The Hahn Banach Theorem

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Abstract

The Hahn-Banach theorem is one of the cornerstones of Functional Analysis. Given a linear functional f on a subspace Z of a vector space X, under certain conditions the theorem asserts the existence of a linear extension to the whole space X without loss of the properties of the functional. The theorem guarantees the existence of "enough" continuous linear functionals on a normed space X for the study of its dual space. Further, the Hahn-Banach theorem implies that if X is a normed space and $f(x) = 0$ for all bounded linear functionals f on X, then $x = 0$. The main objective of this essay is to give detailed proofs of the Hahn-Banach theorem in different contexts for better understanding. We then consider some of the consequences and applications of the theorem.

Keywords: Hahn-Banach, normed space, bounded linear functional.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

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Introduction

The Hahn-Banach theorem is one of the fundamental theorems of Functional Analysis. In particular, it is an important theorem for normed and Banach spaces. There are two forms of the theorem, namely the geometric and analytic form. For our study we consider the analytic version. The theorem states that, under certain conditions, a linear functional on a subspace of a vector space can be extended to the whole vector space whilst retaining its properties. The Hahn-Banach theorem assures us that a normed space is enriched with bounded linear functionals. This assurance is especially important in the study of dual spaces. The theorem is named after H. Hahn and S. Banach who independently discovered it in 1927 and 1929, respectively. It was later generalised to complex vector spaces by H.F. Bohnenblust and A. Sobczyk in 1938 ([5], p. 213). A special case of the theorem, for the function space $C[a,b]$ was proved by Eduard Helly in 1912 ([7]).

The main aim of this essay is to carefully study the Hahn-Banach theorem and give a detailed proof of the theorem in different spaces, that is, vector spaces (real or complex), normed spaces and inner product (Hilbert) spaces. Furthermore, we shall have a glimpse of some applications of the theorem.

The essay consists of 3 chapters. In Chapter 1 we discuss basic definitions, concepts and results which are important for the rest of the work. Section 1.1 presents the notion of normed spaces including linear operators and functionals. In Section 1.2 we discuss the dual space X', that is, the normed space formed by the collection of all bounded linear functionals on a normed space X. Section 1.3 presents the basic theory of inner product spaces.

In Chapter 2 we begin by stating Zorn's lemma, which asserts that a partially ordered set has a maximal element if every chain contained in it is bounded above. This result is very important as it is the main tool in the proof of the Hahn-Banach theorem in real vector spaces. In Section 2.1 we give a detailed proof of this basic Hahn-Banach theorem (Theorem (2.1.1)). In Section 2.2 we generalise the Hahn-Banach theorem by considering vector spaces over a complex field (Theorem (2.2.1)). In the proof of Theorem (2.2.1), the extension of the functional $f(x) = g(x) + ih(x)$ was obtained using the relation $f(x) = g(x) - ig(ix)$ in ([5], $p.219$), that is, $\tilde{f}(x) = \tilde{g}(x) - i\tilde{g}(ix)$. In our case we use $f(x) = h(ix) + ih(x)$ to obtain the extension $\tilde{f}(x) = \tilde{h}(ix) + i\tilde{h}(x)$. In Section 2.3 we discuss the Hahn-Banach theorem in normed spaces and Hilbert spaces. In corollary (2.3.2) we see that Hilbert spaces give a some-what easier proof of the Hahn-Banach theorem for normed spaces. We conclude the section by considering some consequences of the theorem.

Lastly, in Chapter 3 we consider some of the applications of the Hahn-Banach theorem. Section 3.1 presents the notion of the adjoint operator $T^{\times}$ of a bounded linear operator T. We look at Theorem (3.1.3) which states that the norm of the adjoint operator $\|T^{\times}\|$ is equal to the norm $\|T\|$ of T. In Section 3.2 we discuss the application of the Hahn-Banach theorem to separability of spaces. The separability theorem states that if the dual space X' is separable then X is separable. Section 3.3 presents the notion of weak convergence. We use Corollary (2.3.5) to prove Lemma (3.3.3) which states that the weak limit x of a weakly convergent sequence (x_n) is unique. To conclude the chapter we prove Theorem (3.3.8) which states that if (x_n) converges weakly to x, then the sequence $(\|x_n\|)$ is bounded.

1. Preliminaries

We dedicate this chapter to fundamental concepts, definitions and theorems, which shall be used to develop the Hahn-Banach Theorems and consequently their proofs.

1.1 Normed Spaces

1.1.1 Definition. Let X be a vector space over a field K (where K is either $\mathbb{R}$ or $\mathbb{C}$ throughout this document). A $norm$ on X is a function $\|\cdot\| : X \longmapsto \mathbb{R}$ which satisfies the following properties for all $x, y \in X$ and all $\alpha \in K$:

1. $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$.
2. $\|\alpha x\| = |\alpha|\|x\|$.
3. $\|x + y\| \leq \|x\| + \|y\|$. **(Triangle inequality)**

1.1.2 Definition. A $normed\ space$ X is a vector space on which a norm is defined. We say that X is a $Banach\ space$ if it is a complete normed space, that is, every Cauchy sequence in X converges.

1.1.3 Example. The following are some examples of normed spaces:

1. The Euclidean space $\mathbb{R}^n = \{x = (\xi_i)_{i=1}^n : \xi_i \in \mathbb{R}\}$, with norm defined by

$$\|x\| = \sqrt{\sum_{i=1}^{n} |\xi_i|^2}.$$

   Note that the Euclidean space is finite dimensional. Thus, it is a complete normed space, that is, $\mathbb{R}^n$ is a Banach space.

2. The space $l^p = \{x = (\xi_i)_{i=1}^\infty : \xi_i \in \mathbb{R}$ or $\mathbb{C}$ such that $\sum_{i=1}^\infty |\xi_i|^p < \infty\}$, with the norm defined by

$$\|x\| = \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{\frac{1}{p}}.$$

   The l^p space is a complete metric space under the metric

$$d(x, y) = \left(\sum_{i=1}^{\infty} |\xi_i - \nu_i|^p\right)^{\frac{1}{p}}, \text{ where } x = (\xi_i)_{i=1}^\infty \text{ and } y = (\nu_i)_{i=1}^\infty \text{ in } l^p.$$

   Hence, the l^p space is a Banach space.

1.1.4 Definition. A non-negative real-valued function p on a vector space X is said to be a $semi\text{-}norm$ on X, if it satisfies, for all $x, y \in X$ and for all scalars α, the following:

1. $p(x + y) \leq p(x) + p(y)$.
2. $p(\alpha x) = |\alpha| p(x)$.

1.1.5 Definition. Let X and Y be vector spaces over the field K. Then a mapping $T : \mathcal{D}(T) \subseteq X \longmapsto \mathcal{R}(T) \subseteq Y$ is said to be a $linear\ operator$ if for all $x, y \in \mathcal{D}(T)$ and scalars $\alpha, \beta \in K$

1. $T(\alpha x + \beta y) = \alpha(Tx) + \beta(Ty)$.

We now take a look at an example of a linear operator.

1.1.6 Example. Let $Y = C[0,1]$ be the normed space of continuous functions defined on the unit interval $[0,1]$ and let X be the subspace of Y consisting of those elements x which have first and second derivative continuous on $[0,1]$. Let $p, q \in Y$ and define an operator T on X by $Tx = y$ where

$$y(s) = x''(s) + p(s)x'(s) + q(s)x(s).$$

Then T is a linear operator from X to Y. For all $x_1, x_2 \in X$ and all scalars λ_1, λ_2 we have the following:

$$\begin{aligned} T(\lambda_1 x_1 + \lambda_2 x_2) &= (\lambda_1 x_1 + \lambda_2 x_2)'' + p(\lambda_1 x_1 + \lambda_2 x_2)' + q(\lambda_1 x_1 + \lambda_2 x_2) \\ &= \lambda_1 x_1'' + \lambda_2 x_2'' + \lambda_1 p x_1' + \lambda_2 p x_2' + \lambda_1 q x_1 + \lambda_2 q x_2 \\ &= \lambda_1 (x_1'' + p x_1' + q x_1) + \lambda_2 (x_2'' + p x_2' + q x_2) \\ &= \lambda_1 T x_1 + \lambda_2 T x_2. \end{aligned}$$

1.1.7 Definition. Let X and Y be normed spaces and T a linear operator as defined in Definition (1.1.5). Then T is said to be a *bounded linear operator* if there is a real number $c > 0$ such that for all $x \in \mathcal{D}(T)$,

$$\|Tx\| \leq c\|x\|.$$

Let us look at examples of a bounded linear operator.

1.1.8 Example. Let X be a normed space and I the identity operator on X defined by

$$Ix = x.$$

Clearly, I is a bounded linear operator with norm

$$\|I\| = \sup_{\substack{x \in X \\ \|x\|=1}} \|Ix\| = \sup_{\substack{x \in X \\ \|x\|=1}} \|x\| = 1.$$

Let us have a look at another example.

1.1.9 Example. A bounded linear operator T from $C[a,b]$ into itself is defined by

$$Tx(t) = \int_a^t x(\tau)d\tau.$$

Let $x, y \in C[a,b]$ and λ_1, λ_2 be scalars. Then

$$\begin{aligned} T(\lambda_1 x + \lambda_2 y)(t) &= \int_a^t (\lambda_1 x + \lambda_2 y)(\tau)d\tau \\ &= \int_a^t (\lambda_1 x(\tau) + \lambda_2 y(\tau))\, d\tau \\ &= \lambda_1 \int_a^t x(\tau)d\tau + \lambda_2 \int_a^t y(\tau)d\tau \\ &= \lambda_1 Tx(t) + \lambda_2 Ty(t). \end{aligned}$$

Hence, T is linear.

