1, 0) \cdot ((1 + x)x) - (1, 0, 1) \cdot ((2 + x)x) \\ &= (2x, 1 + x, 2 + x) \cdot x - (1 + 2x, 1 + 2x, 0) - (1 + x, 0, 1 + x) \\ &= (-1, 0, 0) \end{aligned}$$

is in $LC_2(\{v_1, v_2\})$.Then for any element $(a, b, c)$ in $R^3$, we have $(a, b, c) = v_1 \cdot b + v_2 \cdot c + v_3 \cdot (-a+b+c)$. Thus $(a, b, c)$ is in $LC_3(\{v_1, v_2\}) \subseteq gen(\{v_1, v_2\})$. Hence, $gen(\{v_1, v_2\}) = R^3$.

In general it is not easy to determine the $gen$ of a given set of elements, an algorithm to do this task is given in (Djagba, 2019b). The result in Example 4.3.3 will be generalized in the following theorem.

**4.3.4 Theorem.** *(Djagba, 2019b) Let $R$ be a proper nearfield and $n$ an integer. Then there exists elements $v_1, \ldots, v_{n-1}$ of $R^n$ such that $gen(\{v_1, \ldots, v_{n-1}\}) = R^n$.*

*Proof.* Suppose $R$ is a proper nearfield. Then by the lack of right distributivity, there exist $\alpha_1, \ldots, \alpha_{n-1}$ in $R$ and $\lambda$ in $R$ such that $\left(\sum_{i=1}^{n-1} \alpha_i\right) \lambda \neq \sum_{i=1}^{n-1} \alpha_i \lambda$.

For $i$ in $1, \ldots, n-1$ in $\mathbb{N}$, let $v_i$ be the element such that the first and the $i+1^{th}$ components are 1 and the others $0$. Then

$$\begin{aligned} v &= ((1,1,\ldots,0)\cdot\alpha_1 + (1,0,1,\ldots,0)\cdot\alpha_2 + \ldots + (1,0,\ldots,1)\cdot\alpha_{n-1})\cdot\lambda \\ &\quad - (1,1,\ldots,0)\cdot\alpha_1\lambda - (1,0,1,\ldots,0)\cdot\alpha_2\lambda - \ldots - (1,0,\ldots,1)\cdot\alpha_{n-1}\lambda \\ &= ((\alpha_1,\alpha_1,\ldots,0) + (\alpha_2,0,\alpha_2,\ldots,0) + \ldots + (\alpha_{n-1},0,\ldots,\alpha_{n-1}))\cdot\lambda \\ &\quad - (\alpha_1\lambda,\alpha_1\lambda,0,\ldots,0) - (\alpha_2\lambda,0,\alpha_2\lambda,0,0,\ldots,0) - \ldots - (\alpha_{n-1}\lambda,0,\ldots,\alpha_{n-1}\lambda) \\ &= (\alpha_1+\alpha_2+\ldots+\alpha_{n-1},\alpha_1,\alpha_2,\ldots,\alpha_{n-1})\cdot\lambda \\ &\quad - (\alpha_1\lambda+\alpha_2\lambda+\ldots+\alpha_{n-1}\lambda,\alpha_1\lambda,\alpha_2\lambda,\ldots,\alpha_{n-1}\lambda) \\ &= ((\alpha_1+\alpha_2+\ldots+\alpha_{n-1})\lambda,\alpha_1\lambda,\alpha_2\lambda,\ldots,\alpha_{n-1}\lambda) \\ &\quad - (\alpha_1\lambda+\alpha_2\lambda+\ldots+\alpha_{n-1}\lambda,\alpha_1\lambda,\alpha_2\lambda,\ldots,\alpha_{n-1}\lambda) \\ &= ((\alpha_1+\alpha_2+\ldots+\alpha_{n-1})\lambda - \alpha_1\lambda - \alpha_2\lambda - \ldots - \alpha_{n-1}\lambda,0,0,\ldots,0) \\ v &= (\gamma,0,0,\ldots,0), \end{aligned}$$

