Asymmetric Topology: A New Framework in Software Engineering

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Abstract

In this dissertation, we look into the possibility of weakening to a maximum the imposed conditions required of a self mapping in the postulation of Kleene's fixed point theorem to guarantee a fixed point, when the undelying partially ordered set comes from a T_0-quasi-metric space. Ultimately, we apply the weakened fixed point technique in asymptotic complexity analysis of algorithms whose execution time of computing follows a recurrence equation.

Keywords: Kleene's fixed point ; self mapping ; partially ordered set ; quasi-metric space ; algorithm; asymptotic complexity analysis ; recurrence equation.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Abraham O Orinda, 14 May 2020

Contents

1. Introduction

When computer programmers write algorithms to solve a certain problem, two considerations are often central to them. On the one hand, the meaning of such an algorithm. On the other hand, the time it takes before the aforesaid algorithm gives the desired output. Fixed point theory, as utility, has provided a foundation for most research work in different fields of Software Engineering such as asymptotic complexity analysis and denotational semantics (Mahmudova et al., 2019). The chain complete partially ordered sets, in addition, presents a natural class of ordered sets within which such fixed point theorems are advanced.

Based on the widely acclaimed Kleene's fixed point theorem, introduced by Stephen Cole Kleene, for monotonic endofunctions in partially ordered sets, Scott D.S. in (Scott, 1970) developed a fixed point induction principle which has remained vital in Denotational Semantics. More succinctly, Scott's inductive principle permits to obtain the meaning of a recursively defined algorithm as a supremum of the non-recursive mapping. The aforementioned mapping models the evolution of the program execution while the partially ordered set ciphers some computational information. Solutions to such fixed point equations are often determined through successive approximations, so that each iterative step provides more computational information than the preceding ones.

The supremum of the increasing sequence contained in the hypothesis of Kleene's fixed point theorem (see Theorem 3.1.1) is the most significant in Scott's framework since it ciphers all the computational information of such a recursive specification. Scott's fixed point technique was however qualitative as they were based on partially ordered sets and not metric spaces.

While ardent to preserve the original Scott ideas, Matthews introduced a quantitative fixed point technique based on an extension of the Banach's contraction principle in a partial metric space (defined in (Matthews, 1994), Definition 3.1). Unlike metric spaces, which are Hausdorff, the generalized metric spaces naturally induce a T_0 topolgy which is essential in inducing an order in such spaces. The quantitative nature of Matthews' fixed point technique added the possibility of providing a measure of the degree of approximation of constituent elements of an analogous to Scott's inductive model. As an application, using Kahn's model of parallel computation, he specified lazy data flow deadlock (see (Matthews, 1995) for a detailed elucidation).

M. P Schellekens raised the possibility of extending Scott's ideas to other fields of Software Engineering other than denotational semantics. Like Matthews, he tapped into the utility of Banach's contraction principle to front a quantitative fixed point technique. His novel generalized distance notion, referred hereafter as a quasi-metric (defined in Definition 2.1.1), presented a framework for Asymptotic complexity analysis of algorithms. His technique allowed him to model the asymptotic upper bounds of complexity of such algorithms whose execution time satisfies a recurrence equation by means of a complexity space defined in (Schellekens, 1995).

In this dissertation, we study the likelihood of weakening to a maximum the necessary conditions of order completeness of the partially ordered sets and order continuity of the self mapping in the statement of the Kleene's fixed point theorem required to guarantee a fixed point. We present a generalized version of the aforementioned fixed point technique that imposes weaker and more local than global conditions on the self mapping in a T_0-quasi-metric space. The framework presented in Corollary 3.3.8 permits to simultaneously determine the upper and lower asymptotic bounds as well as model the meaning of a recursive specification.

The rest of this dissertation is organized as follows:

In Chapter 2, we introduce the pertinent notions in T_0-quasi-metric spaces. Owing to lack of symmetry in the aforementioned spaces, we introduce the different notions of completion necessary in characterizing convergence in such spaces. Ordered sets and their properties and an introduction to complexity analysis are mentioned here given their importance in understanding our object of study in subsequent chapters.

In Chapter 3, we present a weakened version of the Kleene's fixed point theorem in a T_0-quasi-metric framework (see Corollary 3.3.8). Further, we present an illustrative example to show that the assumed conditions in the statement of the aforementioned result cannot be weakened any further.

In Chapter 4, we discuss the utility of Corollary 3.3.8 as a Mathematical tool for discussing the asymptotic complexity analysis of algorithms. We end this chapter by showing, in details, how to obtain the complexity class of Towers of Hanoi algorithm using the new technique presented in Theorem 4.2.3.

2. The Building Blocks

This chapter serves to provide a basic background to the content of this project as well as fix notations. The content follows (Willard, 2004), (Schröder, 2003), (Davey and Priestley, 2002), (Goubault-Larrecq, 2013), (Reilly et al., 1982) and (Brassard and Bratley, 1988).

2.1 T_0-quasi-metric spaces and their topologies

2.1.1 Definition. For a non-empty set X, the real-valued function $d : X \times X \to [0, \infty)$ is a quasi-pseudometric if and only if $\forall x, y, z \in X$:

1. $d(x, x) = 0$ and
2. d satisfies triangle inequality, $d(x, z) \leq d(x, y) + d(y, z)$,

The pair (X, d) is called a **quasi-pseudometric space**, where d defines a quasi-pseudometric on X.

Moreover, we shall say a quasi-pseudometric d is a $\mathbf{T_0}$**-quasi-metric** on X if:

3. $d(x, y) = 0 = d(y, x) \implies x = y, \forall x, y \in X$. This condition is often called the T_0-condition.

The ordered pair (X, d) in this case becomes a $\mathbf{T_0}$**-quasi-metric space**.

A quasi-pseudometric d which further satisfies:

4. $d(x, y) = d(y, x), \forall x, y \in X$, is called a **pseudo-metric** on X .
5. $d(x, y) = 0 \iff x = y, \forall x, y \in X$, is a **quasi-metric** on X.

A T_0-quasi-metric which satisfies 4 is the usual **metric** on X.

Accordingly, the pair (X, d) is called a pseudo-metric, quasi-metric, and a metric space, respectively.

2.1.2 Remark. In T_0-quasi-metric spaces, we can have $d(x, y) = 0$ or $d(y, x) = 0$ for distinct values of $x, y \in X$.

2.1.3 Remark. In some cases, we may assume that a quasi-pseudometric d maps into the set $[0, \infty]$. This describes an **extended quasi-pseudometric** (respectively, an extended T_0-quasi-metric). Here, the quasi-pseudometric d can take the value ∞.

We henceforth denote by (X, d) a T_0-quasi-metric space unless stated otherwise.

2.1.4 Example. Let X be the set of real numbers and let $d : X \times X \to \mathbb{R}^+$ be the map defined by:

$$d(x, y) = \begin{cases} y - x & \text{if } y \geq x \\ x - y & \text{if } y < x \end{cases}$$

Then (X, d) is a T_0-quasi-metric space.

To see that this is a T_0-quasi-metric;

1. $d(x, x) = x - x = 0$

2. For the T_0-condition, we have that,

$$d(x,y)=0 \implies y-x=0 \implies y=x.$$

and

$$d(y,x)=0 \implies x-y=0 \implies x=y.$$

thus, in both cases,

$$x=y$$

3. To show the triangle inequality, without loss of generality, we distinguish the following two cases:
   (a) $y \geq x$. We have that $d(x,y)=y-x$.
   If $z \geq y \geq x$, then $d(x,z)=z-x$ and $d(z,y)=0$.
   If $y \geq z \geq x$, then $d(x,z)=z-x$ and $d(z,y)=y-z$.
   If $y \leq x \leq z$, then $d(x,z)=0$ and $d(z,y)=y-z$.
   (b) $y < x$. We have that $d(x,y)=x-y$.
   If $z < y < x$, then $d(x,z)=x-z$ and $d(z,y)=0$.
   If $y < x < z$, then $d(x,z)=0$ and $d(z,y)=z-y$.
   If $y < z < x$, then $d(x,z)=x-z$ and $d(z,y)=z-y$.
   Therefore, $\forall x,y,z \in \mathbb{R}^+$, we have that $d(x,y) \leq d(x,z)+d(z,y)$.

The next definition is as a consequence of lack of symmetry in T_0-quasi-metric spaces.

2.1.5 Definition. Let d be a quasi-pseudometric on a set X. Then the real-valued function $d^{-1}: X \times X \to [0,\infty)$ defined by

$$d^{-1}(x,y)=d(y,x)$$

whenever $x,y \in X$ is a quasi-pseudometric, called the **dual** or **conjugate quasi-pseudometric** of d.

2.1.6 Remark. For a quasi-pseudometric d on X, d becomes a pseudometric whenever $d=d^{-1}$.

2.1.7 Remark. We note that for any (T_0-)quasi-pseudometric d, the real-valued function d^s on X defined by

$$d^s(x,y)=\max\{d(x,y),d^{-1}(x,y)\}$$

is a metric on X (respectively, pseudometric).

We verify that the real valued function d^s as defined is in fact a metric if d is a T_0-quasi-metric on X.

1.

$$\begin{aligned} d^s(x,x) &= \max\{d(x,x),d^{-1}(x,x)\} \\ &= \max\{d(x,x),d(x,x)\} \\ &= \max\{0,0\} \end{aligned}$$

Therefore, $\forall x \in X$, $d^s(x,x) = 0$.

2.

$$d^s(x,y) = 0 \iff \max\{d(x,y), d^{-1}(x,y)\} = 0$$

But

$$\max\{d(x,y), d(y,x)\} = 0 \iff d(x,y) = 0 \vee d(y,x) = 0$$

Since d is T_0,

$$d(x,y) = 0 \vee d(y,x) = 0 \iff x = y$$

Therefore, $\forall x, y \in X, d^s(x,y) = 0 \iff x = y$

3.

$$\begin{aligned} d^s(x,y) &= \max\{d(x,y), d^{-1}(x,y)\} \\ &= \max\{d(x,y), d(y,x)\} \\ &= \max\{d(y,x), d(x,y)\} \\ &= \max\{d(y,x), d^{-1}(y,x)\} \\ &= d^s(y,x) \end{aligned}$$

Hence symmetric.

4. To show the triangle inequality, without loss of generality, we distinguish the following two cases:
   (a) $\max\{d(x,y), d^{-1}(x,y)\} = d(x,y)$. We have that $d^s(x,y) = d(x,y)$.

   If $x \geq y \geq z$, then $d^s(x,z) = d(x,z)$ and $d^s(z,y) = 0$.

   If $z \geq y \geq x$, then $d^s(x,z) = 0$ and $d^s(z,y) = d(z,y)$.

   If $x \geq z \geq y$, then $d^s(x,z) = d(x,z)$ and $d^s(z,y) = d(z,y)$.

   Therefore, $\forall x, y, z \in \mathbb{R}^+$, we have that

$$\begin{aligned} d^s(x,y) = d(x,y) &\leq d(x,z) + (z,y) \\ &= d^s(x,z) + d^s(z,y). \end{aligned}$$

   (b) $\max\{d(x,y), d^{-1}(x,y)\} = d^{-1}(x,y)$. We have that $d^s(x,y) = d^{-1}(x,y)$.

   If $y \geq x \geq z$, then $d^s(x,z) = 0$ and $d(z,y) = d^{-1}(z,y)$.

   If $z \geq y \geq x$, then $d^s(x,z) = d^{-1}(x,z)$ and $d^s(z,y) = 0$.

   If $y \geq z \geq x$, then $d^s(x,z) = d^{-1}(x,z)$ and $d^s(z,y) = d^{-1}(z,y)$.

   Therefore, $\forall x, y, z \in \mathbb{R}^+$, we have that

$$\begin{aligned} d^s(x,y) = d^{-1}(x,y) &\leq d^{-1}(x,z) + d^{-1}(z,y) \\ &= d^s(x,z) + d^s(z,y). \end{aligned}$$

The following is a well known result useful in characterizing bicompletion of (T_0-)quasi-pseudometric spaces.