We now show that T is bounded. Let $J = [a, b]$ and $\|x\| = \max_{t \in J} |x(t)|$. Then

$$\begin{aligned}
\|Tx\| &= \max_{t\in J} \left| \int_a^t x(\tau) d\tau \right| \\
&\leq \max_{t\in J} \int_a^t |x(\tau)| d\tau \\
&\leq \max_{t\in J} \int_a^t \max_{u\in J} |x(u)| d\tau \\
&= \max_{u\in J} |x(u)| \cdot \max_{t\in J} \int_a^t d\tau \\
&= \|x\| \cdot \max_{t\in J} (t - a) \\
&= (b - a) \|x\|,
\end{aligned}$$

that is, $\|Tx\| \leq (a - b)\|x\|$. Therefore, T is bounded.

We now proceed to give the definition of a linear functional, which is basically a linear operator with range in the scalar field.

1.1.10 Definition. Let X be a vector space over a field K. A $linear\ functional\ f$ is a linear operator with domain $\mathcal{D}(f) \subseteq X$ and range $\mathcal{R}(f) \subseteq K$, that is,

$$f : \mathcal{D}(f) \longmapsto \mathcal{R}(f) \subseteq K.$$

1.1.11 Definition. A linear functional f is said to be a $bounded\ linear\ functional$ if there is a real number $c > 0$ such that for all $x \in \mathcal{D}(f)$,

$$|f(x)| \leq c\|x\|.$$

We now state an important theorem of linear functionals, which happens to be a special case of ([5], Theorem 2.7-9). Basically the theorem asserts that for linear functionals, continuity and boundedness are equivalent concepts. We are going to apply this result in some sections of Chapter 3.

1.1.12 Theorem (Continuity and boundedness)**.** ([5], Theorem 2.8-3). *A linear functional f with domain $\mathcal{D}(f)$ in a normed space X is continuous if and only if it is bounded.*

We conclude the section by looking at one of the fundamental theorem of functional analysis. The Uniform bounded Theorem, gives sufficient conditions for $(\|T_n\|)$ to be bounded, where (T_n) is a sequence of bounded linear operators. This theorem is very important in our study as we shall use it to prove that if (x_n) converges weakly to x, then the sequence $(\|x_n\|)$ is bounded.

1.1.13 Theorem (Uniform Boundedness Theorem)**.** ([5], Theorem 4.7-3). *Let X be a Banach space and Y a normed space. Let (T_n) be a sequence of bounded linear operators $T_n : X \longmapsto Y$ such that $(\|T_n x\|)$ is bounded for every $x \in X$, that is,*

$$\|T_n x\| \leq c_x,$$

where $n \in \mathbb{N}$ and c_x a real number. Then the sequence of the norms $\|T_n\|$ is bounded, that is, there is a $c > 0$ such that

$$\|T_n\| \leq c,$$

where $n \in \mathbb{N}$.

1.2 Dual Spaces

In this section we discuss the basic properties of the set of bounded linear operators (functionals) from one normed space to another. It is intriguing to know that the set of all bounded linear operators on a normed space actually form a normed space.

1.2.1 Theorem. *Let X and Y be normed spaces and let*

$$B(X,Y) = \{T : X \longmapsto Y | \, T \text{ is a bounded linear operator}\}.$$

Then $B(X,Y)$ is a vector space if we define for all $T_1, T_2 \in B(X,Y)$ and all scalars λ the sum and scalar multiplication by

$$(T_1 + T_2)x = T_1x + T_2x \text{ and } (\lambda T)x = \lambda(Tx),$$

respectively.

1.2.2 Theorem ([6], Proposition 5.6)**.** *The following statements are true for normed spaces X and Y.*

1. *The vector space $B(X,Y)$ is a normed space with norm defined by*

$$\|T\| = \sup_{\substack{x\in X \\ \|x\|\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in X \\ \|x\|=1}} \|Tx\|.$$

2. *If Y is a Banach space, then $B(X,Y)$ is a Banach space.*

1.2.3 Theorem. *Let X be a normed space over a field K. Then the set*

$$X' = \{f : X \longmapsto K | \, f \text{ is linear and bounded}\},$$

is a normed space with norm defined by

$$\|f\| = \sup_{\substack{x\in X \\ \|x\|=1}} |f(x)|.$$

We call the space X' the $dual\ space$ of X.

1.2.4 Theorem ([5], Theorem 2.10-4)**.** *Let X be a normed space. Then the dual space X' is a Banach space.*

1.2.5 Remark. Note that X' is a Banach space whether X is a Banach space or not.

1.3 Inner Product Spaces

1.3.1 Definition. An $inner\ product$ on a K-vector space X is a mapping $\langle \cdot, \cdot \rangle : X \times X \longmapsto K$ such that, for all $x, y, z \in X$ and all scalars $\lambda \in K$,

1. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$.
2. $\langle \lambda x, y\rangle = \lambda \langle x, y\rangle$.

3. $\langle x, y\rangle = \overline{\langle y, x\rangle}$.
4. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

An *inner product space* (or *pre-Hilbert space*) is a pair $(X, \langle\cdot,\cdot\rangle)$ where X is a vector space and $\langle\cdot,\cdot\rangle$ is an inner product on X.

1.3.2 Remark. The norm of a vector space X in an inner product space is given by

$$\|x\| = \sqrt{\langle x, x\rangle}$$

and a metric on X is defined by

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle}.$$

1.3.3 Definition. A *Hilbert space* is an inner product space which is a complete metric space with respect to the metric induced by its inner product.

1.3.4 Remark. We notice that all Hilbert spaces are Banach spaces, but the converse is not true.

1.3.5 Theorem ([5], Theorem 3.2-4)**.** *Let Y be a subspace of a Hilbert space H. Then we have the following:*

(a) The subspace Y is complete if and only if Y is closed in H.

(b) If Y is finite dimensional, then Y is complete.

(c) If H is separable, so is Y. In particular, every subset of a separable inner product space is separable.

1.3.6 Theorem (Riesz-Fréchet)**.** ([10], Theorem 6.8). *Let H be a Hilbert space and f a bounded linear functional on H. Then for a uniquely determined $z \in H$, we can represent f in terms of the inner product, that is,*

$$f(x) = \langle x, z\rangle \ \forall x \in H,$$

where z depends on f and has norm

$$\|z\| = \|f\|.$$

1.3.7 Proposition ([5], Problem nr.3 p.194)**.** Let X be an inner product space. If $z \in X$ is any fixed element, then

$$f(x) = \langle x, z\rangle \ \forall x \in X,$$

defines a bounded linear functional f on X, of norm $\|z\|$.

2. The Hahn-Banach Theorem

In this chapter we are going to give a detailed proof of the Hahn-Banach theorem, which is one of the fundamental theorems of normed and Banach spaces. Most of the material discussed in this chapter is based on ([5], Chapter 4).

2.0.1 Definition. A *partially ordered* set M is a set on which a partial order is defined, that is, a binary relation denoted by $\leq$ and which satisfies the following conditions for all $a, b, c \in M$:

1. $a \leq a$ for every a. **(Reflexivity)**
2. If $a \leq b$ and $b \leq a$, then $a = b$. **(Antisymmetry)**
3. $a \leq b$ and $b \leq c$, then $a \leq c$. **(Transitivity)**

2.0.2 Definition. The set M is said to be a *chain* or a *totally ordered* set if every two elements in M are comparable. An element u of a partially ordered set M is said to be an *upper bound* of a subset W of M if $x \leq u$ for every $x \in W$. An element $m \in M$ is said to be a *maximal element* if $m \leq x$ implies $m = x$ for all $x \in M$.

2.0.3 Zorn's lemma. Let a non-empty set M be a partially ordered set. If every chain $C \subset M$ has an upper bound, then M has at least one maximal element.

2.0.4 Definition (Sublinear functional)**.** A real-valued functional p on a vector space X is said to be a *sublinear functional* if it satisfies

1. $p(x + y) \leq p(x) + p(y) \quad \forall\, x, y \in X.$ **(Subadditivity)**
2. $p(\lambda x) = \lambda p(x) \quad \forall\, \lambda \in \mathbb{R},\ \lambda \geq 0$ and $x \in X.$ **(Positive-homogeneity)**

2.0.5 Remark. A norm $\|\cdot\|$ on a vector space X is an example of a sublinear functional since it satisfies axiom (1) and (2) of Definition $(2.0.4)$.

2.1 The Hahn-Banach Theorem in Real Vector Spaces

In this section we discuss the Hahn-Banach theorem in real vector spaces.

2.1.1 Theorem (The Hahn Banach Theorem (Extension of linear functionals))**.** ([5], Theorem 4.2-1*). Let X be a real vector space and let p be a sublinear functional on X. Also, let f be a linear functional defined on a subspace Z of X such that*

$$f(x) \leq p(x) \quad \forall\, x \in Z.$$

Then there exists a linear extension $\tilde{f}$ of f from Z to X satisfying

$$\tilde{f}(x) \leq p(x) \quad \forall\, x \in X.$$

Proof. We are going to use Zorn's lemma for our proof, so we need a partially ordered set from which we shall obtain a maximal element.

