Multi-objective Optimization Problem with Multiple Decision Makers: Group Preference

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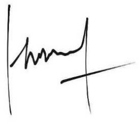
Abstract

The application and use of optimization techniques with multiple (and often conflicting objective functions) attract researchers from different disciplines. Optimization techniques find application in varied fields such as engineering, management and logistics among others. Minimizing cost while maximizing comfort for buying a car is example of multi-objective optimization problem involving two conflicting objective functions. Due to the conflicting nature of the objective functions, choosing a solution often depends on the preference of the human decision maker. Sometimes, there will be multiple decision makers working towards optimizing a multi-objective optimization problem. The decision makers may have their own preference, and the proposed preference may possibly conflict each other. Thus, the study of analysing and producing an agreed (or compromised) preference at the time of group decision making is critical. Hence, how to aggregate the existing multiple preference is the main objective of this report. This report presents a group decision making scenario with multiple decision maker preferences of the same type. New techniques that merge all preferences to produce a single preference which can be used to solve the problem, are proposed. Examples to demonstrate this novel approach are given. The report concludes with a discussion on possible future work in this field.

Keywords: Multi-objective optimization, Multiple decision maker, Group preference

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



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1. Introduction

Our daily life involves various natures of problems which decision needs to be made for. In most cases and under different circumstances, different good candidate solutions can exist for a problem. One question is then, what is the right choice to select among a set of possible solutions? This problem of choosing a solution among many brings in the problem of objective optimization. A decision then refers to right choice to make when having various alternatives. An optimization problem with only one objective function is termed single-objective optimization problem. However, a decision making problem can also have multiple objective functions. The latter problem is termed multi-objective optimization problem. In this work we are interested in the latter problem.

Solution to a multi-objective optimization problem is hard to be found. Many researchers are interested in such problems and have proposed several methods to tackle multi-objective optimization problem. One of the aspects that make multi-objective optimization problem hard to be addressed is that objective functions can be conflicting among themselves. Then finding a solution which aggregates conflictual objective functions may be difficult if not impossible to achieve. For example, let us suppose a person is planning to buy a car. Some criteria can be defined in order to buy a car. It can be low fuel consumption, new brand, cheap price, a specific brand name, number of seats and so on. Now assuming that these criteria cannot be all satisfied. For instance you cannot buy a first-class car with low price. Now requesting different decision makers to set preference on the criteria, their preferences may not be the same. Some decision makers may prefer one criteria but not others. For example, one can prefer to buy a Toyota regardless its level of fuel consumption while another may prefer low fuel consumption regardless of the brand name. Hence, in addition to the objectives, the decision preferences can also be in conflict. Studying the way of combining this preferences to produce a compromised preference is the main aim of this research.

Hence, in this project, different preference structure is studied which will be used to solve the problem using appropriate solution method. Preference based approaches seek for overall optimal solution from conflictual preferences of various decision makers. In other words, preference based approaches look for a solution which satisfies admissible intersection of conflictual preferences based on the given preference. This problem of conflictual preferences are only observed when various decision makers are involved in the process of making decision, but not a single decision maker. That means, with a single decision maker, the problem is much easier to address. It turns out that the main issue could be aggregation of conflictual preferences. Here, we use different existing aggregation schemes to combine these preferences.

Many solution methods to multi-objective optimization problem exists in literature . Solutions to multi-objective optimization problem can be categorised into four groups: priori approach, posteriori approach, interactive (or progressive) approach and non-preference approach. In a priori approach, sufficient information about the right preference is assumed to be known before hand (Andersson, 2000). For posteriori approach, an approach that produces a solution is repeated several times and an aggregating method is considered to get a single solution (Augusto et al., 2012). In interactive approach, preferences are updating along the process of making decision (Thiele et al., 2009; Jaimes et al., 2009). In non-preference method decision maker is not involve to solve the problem. (Wang et al., 2017; Chiandussi et al., 2012). There is also hybrid methods, which combine some of the methods above mentioned. For instance, researchers apply ranking methods and trade-off concept while aggregating conflictual preferences (Tilahun and Ong, 2012; Xiong et al., 2013).

With the aim of preference study in group decision making, this report is structured as follows. Chapter 2 discussed on some concepts of multi-objective optimization problem, solution methods and preference models. On Chapter 3 multiple preference aggregation models of multiple decision maker for multi-objective optimization problem is presented. Some conclusions are provided in the last Chapter including possible future works.

2. Multi-Objective Optimization

2.1 Preliminaries

A multi-objective optimization problem involves a set of objective functions $\mathsf{F}(\mathbf{x})$ to be minimize with some given constraints over the feasible region $\mathcal{X}$. A minimized multi-objective optimization problem with k, $(k \geq 2)$ objective functions over feasible region $\mathcal{X}$, is mathematically expressed by:

$$\min_{\mathbf{x}\in\mathcal{X}\subseteq\mathbb{R}^n} : \quad F(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}), \cdots, f_k(\mathbf{x}))^T, \tag{2.1.1}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ are decision variable vectors. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$, $\mathbf{x}_2$ is dominated by $\mathbf{x}_1$, if $f_j(\mathbf{x}_2) \geq f_j(\mathbf{x}_1)$ for all j and $F(\mathbf{x}_2) \neq F(\mathbf{x}_1)$. For multi-objective optimization problem, the obtained solution is said to be Pareto optimal solution when that solution is not dominated by any other solutions in $\mathcal{X}$ (Marler and Arora, 2004). Due to the conflict of objectives, often, a multi-objective optimization problem has no single solution, which means that there is a set of Pareto optimal solution. This set is called a Pareto optimal set and the outcomes are said to be a Pareto front. In multi-objective optimization problem, a Pareto front is bounded by a so-called ideal point z^{id} and nadir point z^{nad} which can be defined as follows:

2.1.1 Definition. A point $z^{\mathsf{id}} = (z_1^{\mathsf{id}}, z_2^{\mathsf{id}}, \ldots, z_k^{\mathsf{id}})$ is called the ideal point which is defined as:

$$z_i^{id} = \min \{f_i(\mathbf{x}) \mid \mathbf{x} \in \mathcal{X}, \forall\, i = 1, \ldots, k\}.$$

2.1.2 Definition. A point $z^{\mathsf{nad}} = (z_1^{\mathsf{nad}}, z_2^{\mathsf{nad}}, \ldots, z_k^{\mathsf{nad}})$ is called the nadir point which is:

$$z_i^{\mathsf{nad}} = \max \{f_i(\mathbf{x}) \mid \mathbf{x} \in \mathcal{X}, \forall\, i = 1, \ldots, k\}.$$

In practice, the nadir point can only be approximated as, typically, the whole Pareto optimal set is unknown. In addition, a utopia vector z^{uto} is defined by:

$$z_i^{\mathsf{uto}} = z_i^{\mathsf{id}} - \epsilon, \quad \forall\, i = 1, 2, \ldots, k,$$

where $\epsilon > 0$ is a small constant is often defined because of numerical reasons.