2.1.8 Lemma. Consider a (T_0-)quasi-pseudometric space (X, d) and let $a, b, x, y \in X$. Then we have that,

$$|d(x,y) - d(a,b)| \leq d^s(x,a) + d^s(y,b).$$

Proof. From triangle inequality, we have that,

$$d(x,y) \leq d(x,a) + d(a,b) + d(b,y) \tag{2.1.1}$$
$$\implies d(x,y) - d(a,b) \leq d(x,a) + d(b,y) \tag{2.1.2}$$

Similarly,

$$d(a,b) \leq d(a,x) + d(x,y) + d(y,b) \tag{2.1.3}$$
$$\implies d(a,b) - d(x,y) \leq d(a,x) + d(y,b) \tag{2.1.4}$$

From (2.1.2) and (2.1.4), it follows then that $|d(x,y) - d(a,b)| \leq d^s(x,a) + d^s(y,b)$. □

2.1.9 Definition. Let $Y \subset X$ and let $\hat{d} : Y \times Y \to \mathbb{R}^+$ be the restriction $d|_{Y\times Y}$ of d to Y. Then $\hat{d}$ is the metric induced on Y by d. The ordered pair $(Y, \hat{d})$ is called a **subspace** of (X, d)

In the following we discuss the topology of T_0-quasi-metric spaces.

2.1.10 Definition. Let (X, d) be a T_0-quasi-metric space. Given a point $x_0 \in X$ and a real number $r > 0$, the set

$$B_d(x_0; r) := \{x \in X : d(x_0, x) < r\}$$

denotes **d-open balls** centered at x_0 with radius r. We note that the family of d-open balls :

$$\{B_d(x_0; r) : x_0 \in X, r > 0\}$$

yields the base for a topology $\mathcal{T}_d$ on X. The topology $\mathcal{T}_d$ is often referred to as the **forward topology** induced by d on X.

In a similar manner, Given a point $x_0 \in X$ and a real number $r > 0$, the set

$$C_d(x_0; r) := \{x \in X : d(x_0, x) \leq r\}$$

denotes **d-closed balls** centered at x_0 with radius r.

Similarly, the family of d^{-1}-open balls

$$\{B_{d^{-1}}(x_0; r) : x_0 \in X, r > 0\}$$

yields the base for the **backward topology** $\mathcal{T}_{d^{-1}}$ on X induced by d^{-1}

We note that $C_d(x_0; r)$ is $\mathcal{T}_{d^{-1}}$-closed, but not $\mathcal{T}_d$-closed in general.

2.1.11 Remark. The two topologies $\mathcal{T}_d$ and $\mathcal{T}_{d^{-1}}$ makes the T_0-quasi-metric space (X, d, d^{-1}) **bitopological**.

2.1.12 Definition. Let (X, d) be a (T_0-)quasi-pseudometric space. Given a point $x_0 \in X$ and a real number $r > 0$, the set

$$B_{d^s}(x_0; r) := \{x \in X : d(x_0, x) < r\}$$

denotes $\mathbf{d^s}$**-open balls** centered at x_0 with radius r. We note that the family of d^s-open balls :

$$\{B_{d^s}(x_0; r) : x_0 \in X, r > 0\}$$

yields the base for a topology $\mathcal{T}_{d^s}$ induced by the pseudometric (respectively, metric) on X.

2.1.13 Example. On $[0, \infty)$, set $u(x, y) = x \dot{-} y$ whenever $x, y \in [0, \infty)$. Here, $x \dot{-} y$ is interpreted as $\max\{x - y, 0\}$. Then u is the standard T_0-quasi-metric on $[0, \infty)$

Given a point $x \in [0, \infty)$, for any $\epsilon > 0$, the family of intervals $[x, \epsilon)$ forms a basis for open u-neighbourhoods while the family $(x - \epsilon, x]$ forms a basis for the open u^{-1}-neighbourhoods of x. Obviously, $u^s(x, y) = |x - y|, x, y \in [0, \infty)$. $\mathcal{T}_{u^s}$ thus defines the usual Euclidean topology on $[0, \infty)$.

We can then verify that the pair $([0, \infty), u)$ is a T_0-quasi-metric space.

1. $$u(x, x) = \max\{x - x, 0\} = \max\{0, 0\} = 0.$$

   Thus, $\forall x \in [0, \infty), u(x, x) = 0$.

2. $$u(x, y) = 0 \implies \max\{x - y, 0\} = 0 \implies x - y \leq 0 \implies x \leq y,$$

   and

   $$u(y, x) = 0 \implies \max\{y - x, 0\} = 0 \implies y - x \leq 0 \implies y \leq x.$$

   Hence, $x = y$.

   Therefore, $\forall x, y \in [0, \infty), u(x, y) = u(y, x) = 0 \implies x = y$, which verifies the T_0-condition.

3. To show triangle inequality, we distinguish the following two cases:

   (a) $\max\{x - y, 0\} = x - y$. We have that $u(x, y) = x - y$.

      If $z \leq y \leq x$, then $u(x, z) = x - z$ and $u(z, y) = 0$.

      If $y \leq z \leq x$, then $u(x, z) = x - z$ and $u(z, y) = z - y$.

      If $y \leq x \leq z$, then $u(x, z) = 0$ and $u(z, y) = z - y$.

   (b) $\max\{x - y, 0\} = 0$. We have that $u(x, y) = 0$.

      If $z \leq x \leq y$, then $u(x, z) = x - z$ and $u(z, y) = 0$.

      If $x \leq z \leq y$, then $u(x, z) = 0$ and $u(z, y) = 0$.

      If $x \leq y \leq z$, then $u(x, z) = 0$ and $u(z, y) = z - y$.

   Therefore, $\forall x, y, z \in [0, \infty)$, we have that $u(x, y) \leq u(x, z) + u(z, y)$.

2.1.14 Remark. A subspace Y of (X, d) is said to be **open** if for every $x_0 \in Y, \exists r > 0$ such that $B(x_0; r)$ is contained in Y. Let $(X, \mathcal{T})$ be a topological space and let $x \in X$ and $N \subset X$. Consider an open subset Y of X such that $x \in Y \subset N$. N is called the **neighbourhood** of $x \in X$.

2.1.15 Definition. For all distinct pairs $x, y \in X$, a topology is $\mathbf{T_0}$ if either there is a neighbourhood V_x of $x \in X$ such that $y \notin V_x$ or there is a neighbourhood W_y of $y \in X$ such that $x \notin W_y$ and $\mathbf{T_1}$ if there is a neighbourhood V_x of $x \in X$ such that $y \notin V_x$ and there is a neighbourhood W_y of $y \in X$ such that $x \notin W_y$

2.1.16 Remark. Each quasi-metric space (X, d) induces a T_0 topology $\mathcal{T}$, hence the name

T_0-quasi-metric. The T_0 topology is induced as follows

Let $x, y \in X$. For distinct values of x and y, we have that $d(x, y) > 0$ or $d(y, x) > 0$. But $d(x, y) > 0$ implies that there exists a neighbourhood U which contains x but not y. Similarly, $d(y, x) > 0$ implies that there exists a neighbourhood U which contains y but not x.

2.1.17 Example. Let $X = \mathbb{R}$. Consider the function $d : X \times X \to \mathbb{R}^+$ be defined by:

$$d(x, y) = \begin{cases} y - x & y \geq x \\ 1 & y < x \end{cases}$$

Then d is a quasi-metric on $\mathbb{R}$, the **Sorgenfrey quasi metric**. On $\mathbb{R}$, the forward topology often induced is the lower limit topology, and it is not metrizable on $\mathbb{R}$ since the usual distance function is not a metric on this space; it induces the usual topology, not the right-half plane topology. The backward topology, in the same sense, is the upper limit topology. The forward topology d induces a T_1-topology on $\mathbb{R}$, where a basis of open d-neighbourhoods of a point $x \in \mathbb{R}$ is generated by the family $[x, x+\epsilon), 0 < \epsilon < 1$. Clearly, $d^s(x, y) = 1, \forall x \neq y$. Therefore d^s is the discrete metric space in $\mathbb{R}$. Sorgenfrey quasi-metrics are examples of T_1 spaces, which provide counter examples to notions such as continuity in quasi-metric spaces.

We then recall a few properties of mappings between quasi-pseudometric spaces. We first note that;

2.1.18 Definition. Let X and Y be sets, and let $A \subset X$. A **mapping** T from A into Y associates each $x \in A$ with a single $y \in Y$, such that $y = Tx$. More concisely, a mapping T is such that $T : A \to Y, x \mapsto Tx$. The set A is called the domain of the mapping, and the target set Y the codomain of T.

2.1.19 Definition. A mapping $f : (X, d) \to (Y, e)$ between two T_0-quasi-metric spaces (X, d) and (Y, e) is called an **isometric map** provided that

$$d(x, y) = e(f(x), f(y)), \forall x, y \in X$$

2.1.20 Remark. Let $f : (X, d) \to (Y, e)$ be an isometric map. If (X, d) is a T_0-quasi-metric space, then f is **injective.** Indeed if (X, d) is a T_0-quasi-metric space, then for all $x, y \in X$

$$d(x, y) = 0 \implies e(f(x), f(y)) = 0,$$

and

$$d(y, x) = 0 \implies e(f(y), f(x)) = 0.$$

We then have that $f(x) = f(y) \implies x = y$.

Two T_0-quasi-metric spaces (X, d) and (Y, e) will be called **isometric** whenever there exists a bijective isometric map between them.

A mapping $f : (X, d) \to (Y, e)$ between two T_0-quasi-metric spaces (X, d) and (Y, e) will be called **quasi-uniformly continuous** provided that for each $\epsilon, \delta > 0$,

$$d(x, y) < \delta \implies e(f(x), f(y)) < \epsilon, \forall x, y \in X.$$

We then wind up this section by recalling pertinent notions on convergence in T_0-quasi-metric spaces.
2.1.21 Convergence in T_0-quasi-metric spaces.
In this subsection, we recall from the concept of sequences, some definitions that will be principal in understanding convergence and completeness in T_0-quasi-metric spaces.

2.1.22 Definition. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a topological space $(X, \mathcal{T})$ is said to **converge** to an element x, denoted by $x_n \to x$, if for every neighbourhood U of x, $\exists N = N(U)$ such that $\forall n \geq N, x_n \in U$. The point x, defined for every neighbourhood U of x, is called the **limit point** of $\{x_n\}_{n\in\mathbb{N}}$.

If (X, d) is a metric space, the above definition holds if $\mathcal{T}$ is the topology generated by the metric d at x.

2.1.23 Remark. A sequence that does not converge to any limit point is said to **diverge**.

2.1.24 Definition. Let (X, d) be a T_0-quasi-metric space and let $x \in X$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is

1. **right d-Cauchy** if for any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $d(x_n, x) \leq \epsilon, \forall n \geq n_\epsilon$. Accordingly, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is **left d-Cauchy** if $d(x, x_n) \leq \epsilon$.
2. **right K-Cauchy** if for any $\epsilon > 0$, $\exists\ n_\epsilon \in \mathbb{N}$ such that $\forall\ n \geq m \geq n_\epsilon$, $d(x_n, x_m) < \epsilon$. Accordingly, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is **left K-Cauchy** if for any $\epsilon > 0$, $\exists\ n_\epsilon \in \mathbb{N}$ such that $\forall\ n \geq m \geq n_\epsilon$, $d(x_m, x_n) < \epsilon$.

2.1.25 Definition. Let (X, d) be a (T_0-)quasi-pseudometric space and let $x \in X$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be **left d-convergent** to x if for any $\epsilon > 0$, there exists an integer n_ϵ such that $d(x, x_n) \leq \epsilon$, for all $n \geq n_\epsilon$. Accordingly, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is **right d-convergent** id $d(x_n, x) \leq \epsilon$.

2.1.26 Proposition. For the space (X, d), a sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be **$\mathcal{T}_{d^s}$-convergent** to a point $x \in X$ if and only if it is $\mathcal{T}_d$-convergent and $\mathcal{T}_{d^{-1}}$-convergent to $x \in X$.