Firstly, we define a set E of all linear extensions g of f satisfying

$$g(x) \leq p(x) \quad \forall\, x \in \mathcal{D}(g), \tag{2.1.1}$$

where $\mathcal{D}(g)$ denotes the domain of g. E is non-empty since f is its own linear extension and $f(x) = f(x)$ for all $x \in Z$. Let us now define a partial ordering on E by

$$g \leq h \text{ meaning } h \text{ is an extension of } g,$$

that is, $\mathcal{D}(g) \subseteq \mathcal{D}(h)$ and $h(x) = g(x)$ for all $x \in \mathcal{D}(g)$. Consider any chain $C \subset E$. We define a functional $\hat{g}$ on the domain

$$\mathcal{D}(\hat{g}) = \bigcup_{g \in C} \mathcal{D}(g) \subseteq X,$$

by

$$\hat{g}(x) = g(x) \quad \forall\, x \in \mathcal{D}(g). \tag{2.1.2}$$

We need to show that the domain $\mathcal{D}(\hat{g})$ of $\hat{g}$ is a vector space. Since C is a chain, then whenever $x, y \in \mathcal{D}(\hat{g})$ there exist $g_1, g_2 \in C$ such that $x \in \mathcal{D}(g_1)$ and $y \in \mathcal{D}(g_2)$. Now $g_1, g_2 \in C$ implies that $g_1 \leq g_2$ or $g_2 \leq g_1$. If $g_1 \leq g_2$ then $\mathcal{D}(g_1) \subseteq \mathcal{D}(g_2)$ and $g_2(w) = g_1(w)$ for all $w \in \mathcal{D}(g_1)$, so that $x, y \in \mathcal{D}(g_2)$. We know that $\mathcal{D}(g_2)$ is a vector space, hence for scalars $\alpha, \beta \in \mathbb{R}$ we have that $\alpha x + \beta y \in \mathcal{D}(g_2) \subseteq \mathcal{D}(\hat{g})$. Similarly if $g_2 \leq g_1$ then $x, y \in \mathcal{D}(g_1)$ and $\alpha x + \beta y \in \mathcal{D}(g_1) \subseteq \mathcal{D}(\hat{g})$. Therefore, $\mathcal{D}(\hat{g})$ is a vector space. Let $x \in \mathcal{D}(g_1) \cap \mathcal{D}(g_2)$ with $g_1, g_2 \in C$. Then (2.1.2) implies

$$\hat{g}(x) = g_1(x) \text{ and } \hat{g}(x) = g_2(x).$$

Now since $g_1, g_2 \in C$ and C is a chain, $g_1 \leq g_2$ or $g_2 \leq g_1$, thus $x \in \mathcal{D}(g_1) \cap \mathcal{D}(g_2)$ implies $g_1(x) = g_2(x)$. Hence $\hat{g}$ is well defined. Since each $g \in E$ is a linear functional and the domain $\mathcal{D}(g)$ of g is a vector space it follows that for $x, x' \in \mathcal{D}(g)$ and scalars α and β

$$\hat{g}(\alpha x + \beta x') = g(\alpha x + \beta x') = \alpha g(x) + \beta g(x') = \alpha \hat{g}(x) + \beta \hat{g}(x').$$

Thus, $\hat{g}$ is a linear functional. Then by (2.1.1) and (2.1.2) we have that

$$\hat{g}(x) = g(x) \leq p(x) \qquad \forall\, x \in \mathcal{D}(g).$$

This implies $\hat{g} \in E$. Clearly, $\hat{g}$ is an upper bound of C. In particular, $f \leq \hat{g}$. Since the chain $C \subset E$ we chose was arbitrary, by Zorn's Lemma we have that E has a maximal element, say $\tilde{f}$, satisfying

$$\tilde{f}(x) \leq p(x) \quad \forall\, x \in \mathcal{D}(\tilde{f}) \text{ and } \tilde{f}(x) = f(x) \quad \forall\, x \in Z. \tag{2.1.3}$$

Now, we have a linear extension $\tilde{f}$ of f dominated by p as can been seen in (2.1.3) and $\tilde{f}$ has no proper linear extension dominated by p (by its maximality in E). So, we need to show that $\tilde{f}$ is defined on all of X, that is $\mathcal{D}(\tilde{f}) = X$. We shall do this by contradiction.

Suppose $\mathcal{D}(\tilde{f}) \neq X$. Then there is some $y_1 \in X - \mathcal{D}(\tilde{f})$. Consider the subspace $Y_1 \subseteq X$ generated by $\mathcal{D}(\tilde{f})$ and y_1, that is $Y_1 = \text{span}(\mathcal{D}(\tilde{f}) \cup \{y_1\})$. Note that $y_1 \neq 0$ since $0 \in \mathcal{D}(\tilde{f})$. Any x in Y_1 can be represented as

$$x = y + \alpha y_1,$$

and the representation above is unique. Suppose there was another way, say $y + \alpha y_1 = \tilde{y} + \beta y_1$ where $y, \tilde{y} \in \mathcal{D}(\tilde{f})$. Then we should have $y - \tilde{y} = (\beta - \alpha) y_1$. We note that $y, \tilde{y} \in \mathcal{D}(\tilde{f})$ implies that $y - \tilde{y} \in \mathcal{D}(\tilde{f})$ since $\mathcal{D}(\tilde{f})$ is a vector space. Now since $y_1 \notin \mathcal{D}(\tilde{f})$ then we must have that $y - \tilde{y} = 0$ and $\beta - \alpha = 0$, that is, $y = \tilde{y}$ and $\beta = \alpha$. We define a functional g_1 on Y_1 by

$$g_1(x) = g_1(y + \alpha y_1) = \tilde{f}(y) + \alpha c, \tag{2.1.4}$$

where $c \in \mathbb{R}$. The functional g_1 is linear since for $x_1, x_2 \in Y_1$ where $x_1 = w + \alpha y_1$ and $x_2 = w' + \beta y_1$ ($w, w' \in \mathcal{D}(\tilde{f})$ and $\alpha, \beta \in \mathbb{R}$), we have that

$$\begin{aligned} g_1(x_1 + x_2) &= g_1\left((w + \alpha y_1) + (w' + \beta y_1)\right) \\ &= g_1\left((w + w') + (\alpha + \beta) y_1\right) \\ &= \tilde{f}(w + w') + (\alpha + \beta)c \\ &= \tilde{f}(w) + \tilde{f}(w') + \alpha c + \beta c \\ &= \tilde{f}(w) + \alpha c + \tilde{f}(w') + \beta c \\ &= g_1(w + \alpha y_1) + g_1(w' + \beta y_1) \\ &= g_1(x_1) + g_2(x_2). \end{aligned}$$

Also for $\alpha = 0$ we have $g_1(y) = \tilde{f}(y)$. Therefore, we have that $\mathcal{D}(\tilde{f}) \subseteq Y_1$ and $g_1(y) = \tilde{f}(y)$ for all $y \in \mathcal{D}(\tilde{f})$ which implies that g_1 is an extension of $\tilde{f}$. Note that $y_1 \in Y_1 \setminus \mathcal{D}(\tilde{f})$; hence $\mathcal{D}(\tilde{f}) \subsetneq Y_1$. Therefore, g_1 is a proper extension of $\tilde{f}$. It remains to show that

$$g_1(x) \leq p(x) \qquad \forall\, x \in \mathcal{D}(g_1). \tag{2.1.5}$$

Hence $g_1 \in E$, and this contradicts the fact that $\tilde{f}$ is a maximal element of E. Thus, $\mathcal{D}(\tilde{f}) \neq X$ is not a true statement and thus $\mathcal{D}(\tilde{f}) = X$ holds. So we must show that with a suitable c in (2.1.4), g_1 satisfies (2.1.5). Let us consider $y, z \in \mathcal{D}(\tilde{f})$. From (2.1.3) and by the definition of a sublinear functional we have that

$$\begin{aligned} \tilde{f}(y) - \tilde{f}(z) = \tilde{f}(y - z) &\leq p(y - z) \\ &= p(y + y_1 - y_1 - z) \\ &\leq p(y + y_1) + p(-y_1 - z). \end{aligned}$$

The above inequality can be written as

$$-p(-y_1 - z) - \tilde{f}(z) \leq p(y + y_1) - \tilde{f}(y), \tag{2.1.6}$$

for a fixed y_1. Now since the left hand side and the right hand side of the inequality is independent of y and z respectively, it implies that the inequality continues to hold if we take the supremum $m_0 = \sup\{z : z \in \mathcal{D}(\tilde{f})\}$ and infimum $m_1 = \inf\{y : y \in \mathcal{D}(\tilde{f})\}$. Then $m_0 \leq m_1$ and for $m_0 \leq c \leq m_1$ we have from (2.1.6) that

$$-p(-y_1 - z) - \tilde{f}(z) \leq c \qquad \forall\, z \in \mathcal{D}(\tilde{f}) \tag{2.1.7}$$

$$c \leq p(y + y_1) - \tilde{f}(y) \qquad \forall\, y \in \mathcal{D}(\tilde{f}). \tag{2.1.8}$$

We first prove (2.1.5) for $\alpha < 0$ in (2.1.4). Consider equation (2.1.7) and let $z = \alpha^{-1} y$ that is,

$$-p\left(-y_1 - \frac{1}{\alpha} y\right) - \tilde{f}\left(\frac{1}{\alpha} y\right) \leq c.$$

Multiplying by $-\alpha > 0$ gives

$$\alpha p\left(-y_1 - \frac{1}{\alpha} y\right) + \tilde{f}(y) \leq -\alpha c$$

so that

$$\tilde{f}(y) + \alpha c \leq -\alpha p\left(-y_1 - \frac{1}{\alpha} y\right) = p(y + \alpha y_1). \tag{2.1.9}$$