2.1.3 Definition. Two outcomes $y = (y_1, y_1, \ldots, y_k)$ and $z = (z_1, z_2, \ldots, z_k)$ are said to be equivalent written as $y \sim z$, if the two solutions are indifferent solutions for the decision maker.

2.2 Multi-objective Optimization Solution Methods

Numerous studies are concerned with the solutions to multi-objective optimization problem and hence several methods have been studied. These method can be classified into two main categories, namely preference based and non-preference based approaches. Since a multi-objective optimization problem will have multiple Pareto optimal solutions, choosing one depends on the preference of a decision maker(s). Some solution methods for multi-objective optimization problem, incorporate or use preference of a decision maker to solve the problem. Giving a preference is not always an easy task for a decision maker. Hence, some solution methods are proposed to solve the problem without a given preference.

(A) **Preference Based Approach** This approach always involves the role of decision makers to solve the problem. Based on the time the role of decision makers are required. This approach is classified into three main groups.

(i) **Priori Approach**: In a priori method, the decision maker provides a preference before searching for a solution. Which means that this approach needs some priori knowledge about the given problem. This approach is used for converting multi-objective optimization problem into single-objective optimization problem. After that, obtaining a Pareto optimal solution for the given multi-objective optimization problem becomes easer. However, it is not an easy task when the decision maker does not have enough knowledge about the given problem (Wang et al., 2017).

(ii) **Progressive Approach**: In a progressive method, the decision maker should find a Pareto optimal solution. After that, depending on the Pareto optimal solution, the decision maker can adjust their own preference for the problem. This process is continues until the interest of the decision maker is fulfilled (Chiandussi et al., 2012). This means that the decision makers are always required throughout the whole process.

(iii) **Posteriori Approach**: In a posteriori method, by using different algorithms the given multi-objective optimization problem is solved. Thus, the decision maker is provided with a set of Pareto optimal solutions. Therefore, based on their preference they can choose a Pareto optimal solution from the obtained Pareto optimal set (Wang et al., 2017).

Non-preference Based Approach In this method, the decision maker preference is not needed for finding a Pareto optimal solution (Emmerich and Deutz, 2018), rather a single Pareto optimal solution is produced and delivered to the decision maker.

Solution methods for multi-objective optimization problem can further classified as classical versus meta-heuristic based solution methods. The classical solution methods are methods based on mathematical and statistical argument to arrive to a Pareto optimal solution(s) whereas metaheuristic algorithms give an approximate solution using soft computing approaches. The scope and interest of this report is on group decision making and hence the focus is on preference related issues. However, for the sake of completeness, basic and commonly used classical solution methods are discussed below.

2.2.1 Weighted Sum Method. The weighted sum method is the most simple way of obtaining a single unique solution for the given multi-objective optimization problem. In this method each objective function is assigned a scalar weight w_i based on their levels of importance (Augusto et al., 2012). Thus, the given multi-objective optimization problem is transformed into a single-objective optimization problem:

$$\min_{\mathbf{x}\in\mathcal{X}} : \quad \sum_{i=1}^{k} w_i f_i(\mathbf{x}), \tag{2.2.1}$$

where

$$w_i \geq 0, \quad \sum_{i=1}^{k} w_i = 1.$$

This method is a priori method, thus, each w_i's are pre-defined for each objective function (Chiandussi et al., 2012). The challenge will be the proper selection of the weights. If the given problem is convex then this method might be effective and the decision maker can obtain all Pareto solutions by using various coefficients of the weights.

2.2.2 Weighting Metric Methods. This method is used to change multi-objective optimization problem into single-objective optimization problem. The idea of this method is to solve the problem by defining weighted metrics to know the distance from a solution to the ideal point z_i^{id} (Augusto et al., 2012). This method is also called the ideal point method, as it try to find the Pareto solution which produces a solution near to the ideal point. The problem can be stated as follows:

$$\min_{\mathbf{x}\in\mathcal{X}} : \left(\sum_{i=1}^{k}[w_i^*|f_i(\mathbf{x}) - z_i^{\text{id}}|]^p\right)^{\frac{1}{p}}, \tag{2.2.2}$$

where the normalized weight

$$w_i^* = \frac{w_i}{z_i^{\text{nad}} - z_i^{\text{id}}} \quad \text{and } 1 \leq p \leq \infty.$$

When $p = \infty$ then the problem is said to be the weighted min-max problem also known as (Tchebycheff problem). Mathematically this problem can be defined by:

$$\min_{\mathbf{x}\in\mathcal{X}} : \quad \max_{1\leq i\leq k}\{w_i^*|f_i - z_i^{id}|\}. \tag{2.2.3}$$

The main drawback of this method is that it needs the minimum and maximum values of the given problems (Hawe and Sykulski, 2008).

2.2.3 Goal Programming Approach. The idea of this approach is to minimize the deviation between the aspirational levels and their achievement of the goals. For each objective f_i the decision maker sets goal, b_i value to be achieved (Odu and Charles-Owaba, 2013). In the weighted goal programming approach the decision maker assigns weight according to their levels of importance for the unwanted deviations and minimized as an Archimedian sum. The problem is stated mathematically as follows:

$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{X}} : \quad & \sum_{i=1}^{k} w_i^*(d_i^+ + d_i^-), \\ \text{subject to :} \quad & f_i(\mathbf{x}) + d_i^+ + d_i^- = b_i, \quad i = 1, 2, \cdots, k, \\ & d_i^+,\ d_i^- \geq 0 \text{ and } d_i^+ d_i^- = 0, \quad i = 1, 2, \cdots, k, \end{aligned} \tag{2.2.4}$$

where d_i^-, d_i^+ represents under-achievement and over-achievement of the i^{th} goal respectively. The word achievement means that a goal has been reached. This method needs priori information about its aspiration levels and weights which is hard for the decision maker. However, this method has the capacity to handle large-scale problems.

2.2.4 The ϵ-Constraint Method. In the ϵ-constraint method decision maker specify a trade-off in terms of the level of importance among the multiple objectives. That means one of the objective is optimized while the others are treated as constraints. The problem can expressed by:

$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{X}} : \ & f_i(\mathbf{x}), \\ \text{subject to :} \ & f_j \leq \epsilon_j, \ \forall\ j \neq i, \end{aligned} \tag{2.2.5}$$

where ϵ_j represents an upper bounds for the j^{th} objectives set by the decision-maker.

This method can be an interactive method. To get several Pareto optimal solutions the decision maker uses multiple different values for ϵ_j (Jaimes et al., 2009). The decision maker uses this method to solve both convex and non-convex problems. However, to choose the vector ϵ is not an easy task for the decision maker. In addition, this method is not applicable when the problem has more objective functions.