2.1.27 Remark. The concepts of convergence and completeness of metric spaces can be extended to T_0-quasi-metric spaces. However, since T_0-quasi-metric spaces are naturally bitopological, we define such notions as convergence and completeness, with strict allegiance to the forward topology or backward topology induced by the T_0-quasi-metric therein.

Following Remark 2.1.27 above, we define, more contextually, completeness in a T_0-quasi-metric space.

2.1.28 Definition. The space (X, d) is said to be **right K-sequentially complete** provided that every right K-Cauchy sequence converges with respect to $\mathcal{T}_d$.

2.1.29 Definition. The space (X, d) is **right Smyth complete** provided that every right K-Cauchy sequence converges with respect to $\mathcal{T}_{d^s}$ (accordingly, left Smyth complete).

2.1.30 Definition. The space (X, d) is called **weightable** provided that there exists a function

$w_d : X \to \mathbb{R}^+$ such that
$$d(x, y) + w_d(x) = d(y, x) + w_d(y)$$
for all $x, y \in X$.

2.1.31 Definition. A metric space (X, d) in which every Cauchy sequence converges is said to be **complete**

2.1.32 Definition. A T_0-quasi-metric space (X, d) is called **bicomplete** whenever the metric d^s given in Remark 2.1.7 above is complete.

2.2 Order theory

In this section, we introduce some concepts from Order theory that serves to present the natural link there is between order and topology.

2.2.1 Definition. An order on a set X refers to the relation $\leq \subseteq X \times X$. We say that $\leq$ is a **quasiorder** (preorder) if it satisfies the following conditions:

1. $(\forall x \in X)\ x \leq x$, that is, $\leq$ is **reflexive.**
2. $(\forall x, y, z \in X)\ (x \leq y) \wedge (y \leq z) \implies x \leq z$, that is , the quasiorder is **transitive**.

2.2.2 Definition. A quasiorder that further satisfies the property that:

3. $(\forall x, y \in X)\ (x \leq y) \wedge (y \leq x) \implies x = y$, that is, the quasi order is **antisymmetric**.

is said to be a **partial order**.

A set X equipped with a partial order $\leq$ is called a **partially ordered set**, or just a **poset**, denoted by $(X, \leq)$.

We give some usual examples of posets.

2.2.3 Example. The usual orderings on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ are partial orderings.

2.2.4 Example. If $X = \mathbb{N}$ and define $x \leq y$ to mean "x divides y" , that is, $x|y$, then $(X, \leq)$ is a poset.

We can easily check that $(X, \leq)$ is in deed a poset;

1. By definition, $x|x, \forall x \in X$, thus the relation is reflexive.
2. $x|y \implies \exists k_1 \in \mathbb{N}$ such that $y = k_1 x$ and $y|x \implies \exists k_2 \in \mathbb{N}$ such that $x = k_2 y$. This only sufices when $x = y$, hence antisymmetric.
3. $x|y \implies \exists k_1 \in \mathbb{N}$ such that $y = k_1 x$ and $y|z \implies \exists k_2 \in \mathbb{N}$ such that $z = k_2 y \implies z = k_2(k_1 x) \implies z = k_3 x,$ where $k_3 = k_1 k_2$. Therefore $x|z$, hence the relation is transitive.

2.2.5 Example. Suppose X is a set and A the set of all functions $f : X \to \mathbb{R}$. Define $f \leq g$ to mean $f(x) \leq g(x)$ for $x \in X$. Then $(A, \leq)$ is a poset.

2.2.6 Definition. The **dual of a poset** $(X, \leq)$ is the ordered set $(X, \geq)$, that is,$\forall x, y \in X$

$$x \geq y \iff y \leq x.$$

2.2.7 Definition. Let $(X, \leq)$ be a poset. A subset $D \subset X$ is a **downset** if and only if

$$(\forall x \in D)(\forall y \in X)\ y \leq x \implies y \in D.$$

Let $(X, \geq)$ be the dual of a poset. A subset $U \subset X$ is an **upset** if and only if

$$(\forall x \in U)(\forall y \in X)\ y \geq x \implies y \in U.$$

Further, given an element $x \in X$, we will define the **lower closure** of x, denoted by $\downarrow x$ as the set $\downarrow x = \{y \in X : y \leq x\}$ and the **upper closure** of x, denoted by $\uparrow x$ as the set

$$\uparrow x = \{y \in X : y \geq x\}.$$

2.2.8 Definition. Given a subset $Y \subset X$, a **lower bound** of Y is an element $x_1 \in X$ such that $x_1 \leq y, \forall y \in Y$. Similarly, an element x_2 is an **upper bound** of Y such that $x_2 \geq y, \forall y \in Y$.

Moreover, the **greatest lower bound** of Y, if it exists, is an element $x \in X$ such that:

1. $x \in X$ is a lower bound of Y,
2. $x \in X$ is the greatest of such lower bounds of Y, that is, $\forall z \in X : (\forall y \in Y : z \leq y) \implies (z \leq x)$.

Accordingly, the least upper bound, if it exists, is an element $x \in X$ such that:

1. $x \in X$ is a upper bound of Y,
2. $x \in X$ is the least upper bounds of Y, that is, $\forall z \in X : (\forall y \in Y : z \geq y) \implies (x \leq z)$.

Upon their existence, it is a standard result that both the greatest lower bound and the least upper bound are unique.

Further, an element $x \in X$ is **maximal** in $(X, \leq)$ if $\exists y \in X : (x \leq y) \implies (y = x)$

2.2.9 Definition. If $\leq$ is a partial ordering on a set X, we say it is a **total ordering** if in addition it satisfies the connex property, that is, $(\forall x, y \in X)\ x \leq y \vee y \leq x$, that is, all points are comparable.

An ordered set will be called **finite** if and only if the underlying set is finite, or accordingly **infinite**.

2.2.10 Definition. If $\leq$ is a total ordering on $(X, \leq)$, then every non empty finite subset of X has a minimal and a maximal element. We say $\leq$ is a **well ordering** if every non-empty subset of X has a minimal element.

2.2.11 Remark. If $\leq$ is a well ordering of X, then it is a total ordering.

To see that the well ordering $\leq$ is a total ordering, we suppose $x, y \in X$. Consider the set $H = \{x, y\}$. Hypothetically, H has a minimal element. Suppose x is the minimal element, then $x \leq y$, and if y is the minimal element, then $y \leq x$.

2.2.12 Remark. Every total order is a **chain** of $(X, \leq)$

2.2.13 Remark. An **anti-chain** of a poset is a subset $A \subset X$ such that $(\forall x, y \in A)\; x \leq y \implies x = y$. A poset $(X, \leq)$ is an anti-chain if and only if X is an antichain of $(X, \leq)$.

2.2.14 Example. The posets $(\mathbb{N}, \leq), (\mathbb{Z}, \leq), (\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$ are total orderings (chains).

2.2.15 Definition. A **real sequence** is a mapping whose domain is the set $\mathbb{N}$, and the codomain is the set $\mathbb{R}$. We will denote by $\{x_n\}_{n\in\mathbb{N}}$ an arbitrary sequence, where x_n is the n-th term of the sequence, for $n \in \mathbb{N}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a poset $(X, \leq)$ is **decreasing** if $x_{n+1} \leq x_n, \forall n \in \mathbb{N}$. A sequence $\{x_n\}_{n\in\mathbb{N}} \in (X, \leq)$ is **increasing** if $x_n \leq x_{n+1}, \forall n \in \mathbb{N}$.

2.2.16 Definition. A poset $(X, \leq)$ is said to be **chain complete** provided that there exists a least upper bound(supremum) for every increasing sequence.

2.2.17 Definition. A T_0-quasi-metric space is a natural poset in the sense that it induces on X a **specialization partial order**, which we will denote as $\leq_d$.

The specialization partial order is defined as:

$$x \leq_d y \iff d(x, y) = 0.$$

To see that this is a partial order on X,

1. $d(x, x) = 0 \implies x \leq_d x$, hence reflexive.
2. $d(x, y) = 0 \implies x \leq_d y$ and $d(y, x) = 0 \implies y \leq_d x$. By T_0-condition, then $x = y$, hence antisymmetric.
3. $d(x, y) = 0 \implies x \leq_d y$ and $d(y, z) = 0 \implies y \leq_d z$. By triangular inequality, we have that $d(x, z) \leq d(x, y) + d(y, z) \implies d(x, z) = 0 \implies x \leq_d z$, hence transitive.

2.2.18 Definition. A T_0-quasi-metric space (X, d) is **chain complete** provided that the underlying poset $(X, \leq_d)$ is chain complete.

2.2.19 Definition. For every increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ whose supremum x in $(X, \leq)$ exists, a mapping $f : X \to X$ is said to be $\leq$**-continuous** provided that the supremum of the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ is $f(x)$.

2.2.20 Remark. A mapping f from a poset to itself, that is, $f : (X, \leq) \to (X, \leq)$ is **monotonic** if $f(x) \leq f(y)$ whenever $x \leq y$

2.2.21 Definition. For a set X, a self mapping $f : X \to X$ is a **fixed point** if $f(x) = x$, for all $x \in X$.

2.2.22 Remark. We will denote by $\mathbf{fix(f)}$ the set of all fixed points $\{x \in X : f(x) = x\}$.

2.3 Algorithms and complexity analysis

2.3.1 History of algorithms.
Algorithms are widely used today in every sphere of life. More insightfully, the crude definition of an algorithm categorises most activities we partake, as in fact, very likely to be algorithmically motivated. The word 'algorithm' in a more natural sense alludes to sets of computer programs executing complex Mathematical processes, when in reality, they should be understood as a set of instructions motivated towards solving a problem. The origin of algorithms, thus are by no surprise primeval.

The word "algorithm" draws its origin from Latinization of Al-Khwarizmi's name to " algoritm", a term the Latins used to describe the decimal positioning number system presented to them by Al-Khwarizm in 9BC. The first ever systematic solution of linear and quadratic equations, which was an instance of algorithm, is also attributed to Al-Khwarizm.

Early instances of algorithms were also witnessed with the Babylonians (600BC). In their attempts to provide solutions to the ancient mathematical problems, the Babylonians solved most of the Mathematics by laying out informal procedures aimed at getting to a solution. They developed a step-by-step method for finding roots of numbers. Through to Euclid's reputable algorithm for finding the greatest common divisors of numbers, through to Archimedes' approximation of pi, through to Nicomachus of Gerasa's Sieve of Eratosthenes algorithm for extracting prime numbers in the 200 BC, in their rudimentary form, algorithms were often procedures for performing arithmetic and calculations.

In 1943, Stephene Kleene made the concept of algorithms more definitive. In his theory, Kleene set up rules that govern algorithms today- independent, effective, non-ambiguous, well defined sets of functions that would execute operations in a finite set of commands.

The history of algorithms, from its inception to development is so deeply rooted in Mathematics, but algorithms are not always Mathematical processes per se. Problems like sorting employ heavy use of algorithms.

We cannot think of a world without thinking about algorithms; they are ubiquitous. Applications of algorithms are witnessed everywhere; from performing the simplest of tasks like assignment of roles, maintaining queues in banking halls, Google's page rank; a sorting algorithm that keeps tabs on the websites we frequent.

An imprecise view of the classical treatment of algorithms can be found in (Cormen et al., 2009).
2.3.2 Complexity analysis.
When computer programmers write algorithms to solve a certain problem, of central interest to them are considerations on:

- the time it takes for the algorithm to give the desired output.
- the amount of computer space the algorithm takes up.
- the efficiency of the algorithm.

Sometimes, there could be many ways of solving a specific problem and it is often a rule of thumb to adopt the algorithm which requires the least resources to solve the problem. The need for asymptotic analysis has never been more profound, in circumstances where we have to settle for the best algorithm that performs a task among the many that are capable of solving the problem when large inputs are considered.

One way to think of this would be to implement the two algorithms and see which one takes a shorter time to give desired outputs for different inputs. This naive approach however could be quite misleading:

- for some input, the first algorithm could perform better than the second while for some inputs the second could be more efficient than the first.
- for some input, the first algorithm could be more efficient on one machine, and the second could be better on another machine for some inputs.