From (2.1.9) and (2.1.4) with $x = y + \alpha y_1$ we have,

$$g_1(x) = \tilde{f}(y) + \alpha c \leq p(y + \alpha y_1) = p(x). \tag{2.1.10}$$

For $\alpha = 0$ we have $x \in \mathcal{D}(\tilde{f})$ and there is nothing to prove. For $\alpha > 0$ we consider (2.1.8) and by letting $y = \alpha^{-1} y$

$$c \leq p\left(\frac{1}{\alpha} y + y_1\right) - \tilde{f}\left(\frac{1}{\alpha} y\right).$$

Multiplying by $\alpha > 0$ gives

$$\alpha c \leq \alpha p\left(\frac{1}{\alpha} y + y_1\right) - \tilde{f}(y) = p(x) - \tilde{f}(y). \tag{2.1.11}$$

From (2.1.11) and (2.1.4) we have the following;

$$g_1(x) = \tilde{f}(y) + \alpha c \leq p(x). \tag{2.1.12}$$

The inequalities (2.1.10) and (2.1.12) imply that $g_1 \in E$ which contradicts the maximality of $\tilde{f}$. Therefore, $\mathcal{D}(\tilde{f}) = X$. This concludes our proof. □

2.2 The Hahn-Banach Theorem in Complex Vector Spaces

In this section we discuss the Hahn-Banach theorem in the case of complex vector spaces.

In the case of the generalised Hahn-Banach theorem for $f(x) = g(x) + ih(x)$, F.J. Murray proved that the extension exists using the relation $f(x) = g(x) - ig(ix)$ ([7]). In our case we prove this using the equation $f(x) = h(ix) + ih(x)$, as a small contribution to the work.

2.2.1 Theorem (The Hahn-Banach Theorem (Generalised))**.** ([5], Theorem 4.3-1). *Let X be a vector space over a field K and p a semi-norm on X. Let f be a linear functional defined on a subspace Z of X such that*

$$|f(x)| \leq p(x) \qquad \forall\, x \in Z.$$

Then there exists a linear extension $\tilde{f}$ of f from Z to X satisfying

$$|\tilde{f}(x)| \leq p(x) \qquad \forall\, x \in X.$$

Proof. We first consider the case when X is a real vector space. We notice that a semi-norm is a sublinear functional, and $|f(x)| \leq p(x)$ implies that $f(x) \leq p(x)$ for all $x \in Z$. Thus, by the Hahn-Banach theorem (2.1.1) f has a linear extension $\tilde{f}$ from Z to X satisfying

$$\tilde{f}(x) \leq p(x) \qquad \forall\, x \in X. \tag{2.2.1}$$

By (2.2.1) and Definition (1.1.4) we have that $\tilde{f}(-x) \leq p(-x) = (|-1|)p(x)$, which implies that $-\tilde{f}(x) \leq p(x)$ for all $x \in X$, that is,

$$\tilde{f}(x) \geq -p(x). \tag{2.2.2}$$

Therefore, by (2.2.1) and (2.2.2) the inequality

$$|\tilde{f}(x)| \leq p(x) \qquad \forall\, x \in X$$

holds as required.

Now suppose that X is a complex vector space. Then Z is a complex subspace of X. This implies f is a complex-valued functional. Let

$$f(x) = g(x) + ih(x) \qquad \forall\ x \in Z,$$

where g and h are real-valued and are the real and imaginary parts of f, respectively. Let X_r and Z_r be the real vector spaces obtained by restricting multiplication by scalars to real numbers. Then g and h are defined on Z_r, a subspace of X_r. Since f is linear on Z, we have that for all scalars $\alpha, \beta \in \mathbb{R}$ and $x, y \in Z$ the following holds:

$$\begin{aligned} f(\alpha x + \beta y) &= \alpha f(x) + \beta f(y) \\ &= \alpha \left(g(x) + ih(x)\right) + \beta \left(g(y) + ih(y)\right) \\ &= \alpha g(x) + \beta g(y) + i \left(\alpha h(x) + \beta h(y)\right). \end{aligned} \tag{2.2.3}$$

Also since $f(x) = g(x) + ih(x)$ we have that

$$f(\alpha x + \beta y) = g(\alpha x + \beta y) + i\, h(\alpha x + \beta y).$$

When we equate the real and imaginary parts of the equation above and (2.2.3) we have that

$$\begin{aligned} g(\alpha x + \beta y) &= \alpha g(x) + \beta g(y) \\ h(\alpha x + \beta y) &= \alpha h(x) + \beta h(y). \end{aligned}$$

Hence g and h are linear on Z_r. Then

$$h(x) \le |h(x)| \le |g(x) + ih(x)| = |f(x)| \le p(x),$$

that is,

$$h(x) \le p(x) \qquad \forall\ x \in Z_r.$$

By the Hahn-Banach theorem (2.1.1) h has a linear extension $\tilde{h}$ from Z_r to X_r such that

$$\tilde{h}(x) \le p(x) \qquad \forall x \in X_r.$$

We know that for all $x \in Z$

$$g(ix) + ih(ix) = f(ix) = if(x) = i\left(g(x) + ih(x)\right) = ig(x) - h(x).$$

Hence by comparing the imaginary parts we have

$$g(x) = h(ix) \qquad \forall\ x \in Z.$$

We then define

$$\tilde{f}(x) = \tilde{h}(ix) + i\tilde{h}(x) \qquad \forall\ x \in X.$$

We need to show that $\tilde{f}$ defined above is a linear extension of f from Z to X. For $a, b \in \mathbb{R}$, let $a + ib$ be any complex number. Then we have that

$$\begin{aligned} \tilde{f}\left((a+ib)x\right) &= \tilde{h}\left(i(a+ib)x\right) + i\tilde{h}\left((a+ib)x\right) \\ &= a\tilde{h}(ix) - b\tilde{h}(x) + ia\tilde{h}(x) + ib\tilde{h}(ix) \\ &= (a+ib)\tilde{h}(ix) + i(a+ib)\tilde{h}(x) \\ &= (a+ib)(\tilde{h}(ix) + i\tilde{h}(x)) \\ &= (a+ib)\tilde{f}(x). \end{aligned}$$

Therefore, $\tilde{f}$ is a complex linear functional defined on X. Also for $x \in Z$ we have

$$\tilde{f}(x) = \tilde{h}(ix) + i\tilde{h}(x) = g(x) + ih(x) = f(x).$$

Hence $\tilde{f}$ is a linear extension of f from Z to X. Finally we need to show that $\tilde{f}$ satisfies $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$. Let $x \in X$ such that $\tilde{f}(x) \neq 0$ and writing $\tilde{f}(x)$ in polar form as

$$\tilde{f}(x) = re^{i\theta} \text{ with } r = |\tilde{f}(x)|,$$

we have that $|\tilde{f}(x)| = \tilde{f}(x)e^{-i\theta} = \tilde{f}(e^{-i\theta}x)$. Hence $\tilde{f}(e^{-i\theta}x)$ is real and positive, since $|\tilde{f}(x)|$ is real and positive, so that by (2.2.1) and Definition (1.1.4)

$$|\tilde{f}(x)| = \tilde{f}(e^{-i\theta}x) \leq p(e^{-i\theta}x) = |e^{-i\theta}|p(x) = p(x).$$

This completes the proof. □

With the above results we are going to state and prove a corollary.

2.2.2 Corollary. *Let X be a vector space over a field K and p a semi-norm on X. Then for any given $x_0 \in X$ there is a linear functional $\tilde{f}$ on X such that*

$$\tilde{f}(x_0) = p(x_0)$$

and

$$|\tilde{f}(x)| \leq p(x) \ \forall \ x \in X.$$

Proof. Our aim is to use Theorem (2.2.1). So we need a linear functional dominated by the semi-norm p. Let Z be the subspace of X given by $Z = \{x = \alpha x_0 : \ \alpha \in K\}$. We define a functional $f : Z \mapsto K$ by

$$f(\alpha x_0) = \alpha p(x_0).$$

We need to show that f is linear. Let $x(= \alpha x_0), y(= \beta x_0) \in Z$ and λ_1, λ_2 be scalars. Then we have that

$$\begin{aligned} f(\lambda_1 x + \lambda_2 y) &= f(\lambda_1(\alpha x_0) + \lambda_2(\beta x_0)) \\ &= f((\lambda_1 \alpha + \lambda_2 \beta)x_0) \\ &= (\lambda_1 \alpha + \lambda_2 \beta)p(x_0) \\ &= \lambda_1(\alpha(p x_0)) + \lambda_2(\beta p(x_0)) \\ &= \lambda_1 f(x) + \lambda_2 f(y). \end{aligned}$$

By the definition of a semi-norm we have that

$$|f(\alpha x_0)| = |\alpha p(x_0)| = |\alpha|p(x_0) = p(\alpha x_0),$$

that is, $|f(x)| = p(x)$. Thus, by Theorem (2.2.1) we have that f has a linear extension $\tilde{f}$ such that $|\tilde{f}(x)| \leq p(x)$ on X. It also follows that if $x = x_0 \in Z$, then

$$\tilde{f}(x_0) = f(x_0) = p(x_0).$$

□

2.3 The Hahn-Banach Theorem in Normed Vector Spaces

In this section we give the proof of the Hahn-Banach theorem in normed spaces and Hilbert spaces. Thereafter, we consider some consequences and applications of Theorem (2.3.1).