2.2.5 Utility Function Method. In this method the decision maker defines a real valued function from the objective space. That function is called a utility function. The utility function is a function that maps from the decision space into a real number. i.e, $U : (f(\mathbf{x_1}), f(\mathbf{x_2}), \cdots, f(\mathbf{x_k})) \to \mathbb{R}$, it gives a value of utility for decision maker (Andersson, 2000). The problem can be defined by:

$$\min_{\mathbf{x} \in \mathcal{X}} : \quad U(f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_k(\mathbf{x})),$$

where $U : \mathbb{R}^k \to \mathbb{R}$.

This method is a priori approach that requires some knowledge about the problems.

2.3 Decision Maker Preference Modelling Methods

Decision preference model uses for making a good decision based on the preference of a decision maker. Usually, decision makers use two types of preferences: either objective level preference based or solution level preference based. In the former one the decision maker uses a preference before solving the given multi-objective optimization problem. i.e, decision makers use preference to choose which objective function is more important and solves that objective function first. Generally in this preference level decision maker uses a preference for comparing the objective functions. In the later one the decision maker uses a preference to choose the best solution from alternative solutions or give a preference based on the give solution. In this section we discuss some preference models such as goals, weights, utility functions, lexicographical ordering, preference relation, tolerance preference and trade-off preference.

2.3.1 Goals. Providing goal information is an easy and understandable way to elaborate preference. Most of the time, users set some targets for different objectives. The decision makers can use goals as the additional criteria for providing a rank with the preference information (Wang et al., 2017).

Advantage:

- Based on the expectation and target of the decision maker and hence easier.

Disadvantages:

- The decision maker requires information about the problem before setting coherent goals.
- This method requires the knowledge of the behaviour of the objectives to preserve feasibility.
- It may propose solutions under or dominated by other solutions, i.e. non-Pareto solutions.

2.3.2 Weights. By using weight vectors $\mathbf{w} = \{w_1, \cdots, w_k\}$, in weight space the decision maker can assign levels of importance for various criteria. The decision maker uses different aggregation functions to convert a multi-objective optimization problem into a single-objective optimization problem (Wang et al., 2017). Most of the time the decision maker prefers the weighted sum method (see Section 2.2.1) and the Tchebycheff approach (see Section 2.2.2) as an aggregation function. To assign accurate weights without priori information is not a simple task for the decision maker.

Advantage:

- For the decision maker this method is easy to implement.

Disadvantages:

- The decision maker requires information before putting weight vector.
- For each objective function to set weight vector is difficult for the decision maker.

2.3.3 Utility Function. Decision makers can define a utility function as a preference for ranking the solutions from the decision space to real number. This function is used for implicitly ranking a set of solutions (Wang et al., 2017). This approach is a priori approach, each objective function information is predefined. Additionally, in this method all attributes are mutually, preferentially independent (Lewandowski, 2017).

Advantage:

- It is advantageous to obtain a precise rank of solution or objective function.

Disadvantages:

- For decision maker it is very difficult to construct a utility function.
- It require information about the problem before start putting a rank.

2.3.4 Lexicographical Ordering. In this preference, the decision maker sets a rank for each objective function (Talebian and Kareem, 2010) according to their importance (from best to worst). This ordering means that the decision maker selects an objective function to be minimized: decision makers start from the most important objective function to the least important objective function. If the decision maker gets a unique solution for the given problem stop searching. Otherwise, select the second most important objective to minimize and continue until the decision maker gets a Pareto optimal solution (Fishburn, 1974).

Advantage:

- For the decision maker it is easy to set a rank.

Disadvantage:

- It difficult to specify an absolute order of importance for each objective function.

2.3.5 Binary Preference Relation. For conflicting objectives decision makers can put different preferences. During the process of decision making, some objectives may not have equal importance. The idea of this preference is first the importance of objectives express quantitatively and then transforming into weights (see Table 2.1 below).

Table 2.1: Objective functions can expressed by those symbols (Wang et al., 2017).

Relation	Meaning	Relation	Meaning
$\prec$	Less important	$\succ$	More important
$\lll$	Much less important	$\ggg$	Much more important
$\approx$	Equally important	$\#$	Do not care
$\neg$	Not important	$\neg$	Important

As shown in Table 2.1, some of the preferences are binary (like 'more important' and 'less important') whereas some are unitary (like 'do not care'). This preference is also similar with those preferences that require information about the problem before starting the process.

This preference can be converted to with weighting preference or lexicographic ranking. Hence, solution methods and analysis which can be used for weighting and ranking can also be used here.

Advantage:

- Easy to provide by the decision maker.

Disadvantage:

- It is difficult to assigning weights after converting.

2.3.6 Tolerance Preference. In this preference approach the decision maker determines the maximum possible tolerance level for almost all objectives. However, if the decision maker gives unreasonable tolerance this may mislead the search process.

Advantage:

- It converts the multi-objective optimization problem into a single-objective optimization problem, if the problem is linear it preserve linearity.

Disadvantages:

- It is a very time-consuming approach.
- Deciding which objectives to be in the constraint is subjective and affect the results.

2.3.7 Trade-off. In this preference the decision maker to set a rank depends on the given value $(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_i, \cdots, \mathbf{x}_j, \cdots, \mathbf{x}_k)$. The idea of this preference is the decision maker willing to give up on objective function i for a unit improvement of objective function j (Tilahun and Ong, 2012). This means that

$$(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_i, \cdots, \mathbf{x}_j, \cdots, \mathbf{x}_k) \sim (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_i - 1, \cdots, \mathbf{x}_{j+l}, \cdots, \mathbf{x}_k), \text{ for } \ l \leq c + d,$$

and this equation is equivalent. Thus, this preference is hard to the decision maker.

Advantage:

- Easy to decide which solution is best.

Disadvantages:

- It is very time consuming because each pair of solution must compared to each other.
- If there is no priori information about the problem it is hard for the decision maker to chose the best solution.

3. Multiple Decision Makers

The process of group decision making involves more than one individual decision maker. Multiple decision makers are a group of people who are interested into optimizing the objectives based on their own preferences. Each of the decision maker has their own priorities and objectives. However, to obtain a solution based on the preferences, a negotiation on preference setting needs to be done. In this study, we consider independent preference setting where each decision maker set and describe their preference separately, independent of the other decision maker's preference. The main goal of multiple decision maker is to get a final solution which is a compromise solution for each the given problem. The chosen solution will take into consideration that, the decision maker have different levels of authority. This authority level is called voting power and represented by v_i, $\sum_{i=1}^{n} v_i = 1$, where n is number of decision makers.

In this chapter, we discuss how to aggregate the given group decision maker preference in a multi-objective optimization problem and analyse existing approaches when all decision makers use the same categorise of preferences modelling methods. Additionally, for the given multi-objective optimization problem each decision maker must use identical preference modelling method.