Complexity analysis therefore focuses on comparisons of two or more algorithms on idea level, and not low level details such as hardware or processors. Complexity analysis defines the rate at which an algorithm needs resources to complete as function of its input, often viewed in terms of time complexity and space complexity.

2.3.3 Definition. Time complexity gives a measure of the time it takes for an algorithm to run as a function of its input size while **space complexity** gives a measure of the amount of memory taken by an algorithm to run as a function of its input size.

In this essay, we however focus on time complexity.

Suppose for some input, an algorithm gives the desired output in t seconds

$$\underbrace{\longrightarrow \boxed{\text{Algorithm}} \longrightarrow}_{t_s}$$

The time t here does not define the time complexity, but rather an execution time. The table below gives different execution time for different input sizes.

Input	Execution time
I_1	t_1
I_2	t_2
I_3	t_3
$\vdots$	$\vdots$
I_n	t_n

Input for an algorithm is often measurable, say the number of users, for instance, for 10 users, an algorithm takes 5 seconds to execute.

For different values of input and execution time, we can make a two dimensional plot. Suppose (Input, time) $= [(10,3),(20,5),(30,6),(40,5),(50,7),(60,9)]$

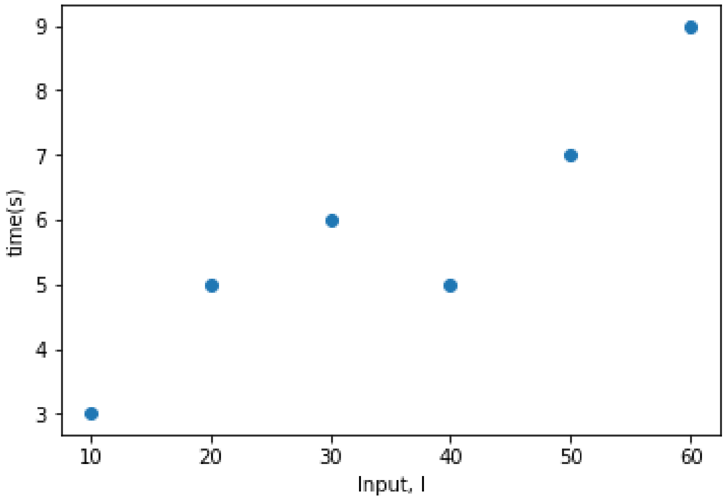

Figure 2.1: A plot of execution time as a function of input.

A curve through the the points defines a function $f : \mathbb{N} \to]0, \infty]$ of input values,

$$f(x = \text{Input, I}) = y(\text{time})$$

Suppose this function is given by

$$f(I) = 2I^3 + 5I^2 + 7I + 2. \tag{2.3.1}$$

The function f defines the amount of time an algorithm requires to execute for some given input of size I.

Since the running time is not only dependent on the input size, but on the nature and distribution of a particular input, we distinguish, in literature, three possible behaviours associated with running time of an algorithm. These are the best case, average case and worst case.

2.3.4 Definition. Order of growth describes the asymptotic behaviour of a function.

Conventionally, determining explicitly the function which describes the execution time of an algorithm is an onerous task. The order of growth therefore serves to describe the running time of algorithms in an approximate way.

2.3.5 Asymptotic Notations.

The following notions are central in establishing the running time of algorithms.

1. $\mathcal{O}$-Notation

   The $\mathcal{O}$-notation, denoted as $\mathcal{O}(f(I))$, for a function f measures the worst case time complexity, the maximal time an algorithm takes to execute. It therefore gives an asymptotic upper bound of an algorithm's execution time. More formally:

   $$\mathcal{O}(f(I)) = \{g(I) : \exists k, I_0 \in \mathbb{N} \text{ such that } g(I) \leq_\infty kf(I),\ \forall I \geq I_0\}$$

   where $g : \mathbb{N} \to]0, \infty]$ denotes the execution time of computing of an algorithm and $\leq_\infty$ denotes the usual order on the extended real line. We therefore write $g \in \mathcal{O}(f)$.

2. $\Omega-$Notation

   It measures the best case scenario or the minimal time an algorithm takes to execute. It gives the asymptotic lower bound of an algorithm's execution time.

   $$\Omega(f(I)) = \{g(I) : \exists k, I_0 \in \mathbb{N} \text{ such that } kf(I) \leq_\infty g(I),\ \forall I \geq I_0\}$$

   where $g : \mathbb{N} \rightarrow]0, \infty]$ denotes the execution time of computing of an algorithm. When f is the asymptotic lower bound, we write $g \in \Omega(f)$.

3. $\Theta-$Notation

   This expresses both the lower and upper bound of execution time of an algorithm. It therefore gives an asymptotic tight bound

   $$\Theta(f(I)) = \{g(I) : \exists k_1, k_2, I_0 \in \mathbb{N} \text{ such that } k_1 f(I) \leq_\infty g(I) \leq_\infty k_2 f(I),\ \forall I \geq I_0\}$$

   Therefore, $g \in \Omega(f) \cap \mathcal{O}(f)$, denoted by $g \in \Theta(f)$. We say g belongs to the asymptotic complexity class of f whenever $g \in \Theta(f)$.

In principle, we consider the worst case execution time in complexity analysis. To compute the $\mathcal{O}$-notation, we ignore the low order terms because in complexity analysis, we are interested in analysing the usefulness of an algorithm for large inputs.

For the function $f(I) = 2I^3 + 5I^2 + 7I + 2$ introduced in Equation (2.3.1), since $2I^3 \supset 5I^2 \supset 7I \supset 2$, then:

$$\mathcal{O}(f(I)) = \mathcal{O}(2I^3 + 5I^2 + 7I + 2) = \mathcal{O}(I^3)$$

The function $\mathcal{O}(I^3)$ is the time complexity of the function $f(I)$.

Depending on what an algorithm intends to achieve, asymptotic behaviours or order of growth are important in selecting the best algorithm for a specific task.

2.3.6 Limitations of traditional asymptotic complexity analysis.

1. Since we ignore the constant terms in our analysis, it is often hard to pick the best algorithm from two algorithms with the same order of growth.

2. It is tailored for large inputs. An asymptotically slower algorithm could just be efficient for some finite, practical input.

3. Weakening Kleene's Fixed Point Theorem

Fixed point theorems play an important role in Mathematics. Wide applications of such theorems are readily seen in semantic models in Computability theory, where partially ordered sets provide a framework for obtaining a solution to a fixed point equation through successive approximation, should one exist.

This Chapter therefore serves to introduce the celebrated Kleene's fixed point theorem, a fixed point theorem due to Stephen Cole Kleene, and ascribe to it, the minimum conditions required on the self mapping to guarantee the existence of at least one fixed point in partially ordered sets.

To this end, we present the original Kleene's fixed point theorem.

3.1 The original Kleene's fixed point theorem

3.1.1 Theorem. *((Stoltenberg-Hansen et al., 1994) p.24)*

Let $(X, \leq)$ be a chain complete poset and let $f : (X, \leq) \to (X, \leq)$ be a $\leq$-continuous mapping. Then f has a fixed point, say x^* which is the supremum of the sequence $\{f^n(x)\}$, for some $x \in X$.

Proof. We first show the existence of an an ascending Kleene's chain. We note that

$$x^* \in \uparrow x.$$

By chain completeness property of the poset and monotonicity of the endofunction f, we have that $x \leq f(x)$. Applying the n-fold composite function f^n, we obtain

$$f^n(x) \leq f^{n+1}(x), \ \forall n \in \mathbb{N}.$$

Without loss of generality, we realize the chain

$$x \leq f(x) \leq \cdots \leq f^n(x) \leq f^{n+1}(x) \leq \cdots \in X$$

Since $(X, \leq)$ is chain complete, then the above chain has a supremum, say x^*. Thus,

$$x^* = \sup(f^n(x))$$

Now, to show that x^* is a fixed point, by definition of x^*,

$$\begin{aligned}
f(x^*) &= f(\sup(f^n(x))) \\
&= \sup(f(f^n(x))), \quad \text{by continuity of } f \\
&= \sup(f^{n+1}(x)) \\
&= \sup(f^{n+1}(x)) \cup \{x\} \\
&= \sup(f^{n+1}(x)) \cup f^0(x) \\
&= \sup(f^n(x)), \quad \text{since } x \leq f^n(x), \forall n \\
&= x^*
\end{aligned}$$

Therefore, x^* is a fixed point.

We then show that x^* is the least of such fixed points in X. Suppose $y \in X$ be a fixed point of f. By definition, $f(y) = y$, and consequently, by induction, $f^n(y) = y$. We then have that $x \leq y$, and by applying f^n, we obtain

$$f^n(x) \leq f^n(y) = y.$$

By property of supremum, $\sup(f^n(x)) \leq y \implies x^* \leq y$. Thus, x^* is the least fixed point. □

In the following, we present a version of the Kleene's fixed point theorem due to (Estevan et al., 2019).

3.1.2 Theorem. *(See (Estevan et al., 2019), Theorem 1)*

Let $(X, \leq)$ be a chain complete poset and let $f : (X, \leq) \to (X, \leq)$ be a $\leq$-continuous mapping. Assume existence of an $x_0 \in X$ such that $x_0 \leq f(x_0)$. Then there exists a fixed point $x^* = sup\{f^n\{x_0\}\}_{n\in\mathbb{N}}$, that is, $x^* \in \uparrow_{\leq} x_0$. Moreover, $x^* \in \downarrow_{\leq} y_0$ provided that $y_0 \in X$ such that $x_0 \leq y_0$ and $f(y_0) \leq y_0$. Furthermore, x^* is the minimum fixed point in $\uparrow_{\leq} x_0$.

3.1.3 Remark. Kleene's fixed point theorem, as presented above, puts restrictions of order completeness on the poset and order continuity on the self mapping. However, in a practical framework, for example, in denotational semantics, one of the two restrictions could be unsatisfied. From a practical application perspective, locally characterized conditions would be more suited.

The next section, thus, lays a framework for developing a generalized Kleene's fixed point theorem.

3.2 A reformulation in partially ordered sets

In this section, we seek to present a more generalized version of the Kleene's fixed point theorem in posets by characterizing the necessary and sufficient conditions required of a self mapping to yield at least a fixed point.

We give a few examples that ratifies a possibility of presenting a weakened version of the aforesaid theorem that still satisfies the conclusions of the original Kleene's theorem.

We first note that

3.2.1 Remark. It is a standard result that every $\leq$-continuous map is monotonic. It is therefore natural to think that Kleene's fixed point theorem would not hold for non-monotone endofunctions.

In the following, we present an example of a non-monotonic self mapping on a chain complete poset, hence not $\leq$-continuous, that meets the conclusions of Theorem 3.1.2.

3.2.2 Example. (Compare (Estevan et al., 2019), Example 1)

The poset $([0, 2], \leq)$ is chain complete. Consider the self mapping $f : [0, 2] \to [0, 2]$ defined as

$$f(x) = \begin{cases} \frac{x+1}{2} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

The self mapping f is not monotonic on $([0, 2], \leq)$ and has a fixed point $x = 1$.

We first verify that the poset $([0,2],\leq)$ is indeed chain complete. Pick a sequence whose n^{th} term is given by $x_n = \frac{n}{n+1}$. Clearly, x_n is increasing in $[0,2]$ and its supremum is 1.

Pick $x = 0$ and $y = 2$. Then $f(0) = 2 \nleq f(2) = \frac{3}{2}$, therefore f is not monotone, and consequently not $\leq$-continuous.

We then verify the conclusions of the Kleene's fixed point theorem. Pick $x_0 = \frac{1}{2}$. It follows that $\frac{1}{2} \leq f(\frac{1}{2}) = \frac{3}{4}$.

Consider the sequence $\{f^n(\frac{1}{2})\}$. Clearly, $\{f^n(\frac{1}{2})\}$ is increasing in $[0,2]$ and has a supremum say, $x^* = 1$. We further note that $x^* = 1$ is a fixed point since $f(1) = 1$. Therefore, $1 \in\uparrow_{\leq} \frac{1}{2}$.