2.3.1 Theorem (The Hahn-Banach Theorem (Normed spaces)). ([5], Theorem 4.3-2). *Let X be a normed vector space and f a continuous linear functional on a subspace Z of X. Then there exists a continuous linear functional $\tilde{f}$ on X which is an extension of f to X and*

$$\|\tilde{f}\|_X = \|f\|_Z$$

where

$$\|\tilde{f}\|_X = \sup_{\substack{x\in X\\ \|x\|=1}} |\tilde{f}(x)| \text{ and } \|f\|_Z = \sup_{\substack{x\in Z\\ \|x\|=1}} |f(x)|.$$

Proof. If $Z = \{0\}$ the proof follows immediately from the fact that $f = 0$, so that $\tilde{f} = 0$. Suppose $Z \neq \{0\}$. Let f be a continuous linear functional on a subspace Z of X. By Theorem (1.1.12) f is a bounded linear functional. For all $x \in Z$ we have

$$|f(x)| \leq c\|x\|.$$

With $c = \|f\|_Z$ the equation above becomes

$$|f(x)| \leq \|f\|_Z\|x\|, \quad \forall\, x \in Z.$$

We define a function p by

$$p(x) = \|f\|_Z\|x\|,$$

for all $x \in X$. Then p is a semi-norm since for all $x, x' \in X$ and all scalars λ we have that

$$\begin{aligned} p(x + x') = \|f\|_Z\|x + x'\| &\leq \|f\|_Z\,(\|x\| + \|x'\|) \qquad \text{(by the triangle inequality)} \\ &= \|f\|_Z\|x\| + \|f\|_Z\|x'\| \\ &= p(x) + p(x') \end{aligned}$$

and

$$p(\lambda x) = \|f\|_Z\|\lambda x\| = \|f\|_Z(|\lambda|\|x\|) = |\lambda|(\|f\|_Z\|x\|) = |\lambda|p(x).$$

Clearly, $|f(x)| \leq p(x)$ for all $x \in Z$. Therefore, by Theorem (2.2.1) there exists a linear extension $\tilde{f}$ of f from Z to X which satisfies

$$|\tilde{f}(x)| \leq p(x) = \|f\|_Z\|x\| \qquad \forall \quad x \in X.$$

Taking the supremum over all $x \in X$ of norm 1 we have the following inequality:

$$\|\tilde{f}\|_X = \sup_{\substack{x\in X\\ \|x\|=1}} |\tilde{f}(x)| \leq \|f\|_Z.$$

Now since $\tilde{f}(x) = f(x)$ for all $x \in Z$ then we must have

$$\|\tilde{f}\|_X \geq \|f\|_Z.$$

Therefore, we have that

$$\|\tilde{f}\|_X = \|f\|_Z.$$

□

It is worth noting that, if the normed space in Theorem (2.3.1) is a Hilbert space, then the proof follows immediately from the inherent properties of Hilbert spaces. That is, we can prove Theorem (2.3.1) without using Theorem (2.2.1). To vindicate the claim, we are going to state Theorem (2.3.1) as a corollary in terms of Hilbert spaces and consequently prove it.

2.3.2 Corollary. *Let $X = H$ be a Hilbert space and f a bounded linear functional on a closed subspace Z of H. Then there exists a continuous linear functional $\tilde{f}$ on H which is an extension of f to H and $\|\tilde{f}\|_H = \|f\|_Z$.*

Proof. Suppose that Z is a closed subspace of H. By Theorem (1.3.5) Z is a Hilbert space. This implies that f is a bounded linear functional on a Hilbert space. Therefore, by Theorem (1.3.6) f has a Riesz representation, that is, there exists a unique $z \in Z$ such that for all $x \in Z$ we have

$$f(x) = \langle x, z\rangle \text{ and } \|f\|_Z = \|z\|,$$

where z depends on f. Since the inner product is defined on all of H, then we can define

$$\tilde{f}(x) = \langle x, z\rangle \ \forall \ x \in H.$$

We also note that the unique element $z \in H$. Clearly, $\tilde{f}$ is an extension of f from Z to H and

$$\|\tilde{f}\|_H = \|z\| = \|f\|_Z.$$

□

The following result is obtained from Theorem (2.3.1), it is very important as it guarantees the existence of bounded linear functionals on a normed space. Theorem (2.3.3) states that in a normed space X for a non-zero x_0 there is an $\tilde{f} \in X'$ such that $\tilde{f}(x_0) = \|x_0\|$.

2.3.3 Theorem (Bounded Linear functionals)**.** ([5], Theorem 4.3-3). *Let X be a normed space over a field K and $x_0 \neq 0$ an arbitrary element of X. Then there exists a bounded linear functional $\tilde{f}$ on X such that*

$$\tilde{f}(x_0) = \|x_0\| \textit{ and } \|\tilde{f}\| = 1.$$

Proof. We shall use Theorem (2.3.1) for our proof. All we need is a subspace of X and a continuous linear functional defined on the subspace. Let Z be the subspace of X given by $\{x = \alpha x_0 : \ \alpha \in K\}$. We define a functional on Z by

$$f(x) = f(\alpha x_0) = \alpha\|x_0\|.$$

Clearly, f is linear since for all $x(=\alpha x_0), y(=\beta x_0) \in Z$ and all scalars λ_1, λ_2 we have that

$$\begin{aligned} f(\lambda_1 x + \lambda_2 y) &= f(\lambda_1(\alpha x_0) + \lambda_2(\beta x_0)) \\ &= f((\lambda_1\alpha + \lambda_2\beta)x_0) \\ &= (\lambda_1\alpha + \lambda_2\beta)\|x_0\| \\ &= \lambda_1(\alpha\|x_0\|) + \lambda_2(\beta\|x_0\|) \\ &= \lambda_1 f(x) + \lambda_2 f(y), \end{aligned}$$

as required. Also, we have that if $x = \alpha x_0$, then

$$|f(x)| = |f(\alpha x_0)| = |\,\alpha\|x_0\|\,| = |\alpha|\,\|x_0\| = \|\alpha x_0\| = \|x\|.$$

Clearly, f is a bounded linear functional on Z and $\|f\| = 1$. Therefore, by Theorem (2.3.1) there exists a continuous linear extension $\tilde{f}$ of f from Z to X with norm

$$\|\tilde{f}\| = \|f\| = 1.$$

We note that if $\alpha = 1$ then equation (2.3) reduces to $f(x_0) = \|x_0\|$. Since $\tilde{f}$ is an extension of f, $f(x_0) = \|x_0\|$ implies that

$$\tilde{f}(x_0) = f(x_0) = \|x_0\|.$$

□

In what follows, we are going to restate and prove Theorem (2.3.3) in the case of Hilbert spaces.

2.3.4 Corollary. *Let $X = H$ be a Hilbert space and let $x_0 \neq 0$ be any element in H. Then there exists a bounded linear functional $\tilde{f}$ on H such that*

$$\|\tilde{f}\| = 1 \text{ and } \tilde{f}(x_0) = \|x_0\|.$$

Proof. Let z be any fixed element in H. Then Proposition (1.3.7) implies that $\tilde{f}$ defined by

$$\tilde{f}(x) = \langle x, z\rangle \ \forall \ x \in H,$$

is a bounded linear functional on H and has norm

$$\|\tilde{f}\| = \|z\|. \tag{2.3.1}$$

Now for an $0 \neq x_0 \in H$ we set $z = \frac{x_0}{\|x_0\|}$. Using Equation (2.3.1), we can write

$$\|\tilde{f}\| = \|z\| = \left\|\frac{x_0}{\|x_0\|}\right\| = \frac{\|x_0\|}{\|x_0\|} = 1.$$

It also follows that

$$\tilde{f}(x_0) = \langle x_0, z\rangle = \left\langle x_0, \frac{x_0}{\|x_0\|}\right\rangle = \frac{1}{\|x_0\|}\langle x_0, x_0\rangle = \frac{\|x_0\|^2}{\|x_0\|} = \|x_0\|.$$

This completes the proof. □

2.3.5 Corollary ([5], Corollary 4.3-4)**.** *For every x in a normed space X we have*

$$\|x\| = \sup_{\substack{f\in X' \\ f\neq 0}} \frac{|f(x)|}{\|f\|}. \tag{2.3.2}$$

Hence if x_0 is such that $f(x_0) = 0$ for all $f \in X'$, then $x_0 = 0$.

Proof. Let X be a normed space and let x be any element in X. For $x = 0$ it is easy to see that (2.3.2) holds. Let $x \neq 0$. Then Theorem (2.3.3) implies that there exists an $\tilde{f} \in X'$ such that $\tilde{f}(x) = \|x\|$ and $\|\tilde{f}\| = 1$. Now, we define a set $S = \left\{\frac{|f(x)|}{\|f\|} : 0 \neq f \in X'\right\}$. We need to show that S is bounded above and thus the supremum $\sup S$ exists. For this purpose, we let $\|x\| = M \in \mathbb{R}$. Consider an arbitrary $f \in X'$. Then we have that

$$|f(x)| \leq \|f\|\|x\|. \tag{2.3.3}$$

Therefore,

$$\frac{|f(x)|}{\|f\|} \leq \|x\| = M < \infty,$$

that is, the set S is bounded above. This implies that

$$\sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|} \leq \|x\|. \tag{2.3.4}$$

On the other hand we have that,

$$\sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|} \geq \frac{|\tilde{f}(x)|}{\|\tilde{f}\|} = \frac{\|x\|}{1} = \|x\|. \tag{2.3.5}$$

Therefore, by (2.3.4) and (2.3.5) we obtain

$$\|x\| = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|}.$$

Now, suppose $f(x_0) = 0$ for all $f \in X'$. Then we have that

$$\|x_0\| = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x_0)|}{\|f\|} = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{0}{\|f\|} = 0,$$

that is, $\|x_0\| = 0$. Hence, $x_0 = 0$. □

The following result asserts that for every non-trivial normed space there exists a bounded linear functional.