The preference of a decision maker can basically be categorized as either an objective level preference approach or a solution level preference approach based on how the preference is obtained. Some examples of the former approach are: multiple goals, multiple weights, multiple utility functions, multiple lexicographic orders, multiple tolerance preference and an example of the latter approach is multiple trade-off preference.

3.1 Objective Level Preference Approach

In this approach each decision maker puts a preference based on the given objective functions. The preference given does not depend on a particular solution, rather it is a preference of how the objectives are ranked, compared or weighted in the eyes of the decision maker. As discussed above, after obtaining the preference of each of the decision maker, the next concern will be how to aggregate the preferences so that a single preference will be used to compute a solution. Suppose there are n decision makers for an integer $n > 1$.

3.1.1 Multiple Goals. Suppose the decision makers have different levels of power for dictating their preference: $DM_i,\ i \in \{1, 2, \cdots, n\}$. DM_1 has the greatest amount of power, so their authority from DM_1 to DM_n decreases. A weight can be used to describe the authority level of each decision maker in forcing their preference. The weight will be directly proportional to the decision maker's level of authority, i.e. high level of authority means higher weight. Due to that, the decision will get a higher vote if he/she has a high level of authority. Let B_i represent the goals of each objective function.

For each decision maker DM_i, we have:

$$B_i = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix},$$

then for each objective function to select the final goal B_i', the decision maker can use the following sum:

$$B_i' = \frac{\sum_{i=1}^{n} v_i B_i}{n}, \tag{3.1.1}$$

where n be number of decision makers.

3.1.2 Example. Consider there are three decision makers and three objective functions.

Suppose the voting power weight for each decision maker is given by: $v_i = \begin{bmatrix} 0.7 \\ 0.2 \\ 0.1 \end{bmatrix}$,

and consider the following vector B_i represents the goals for each objective function corresponding to each decision maker:

$$\begin{aligned} &DM_1 \quad \text{assign a goal for each objective function,} \quad B_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \\ &DM_2 \quad \text{assign a goal for each objective function,} \quad B_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \\ &DM_3 \quad \text{assign a goal for each objective function,} \quad B_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}. \end{aligned}$$

Now by using Equation 3.1.1 decision makers can obtain one final goal for each objective function.

For objective function 1:

$$\begin{aligned} B_1' &= \frac{v_1 b_{11} + v_2 b_{12} + v_3 b_{13}}{3}, \\ &= \frac{(0.7)(1) + (0.2)(3) + (0.1)(3)}{3}, \\ &= 0.5333. \end{aligned}$$

For objective function 2:

$$\begin{aligned} B_2' &= \frac{v_1 b_{21} + v_2 b_{22} + v_3 b_{23}}{3}, \\ &= \frac{(0.7)(3) + (0.2)(1) + (0.1)(2)}{3}, \\ &= 0.8333. \end{aligned}$$

For objective function 3:

$$\begin{aligned} B_3' &= \frac{v_1 b_{31} + v_2 b_{32} + v_3 b_{33}}{3}, \\ &= \frac{(0.7)(2) + (0.2)(2) + (0.1)(1)}{3}, \\ &= 0.6333. \end{aligned}$$

Therefore, the final goal for those three objective functions will be: $B_i^{'} = \begin{bmatrix} 0.5333 \\ 0.8333 \\ 0.6333 \end{bmatrix}$.

After doing this the given problem can be solved using one of the solution methods mentioned in Section 2.2. In some cases the aggregated goal can be infeasible. An ideal point approach can be used to solve the problem with an aggregate infeasible goal.

3.1.3 Multiple Weights. Here the decision makers assign their own weight vectors $w_i = [w_{i,1}, w_{i,2}, \ldots, w_{i,k}]$ for each objective function in terms of their levels of importance. Which means that for one objective function each decision maker can assign different weights. According to their level of authority they can use the average sum of each decision maker weight for obtaining a single weight vector. Since there are n decision makers $DM_1, DM_2, DM_3, \cdots, DM_n$ for each objective function they assign different weight vectors w_i. Then the problem can be mathematically expressed as follows:

$$W^{'} = \sum_{i=1}^{n} v_i W_i. \tag{3.1.2}$$

Whenever the voting power of DM_i is larger than DM_j, the aggregated weights will be favour decision maker i, see Example 3.1.4 below.

3.1.4 Example. Consider

$$\min_{\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^2} \; F(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x})),$$

$$\text{where} \quad f_1 = \frac{1}{2}\sum_{i=1}^{2} \mathbf{x}_i^2, \;\; f_2 = \frac{1}{2}\sum_{i=1}^{2} (\mathbf{x}_i - 2)^2 \quad \text{and } \mathcal{X} = \{(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^2 \mid -1 \le \mathbf{x}_1, \mathbf{x}_2 \le 1\}$$

(Tilahun and Ong, 2012). The Pareto front simulation for this function as shown in Figure 3.1:

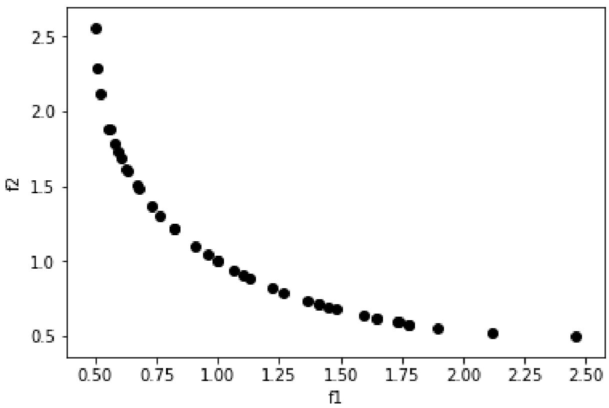

Figure 3.1: A Pareto front for the given functions f_1 and f_2 is obtained by the weighting method.

Suppose there are two decision maker. Let $w_1 = [0.3, 0.7]$, $w_2 = [0.8, 0.2]$ be a weight vector for objective function one and objective function two defined by decision maker one and decision maker two, respectively. The illustration of each weight vector over the Pareto front is as follow:

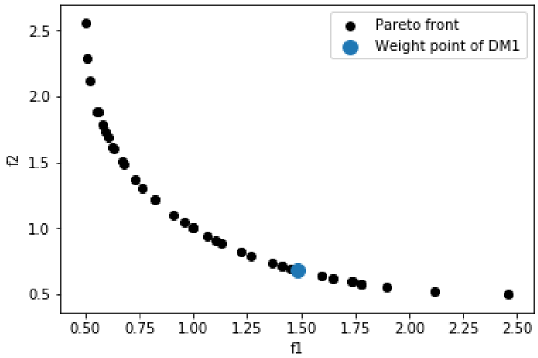

Figure 3.2: A Pareto front for the given functions. The blue point is the point which is obtained from the weight of decision maker one.