Further, pick $y_0 = 2 \in X$,then $f(y_0) \leq \frac{3}{2} \leq 2$ and $x_0 = \frac{1}{2} \leq y_0 = 2$. Therefore, $1 \in\downarrow_{\leq} 2$. And since $x^* = 1 \leq y_0 = 2$, then $x^* = 1$ is the minimum fixed point in $\uparrow_{\leq} \frac{1}{2}$.

The example above indicates a likelihood of relaxing conditions required of the self mapping to yield a fixed point. We introduce the following generality

3.2.3 Definition. (See (Estevan et al., 2019), Definition 1) Let $(X,\leq)$ be a poset and let $x_0 \in X$. A mapping $f : X \to X$ is orbitally $\leq$-continuous at x_0 if $f(x)$ is the supremum of the sequence $\{f^{n+1}(x)\}$ whenever x is the supremum of the sequence $\{f^n(x)\}$.

As was pointed out earlier, the original statement of Kleene's fixed point theorem assumes conditions with global, rather than local character. Orbital $\leq$-continuity presents a more general notion of continuity in that we only require the self mapping to be continuous for some fixed point, say x_0. This notion, unlike $\leq$-continuity which has global properties, is more suitable for practical applications.

The following example illustrates the generality presented above.

3.2.4 Example. The endofunction introduced in Example 3.2.2 is orbitally $\leq$-continuous at $x_0 = \frac{1}{2}$. We had shown that the sequence $\{f^n(\frac{1}{2})\}$ has a supremum at 1. The sequence $\{f^{n+1}(\frac{1}{2})\}$ also has a supremum at 1. And since $f(1) = 1$, we therefore conclude that f is orbitally $\leq$-continuous at $\frac{1}{2}$.

In the following example, we present self mappings which are $\leq$-continuous but not orbitally $\leq$-continuous.

3.2.5 Example. Let $([0,1],\leq)$ be a poset. Define an endofunction $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0,\frac{1}{2}[\\ \frac{1}{2} & \text{if } x \in [\frac{1}{2},1] \end{cases}$$

Clearly, f is monotone. Pick $x = 0$ and $y = 1$. Then $f(x) = 0 \leq f(y) = \frac{1}{2}$. Further, we have that every sequence $\{x_n\}_{n\in\mathbb{N}}$ is increasing whenever $x_n = x_{n+1}$. The sequence $\{f^n(\frac{1}{4})\}$ has a supremum 1 while the sequence $\{f^{n+1}(\frac{1}{4})\}$ has a supremum 1 in $[0,1]$. Since $f(1) \neq 1$, then f is not orbitally $\leq$-continuous at $\frac{1}{4}$.

3.2.6 Remark. Generally, for a poset $(X,\leq)$, if a self mapping f is $\leq$-continuous, then f is orbitally $\leq$-continuous at $x \in X$ provided $x \leq f(x)$. The converse however, is not generally true.

In a light of the above examples,(a detailed treatment can be found in (Estevan et al., 2019)) it is evident that a more generalized reformulation of Kleene's fixed point theorem can be achieved by only imposing conditions on the sequence $\{f^n(x_0)\}$ without assuming restrictions of $\leq$-completeness and $\leq$-continuity. To this end, we introduce a refined variant of the Kleene's fixed point theorem as proposed by (Estevan et al., 2019).

3.2.7 Theorem. *(See (Estevan et al., 2019), Theorem 2)*

Let $(X, \leq)$ be a poset and let $f : X \to X$ be a self mapping. Then the following are equivalent:

1. $x^* \in fix(f) \neq \emptyset$.

2. There exists $x_0 \in X$ such that

   (a) The sequence $\{f^n(x_0)\}$ is increasing in the poset,

   (b) x^* is the supremum of $\{f^n(x_0)\}$ and, thus, $x^* \in \uparrow_{\leq} x_0$,

   (c) f is orbitally $\leq$-continuous at x_0.

3. There exists $y_0 \in X$ satisfying

   (a) $y_0 \leq f(y_0)$ in the poset,

   (b) x^* is the supremum of $\{f^n(y_0)\}$ and, thus, $x^* \in \uparrow_{\leq} y_0$,

   (c) f is orbitally $\leq$-continuous at y_0.

Proof. $(1) \implies (2)$.

Let $x^* \in fix(f)$. Then $f(x^*) = x^*$ by definition. Applying the composite map f^n and setting $x^* = x_0$, we get

$$f^n(x_0) = x_0.$$

The sequence $\{f^n(x_0)\}$ is therefore increasing in $(X, \leq)$ whenever $\{x_n\} = \{x_{n+1}\}$.

Since $f^n(x_0) = x_0 = x^* \implies x^*$ is the supremum of $f^n(x_0) = x_0$. Further, $f^{n+1}(x_0) = x_0 = x^* \implies x^*$ is the supremum of $f^{n+1}(x_0)$, thus f is orbitally $\leq$-continuous at x_0.

$(2) \implies (3)$.

Assume an x_0 that satisfies all the conditions in 2, and set $y_0 = x_0$. Without loss of generality, since $f^n(y_0)$ is increasing in $(X, \leq)$, then $y_0 \leq f(y_0)$.

$(3) \implies (1)$,

Suppose there exists $y_0 \in X$ that satisfies all the conditions of 3 in Theorem 3.2.7 above. Given that x^* is the supremum of $\{f^n\{y_0\}\}$ and since $y_0 \leq f(y_0)$, then

$$x^* = \sup(\{f^{n+1}(y_0)\}) \tag{3.2.1}$$

Further, since f is orbitally $\leq$-continuous at y_0, we have that

$$f(x^*) = \sup(\{f^{n+1}(y_0)\}) \tag{3.2.2}$$

From (3.2.1) and (3.2.2), it suffices that $f(x^*) = x^* \implies x^* \in fix(f)$.

□

We can refine Theorem 3.2.7 above by employing the chain completeness property of posets, and assuming a monotonic self mapping which is not $\leq$-continuous to get the following result

3.2.8 Corollary. (See (Estevan et al., 2019) Corollary 5)

Let $(X, \leq)$ be a chain complete poset and let $f : X \to X$ be a monotonic self mapping. Then the following are equivalent.

1. $fix(f) \neq \emptyset$,
2. $\exists\ x_0 \in X$ satisfying;
   (a) $x_0 \leq f(x_0)$,
   (b) f is orbitally $\leq$-continuous at x_0.

Further, $\exists\ x^* \in fix(f)$ such that $x^* = \sup(f^n(x_0)) \implies x^* \in \uparrow_{\leq} x_0$. Moreover, $x^* \downarrow_{\leq} y_0$ provided that $y_0 \in X$ such that $x_0 \leq y_0$ and $f(y_0) \leq y_0$. Further, x^* is the minimum of $fix(f) \cap \uparrow_{\leq} x_0$ in $(X, \leq)$.

Proof. To show that $(1) \implies (2)$, pick $x_0 \in fix(f)$. By definition, $f(x_0) = x_0$. And by induction, $f^n(x_0) = x_0$. The sequence $f^n(x_0)$ is therefore increasing with a supremum at x_0. Since f is monotone, we have that $x_0 \leq f(x_0)$. Further, x_0 is the supremum of the sequence $f^{n+1}(x_0)$, thus f is orbitally $\leq$-continuous at x_0.

To show that $(2) \implies (1)$, assume an $x_0 \in X$ which satisfies all the properties of 2. above. Since f is monotone and $x_0 \leq f(x_0)$, the sequence $f^n(x_0)$ is increasing. By chain completeness of $(X, \leq)$, there exists an $x^* = \sup(f^n(x_0)) \implies x^* \in \uparrow_{\leq} x_0$.

Further, since f is orbitally $\leq$-continuous at x_0, then by Theorem 3.2.7, $x^* \in fix(f)$.

Pick $y_0 \in X$ such that $x_0 \leq y_0$ and $f(y_0) \leq y_0$. Then $f^n(x_0) \leq y_0 \implies x^* \leq y_0$. Therefore, y_0 is an upper bound of $\uparrow_{\leq} x_0$.

To show that x^* is the minimum, pick $l \in fix(f) \cap \uparrow_{\leq} x_0$. By definition, $f(l) = l$ and by induction, $f^n(l) = l$.

And since $x_0 \leq l \implies f^n(x_0) \leq f^n(l) = l$. By property of supremum, since $x^* = sup(f^n(x_0))$, then $x^* \leq l$, which proves the claim of x^* being the minimum fixed point. □

3.2.9 Remark. The monotone property of the self mapping is necessary for the existence of $y_0 \in X$ with $x_0 \leq y_0$ and $f(y_0) \leq y_0$ such that $x^* \in \downarrow_{\leq} y_0$, and thus cannot be omitted from Corollary 3.2.8.

In what follows, we consider when the underlying poset comes from a T_0-quasi-metric space.

3.3 The T_0-quasi-metric case

In this section, we extend the generalized Kleene's fixed point theorem presented as Corollary 3.2.8 to T_0-quasi-metric spaces, a class of posets which have proven instrumental in denotational specification and asymptotic analysis of recursive algorithms.

3.3.1 Remark. In the sense of a T_0-quasi-metric space, the underlying partial order is the specialization partial order introduced in Definition 2.2.17.

From Corollary 3.2.8 , we get the following result

3.3.2 Corollary. (Estevan et al., 2019) Let (X, d) be a chain complete T_0-quasi-metric space and let $f : X \to X$ be a monotonic self mapping. Then the following are equivalent.

1. $fix(f) \neq \emptyset$,
2. $\exists\ x_0 \in X$ satisfying;
   (a) $x_0 \leq_d f(x_0)$,
   (b) f is orbitally $\leq_d$-continuous at x_0.

Further, $\exists\ x^* \in fix(f)$ such that $x^* = sup(f^n(x_0)) \implies x^* \in \uparrow_{\leq_d} x_0$. Moreover, $x^* \downarrow_{\leq_d} y_0$ provided that $y_0 \in X$ such that $x_0 \leq_d y_0$ and $f(y_0) \leq_d y_0$. Further, x^* is the minimum of $fix(f) \cap \uparrow_{\leq_d} x_0$ in $(X, \leq_d)$.

3.3.3 Remark. The proposed weakened Kleene's fixed point theorem presented herein as Corollary 3.3.2 does not guarantee uniqueness of the fixed point. It however improves Theorem 3.1.1 since it gives specific properties of a fixed point when the conditions on the existence of a $y_0 \in X$ such that $x_0 \leq y_0$ and $f(y_0) \leq y_0$ are met. Further, inception of orbital $\leq$-continuity puts weaker restrictions on the self mapping.

In principle, there are several instances of chain complete T_0-quasi-metric spaces drawn from the different notions of completeness that comes with a quasi-metric framework. We, however, restrict the scope of this essay to $\leq_d$-complete T_0-quasi-metric spaces, because they preserve the topological properties of the concept of Smyth completeness.

On account of (Estevan et al., 2019), we first recall that

3.3.4 Definition. The space (X, d) is **$\leq_{\mathbf{d}}$-complete** if every increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(X, \leq)$ converges with respect to $\mathcal{T}_{d^s}$.

The following result will be central in sections hereafter

3.3.5 Proposition. (See (Estevan et al., 2019) Lemma 1)

For the space (X, d), if $x \in X$ and the increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(X, \leq)$ converges to x with respect to $\mathcal{T}_{d^s}$, then x is the supremum of $\{x_n\}_{n\in\mathbb{N}}$.

Proof. By Proposition 2.1.26, we consider convergence with respect to both $\mathcal{T}_d$ and $\mathcal{T}_{d^{-1}}$.

Suppose $\{x_n\}_{n\in\mathbb{N}}$ converges to $x, y \in X$ with respect to $\mathcal{T}_d$, then it is left d-convergent. Therefore, for any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $d(x, x_n) \leq \epsilon$ and $d(y, x_n) \leq \epsilon,\ \forall n \geq n_\epsilon$.