2.3.6 Corollary. *Let X be a normed space and X' its dual space. If X is non-trivial, that is $X \neq \{0\}$, then X' is also non-trivial.*

Proof. Suppose $X \neq \{0\}$, and let $0 \neq x \in X$. Then, by Theorem (2.3.3) there exists a bounded linear functional $\tilde{f}$ on X such that

$$\tilde{f}(x) = \|x\| \neq 0 \text{ and } \|\tilde{f}\| = 1. \tag{2.3.6}$$

This implies that $\tilde{f} \neq 0 \in X'$, that is, X' is non-trivial. □

2.3.7 Corollary. *Let X be a normed space and X' its dual space. If $f(x_0) = 0$ for all $f \in X'$, then $x_0 = 0$.*

Proof. Suppose that $0 \neq x_0 \in X$. Then by Theorem (2.3.3) there exists an $f \in X'$ such that $f(x_0) = \|x_0\| \neq 0$. □

2.3.8 Corollary. *If $f(x) = f(y)$ for every bounded linear functional f on a normed space X, then $x = y$.*

Proof. Suppose $f(x) = f(y)$. Then by linearity of f we have that $f(x - y) = 0$. Hence, by Corollary (2.3.5) we have that $x - y = 0$ which gives $x = y$. □

2.3.9 Corollary. *Let X be a normed space over a field K and $x_0 \neq 0$ be any element in X. Then there is a bounded linear functional $\hat{f}$ on X such that $\|\hat{f}\| = \|x_0\|^{-1}$ and $\hat{f}(x_0) = 1$.*

Proof. Let X be a normed space. Then for a non-zero x_0 in X by Theorem (2.3.3) we have that there exists a bounded linear functional $\tilde{f}$ such that

$$\|\tilde{f}\| = 1 \text{ and } \tilde{f}(x_0) = \|x_0\|.$$

We set $\hat{f} = \|x_0\|^{-1}\tilde{f}$. Thus, applying $\hat{f}$ to x_0 we obtain

$$\hat{f}(x_0) = \|x_0\|^{-1}\tilde{f}(x_0) = \|x_0\|^{-1}\|x_0\| = 1.$$

It also follows that the norm of $\hat{f}$ is

$$\|\hat{f}\| = \| \, \|x_0\|^{-1}\tilde{f} \, \| = \|x_0\|^{-1}\|\tilde{f}\| = \|x_0\|^{-1}.$$

□

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3. Applications of The Hahn-Banach Theorem

The previous chapter consisted of the development of the Hahn-Banach theorem. In this chapter we are going to look at some of the applications to the adjoint operator, separability and strong and weak convergence.

3.1 Adjoint Operator

In this section we study the application of the Hahn-Banach theorem, we are going to use Theorem (2.3.3) to prove an important and interesting result which states that the norm of the adjoint operator is the same as the norm of the "normal" operator defined in Chapter 2.

We begin by giving the definition of the adjoint operator.

3.1.1 Definition (Adjoint Operator)**.** Let X and Y be normed spaces and let $T : X \longmapsto Y$ be a bounded linear operator. Then we define the adjoint operator $T^{\times} : Y' \longmapsto X'$ of T by

$$(T^{\times}g)(x) = f(x) = g(Tx) \ \forall \ g \in Y', \tag{3.1.1}$$

where X' is the dual space of X and Y' the dual space of Y.

3.1.2 Remark. We notice that f given in Equation (3.1.1) is a bounded linear functional and its domain is X. Since g and T are both linear, then for all $x, y \in X$ and all scalars α, β we have the following:

$$\begin{aligned} f(\alpha x + \beta y) &= g(T(\alpha x + \beta y)) \\ &= g(\alpha Tx + \beta Ty) \\ &= \alpha g(Tx) + \beta g(Ty) \\ &= \alpha f(x) + \beta f(y). \end{aligned}$$

Therefore, f is linear.

Also, by the boundedness of g and T we obtain

$$|f(x)| = |g(Tx)| \leq \|g\| \, \|Tx\| \leq \|g\| \, \|T\| \|x\|.$$

Hence,

$$\sup_{\substack{x \in X \\ \|x\|=1}} |f(x)| = \|f\| \leq \sup_{\substack{x \in X \\ \|x\|=1}} \|g\| \, \|T\| \|x\| = \|g\| \, \|T\|,$$

that is,

$$\|f\| \leq \|g\| \, \|T\|. \tag{3.1.2}$$

Therefore, we have that $f \in X'$.

The following result is interesting since it asserts that the norm of the adjoint operator $T^{\times}$ and the norm of the operator T are equivalent.

3.1.3 Theorem (Norm of the adjoint operator)**.** ([5], Theorem 4.5-2). *The adjoint operator* $T^\times$ *is linear and bounded, and*

$$\|T^\times\| = \|T\|. \tag{3.1.3}$$

Proof. To obtain the desired results we shall use Theorem (2.3.3). Firstly, we need to show that $T^\times$ is linear and bounded.

The domain $\mathcal{D}(T^\times)$ of $T^\times$ is Y' by Definition (3.1.1), which is a vector space. Let $g_1, g_2 \in Y'$ and α, β be scalars. Then it follows that $\alpha g_1 + \beta g_2 \in Y'$. Thus,

$$\begin{aligned}(T^\times(\alpha g_1 + \beta g_2))(x) &= (\alpha g_1 + \beta g_2)(Tx)\\ &= \alpha g_1(Tx) + \beta g_2(Tx)\\ &= \alpha(T^\times g_1)(x) + \beta(T^\times g_2)(x)\end{aligned}$$

Therefore, $T^\times$ is linear.

We now proceed to show that $T^\times$ is bounded. By Definition (3.1.1) we have that $(T^\times g)(x) = f(x)$. Thus, we can write $f = T^\times g$. It then follows from the Inequality (3.1.2) that

$$\|T^\times g\| = \|f\| \le \|T\|\,\|g\|.$$

Therefore, $T^\times$ is bounded.

Now, from the inequality above we obtain

$$\sup_{\substack{g\in Y'\\ \|g\|=1}} \|T^\times g\| = \|T^\times\| \le \sup_{\substack{g\in Y'\\ \|g\|=1}} \|T\|\,\|g\| = \|T\|,$$

that is,

$$\|T^\times\| \le \|T\|. \tag{3.1.4}$$

We need to show that $\|T\| \le \|T^\times\|$. Now, since $y = Tx$, let $Tx_0 \neq 0$. Theorem (2.3.3) implies that there is a $g_0 \in Y'$ such that

$$\|g_0\| = 1 \text{ and } g_0(Tx_0) = \|Tx_0\|.$$

On the other hand we have that $g_0(Tx_0) = (T^\times g_0)(x_0)$, by Definition (3.1.1). So if we set $f_0 = T^\times g_0$, we obtain the following:

$$\begin{aligned}\|Tx_0\| = g_0(Tx_0) &= f_0(x_0)\\ &\le |f_0(x_0)|\\ &\le \|f_0\|\|x_0\|\\ &= \|T^\times g_0\|\|x_0\|\\ &\le \|T^\times\|\,\|g_0\|\|x_0\|.\end{aligned}$$

Now since $\|g_0\| = 1$, then it follows that for all $x_0 \in X$

$$\|Tx_0\| \le \|T^\times\|\|x_0\|. \tag{3.1.5}$$

Clearly, (3.1.5) holds if $Tx_0 = 0$. Now T being bounded implies that

$$\|Tx_0\| \le c\|x_0\|,$$

and the smallest c for which the inequality holds is $\|T\|$. Therefore, (3.1.5) implies that

$$\|T\| \le \|T^\times\| \tag{3.1.6}$$

Therefore, by (3.1.4) and (3.1.6) we have $\|T\| = \|T^\times\|$. □

3.2 Separability

In this section we look at applications of the Hahn-Banach theorem to separability. Here we are going to show that if the dual space X' of a normed space X is separable then X is separable.

3.2.1 Definition. A subset M of a metric space X is said to be *dense* in X if its closure equals X, that is,

$$\overline{M} = X.$$

X is *separable* if it has a countable dense subset.

3.2.2 Definition. Let X be a normed space and $M \neq \emptyset$ be a subset of X. The *annihilator* M^a of M is the set of all bounded linear functionals on X such that

$$f(x) = 0 \quad \forall\, x \in M.$$

We will need the following lemma for the proof of the separability theorem.

3.2.3 Lemma (Existence of a functional)**.** ([5], Lemma 4.6-7). *Let X be a normed space over a field K and Y a proper closed subspace of X. Let $x_0 \in X - Y$ be arbitrary and*

$$\delta = \inf_{\tilde{y} \in Y} \|\tilde{y} - x_0\|, \tag{3.2.1}$$

the distance from x_0 to Y. Then there exists an $\tilde{f} \in X'$ such that

$$\|\tilde{f}\| = 1,\ \ \tilde{f}(y) = 0\ \forall\ y \in Y \text{ and } \tilde{f}(x_0) = \delta.$$

Proof. We shall use Theorem (2.3.1) for the proof. So we need to find a bounded linear functional which is defined on a subspace of X.