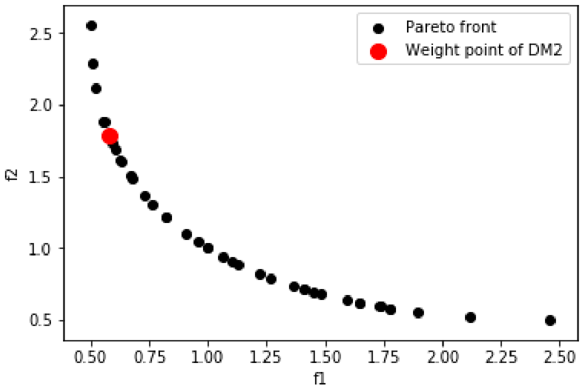

Figure 3.3: A Pareto front for the given functions. The red point is the point which obtained from the weight of decision maker two.

When decision maker one has more power than two, the aggregated weight will be:

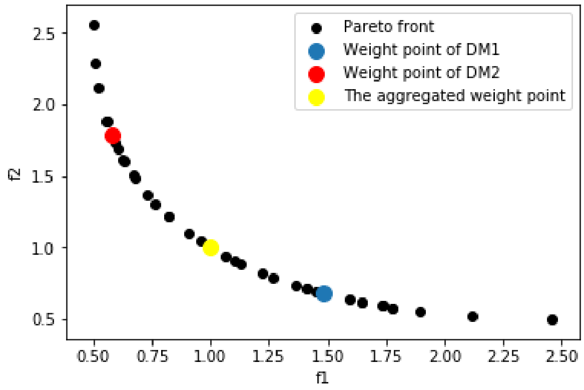

Figure 3.4: A Pareto front for the given functions. The yellow point is the point which is obtained from the aggregated weight.

When decision maker two has more power than one, the aggregated weight point will be:

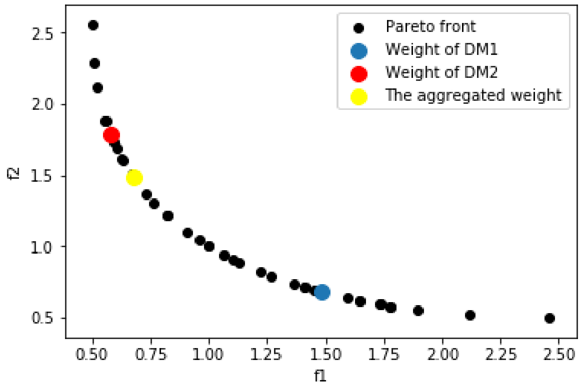

Figure 3.5: A Pareto front for the given functions. The yellow point is the point which is obtained from the aggregated weight.

When all decision maker has equal power, then the aggregated weight point will be:

3.1.5 Multiple Utility Functions. Each decision maker define a utility function for the given objective functions. Suppose there are $U_1, U_2, \ldots, U_n$ functions defined by n decision makers with different levels of power dictating their preferences which decrease from DM_1 to DM_n.

For each decision maker, we have:

$$U_i : \mathbb{R}^k \to \mathbb{R}, \qquad i = 1, 2, \ldots, n.$$

Then, by using the following equation, all decision makers can have one utility function for the given problem:

$$\bar{U}_i = \frac{\sum_{i=1}^{n} v_i U_i}{n}, \tag{3.1.3}$$

where n is the number of decision makers.

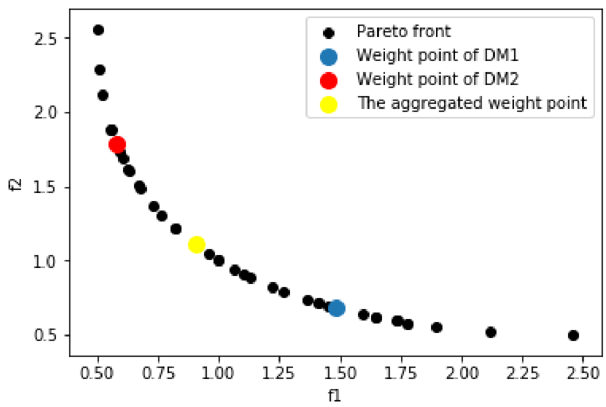

Figure 3.6: A Pareto front for the given functions. The yellow point is the point which obtained from the aggregated weight.

Consider there are two utility functions U_1 and U_2 defined on $\mathbb{R}^2$:

1. If decision maker one has more power than decision maker two the aggregated function $\bar{U}$ will closer to function U_1 defined by decision maker one.
2. If decision maker two has more power than decision maker two the aggregated function $\bar{U}$ will closer to function U_2 defined by decision maker one.
3. If both decision makers have equal power then the aggregated function $\bar{U}$ will be in the middle of the two function.

3.1.6 Example. Consider

$$\min_{\mathbf{x}\in\mathcal{X}\subseteq\mathbb{R}^2} F(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x})),$$

$$f_1 = \frac{1}{2}\sum_{i=1}^{2}\mathbf{x}_i^2,\ f_2 = \frac{1}{2}\sum_{i=1}^{2}(\mathbf{x}_i - 2)^2 \quad \text{and } \mathcal{X} = \{(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^2 \mid -1 \le \mathbf{x}_1, \mathbf{x}_2 \le 1\}$$

(Tilahun and Ong, 2012).

Suppose there are two decision makers. Let's consider each decision maker define a utility function U_1 and U_2 respectively

$$\begin{aligned} U_1 &= 5f_1 + f_2 \\ &= 3\mathbf{x}_1^2 + 3\mathbf{x}_2^2 - 2\mathbf{x}_1 - 2\mathbf{x}_2 + 4, \\ U_2 &= 2f_1 + 3f_2 \\ &= \frac{5}{3}\mathbf{x}_1^2 + \frac{5}{3}\mathbf{x}_2^2 - 6\mathbf{x}_1 - 6\mathbf{x}_2 + 12. \end{aligned}$$

Consider $U_1(\mathbf{x}_1, \mathbf{x}_2)$ then,

$$\nabla U_1 = \begin{pmatrix} 6\mathbf{x}_1 - 2 \\ 6\mathbf{x}_2 - 2 \end{pmatrix} = 0,$$

$$\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 0.3333 \\ 0.3333 \end{pmatrix},$$

since $\nabla^2 U_1$ is positive definite $\begin{pmatrix} 0.3333 \\ 0.3333 \end{pmatrix}$ is minimum of U_1.

$$\therefore\ f^*_{\mathsf{U1}} \begin{pmatrix} f_1^* \\ f_2^* \end{pmatrix} = \begin{pmatrix} 0.1111 \\ 2.778 \end{pmatrix}.$$

Consider $U_2(\mathbf{x}_1, \mathbf{x}_2)$ then,

$$\nabla U_2 = \begin{pmatrix} \frac{10}{3}\mathbf{x}_1 - 6 \\ \\ \frac{10}{3}\mathbf{x}_2 - 6 \end{pmatrix} = 0,$$

$$\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 1.8 \end{pmatrix}.$$

is the minimum of U_2.