We then have that

$$d(x, y) \leq d(x, x_n) + d(x_n, y)$$

But $d(x_n, y) = 0$ since $\{x_n\}_{n\in\mathbb{N}}$ is left d-convergent. Thus, $d(x, y) \leq d(x, x_n) < \epsilon$. Without loss of generality, $d(x, y) = 0 \implies x \leq_d y$.

Suppose $\{x_n\}_{n\in\mathbb{N}}$ converges to $x, y \in X$ with respect to $\mathcal{T}_{d^{-1}}$, then it is right d-convergent. Therefore, for any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $d(x_n, x) \leq \epsilon$ and $d(x_n, y) \leq \epsilon \forall n \geq n_\epsilon$.

We then have that

$$\begin{aligned} d^{-1}(y, x) &\leq d^{-1}(y, x_n) + d^{-1}(x_n, x) \\ &\leq d(x_n, y) + d(x, x_n) \end{aligned}$$

But $d(x, x_n) = 0$ since $\{x_n\}_{n\in\mathbb{N}}$ is right d-convergent. Thus we have that $d^{-1}(y, x) \leq d(x_n, y) < \epsilon$. Without loss of generality, $d^{-1}(y, x) = 0 \implies x \leq_d y$.

Therefore, x is the supremum of $\{x_n\}_{n\in\mathbb{N}}$ in $(X, \leq_d)$. □

We then give a few remarks on completeness of T_0-quasi-metric spaces due to (Künzi, 2001).

3.3.6 Lemma. Let (X, d) be a T_0-quasi-metric space. If (X, d) is left Smyth complete, then (X, d) is $\leq_d$-complete and chain complete.

Proof. Consider an increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(X, \leq_d)$. Then there exists an $n_\epsilon \in \mathbb{N}$ such that for all $n \geq m \geq n_\epsilon$, $d(x_m, x_n) = 0$. Without loss of generality, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is left K-Cauchy. Moreover, since the T_0-quasi-metric space is left Smyth complete, there exists some $x \in X$ such that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x with respect to $\mathcal{T}_{d^s}$. By Proposition 3.3.5, we get that x is the supremum of $\{x_n\}_{n\in\mathbb{N}}$ in $(X, \leq_d)$. □

3.3.7 Lemma. (See (Estevan et al., 2019) Proposition 1)

Let (X, d) be a T_0-quasi-metric space. If (X, d) is $\leq_d$-complete, then (X, d) is chain complete.

Proof. Consider an increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(X, \leq_d)$. By $\leq_d$-completeness of the T_0-quasi-metric space (X, d), there exists some $x \in X$ such that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x with respect to $\mathcal{T}_{d^s}$. By Proposition 3.3.5, we get that x is the supremum of $\{x_n\}_{n\in\mathbb{N}}$ in $(X, \leq_d)$. □

To this end, we present Corollary 3.3.2 with the restriction that the T_0-quasi-metric space be $\leq$-complete.

3.3.8 Corollary. (See (Estevan et al., 2019) Corollary 10)

Let (X, d) be a $\leq$-complete T_0-quasi-metric space and let $f : X \to X$ be a monotonic self mapping. Then the following are equivalent.

1. $fix(f) \neq \emptyset$,
2. $\exists\ x_0 \in X$ satisfying;
   (a) $x_0 \leq_d f(x_0)$,
   (b) f is orbitally $\leq_d$-continuous at x_0.

Further, $\exists\ x^* \in fix(f)$ such that $x^* = sup(f^n(x_0)) \implies x^* \in \uparrow_{\leq_d} x_0$. Moreover, $x^* \downarrow_{\leq_d} y_0$ provided that $y_0 \in X$ such that $x_0 \leq_d y_0$ and $f(y_0) \leq_d y_0$. Further, x^* is the minimum of $fix(f) \cap \uparrow_{\leq_d} x_0$ in $(X, \leq_d)$.

Proof. (1) $\implies$ (2)

Let $x_0 \in fix(f)$. By definition, $f(x_0) = x_0$. By the self distance property, we get $d(x_0, f(x_0)) = 0 \implies x_0 \leq_d f(x_0)$.

Since f is monotone, and $x_0 \leq_d f(x_0)$, then f is orbitally $\leq_d$-continuous at x_0.

(2) $\implies$ (1)

Assume existence of an $x_0 \in X$ that satisfies 2a and 2b. Since f is monotonic and $x_0 \leq_d f(x_0)$, then $f^n(x_0)$ is increasing in $(X, \leq_d)$. Further, since (X, d) is $\leq_d$-complete, then $\exists\, x^*$ such that $f^n(x_0)$ converges to x^* with respect to $\mathcal{T}_{d^s}$. By Proposition 3.3.5,

$$x^* = sup(f^n(x_0)).$$

To see that x^* is a fixed point, since $x_0 \leq_d f(x_0)$, then

$$x^* = \sup\{f^{n+1}(x_0)\}.$$

Moreover, f being orbitally $\leq_d$-continuous at x_0 implies that

$$f(x^*) = \sup\{f^{n+1}(x_0)\}.$$

Therefore, $f(x^*) = x^* \implies x^* \in fix(f)$.

Pick $y_0 \in X$ such that $x_0 \leq_d y_0$ and $f(y_0) \leq_d y_0$, then $f^n(x_0) \leq_d y_0 \implies x^* \leq y_0$, so y_0 is an upper bound. Thus, $x^* \in \downarrow_{\leq_d} y_0$.

To show that x^* is the minimum fixed point in $fix(f) \cap \uparrow_{\leq_d} x_0$ in $(X, \leq_d)$, pick $y^* \in fix(f) \cap \uparrow_{\leq_d} x_0$ in $(X, \leq_d)$. By definition, $f(y^*) = y^*$, and by induction, $f^n(y^*) = y^*$. Since y^* is an upper bound, we have that $x_0 \leq_d y^* \implies f^n(x_0) \leq_d y^*$. From $x^* = sup(f^n(x_0)$, we get that, by property of supremum, $x^* \leq y^*$. Hence, x^* is the minimum fixed point in $fix(f) \cap \uparrow_{\leq_d} x_0$ in $(X, \leq_d)$.

□

The following example shows that monotonicity of the self mapping is necessary in characterizing the fixed point in Corollary 3.3.8

3.3.9 Example. Let $([0,2], d_k)$ be an $\leq_d$-complete T_0-quasi-metric space such that $d_k(x, y) = \max\{y - x, 0\}$. Define a self mapping $f : [0,2] \to [0,2]$ by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

Then f is clearly not monotone. We first verify $\leq_d$-completeness, we note that every increasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $([0,2], \leq_d)$ has a supremum in $([0,2], \leq_d)$. And since $[0,2]$ is a finite subset of $\mathbb{R}$, by Proposition 3.3.5, the sequence converges to the supremum with respect to $\mathcal{T}_{d^s}$, hence $\leq_d$-complete.

Moreover, $d_k(x, y) = 0 \implies y \leq_{d_k} x$. Pick $x_0 = 2$ and $y_0 = 1$. Then $2 \leq_{d_k} 1$ but $f(2) = 1 \nleq_{d_k} f(1) = 2$, hence f is not monotone.

Clearly, $0 \in fix(f)$. By Theorem 3.2.7, $\exists\, x_0 \in]1, 2]$ such that $x_0 \leq_d f(x_0)$. 0 is the supremum of $f^n(x_0)$ and f is orbitally $\leq_d$-continuous at x_0.

Pick $y_0 = 1$. Clearly, $f(y_0) = 2 \leq_{d_k} 1$ and $x_0 \leq_{d_k} y_o$ but $x^* = 0 \nleq_{d_k} y_0 = 1$.

4. The Application

We exploit the usefulness of the weakened Kleene's fixed point theorem in discussing the asymptotic upper and lower bounds of algorithms.

4.1 Schellekens' complexity space

Introduced by Schellekens in 1995, the complexity space presented a topological foundation for obtaining asymptotic upper bounds of algorithms whose execution time of computing satisfies a recurrence equation.

Schellekens taps into the utility of a well known bicomplete quasi-metric space $(]0, \infty], d)$, where $d(x, y) = \max\{\frac{1}{y} - \frac{1}{x}, 0\}$ to advance his theory on complexity distance via function space approach. On account of Schellekens, the complexity space is the pair $(\mathcal{C}, d_{\mathcal{C}})$, where

$$\mathcal{C} = \left\{ f : \mathbb{N} \to (0, \infty] : \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \right\}$$

and the bicomplete quasi-metric $d_{\mathcal{C}}$ on $\mathcal{C}$ is given by

$$d_{\mathcal{C}}(f, g) = \sum_{n=1}^{\infty} 2^{-n} \max \left\{ \frac{1}{g(n)} - \frac{1}{f(n)}, 0 \right\},$$

for all $f, g \in \mathcal{C}$ (see (Schellekens, 1995)). The quasi-metric $d_{\mathcal{C}}$ is normalized by 2^{-n} to ensure convergence of the series. We uphold the rule that $\frac{1}{\infty} = 0$.

From a complexity analysis vantage point, it is possible to relate a function $f_P \in \mathcal{C}$ to an algorithm P, such that $f_P(n)$ represents the execution time of computing of getting the desired output from algorithm P when inputs of size n are considered. The elements of $\mathcal{C}$ are thus referred to as complexity functions.

4.1.1 Remark. Given any two functions $f_P, f_Q \in \mathcal{C}$, the complexity distance from f_P to f_Q, denoted by $d_{\mathcal{C}}(f_P, f_Q)$, can be thought of as the comparative progress made in lowering the complexity by substituting any algorithm P whose complexity function is f_P by an algorithm Q whose complexity function is f_Q.

Therefore, $d_{\mathcal{C}}(f_P, f_Q) = 0 \implies f_P(n) \leq_d f_Q(n), \forall n \in \mathbb{N}$ and $f_P \neq f_Q$, can be interpreted as algorithm P is at least as efficient as algorithm Q. Moreover, for any $g \in \mathcal{C}$,

$$d_{\mathcal{C}}(f_P, g) = 0 \implies f_P \in \mathcal{O}(g)$$

and

$$d_{\mathcal{C}}(g, f_P) = 0 \implies f_P \in \Omega(g)$$

We note that the asymmetric nature of the complexity distance $d_{\mathcal{C}}$ is important since it provides information about the increase in complexity when an algorithm is substituted by another one.

The utility of the complexity space $(\mathcal{C}, d_{\mathcal{C}})$, together with an extension of Banach's contraction principle in quasi-metric spaces was used by Schellekens to obtain asymptotic upper bounds of algorithms whose execution time follows a Divide and Conquer recurrence equation.

A Divide and Conquer algorithm solves a problem of size n by recursively splitting it into subproblems of size $\frac{n}{b}$ for some $b \in \mathbb{N}$ with $b > 1$. Each subproblem is solved separately by the same algorithm. A global solution of the original problem is achieved through combining all solutions of the subproblems.

The recursive form of a Divide and Conquer algorithm leads to a recurrence equation is of the form

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ aT(\frac{n}{b}) + h(n) & \text{if } n \in \mathbb{N}_b \end{cases} \tag{4.1.1}$$

where $\mathbb{N}_b = \{b^k : k \in \mathbb{N}\},\ c \in (0, \infty]$, represents the complexity of the base case, $a \in \mathbb{N}$ such that $a > 1$ represents the number of subproblems a problems is divided into, and $h \in \mathcal{C}$ such that $h(n) \leq \infty$ denotes the time taken to split the problem into a subproblems and recombine individual solutions to a global one. (Detailed exposition can be found at (Brassard and Bratley, 1988) pg 105).

Denote by $\mathcal{C}_{b,c}$ a subset of $\mathcal{C}$ defined by

$$\mathcal{C}_{b,c} = \{f \in \mathcal{C} : f(1) = c \text{ and } f(n) = \infty,\ \forall n \in \mathbb{N}/\mathbb{N}_b, n > 1\}$$

and let $\hat{d_{\mathcal{C}}}$ be the restriction of $d_{\mathcal{C}}$ to the subset $\mathcal{C}_{b,c}$.