Consider the subspace $Z \subseteq X$ given by $Z = \text{span}(Y \cup \{x_0\})$. We notice that $x_0 \neq 0$ since $0 \in Y$. Any element z in Z has a unique representation

$$z = y + \alpha x_0,$$

where $y \in Y$ and $\alpha \in K$. Indeed, suppose there was another representation of z, say $y_1 + \alpha x_0 = y_2 + \beta x_0$ where $y_1, y_2 \in Y$. This implies that $y_1 - y_2 = (\beta - \alpha)x_0$. Since Y is a vector space, $y_1 - y_2 \in Y$. It follows from $x_0 \notin Y$ that $y_1 - y_2 = 0$ and $\beta - \alpha = 0$, that is, $y_1 = y_2$ and $\beta = \alpha$. We define a functional f on Z by

$$f(z) = f(y + \alpha x_0) = \alpha\delta.$$

Clearly, f defined above is linear since for all $z_1, z_2 \in Z$ where $z_1 = y_1 + \alpha x_0$ and $z_2 = y_2 + \beta x_0$ ($y_1, y_2 \in Y$ and $\alpha, \beta \in K$), and all scalars $\lambda_1, \lambda_2 \in K$ we have the following:

$$\begin{aligned} f(\lambda_1 z_1 + \lambda_2 z_2) &= f(\lambda_1 y_1 + \lambda_1 \alpha x_0 + \lambda_2 y_2 + \lambda_2 \beta x_0) \\ &= f(\lambda_1 y_1 + \lambda_2 y_2 + (\lambda_1 \alpha + \lambda_2 \beta) x_0) \\ &= (\lambda_1 \alpha + \lambda_2 \beta)\delta \\ &= \lambda_1(\alpha\delta) + \lambda_2(\beta\delta) \\ &= \lambda_1 f(z_1) + \lambda_2 f(z_2). \end{aligned}$$

We are now going to show that $\delta > 0$. Suppose $\delta = 0$. Then there is a sequence in Y, say (y_n), such that

$$\lim_{n\to\infty} \|y_n - x_0\| = 0,$$

that is, $y_n \longrightarrow x_0$ as $n \longrightarrow \infty$. This is a contradiction since Y is closed and $x_0 \notin Y$. Now, for $\alpha = 0$ we have $f(y) = 0$ for all $y \in Y$. Also if $\alpha = 1$ and $y = 0$, then $f(x_0) = \delta$.

We now need to show that f is bounded, that is, there exists a constant c such that $|f(z)| \leq \|z\|$. We notice that when $\alpha = 0$, $|f(z)| \leq c\|z\|$ clearly holds. So we let $\alpha \neq 0$, then by (3.2.1) and the fact that $-(\frac{1}{\alpha})y \in Y$ we have

$$\begin{aligned}
|f(z)| = |\alpha\delta| = |\alpha|\delta = |\alpha| \inf_{\tilde{y}\in Y} \|\tilde{y} - x_0\| \\
&\leq |\alpha| \left\| -\frac{1}{\alpha}y - x_0 \right\| \\
&= |\alpha| \left| -\frac{1}{\alpha} \right| \|y + \alpha x_0\| \\
&= \|z\|,
\end{aligned}$$

that is, $|f(z)| \leq \|z\|$. Therefore, f is bounded and $\|f\| \leq 1$. It remains to show that $\|f\| \geq 1$. By the definition of infimum, Y contains a convergent sequence (y_n) such that $\|y_n - x_0\| \longrightarrow \delta$ as $n \longrightarrow \infty$. Let $z_n = y_n + \alpha x_0$ with $\alpha = -1$, that is $z_n = y_n - x_0$. Then by the definition of f we have $f(z_n) = -\delta$. Consider the set $S = \left\{ \frac{|f(z)|}{\|z\|} : 0 \neq z \in Z \right\}$. Clearly, S is bounded above since f is bounded. Thus,

$$\|f\| = \sup_{\substack{z\in Z \\ z\neq 0}} \frac{|f(z)|}{\|z\|} \geq \frac{|f(z_n)|}{\|z_n\|} = \frac{\delta}{\|z_n\|}.$$

But $\|z_n\| = \|y_n - x_0\| \longrightarrow \delta$ as $n \longrightarrow \infty$. Hence, $\|f\| \geq \frac{\delta}{\delta} = 1$. From this and $\|f\| \leq 1$ above we obtain $\|f\| = 1$. Therefore, by Theorem (2.3.1) there exists an extension $\tilde{f}$ of f from Z to X such that

$$\|\tilde{f}\| = \|f\| = 1, \quad \tilde{f}(y) = f(y) = 0 \text{ for all } y \in Y \subseteq Z \text{ and } \tilde{f}(x_0) = f(x_0) = \delta, \text{ since } x_0 \in Z - Y \subseteq X.$$

□

Next we discuss some of the immediate applications of Lemma (3.2.3). These applications are listed as corollaries.

3.2.4 Corollary. *Let X be a normed space and let Y_1 and Y_2 be closed subspaces of X such that $Y_1 \neq Y_2$. Then Y_1 and Y_2 have different annihilators.*

Proof. Suppose Y_1 and Y_2 are closed subspaces of a normed space X such that $Y_1 \neq Y_2$. Then there is, say, an $x_0 \in Y_2 \subseteq X - Y_1$. We denote the distance from x_0 to Y_1 by

$$\delta = \inf_{y\in Y_1} \|y - x_0\|.$$

Hence, Lemma (3.2.3) implies that there exists an $\tilde{f} \in X'$ such that

$$\tilde{f}(y) = 0 \ \forall \ y \in Y_1 \text{ and } \tilde{f}(x_0) = \delta.$$

It follows from Definition (3.2.2) that $\tilde{f}$ belongs to the annihilator Y_1^a of Y_1. We also notice that $\tilde{f} \notin Y_2^a$ since $x_0 \in Y_2$ and $\tilde{f}(x_0) = \delta \neq 0$. Therefore, $Y_1^a \neq Y_2^a$. □

3.2.5 Corollary. *Let X be a normed space and Y a closed subspace of X. If every $f \in X'$ which is zero everywhere on Y is zero everywhere on the whole space X, then $Y = X$.*

Proof. Suppose that every $f \in X'$ which is zero everywhere on Y is zero everywhere on X, and suppose that $Y \neq X$. Then there exists an $x_0 \in X - Y$. Let

$$\delta = \inf_{y \in Y} \|y - x_0\|$$

be the distance from x_0 to Y. Then by Lemma (3.2.3) there exists an $\tilde{f} \in X'$ such that

$$\tilde{f}(y) = 0 \ \forall y \in Y \text{ and } \tilde{f}(x_0) = \delta.$$

This contradicts our earlier assumption that every bounded linear function f which is zero everywhere on Y is zero everywhere on X. □

3.2.6 Corollary. *Let X be a normed space and M a subset of X. Then $x_0 \in X$ is an element of $A = \overline{\text{Span } M}$ if and only if $f(x_0) = 0$ for every $f \in X'$ such that $f|_M = 0$.*

Proof. We need to show that $x_0 \in A$ if and only if $f(x_0) = 0$ for all $f \in X'$ satisfying $f|_M = 0$.

We note that A is a closed subspace of X. Let $x_0 \in X - A$. Let the distance from x_0 to A be given by

$$\delta = \inf_{y \in A} \|y - x_0\|.$$

Then by Lemma (3.2.3) there exists an $f \in X'$ such that

$$f(y) = 0 \ \forall y \in A \text{ and } f(x_0) = \delta.$$

Now since $M \subseteq A$ we have that $f(y) = 0$ for all $y \in M$, that is, $f|_M = 0$. Also, $f(x_0) = \delta \neq 0$ since $x_0 \in X - A$ and A is closed.

Now we need to show that if $x_0 \in A$ then $f(x_0) = 0$ for all $f \in X'$ such that $f|_M = 0$.

Let $x_0 \in A$ and f any bounded linear functional satisfying $f(y) = 0$ for all $y \in M$. By linearity of f we have that

$$f|_M = 0 \text{ implies } f|_{\text{Span } M} = 0.$$

Since the Span M is dense in A and f is continuous on A, it follows that $f|_A = 0$, so that $f(x_0) = 0$. □

We now state and prove the separability theorem.

3.2.7 Theorem (Separability)**.** ([5], Theorem 4.6-8). *If the dual space X' of a normed space X is separable, then we have that X is separable.*

Proof. Suppose X' is separable. Then X' contains a countable dense set, say M. We can then write $\overline{M} = X'$. Consider the unit sphere $U' = \{f : \|f\| = 1\} \subseteq X'$. We need to show that U' is separable. For this purpose we define a set $N = \left\{\frac{f}{\|f\|} : 0 \neq f \in M\right\}$. Clearly, N is a countable set. It remains to show that $\overline{N} = U'$. Let g be any element in U'. Since $U' \subseteq X'$, we have that $g \in X'$. Now $\overline{M} = X'$ implies there exists a sequence, say (g_n), in M such that $g_n \longrightarrow g$ as $n \longrightarrow \infty$. Since $\|g\| = 1$, we have that $g \neq 0$ and therefore we may assume that $g_n \neq 0$. Then $\left(\frac{g_n}{\|g_n\|}\right)$ is a sequence in N. Since $g_n \longrightarrow g$ and $\|g\| = 1$, it follows that

$$\frac{g_n}{\|g_n\|} \longrightarrow \frac{g}{\|g\|} = g \text{ as } n \longrightarrow \infty.$$

Therefore, $\overline{N} = U'$ since $g \in U'$ was arbitrary.