where $\gamma = (\alpha_1+\alpha_2+\ldots+\alpha_{n-1})\lambda - \alpha_1\lambda - \alpha_2\lambda - \ldots - \alpha_{n-1}\lambda$, is in $LC_2$. We see that $v\cdot\gamma^{-1} = (1,0,0,\ldots,0)$ and for $(x_1,x_2,\ldots,x_n)$ in $R^n$, we have:

$$\begin{aligned} (x_1,x_2,\ldots,x_n) &= (1,1,\ldots,0)\cdot x_2 + (1,0,1,\ldots,0)\cdot x_3 + \ldots + (1,0,\ldots,1)\cdot x_{n-1} \\ &\quad - (1,0,\ldots,0)\cdot(x_n + x_{n-1} + \ldots + x_2 - x_1) \\ &= v_1\cdot x_2 + v_2\cdot x_3 + \ldots + v_{n-1}\cdot x_{n-1} - v\cdot\gamma^{-1}(x_n + x_{n-1} + \ldots + x_2 - x_1). \end{aligned}$$

We see from there that $(x_1,x_2,\ldots,x_n)$ is in $LC_3(\{v_1,v_2,\ldots,v_{n-1}\}) \subseteq gen(\{v_1,v_2,\ldots,v_{n-1}\})$. Hence, $gen(\{v_1,v_2,\ldots,v_{n-1}\}) = R^n$. □

The following corollary states that the increasing set $\bigcup_{i=0}^{\infty} LC_i$ will become stationary from an integer $p$.

**4.3.5 Corollary.** (Djagba, 2019b) Let $R$ be a nearfield, let us consider the near vector space $R^n$ for an integer $n$ and let $V$ be a finite set of elements in $R^n$. If $gen(V) = R^n$, then there exists a $p$ in $\mathbb{N}$ such that $LC_p = R^n$.

*Proof.* From Theorem 4.3.2, $R^n = \bigcup_{i=0}^{\infty} LC_i$. Let us consider the elements $v_1,\ldots,v_{n-1},v$ in the proof of Theorem 4.3.4. As they are in $R^n$, there exists $n$ integers $p_1,\ldots,p_n$ such that $v_1,\ldots,v_{n-1},v$ will be respectively in $LC_{p_1},\ldots,LC_{p_n}$. Then all of them will be in $LC_{max(p_1,\ldots,p_n)}$. Since every element of $R^n$ is a linear combination of $v_1,\ldots,v_{n-1},v$, they will be in $LC_{max(p_1,\ldots,p_n)+1}$. So $R^n = LC_p$ where $p = max(p_1,\ldots,p_n)+1$.

In particular for $\{v_1,v_2\}$ described in Example 4.3.3 we see that $p \leq 3$. □

This last lemma gives the description of the $gen$ of a singleton in $R^n$.

**4.3.6 Lemma.** (Djagba, 2019b) Let $R$ be a nearfield and $M_R$ a near vector space over $R$. For an element $v$ in $M$, $gen(\{v\}) = v\cdot R$.

*Proof.* For all positive integers $p$, $LC_p(\{v\}) = v\cdot R$. So from Theorem 4.3.2, $gen(\{v\}) = v\cdot R$. □

# 5. Conclusion and future work

## 5.1 Conclusion

In this essay, after giving the preliminary material, we defined a near vector space as introduced by Beidleman (1966). We have seen some characteristics of vector spaces transferred to near vector spaces such as basis, dimension, subspaces, linear mappings. By this transfer, additional requirements were added but they collapse in the original definition for the case of vector spaces. Some similarities with the theory of vector spaces were shown.