$$\therefore\ f^*_{\mathsf{U2}} \begin{pmatrix} f_1^* \\ f_2^* \end{pmatrix} = \begin{pmatrix} 3.24 \\ 0.04 \end{pmatrix}.$$

Then by using Equation 3.1.3 the aggregation function will be:

1. Let $v_1 = 0.2, v_2 = 0.8$ and $n = 2$ then,

$$\bar{U}_1 = 0.99667\mathbf{x}_1^2 + 0.99667\mathbf{x}_2^2 - 2.60.99667\mathbf{x}_1 - 2.60.99667\mathbf{x}_2 + 6.8,$$

$$\nabla \bar{U}_1 = \begin{pmatrix} 1.9334\mathbf{x}_1 - 2.6 \\ 1.9334\mathbf{x}_2 - 2.6 \end{pmatrix} = 0 \implies \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 1.3447 \\ 1.3447 \end{pmatrix},$$

$$\therefore f^*_{\bar{U}_1} \begin{pmatrix} f_1^* \\ f_2^* \end{pmatrix} = \begin{pmatrix} 1.8082 \\ 0.4294 \end{pmatrix}.$$

2. Let $v_1 = 0.5, v_2 = 0.5$ and $n = 2$ then,

$$\bar{U}_2 = 1.1666\mathbf{x}_1^2 + 1.1667\mathbf{x}_2^2 - 2\mathbf{x}_1 - 2\mathbf{x}_2 + 4,$$

$$\nabla \bar{U}_2 = \begin{pmatrix} 2.334\mathbf{x}_1 - 2 \\ 2.3334\mathbf{x}_2 - 2 \end{pmatrix} = 0 \implies \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 0.8571 \\ 0.8571 \end{pmatrix},$$

$$\therefore f^*_{\bar{U}_2} \begin{pmatrix} f_1^* \\ f_2^* \end{pmatrix} = \begin{pmatrix} 0.7346 \\ 1.3062 \end{pmatrix}.$$

3. Let $v_1 = 0.7, v_2 = 0.3$ and $n = 2$ then,

$$\bar{U}_3 = 1.3\mathbf{x}_1^2 + 1.3\mathbf{x}_2^2 - 1.6\mathbf{x}_1 - 1.6\mathbf{x}_2 + 3.2,$$

$$\nabla \bar{U}_3 = \begin{pmatrix} 2.6\mathbf{x}_1 - 1.6 \\ 2.6\mathbf{x}_2 - 1.6 \end{pmatrix} = 0 \implies \begin{pmatrix} \mathbf{x}_1 \\ \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 0.6153 \\ 0.6153 \end{pmatrix},$$

$$\therefore f^*_{\bar{U}_3} \begin{pmatrix} f_1^* \\ f_2^* \end{pmatrix} = \begin{pmatrix} 0.3785 \\ 1.9173 \end{pmatrix}.$$

The illustration of those above three cases are as show in Figure 3.7:

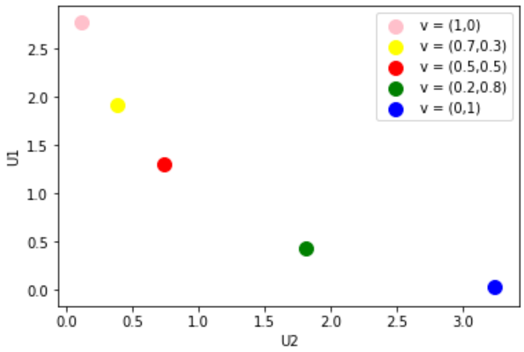

Figure 3.7: A Pareto front with different utility functions defined by decision maker one and decision maker two with different voting powers. Pink point and blue point represents utility functions defined by decision maker one and decision maker two. The yellow, green and red colours means that decision maker one has more power than decision maker two, decision maker two has more power than decision maker one, and both have equal power.

3.1.7 Multiple Lexicographic Ordering. Here each decision maker set different ranks for each objective function based on their importance. Hence to aggregate all those ranks obtained by each decision maker and for obtaining one, the decision maker can use majority rule. Majority rule means that a process to find a rank the decision makers count the number of decision makers who select that objective function is the most important. However, according to their power, decision makers set a rank for each objective function. The algorithm can be summarized as follows:

Step-1 Input: n by k matrix N, where N_{ij} represents the rank of objective j by the decision maker i.

Step-2 Construct a k by k matrix M, in such a way that M_{ij} represents the number of decision makers ranking objective i in rank j.

Step-3 Iteratively construct the aggregated rank: for each rank j from 1 to k, collect the objective with the highest entry $M_{i,j}$ (i.e. rank of objective i will be j). If there is a tie between objective i and p (two objectives without loosing generality).

Step-3.1 Check the sum of the votes of the decision makers for objective i and p based on matrix N and take the one with highest vote to be ranked j and the other $j + 1$.

Step-3.2 If the tie still there: compare $M_{i,j+1}$ and $M_{p,j+1}$ continue comparing in the next column if the tie still exist based on the number of ranking and sum of voting power.

Step-3.3 After comparing in all columns if the tie is not broken, assign rank j to either objective p or i and rank $j + 1$ to either objective i or p, randomly or ask further assistance from the decision makers.

Step-4 Report the result.

3.1.8 Example. Consider the following

$$\min_{\mathbf{x}\in\mathcal{X}} \ (f_1, f_2, f_3, f_4, f_5).$$

Suppose there are three decision makers. Assume $v_1 = 0.5, v_2 = 0.4$ and $v_3 = 0.1$ is the three decision makers voting power. The following matrix N represents the rank of each objective function defined by each decision maker.

$N =$

	Obj1	Obj2	Obj3	Obj4	Obj5
$DM1$	1	4	2	5	3
$DM2$	3	2	1	4	5
$DM3$	4	5	3	2	1

Construct a rank matrix M which represents the number of decision makers ranking objective i in rank j.

$M =$

	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
Obj 1	1	0	1	0	1
Obj 2	0	1	0	1	1
Obj 3	1	1	1	0	0
Obj 4	0	1	0	1	1
Obj 5	1	0	1	0	1

In this example: in the first column objective function one, objective function three and objective function five have equal number of votes. By considering step-3.1 Section 3.1.7 from matrix N the sum of the votes of the three decision makers for objective function one is 0.5, for objective function three is 0.4 and for objective function five is 0.1. Which means objective function one is in the first rank, objective function three is in the second rank and objective function five is in the third rank.