Consider $U' = \overline{\{f_n : n \in \mathbb{N}\}}$. For each $n \in \mathbb{N}$ we define a set $S_n = \{|f_n(x)| : \|x\| = 1\}$. We notice that

$$\sup S_n = 1.$$

Now, we have that $\frac{1}{2}$ is smaller than the supremum of S_n. This implies that some elements of S_n must be greater than $\frac{1}{2}$, that is,

$$|f_n(x_n)| \geq \frac{1}{2}, \tag{3.2.2}$$

where $\|x_n\| = 1$. Let $A = \text{Span}\{x_n : n \in \mathbb{N}\}$ and Y the closure of A. Consider the set of all linear combinations of $x_n's$ whose coefficients are Gaussian rationals[1], say, $B = \{\alpha_1 x_1 + \cdots + \alpha_m x_m : m \in \mathbb{N}, \alpha_j \text{ Gaussian rationals}\}$. Clearly, the set B is countable. We notice that $A \subseteq \overline{B}$ and $B \subseteq A$. Since the closure $\overline{A}$ is the smallest closed set containing A, we have that $\overline{A} \subseteq \overline{B}$ and $\overline{B} \subseteq \overline{A}$. Therefore, $\overline{A} = \overline{B}$, that is, $Y = \overline{B}$. Hence, Y is separable. Now we need to show that $Y = X$. With $Y \neq X$, Y is a proper closed subspace of X. Hence, Lemma (3.2.3) implies that there exists an $\tilde{f} \in X'$ with norm $\|\tilde{f}\| = 1$ and $\tilde{f}(y) = 0$ for all $y \in Y$. Now $x_n \in Y$ implies that $\tilde{f}(x_n) = 0$. From (3.2.2) we have that for all n,

$$\begin{aligned}\frac{1}{2} \leq |f_n(x_n)| &= |f_n(x_n) - \tilde{f}(x_n)| \\ &= |(f_n - \tilde{f})(x_n)| \\ &\leq \|f_n - \tilde{f}\| \, \|x_n\|,\end{aligned}$$

where $\|x_n\| = 1$. Hence, $\|f_n - \tilde{f}\| \geq \frac{1}{2}$. This implies that for $\epsilon < \frac{1}{2}$ the open ball with centre $\tilde{f}$ does not contain an element of $\{f_n : n \in \mathbb{N}\}$. Now since $\|\tilde{f}\| = 1$, we have that $\tilde{f} \in U'$. Therefore every open ball centred at $\tilde{f}$ contains an element of $\{f_n : n \in \mathbb{N}\}$. This is a contradiction. Therefore, $X = Y$. □

3.3 Strong and Weak Convergence

To conclude the chapter we consider the application of the Hahn-Banach theorem to weak convergence. Among the various applications of weak convergence is the general theory of differential equations. We shall focus our attention on two interesting results, that is, the uniqueness of the weak limit x of a weakly convergent sequence (x_n) and the boundedness of the sequence $(\|x_n\|)$.

To start with, we distinguish weak convergence from the commonly know convergence which we shall call strong convergence.

3.3.1 Definition (Strong convergence)**.** A sequence (x_n) in a normed space X is *strongly convergent* (or *convergent* in the norm) if there is an $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

3.3.2 Definition (Weak convergence)**.** A sequence (x_n) in a normed space X is weakly convergent if there is an $x \in X$ such that for every $f \in X'$

$$\lim_{n\to\infty} f(x_n) = f(x).$$

[1] A Gaussian rational number is a complex number whose real and imaginary part are rational. Note that the set of all Gaussian rationals is countably infinite. Hence, it is countable.

We denote this by

$$x_n \xrightarrow{w} x.$$

Having given the definitions above, we now prove the uniqueness of the weak limit.

3.3.3 Lemma (Weak convergence)**.** ([5], Lemma 4.8-3(a)). *If (x_n) is a weakly convergent sequence in a normed space X, say,*

$$x_n \xrightarrow{w} x.$$

Then (x_n) has a unique weak limit.

Proof. We are going to assume (x_n) has two weak limits and show that they are equal.

Let x and y be weak limits of (x_n). Then for every $f \in X'$

$$\lim_{n\to\infty} f(x_n) = f(x) \text{ and } \lim_{n\to\infty} f(x_n) = f(y).$$

Consider the sequence $(f(x_n))$. Clearly, its limit is unique since it is a sequence of numbers. Hence, $f(x) = f(y)$. Then for every bounded linear functional the following holds:

$$f(x) - f(y) = f(x - y) = 0.$$

Therefore, by Corollary (2.3.5) we have that $x - y = 0$, that is, $x = y$. □

Next we discuss an interesting lemma which states that a functional on the dual space X' is of the same norm as $x \in X$. We shall need this result to prove the boundedness of $(\|x_n\|)$. We begin our discussion with a remark.

3.3.4 Remark. Let X be a normed space. Then we define a function g_x on X' for a fixed $x \in X$ by

$$g_x(f) = f(x) \quad \forall f \in X'. \tag{3.3.1}$$

3.3.5 Lemma (Norm of g_x)**.** *Let X'' be the second dual space of a normed space X and g_x the functional defined by (3.3.1). Then g_x is a bounded linear functional on X', that is, $g_x \in X''$, and has norm*

$$\|g_x\| = \|x\|.$$

Proof. Firstly, we shall show that g_x is a bounded linear functional. Then we shall show that $\|g_x\| = \|x\|$. Let $f_1, f_2 \in X'$ and let λ_1, λ_2 be scalars. Then we have the following:

$$\begin{aligned} g_x(\lambda_1 f_1 + \lambda_2 f_2) &= (\lambda_1 f_1 + \lambda_2 f_2)(x) \\ &= \lambda_1 f_1(x) + \lambda_2 f_2(x) \\ &= \lambda_1 g_x(f_1) + \lambda_2 g_x(f_2). \end{aligned}$$

Hence, g_x is linear. Since $f \in X'$ is bounded, we have that

$$|g_x(f)| = |f(x)| \leq \|f\| \, \|x\|.$$

Therefore, g_x is a bounded linear functional.

We now show that $\|g_x\| = \|x\|$. By definition of g_x and Corollary (2.3.5) we have that

$$\|g_x\| = \sup_{\substack{f\in X' \\ f\neq 0}} \frac{|g_x(f)|}{\|f\|} = \sup_{\substack{f\in X' \\ f\neq 0}} \frac{|f(x)|}{\|f\|} = \|x\|.$$

□

3.3.6 Definition. The canonical mapping is the mapping $C : X \longmapsto X''$ defined by

$$Cx = g_x,$$

where $g_x \in X''$ is unique for every $x \in X$ and is given by (3.3.1).

We shall need the following theorem in the proof of Theorem (3.3.8) below. **Note:** The theorem is stated to suit our work.

3.3.7 Theorem ([3], Theorem 7.5)**.** *Every convergent sequence (a_n) of elements in a normed space X is bounded.*

We now state and prove the Lemma that asserts the convergence of $(\|x_n\|)$ if the sequence (x_n) converges weakly to x.

3.3.8 Theorem ([5], Lemma 4.8-3(c))**.** *If (x_n) is a weakly convergent sequence in a normed space X, say,*

$$x_n \xrightarrow{w} x,$$

then the $(\|x_n\|)$ is bounded.

Proof. Let f be a bounded linear functional. Consider the sequence $(f(x_n))$ of numbers. Since the sequence is convergent, we have that by Theorem (3.3.7) $(f(x_n))$ is bounded. Hence, for all $n \in \mathbb{N}$ we can write

$$|f(x_n)| \leq c_f,$$

where c_f depends on f and not on n. By Definition (3.3.6) we define g_n for each $x_n \in X$ by

$$g_n(f) = f(x_n) \ \forall \ f \in X'.$$

This implies that for all $n \in \mathbb{N}$ we have

$$|g_n(f)| = |f(x_n)| \leq c_f,$$

that is, the sequence $(|g_n(f)|)$ is bounded for every $f \in X'$. Since X' is a Banach space by (1.2.4), that is, X' is complete, it follows from Theorem (1.1.13) that the sequence $(\|g_n\|)$ is bounded. By Lemma (3.3.5) we obtain $\|g_n\| = \|x_n\|$.

This completes the proof. □

Conclusion

We see that Zorn's lemma is very vital in the proof of the Hahn-Banach theorem for real vector spaces. For Theorem (2.2.1), the extension of the functional $f(x) = g(x) + ih(x)$ was obtained using the relation $f(x) = g(x) - ig(ix)$ in ([5], $p.219$), that is, $\tilde{f}(x) = \tilde{g}(x) - i\tilde{g}(ix)$. In our case we used $f(x) = h(ix) + ih(x)$ to obtain the extension $\tilde{f}(x) = \tilde{h}(ix) + i\tilde{h}(x)$. Among the interesting results we encountered is Theorem (2.3.3), which asserts that for every normed space X and $0 \neq x \in X$ there exists a bounded linear functional $\tilde{f}$ on X such that $\tilde{f}(x) = \|x\|$ and has norm $\|\tilde{f}\| = 1$. We then used the theorem to prove Corollary (2.3.6) which asserts that every non-trivial normed space X has a non-trivial dual space X'. Another application of Theorem (2.3.3) is the result presented in Corollary (2.3.7), which states that if X is a normed space and $f(x) = 0$ for all bounded linear functionals f on X, then $x = 0$. Furthermore, we use the same theorem to obtain the result that an operator and its adjoint have the same norms, that is, $\|T^{\times}\| = \|T\|$, in Theorem (3.1.3).

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