By the work of Djagba (2019b), we can characterize every finite dimensional near vector space to be $R$-isomorphic to a power of nearfields. Using Beidleman (1966) and Djagba (2019b), we showed that the image of a near vector space under a linear mapping is not necessarily a subspace of the target near vector space.This motivates the study of the normal linear mappings which map a near vector space to a subspace of itself. We gave a sufficient condition for a linear mapping to be normal and a similar notion of projection as in vector space theory. Since every image of a near vector space by a linear mapping is an $R$-subgroup, this motivates the study of $R$-subgroups. We have shown that a proper near vector space $R^n$ can be generated as an $R$-subgroup by $n-1$ elements which is not the case for a vector space.

## 5.2 Future work

As future work, we would like to study from which integer the increasing sequence $LC_i$ defined in 4.3.2 become stationary according to Corollary 4.3.5. We also want to study the structure of the quotient of a near vector space by a subspace.

# Acknowledgements

I am very grateful to God, the creator of the universe, who gives me strength to finish this work under this trouble of the pandemic COVID-19.

Next, I appreciate my supervisor Dr. Karin-Therese HOWELL for her guidance to make this work fruitful. She always motivated me through her constructive comments and talking to her was always so nice. Thank you Karin.

I would also like to thank Prof. Neil TUROK founder of AIMS, Prof. Barry GREEN the Director of AIMS South Africa, Simukai UTETE the Academic Director of AIMS South Africa, all lecturers and all staffs of AIMS South Africa.

A special thanks to Dr. Prudence DJAGBA who introduced me to this area, gave me the materials needed for the research, carefully followed my work, commented and advised me from the beginning of this essay. Thank you so much.

I also appreciate my tutor Rahinatou Yuh NJAH who helped me a lot during the realization of the work. Through her corrections, advice, support and follow-up, my work was improved. You are a good tutor Rahina.

Thank you to our group members Tamara, Fameno, Abraham, my friends Rojo, Patrick, Hirmans, Sitraka, as well and all my classmates.

I want to remember also all my teachers form ENS Antananarivo.

This essay is dedicated to my family in Madagascar: Fanambinantsoa, Faniriantsoa, Antso and my aunt Hajamino.

# References


J. André. Lineare algebra über fastkörpern. *Mathematische Zeitschrift*, 136:295–313, 1974.

J. C. Beidleman. *On near-rings and near-ring modules*. University Microfilms, 1966.

G. K. Betsch. Ein radical für fastringe. In *Math. Zeitschr*, volume 78, pages 86–90. Springer, 1962. doi: 10.1007/BF01195153.

D. Blackett. Simple and semi-simple nearrings. In *American Mathematical Society*, volume 4, pages 772–785. American Mathematical Society, 1959.

L. E. Dickson. On finite algebras. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1905:358–393, 1905.

P. Djagba. On the generalized distributive set of a finite nearfield. *Journal of Algebra*, page 542, 10 2019a. doi: 10.1.1016/j.jalgebra.2019.09.020.

P. Djagba. *Contributions to the theory of Beidleman near vector spaces*. PhD thesis, Stellenbosch: Stellenbosch University, 2019b.

P. Djagba and K.-T. Howell. The subspace structure of finite dimensional near-vector spaces. *Linear and Multilinear Algebra*, pages 1–21, 02 2019. doi: 10.1.80/03081087.2019.1582610.

H. Karzel and G. Kist. Determination of all near vector spaces with projective and affine fibrations. *J.Geom*, 23:124–127, 1984.

J. D. Meldrum. *Near rings and their links with groups*. Number 134. Pitman Advanced Publishing Program, 1985.

G. Pilz. *Near-rings: the theory and its applications*, volume 23. Elsevier, 2011.

G. Pilz. The development of nearrings and nearfields with greetings to the 24th nearring conference. In *Nearrings, Nearfields and Related Topics*, volume 1, pages 1–5. WORLD SCIENTIFIC, 2017.

H. Zassenhaus. über endliche fastkörper. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 11, pages 187–220. Springer, 1935.