4.1.2 Lemma. The quasi-metric space $(\mathcal{C}_{b,c}, \hat{d_{\mathcal{C}}})$ is bicomplete.

Proof. Since the quasi-metric space $(\mathcal{C}, d_{\mathcal{C}})$ is bicomplete, it suffices to show that the set $\mathcal{C}_{b,c}$ is closed in $(\mathcal{C}, d^s_{\mathcal{C}})$.

Pick a function $g \in \overline{\mathcal{C}_{b,c}}^{d^s_{\mathcal{C}}}$ and a sequence $\{f_m\}_{m \in \mathbb{N}} \subset \mathcal{C}_{b,c}$ such that $\lim\limits_{m \to \infty} d^s_{\mathcal{C}}(g, f_m) = 0$.

Thus, for any $\epsilon > 0, \exists m_\epsilon, n_\epsilon \in \mathbb{N}$ such that $d^s_{\mathcal{C}}(g, f_m) < \epsilon,\ \forall\ m \geq m_\epsilon$. Moreover, $\sum_{n_\epsilon+1}^{\infty} 2^{-n} \frac{1}{f_{m_\epsilon}(n)} < \epsilon$. Therefore,

$$\begin{aligned}
\sum_{n_\epsilon+1}^{\infty} 2^{-n} \frac{1}{g(n)} &= \sum_{n_\epsilon+1}^{\infty} 2^{-n} \left(\frac{1}{g(n)} - \frac{1}{f_{m_\epsilon}(n)} + \frac{1}{f_{m_\epsilon}(n)} \right) \\
&\leq \sum_{n_\epsilon+1}^{\infty} 2^{-n} \left| \frac{1}{g(n)} - \frac{1}{f_{m_\epsilon}(n)} \right| + \sum_{n_\epsilon+1}^{\infty} 2^{-n} \frac{1}{f_{m_\epsilon}(n)} \\
&\leq 2 d^s_{\mathcal{C}}(g, f_{m_\epsilon}) + \sum_{n_\epsilon+1}^{\infty} 2^{-n} \frac{1}{f_{m_\epsilon}(n)} \\
&< 3\epsilon.
\end{aligned}$$

Thus, $g \in \mathcal{C}$.

Now suppose by contradiction $g \notin \mathcal{C}_{b,c}$. It then follows that $g(1) \neq c$ and $g(n) \neq \infty$.

Therefore, for any $\epsilon > 0$ and $m_\epsilon \in \mathbb{N}$, we have that $d^s_{\mathcal{C}}(g, f_m) < \epsilon,\ \forall m \geq m_\epsilon$.

Therefore, we have that

$$\sum_{n=1}^{\infty} 2^{-n}\left|\frac{1}{g(n)} - \frac{1}{f_m(n)}\right| < \epsilon.$$

Suppose we choose $\epsilon = 2^{-1}\left|\frac{1}{g(1)} - \frac{1}{c}\right|$, then, without loss of generality, we have that

$$\epsilon = 2^{-1}\left|\frac{1}{g(1)} - \frac{1}{c}\right| \leq \sum_{n=1}^{\infty} 2^{-n}\left|\frac{1}{g(n)} - \frac{1}{f_{m_\epsilon}(n)}\right| < \epsilon,$$

which gives a contradiction. Hence, $g(1) = c$.

Again, suppose we choose $\epsilon = 2^{-n}\left|\frac{1}{g(n)} - 0\right|$, then, without loss of generality, we have that

$$\epsilon = 2^{-n}\left|\frac{1}{g(n)} - \frac{1}{\infty}\right| \leq \sum_{n=1}^{\infty} 2^{-n}\left|\frac{1}{g(n)} - \frac{1}{f_{m_\epsilon}(n)}\right| < \epsilon,$$

which gives a contradiction. Hence, $g(n) = \infty$.

Thus, $g \in \mathcal{C}_{b,c}$, and therefore $\overline{\mathcal{C}_{b,c}}^{d^s_{\mathcal{C}}} = \mathcal{C}_{b,c}$. □

We then give the following result due to (Schellekens, 1995)

4.1.3 Proposition. The quasi-metric space $(\mathcal{C}_{b,c}, \hat{d_{\mathcal{C}}})$ is weightable via the weighting function $w : \mathcal{C} \to]0, \infty]$ given by $w(g) = \sum_{n=1}^{\infty} 2^{-n}\frac{1}{g(n)}$.

4.1.4 Remark. On account of (Künzi, 2001), every bicomplete weightable quasi-metric space is left Smyth complete. The quasi-metric space $(\mathcal{C}_{b,c}, \hat{d_{\mathcal{C}}})$ is thus left Smyth complete.

We then associate a self mapping $\Psi_T : \mathcal{C}_{b,c} \to \mathcal{C}_{b,c}$ to the recurrence Equation (4.1.1) defined as

$$\Psi_T(f)(n) = \begin{cases} c & \text{if } n = 1 \\ af(\frac{n}{b}) + h(n) & \text{if } n \in \mathbb{N}_b \\ \infty & \text{if } n > 1, n \notin \mathbb{N}_b \end{cases} \tag{4.1.2}$$

4.1.5 Remark. A function $f \in \mathcal{C}_{b,c}$ is a solution to Equation (4.1.1) if and only if $\Psi_T(f) = f$.

In order to classify the asymptotic upper and lower bound, we give the following generality

4.1.6 Definition. Let $\Psi_T : \mathcal{C}_{b,c} \to \mathcal{C}_{b,c}$ be a monotone self mapping. Then Ψ_T is called an **improver** with respect to $f \in \mathcal{C}_{b,c}$ if and only if $\Psi_T(f) \leq_{d_{\mathcal{C}}} f,\ \forall n$. Accordingly, Ψ_T is a **worsener** if and only if $f \leq_{d_{\mathcal{C}}} \Psi_T(f),\ \forall n$.

The following fixed point technique due to Schellekens in (Schellekens, 1995) allows to show that the Divide and Conquer recurrence equation type (4.1.1) has a unique solution.

4.1.7 Theorem. *The quasi-metric space $(\mathcal{C}_{b,c}, \hat{d}_{\mathcal{C}})$ is left Smyth complete and the self mapping Ψ_T satisfies that $\hat{d}_{\mathcal{C}}(\Psi_T(f), \Psi_T(g)) \leq_{d_{\mathcal{C}}} \frac{1}{a}\, \hat{d}_{\mathcal{C}}(f, g),\ \forall f, g \in \mathcal{C}_{b,c}$. Hence, a Divide an Conquer recurrence Equation (4.1.1) has a unique solution $g_T \in \mathcal{C}_{b,c}$. Furthermore, the following postulations holds:*

1. *If Ψ_T is an improver with respect to $f \in \mathcal{C}_{b,c}$, then $g_T \in \mathcal{O}(f)$.*
2. *If Ψ_T is a worsener with respect to $f \in \mathcal{C}_{b,c}$, then $g_T \in \Omega(f)$.*

Proof. (Sketch) If Ψ_T is an improver with respect to $f \in \mathcal{C}_{b,c}$, then

$\Psi_T(f) \leq_{d_{\mathcal{C}}} f$, then $\hat{d}_{\mathcal{C}}(\Psi_T(f), f) = 0$.

We then have to show that $\hat{d}_{\mathcal{C}}(g_T, f) = 0$ to complete the proof. Assume for contradiction that $\hat{d}_{\mathcal{C}}(g_T, f) > 0$, by triangle inequality,

$$\begin{aligned}
\hat{d}_{\mathcal{C}}(g_T, f) &\leq_{d_{\mathcal{C}}} \hat{d}_{\mathcal{C}}(g_T, \Psi_T(f)) + \hat{d}_{\mathcal{C}}(\Psi_T(f), f) = \hat{d}_{\mathcal{C}}(g_T, \Psi_T(f)) \\
&\leq_{d_{\mathcal{C}}} \hat{d}_{\mathcal{C}}(g_T, \Psi_T(g_T)) + \hat{d}_{\mathcal{C}}(\Psi_T(g_T), \Psi_T(f)) = \hat{d}_{\mathcal{C}}(\Psi_T(g_T), \Psi_T(f)) \\
&\leq_{d_{\mathcal{C}}} \frac{1}{a}\, \hat{d}_{\mathcal{C}}(g_T, f)
\end{aligned}$$

Therefore, $\hat{d}_{\mathcal{C}}(g_T, f) \leq_{d_{\mathcal{C}}} \frac{1}{a}\, \hat{d}_{\mathcal{C}}(g_T, f) \implies 1 \leq_{d_{\mathcal{C}}} \frac{1}{a}$, which gives a contradiction since $a > 1$. Hence, we have that $\hat{d}_{\mathcal{C}}(g_T, f) = 0 \implies g_T \in \mathcal{O}(f)$

To prove 2, we note that if Ψ_T is a worsener with respect to $f \in \mathcal{C}_{b,c}$, then $f \leq_{d_{\mathcal{C}}} \Psi_T(f)$. Consequently, $\hat{d}_{\mathcal{C}}(f, \Psi_T(f)) = 0$. Then rest of the arguments given for 1 hold for 2.

□

Using the fixed point technique given by Theorem 4.1.7 above, Schellekens was able to classify the asymptotic upper bound of the Mergesort algorithm, which is a particular case of Equation (4.1.1). Concretely, the running time of computing of the aforementioned algorithm follows:

$$T_M(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(\frac{n}{2}) + \frac{n}{2} & \text{if } n \in \mathbb{N}_b \end{cases} \tag{4.1.3}$$

The self mapping $\Psi_T : \mathcal{C}_{2,c} \to \mathcal{C}_{2,c}$ given by Equation (4.1.2) for the particular case recurrence Equation (4.1.3) is an improver with respect to $h \in \mathcal{C}_{b,c}$, where

$$h(n) = \begin{cases} c & \text{if } n = 1 \\ \frac{1}{2} n \log_2(n) & \text{if } n \in \mathbb{N}_b \\ \infty & \text{if } n > 1, n \notin \mathbb{N}_b. \end{cases}$$

By Theorem 4.1.7, we have that $g_T \in \mathcal{O}(n \log_2(n))$.

4.2 The fixed point technique for asymptotic complexity analysis

In this section, we invoke the fixed point technique introduced in Section 3.3 in asymptotic complexity analysis for a wider class of recursive algorithms.

Romaguera in (Romaguera et al., 2012), made an exposé on extending Theorem 4.1.7 to cases where the execution time of computing follows a family of recurrence equation of the form

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ aT(n-1) + h(n) & \text{if } n > 1 \end{cases} \tag{4.2.1}$$

where $c \in (0, \infty],\ a \in \mathbb{N}$ such that $a \geq 1$ and $h \in \mathcal{C}$ such that $h(n) \leq \infty$.

4.2.1 Remark. The Divide and Conquer recurrence equation type (4.1.1) can be retrieved as a particular case of recurrence equation given in (4.2.1) using the following transformation

$$S(m) = \begin{cases} c & \text{if } m = 1 \\ aS(m-1) + r(m) & \text{if } m > 1 \end{cases} \tag{4.2.2}$$

where $S(m) = T(b^{m-1})$ and $r(m) = h(b^{m-1}), \forall m \in \mathbb{N}$.

We then denote by $\mathcal{C}_c$ a subset of $\mathcal{C}$ defined by

$$\mathcal{C}_c = \{f \in \mathcal{C} : f(1) = c\}$$

and let $\hat{d_{\mathcal{C}}}$ be the restriction of $d_{\mathcal{C}}$ to the subset $\mathcal{C}_c$.