Now go back on matrix M and check the fourth column for putting the objective function on rank four. Objective function two and objective function four have equal number of votes. Again by considering step-3.1 Section 3.1.7 the sum of the votes of the decision maker for the objective function two and objective function four is 0.5 and 0.4 respectively, this means that objective function two will be in rank four and objective function four be in rank five. From this process, decision makers can decide the rank of the given objective functions be: firstly f_1 trait secondly f_3, thirdly f_5 fourthly f_2, and fifth f_4.

3.1.9 Multiple Tolerance Preferences. Each decision maker sets their own tolerance for all objective functions. To obtain single tolerance for each objective functions the decision makers can choose the minimum among the obtained tolerance. Consider there are n decision makers, for each decision maker DM_i, we have: ϵ_i which represents the tolerance of each objective function. Mathematically expressed by:

$$\epsilon_j = \min_i \ \{\epsilon_{ij}\}, \ \forall \ j \neq j', \tag{3.1.4}$$

where j' is the index of the objective which stays as the objective function.

Finally all decision makers decide and obtain minimum tolerance. However, if that tolerance is produce empty feasible region then either the decision makers are asked to relax their tolerance or the next minimum tolerance will be take for the objective which is violating the feasibility.

3.1.10 Example. Consider there are three decision makers. Suppose there are five objective functions and the first objective function is selected to be minimized. Each decision maker put their own tolerance for objective function one, objective function two, objective function three, objective function four and function five as follows:

$$\epsilon_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1.5 \\ 4 \end{bmatrix}, \quad \epsilon_2 = \begin{bmatrix} 0.2 \\ 1 \\ 0.5 \\ 7 \\ 10 \end{bmatrix} \text{ and } \epsilon_3 = \begin{bmatrix} 1.2 \\ 3.7 \\ 3 \\ 0.1 \\ 6 \end{bmatrix}.$$

By using Equation 3.1.4 they can decide which tolerance is better for the corresponding objective function, when they consider objective function two they will take

$$\begin{aligned} \epsilon_2 &= \min\{\epsilon_{12}, \epsilon_{22}, \epsilon_{32}\}, \\ &= \min\{2, 1, 3.7\}, \\ &= 1. \end{aligned}$$

Therefore objective function two is less than 1 ($f_2 \leq 1$).

Consider objective function three:

$$\begin{aligned} \epsilon_3 &= \text{min}\{\epsilon_{13}, \epsilon_{23}, \epsilon_{33}\}, \\ &= \min\{1, 0.5, 3\}, \\ &= 0.5. \end{aligned}$$

Therefore objective function two is less than 0.5 ($f_3 \leq 0.5$).

Consider objective function four:

$$\begin{aligned} \epsilon_4 &= \min\{\epsilon_{14}, \epsilon_{24}, \epsilon_{34}\}, \\ &= \min\{1.5, 7, 0.1\}, \\ &= 0.1. \end{aligned}$$

Therefore objective function four is less than 0.1 ($f_4 \leq 0.1$).

Consider objective function five:

$$\begin{aligned} \epsilon_5 &= \min\{\epsilon_{15}, \epsilon_{25}, \epsilon_{35}\}, \\ &= \min\{4, 10, 6\}, \\ &= 4. \end{aligned}$$

Therefore objective function five is less than 4 ($f_5 \leq 4$).

The decision maker can solve by using one of the solving method for multi-objection as mentioned in (see Section 2.2):

$$\begin{aligned} \min_{\mathbf{x} \in \mathcal{X}} \quad & f_1(\mathbf{x}), \\ \text{subject to} \quad & f_2(\mathbf{x}) \leq 1, \\ & f_3(\mathbf{x}) \leq 0.5, \\ & f_4(\mathbf{x}) \leq 0.1, \\ & f_5(\mathbf{x}) \leq 4. \end{aligned}$$

If the feasible region is happen to be empty and suppose the constraint made from the second objective is the bounding constraint, it will be relaxed by taking the next minimum $\epsilon_{i,2} > 1$, that will be 2 hence $f_2(\mathbf{x}) \leq 1$ becomes $f_2(\mathbf{x}) \leq 2$. Since minimum tolerance values are taken it doesn't matter if there is a certain decision making hierarchy or voting structure.

3.2 Solution Level Preference Approach

In this approach each decision maker make a decision based on the given solutions of the problem.

3.2.1 Multiple Trade-off. Each decision maker will be given a solution and asked to set a trade-off, that is how much they are willing to give up on objective function i for a unit improvement of objective function j. Hence, each decision maker can give a k by k matrix T where T_{ij} represents the amount of the decision maker is willing to give up in objective function i for a unit improvement in objective function j while the rest of the objective are kept constant. T_{ii} can be set to be zero as it is meaningless to discuss the trade-off an objective to a unit improvement of itself.

Mathematically expressed by:

$$T_m = \begin{bmatrix} 0 & T_{12} & T_{13} & \dots & T_{1k} \\ T_{21} & 0 & T_{23} & \dots & \vdots \\ \vdots & \ddots & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & 0 & T_{k-1k} \\ T_{k1} & T_{k2} & \dots & T_{kk-1} & 0 \end{bmatrix}, \quad m = 1, 2, \dots, n.$$

To obtain the final trade-off of the given function the decision makers can choose the minimum trade-off among the given solutions,

$$T_{ij}^{'} = \min_{i}\{T_{ij}\}, \quad \forall i \neq j, \tag{3.2.1}$$

where $T_{ij}^{'}$ is the final trade-off of the given solution expressed by each decision maker.

3.2.2 Example. Consider there are two decision makers and three objective functions.

Suppose $(2, 5, 4)$ be the obtained solution for the given problem. Assume decision maker one and decision maker two define a matrix $\hat{T}$ and $\bar{T}$ which consists of the trade-off solution for each objective function respectively.

$$\hat{T} = \begin{bmatrix} 0 & 1 & 1.5 \\ 1 & 0 & 1.5 \\ 0.6 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad \bar{T} = \begin{bmatrix} 0 & 0.5 & 2 \\ 2.2 & 0 & 1.5 \\ 0.4 & 0.2 & 0 \end{bmatrix}.$$

After that by using Equation 3.2.1 the decision makers can compare each solutions:

$$
\begin{aligned}
T_{12}^{'} &= \min\ \{\hat{T}_{12}, \bar{T}_{12}\}, \\
&= \min\ \{1, 0.5\}, \\
&= 0.5. \\
T_{13}^{'} &= \min\ \{\hat{T}_{13}, \bar{T}_{13}\}, \\
&= \min\ \{1.5, 2\}, \\
&= 1.5. \\
T_{21}^{'} &= \min\ \{\hat{T}_{21}, \bar{T}_{21}\}, \\
&= \min\ \{1, 2.2\}, \\
&= 1. \\
T_{23}^{'} &= \min\ \{\hat{T}_{23}, \bar{T}_{23}\}, \\
&= \min\ \{1.5, 1.5\}, \\
&= 1.5. \\
T_{31}^{'} &= \min\ \{\hat{T}_{31}, \bar{T}_{31}\}, \\
&= \min\ \{0.6, 0.4\}, \\
&= 0.4. \\
T_{32}^{'} &= \min\ \{\hat{T}_{32}, \bar{T}_{32}\}, \\
&= \min\ \{0.5, 0.2\}, \\
&= 0.2.
\end{aligned}
$$

Therefore the aggregated trade-off matrix will be

$$
T^{'} = \begin{bmatrix} 0 & 0.5 & 1.5 \\ 1 & 0 & 1.5 \\ 0.4 & 0.2 & 0 \end{bmatrix}
$$

Now, the decision makers can change this matrix into weight preference and can find equivalent solution for the given problem.