It was shown that the self mapping $\Psi_T : \mathcal{C}_c \to \mathcal{C}_c$ defined by

$$\Psi_T(f)(n) = \begin{cases} c & \text{if } n = 1 \\ af(n-1) + h(n) & \text{if } n > 1 \end{cases} \tag{4.2.3}$$

for all $f \in \mathcal{C}_c$, satisfies

$$\hat{d_{\mathcal{C}}}(\Psi_T(f), \Psi_T(g)) \leq \frac{1}{2a}\ \hat{d_{\mathcal{C}}}(f, g),\ \forall f, g \in \mathcal{C}_c. \tag{4.2.4}$$

Indeed

$$\begin{aligned}
\hat{d}_{\mathcal{C}}(\Psi_T(f), \Psi_T(g)) &= \sum_{n=1}^{\infty} 2^{-n} \max\left\{\frac{1}{\Psi_T(g)(n)} - \frac{1}{\Psi_T(f)(n)}, 0\right\} \\
&= \sum_{n=2}^{\infty} 2^{-n} \max\left\{\frac{1}{ag(n-1)+h(n)} - \frac{1}{af(n-1)+h(n)}, 0\right\} \\
&= \sum_{n=2}^{\infty} 2^{-n} \max\left\{\frac{af(n-1)+h(n) - ag(n-1)+h(n)}{(ag(n-1)+h(n))(af(n-1)+h(n))}, 0\right\} \\
&\leq \sum_{n=2}^{\infty} 2^{-n} \max\left\{\frac{af(n-1) - ag(n-1)}{(ag(n-1))(af(n-1))}, 0\right\} \\
&\leq \frac{1}{a}\sum_{n=2}^{\infty} 2^{-n} \max\left\{\frac{f(n-1) - g(n-1)}{(g(n-1))(f(n-1))}, 0\right\} \\
&\leq \frac{1}{a}\sum_{n=1}^{\infty} 2^{-n-1} \max\left\{\frac{f(n) - g(n)}{(g(n))(f(n))}, 0\right\} \\
&\leq \frac{1}{2a}\hat{d}_{\mathcal{C}}(f, g).
\end{aligned}$$

4.2.2 Remark. Following Lemma 4.1.2 and Remark 4.1.4, the quasi-metric space $(\mathcal{C}_c, \hat{d}_{\mathcal{C}})$ is left Smyth complete.

A function $f \in \mathcal{C}_c$ is a solution to Equation (4.2.1) if and only if $\Psi_T(f) = f$, and thus the recurrence Equation (4.2.1) has a unique solution, say $g_T \in \mathcal{C}_c$.

In the following, we show that the generalized Kleene's fixed point theorem given as Corollary 3.3.8 provides a platform for discussing asymptotic complexity analysis of algorithms whose execution time of computing satisfies the wider class of recursive algorithms.

We give the following result due to (Estevan et al., 2019)

4.2.3 Theorem. *Let $\mathcal{R} \subseteq \mathcal{C}$ such that the quasi-metric space $(\mathcal{R}, \hat{d}_{\mathcal{C}})$ is $\leq_{d_{\mathcal{C}}}$-complete. Let $\Psi : \mathcal{R} \to \mathcal{R}$ be a monotone mapping. If there exists $f, g \in \mathcal{R}$ such that the following postulations holds:*

1. *Ψ is a worsener with respect to g and Ψ is orbitally $\leq_{d_{\mathcal{C}}}$-continuous at g,*
2. *Ψ is an improver with respect to f and $g \leq_{d_{\mathcal{C}}} f$.*

Then $\exists\ f^* \in \mathcal{R}$ such that $f^* \in fix(\Psi)$ and $f^* \in \Omega(g) \cap \mathcal{O}(f)$.

Proof. From Theorem 3.3.8, we get that $fix(\Psi_T) \neq \emptyset$. Therefore, there exists $f^* \in fix(\Psi_T)$ and $f^* \in \Omega(g) \cap \mathcal{O}(f)$. □

Estevan et al. in (Estevan et al., 2019) showed that the appropriate extension of the Banach's contraction principle for the wider class of recursive algorithms whose recurrence equations lead to those given in equation (4.2.1) and Theorem 4.1.7 could be recovered from our proposed fixed point technique presented in Theorem 4.2.3.

Therefore, the aforementioned fixed point technique, Theorem 4.2.3, provides an all-round chassis for discussing asymptotic upper and lower bounds of algorithms in ways that conserve the original Schellekens ideas.

4.2.4 Remark. From a complexity analysis perspective, uniqueness of the solutions to the recurrence Equations 4.1.1 and 4.2.1 is often guaranteed by the theory of finite difference equations at initial conditions. Therefore, the aforesaid fixed point technique presented in Theorem 4.2.3, in as much as it does not guarantee uniqueness of such solutions, still remains a viable technique in determining asymptotic upper and lower bounds of algorithms.

4.3 Complexity analysis of Towers of Hanoi algorithm

In this section, we provide, more cogently, the asymptotic complexity class of the acclaimed Towers of Hanoi algorithm using the fixed point technique introduced in Theorem 4.2.3.

4.3.1 Towers of Hanoi.

The Towers of Hanoi, under the assumption of uniform cost, follows a recurrence equation of type (4.2.1). More particularly, the execution time of computing of the aforementioned algorithm is the solution of the recurrence equation $T : \mathbb{N} \to (0, \infty]$ defined by

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(n-1) + d & \text{if } n > 1 \end{cases} \tag{4.3.1}$$

where $c, d > 0$ and c represents the time taken by the algorithm to solve the base case.

The endofunction $\Psi_T : \mathcal{C}_c \to \mathcal{C}_c$ associated with the recurrence equation (4.3.1) is given by

$$\Psi_T(f)(n) = \begin{cases} c & \text{if } n = 1 \\ 2f(n-1) + d & \text{if } n > 1 \end{cases} \tag{4.3.2}$$

We then establish the asymptotic upper and lower bounds of Hanoi algorithm through the fixed point technique presented in Section 4.2 above.

4.3.2 Lemma. The endofunction $\Psi_T : \mathcal{C}_c \to \mathcal{C}_c$ given by Equation (4.3.2) is a worsener with respect to $g \in \mathcal{C}_c$ if and only if $k \leq_{\hat{d_C}} d$ and an improver with respect to $f \in \mathcal{C}_c$ if and only if $k \geq_{\hat{d_C}} d$, where

$$g(n) = \begin{cases} c & \text{if } n = 1 \\ k(2^n - 1) & \text{if } n > 1 \end{cases}$$

Proof. We first check that $\Psi_T(f)$ as defined is monotone. Pick $n = 1 \leq m = 2$. Then, $\Psi_T(f)(1) = c$ and $\Psi_T(f)(2) = 2c + d$. Thus, $\Psi_T(f)(1) \leq_{\hat{d_C}} \Psi_T(f)(2)$.

Suppose Ψ_T is a worsener with respect to g, and since Ψ_T is monotone, we have that $g \leq_{\hat{d_C}} \Psi_T(g)$. Therefore, $g(1) = c = \Psi_T(g)(1)$ and for all $n > 1$:

$$\begin{aligned} g(n) \leq_{\hat{d_{\mathcal{C}}}} \Psi_T(g)(n) &\iff k(2^n - 1) \leq_{\hat{d_{\mathcal{C}}}} 2k(2^{n-1} - 1) + d \\ &\iff k2^n - k \leq_{\hat{d_{\mathcal{C}}}} k2^n - 2k + d \\ &\iff -k \leq_{\hat{d_{\mathcal{C}}}} -2k + d \\ &\iff -k + 2k \leq_{\hat{d_{\mathcal{C}}}} d \\ &\iff k \leq_{\hat{d_{\mathcal{C}}}} d \end{aligned}$$

Suppose Ψ_T is an improver with respect to f, and since Ψ_T is monotone, we have that $\Psi_T(f) \leq_{\hat{d_{\mathcal{C}}}} f$. Therefore, for all $n > 1$:

$$\begin{aligned} \Psi_T(f)(n) \leq_{\hat{d_{\mathcal{C}}}} f(n) &\iff 2k(2^{n-1} - 1) + d \leq_{\hat{d_{\mathcal{C}}}} k(2^n - 1) \\ &\iff k2^n - 2k + d \leq_{\hat{d_{\mathcal{C}}}} k2^n - k \\ &\iff -2k + d \leq_{\hat{d_{\mathcal{C}}}} -k \\ &\iff d \leq_{\hat{d_{\mathcal{C}}}} -k + 2k \\ &\iff d \leq_{\hat{d_{\mathcal{C}}}} k \end{aligned}$$

□

By Remark 4.1.4, the quasi-metric space $(\mathcal{C}_c, \hat{d_{\mathcal{C}}})$ is left Smyth complete. Therefore, from Lemma 3.3.6, the quasi-metric space $(\mathcal{C}_c, \hat{d_{\mathcal{C}}})$ is also $\leq_{\hat{d_{\mathcal{C}}}}$-complete and Chain complete.

Since Ψ_T is monotone and $g \leq_{\hat{d_{\mathcal{C}}}} \Psi_T(g)$, we have that the sequence $\{\Psi_T^n(g)\}_{n\in\mathbb{N}}$ is increasing in $(\mathcal{C}_c, \leq_{d_{\mathcal{C}}})$. By $\leq_{d_{\mathcal{C}}}$-completeness property, we deduce the existence of $f \in \mathcal{C}_c$ such that $\{\Psi_T^n(g)\}_{n\in\mathbb{N}}$ converges to f with respect to $\mathcal{T}_{d^s_{\mathcal{C}}}$. By Proposition 3.3.5,

$$f = \sup(\{\Psi_T^n(g)\}_{n\in\mathbb{N}}).$$

Without loss of generality, Ψ_T is orbitally $\leq_{d_{\mathcal{C}}}$-continuous at g.

Since $g \leq_{\hat{d_{\mathcal{C}}}} \Psi_T(g)$, by induction, we have that $g \leq_{d_{\mathcal{C}}} \Psi_T^n(g)$. And by property of supremum, we get that $g \leq_{\hat{d_{\mathcal{C}}}} f$.

From Lemma 4.3.2 and Theorem 4.2.3, there exists an $f^* \in fix(\Psi_T)$ and $f^* \in \Omega(g) \cap \mathcal{O}(f)$.

Therefore,

$$f^* \in \Omega(\{g(n) : k \leq_{\hat{d_{\mathcal{C}}}} d\}) \cap \mathcal{O}(\{f(n) : k \geq_{\hat{d_{\mathcal{C}}}} d\}).$$

The complexity class of Towers of Hanoi algorithm is therefore given as $f \in \Theta(2^n)$.

5. Conclusion

Fixed point techniques for self mappings on posets are pivotal in characterizing complexity classes of algorithms. However, in literature, most of these aforementioned techniques impose conditions of order continuity of the self mapping and order completeness of the underlying partially ordered set. From an application point of view, the necessary and sufficient conditions demanded to yield a fixed point could not all be satisfied. Faced with these setbacks, several attempts have been made to develop less demanding fixed point techniques, that could still be applied in complexity analysis of algorithms (see, as an example (Fomenko and Podoprikhin, 2017)).

The objective of this dissertation was twofold. On the one hand, present a weakened framework of the Kleene's fixed point theorem, which does not assume global conditions over all the elements of the underlying poset. On the other hand, illustrate its practicability in asymptotic complexity analysis of algorithms whose execution time follows the Divide and Conquer recurrence equation of type 4.1.1 and recurrence equation of type 4.2.1.

In Chapter 2, background from generalized metric spaces, order theory and complexity analysis, which were important in developing the succeeding chapters were introduced. Because of the asymmetric nature of T_0-quasi-metric spaces (X, d), the appropriate generalizations of the metric results were no longer trivial. The metric space (X, d^s), introduced in Remark 2.1.7 was foremost in generalizing the notion of completeness in a quasi-metric framework. The different types of Cauchy sequences in a quasi-metric space were also central in defining convergence in such spaces.

The concept of orbitally order continuity poses a more relaxed and local notion of order continuity. The aforesaid generality was crucial in developing the intended framework given in Section 3.3 as Corollary 3.3.8. In the spirit of Schellekens, we opted for order complete posets, an instance of chain complete posets, which, generally preserves the topology of Smyth completion. Using Proposition 3.3.5, Lemma 3.3.6 and Lemma 3.3.7, we were able to strike correspondences between these theories of completeness in quasi-metric spaces.

Chapter 4 served to set the ground work for complexity analysis. We presented an apropriate extension, via the Schelleken's complexity space, of Corollary 3.3.8, given herein as Theorem 4.2.3. Theorem 4.2.3 and Lemma 4.3.2 were crucial in obtaining the complexity class of the Towers of Hanoi algorithm to be $\Theta(2^n)$.

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