4. Conclusion and Future Work

This project applied group preference modelling methods to solve multi-objective optimization problems where preferences are used to convert a multi-objective optimization problem to a single-objective optimization problem. The motivation of such conversion was due to various existing approaches for single-objective optimization problem.

The hard part of this problem was nature of preferences of decision makers. We considered the case where their preferences can be conflictual. A solution to the optimization problem with conflictual preferences depends on how they are handled.

Many approaches exist in literature to aggregate preferences when there is single decision maker. For our case, to combine conflictual preferences, we applied various group decision maker preference models. Experiments indicated good features of such methods of solution.

Decision making exists in every activity we experience. This often involves multiple and usually conflicting objectives. Due to that, the study of multi-objective optimization is an active research avenue. Group decision making is very common in companies and national/international deals and agreement. This study focused on the aggregation of preference structure and hence is not considering other options which can be taken as possible future works. Some of them are listed below:

1. The preferences of multiple decision makers are considered to be of the same type. However, studying different preferences or mixed preference type is one of the future works. How would we combine mixed preference, i.e. for example one decision maker gives lexicographic ordering, the other gives tolerance and so on?
2. Theoretical studies on the proposed aggregation need to be done. What are the condition for the aggregation to be acceptable by the decision makers?
3. Testing the existing approaches on more complex and multiple objective (more than two) problems need to be done.
4. Classifying solution methods for the preference type need also to be explored, especially if we have mixed preference type.

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References

Abounacer, R., Rekik, M., and Renaud, J. An exact solution approach for multi-objective location–transportation problem for disaster response. *Computers & Operations Research*, 41:83–93, 2014.

Andersson, J. A survey of multiobjective optimization in engineering design. *Department of Mechanical Engineering, Linktjping University. Sweden*, 2000.

Augusto, O. B., Bennis, F., and Caro, S. A new method for decision making in multi-objective optimization problems. *Pesquisa Operacional*, 32(2):331–369, 2012.

Bjornson, E., Jorswieck, E. A., Debbah, M., and Ottersten, B. Multiobjective signal processing optimization: The way to balance conflicting metrics in 5g systems. *IEEE Signal Processing Magazine*, 31(6):14–23, 2014.

Bui, L. T. and Alam, S. An introduction to multi-objective optimization. In *Multi-objective optimization in computational intelligence: Theory and practice*, pages 1–19. IGI Global, 2008.

Bui, L. T., Abbass, H. A., Barlow, M., and Bender, A. Robustness against the decision-maker's attitude to risk in problems with conflicting objectives. *IEEE Transactions on Evolutionary Computation*, 16 (1):1–19, 2010.

Chiandussi, G., Codegone, M., Ferrero, S., and Varesio, F. E. Comparison of multi-objective optimization methodologies for engineering applications. *Computers & Mathematics with Applications*, 63(5):912–942, 2012.

Coello, C. A. C., Lamont, G. B., Van Veldhuizen, D. A., et al. *Evolutionary algorithms for solving multi-objective problems*, volume 5. Springer, 2007.

Deb, K. Multi-objective optimization. In *Search methodologies*, pages 403–449. Springer, 2014.

Deb, K. and Algorithms, M.-o. O. U. E. John wiley and sons ltd. *Chichester, England*, 2001.

Emmerich, M. T. and Deutz, A. H. A tutorial on multiobjective optimization: fundamentals and evolutionary methods. *Natural computing*, 17(3):585–609, 2018.

Fishburn, P. C. Exceptional paper—lexicographic orders, utilities and decision rules: A survey. *Management science*, 20(11):1442–1471, 1974.

Hawe, G. and Sykulski, J. Scalarizing cost-effective multi-objective optimization algorithms made possible with kriging. *COMPEL-The international journal for computation and mathematics in electrical and electronic engineering*, 27(4):836–844, 2008.

Jaimes, A. L., Martınez, S. Z., and Coello, C. A. C. An introduction to multiobjective optimization techniques. *Optimization in Polymer Processing*, pages 29–57, 2009.

Kirlik, G. and Sayın, S. Computing the nadir point for multiobjective discrete optimization problems. *Journal of Global Optimization*, 62(1):79–99, 2015.

Lewandowski, M. Prospect theory versus expected utility theory: Assumptions, predictions, intuition and modelling of risk attitudes. *Central European Journal of Economic Modelling and Econometrics*, 2017.

Marler, R. T. and Arora, J. S. Survey of multi-objective optimization methods for engineering. *Structural and multidisciplinary optimization*, 26(6):369–395, 2004.

Odu, G. and Charles-Owaba, O. Review of multi-criteria optimization methods–theory and applications. *IOSR Journal of Engineering (IOSRJEN)*, 3(10):1–14, 2013.

Talebian, S. H. and Kareem, S. A. A lexicographic ordering genetic algorithm for solving multi-objective view selection problem. In *2010 Second International Conference on Computer Research and Development*, pages 110–115. IEEE, 2010.

Thiele, L., Miettinen, K., Korhonen, P. J., and Molina, J. A preference-based evolutionary algorithm for multi-objective optimization. *Evolutionary computation*, 17(3):411–436, 2009.

Tilahun, S. L. and Ong, H. C. Fuzzy preference of multiple decision-makers in solving multi-objective optimisation problems using genetic algorithm. *Maejo International Journal of Science and Technology*, 6(2):224, 2012.

Wang, H., Olhofer, M., and Jin, Y. A mini-review on preference modeling and articulation in multi-objective optimization: current status and challenges. *Complex & Intelligent Systems*, 3(4):233–245, 2017.

Xiong, J., Tan, X., Yang, K.-w., and Chen, Y.-w. Fuzzy group decision making for multiobjective problems: tradeoff between consensus and robustness. *Journal of Applied Mathematics*, 2013, 2013.

Xiujuan, L. and Zhongke, S. Overview of multi-objective optimization methods. *Journal of Systems Engineering and Electronics*, 15(2):142–146, 2004.

Yu, X. and Gen, M. *Introduction to evolutionary algorithms*. Springer Science & Business Media, 2010.

Yue, Z. An avoiding information loss approach to group decision making. *Applied Mathematical Modelling*, 37(1-2):112–126, 2013.