The Rogers-Ramanujan Identities

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Abstract

The Rogers-Ramanujan identities, identities between infinite series and infinite products, are fascinating discoveries in the theory of q-series. Their importance often arises in various domains of mathematics especially in the theory of integer partitions. The present work aims to emphasize their combinatorial aspect in partition theory. In addition, other immediate consequences and some of their classical generalizations will be introduced, based principally on the works of Andrews (1976) and Sills (2017). Finally, motivated by the partition identities resulted from the Rogers-Ramanujan identities, we look at a generalization of a well-known result of Alladi-Schur and give a bijective proof of this result.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Haingoharijao Faniriniaina RAMANDIAMANANA, 23 May 2019

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1. Introduction

1.1 Basic notion in the theory of partitions

Given a positive integer n, one may wonder on the number of way of writing n as sum of other positive integers. This arises initially from the question asked by Leibniz to Bernoulli in 1674 to compute the number of partitions of a number n (Dousse, 2015).

1.1.1 Definition. Let n be an integer.

A partition of n is a finite sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers such that $\sum_{i=1}^{k} \lambda_i = n$. The λ_i's are called parts.

We denote by $p(n)$ the number of partitions of n.

1.1.2 Example. The partitions of 4 are precisely

$$(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1),$$

so $p(4) = 5$.

1.1.3 Remark. We adopt the following conventions:

- The order is irrelevant in partition, for instance $1+2$ and $2+1$ represent the same partition. Thus we admit that the partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is written non-increasingly, that is $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k$.
- The parts λ_i's need not to be distinct. Hence, we can as well consider the notation $(1^{f_1} 2^{f_2} \ldots)$ to represent the partition λ where f_i is the number of times i appears in the partition. Though, we will omit i when $f_i = 0$.
- $p(n) = 0$ when n is negative and $p(0) = 1$ (we agree that the empty-tuple is the only partition of zero).

Having this sequence of number $(p(n))_{n \in \mathbb{N}}$ in hand, many questions come into mind on its behaviour: is it always prime? how much proportion of odd or even numbers does it contain? how fast does it grow? etc. These questions even become more interesting when we add more conditions on the partitions. We say that these partitions are restricted partitions.

1.1.4 Definition. Let S be a set of positive integers.

We define the following:

- $p(S, n)$: number of partitions of n whose parts are lying in S,
- $p(S(\leqslant d), n)$: number of partitions of n whose parts are lying in S and each part appears at most d times (i.e $f_i \leqslant d$ for all i),
- $p(S, m, n)$: number of partitions of n which have exactly m parts and they are lying in S.

1.1.5 Definition. A partition identity is a proposition stating the equality between the cardinalities of two sets of restricted partitions for a given integer n.

Leibniz builds the fundamental notion in the partition theory but Euler was one of the first who came up with some deep and significant results. He started his investigation on integer partition in 1740 after Naudé's enquiry to him to compute the number of partitions of 50 into 7 distinct parts (Dousse, 2015).

Determining all the partitions of 50 will be one possible manner to solve the problem. However, it would not help to solve the general problem. An attempt to solve the problem generally leads to the analytic methods in partition theory, namely the use of generating functions.

1.2 Motivation

Ramanujan, discovered by G. H. Hardy, was another mathematician who made numerous contributions to the theory of partition. The Rogers-Ramanujan identities are among these contributions which will be the main focus of this essay. Rogers was the first who found, with proof, these identities in 1894 but unfortunately, he was not well-known and thus the identities are attributed to Ramanujan because of his independent rediscovery of the result (Andrews, 1976). Ramanujan derived immediately from the identities two continued fractions described by Hardy as breathtaking. They are given in the following

$$\cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{\vdots}}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right) e^{2\pi/5}$$

and

$$1-\cfrac{e^{-\pi}}{1+\cfrac{e^{-2\pi}}{1-\cfrac{e^{-3\pi}}{\vdots}}} = \left(\sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}\right) e^{\pi/5}.$$

Furthermore, these identities were interpreted combinatorially as partition identity by MacMahon and Schur (Sills, 2017). Namely, we have that "For any integer n, the partitions of n in which the difference between any two parts is at least 2 are equinumerous with its partitions into parts $\equiv 1$ or $4 \mod 5$" and "The partitions of an integer n in which each part exceeds 1 and the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts $\equiv 2$ or $3 \mod 5$".

These discoveries are surprising in the sense that one may not believe that these results are actually the consequences of only two identities. Moreover, more mathematicians found later on that there exists many identities of this type.

1.2.1 Definition. An identity type Rogers-Ramanujan is an infinite "series-product" identity where the product is restricted under some conditions. It is often interpreted combinatorially as the following statement:"For all integers n, the number of partitions of n under some difference conditions is the same as the number of partitions of n under some congruence conditions".

According to Sills (2017), there are more than 200 Rogers-Ramanujan type identities. Throughout this essay, we will give some of these identities after a general analytic approach. We will also state the Rogers-Ramanujan identities themselves and their applications. Finally, we will further explore some bijective proofs of the Alladi-Schur partition identity after giving its generalization.

2. Generating function

Most results in theory of partitions (generally in combinatorics) turn out to be difficult to prove. A smart use and manipulation of some tools are needed. The techniques come from Euler's observation. He remarks, in expanding the following expression in geometric series, that

$$\begin{aligned} \frac{1}{1-q} \times \frac{1}{1-q^2} &= (1 + q + q^{1+1} + q^{1+1+1} + \cdots)(1 + q^2 + q^{2+2} + q^{2+2+2} + \cdots) \\ &= 1 + q + q^{1+1} + q^2 + q^{2+1} + \cdots \end{aligned} \tag{2.0.1}$$

for $|q| < 1$. If we do not simplify explicitly, we remark that the exponents of q are the partition of number using only 1 and 2. We shall see later that this series falls in the notion of generating function.

2.1 Euler partition identity

2.1.1 Definition. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of numbers. The power series

$$f(q) = \sum_{n\geqslant 0} a_n q^n$$

is called the (one-variable) generating function of $(a_n)_{n\in\mathbb{N}}$.

More formally, it is called the ordinary generating function of $(a_n)_{n\in\mathbb{N}}$.

2.1.2 Remark. Notice that:

1. We are dealing with formal power series: the notion of convergence is not required. However, in some part of our work, it is necessary to consider the analyticity of f, especially in all asymptotic work. The theorems in this essay are given in their analytic version.

2. This definition can be extended to multi-variable generating function of the multi-sequence of numbers $(a_{n_1,\ldots,n_m})_{n_1,\ldots,n_m\in\mathbb{N}}$.

2.1.3 Theorem (Andrews (1976))**.** *Let S be a set of positive integers.*

If we denote by:

- $f(q)$ *the generating function of* $(p(S,n))_{n\in\mathbb{N}}$ *i.e.*

$$f(q) = \sum_{n\geqslant 0} p(S,n)q^n,$$

- $f_d(q)$ *the generating function of* $(p(S(\leqslant d),n))_{n\in\mathbb{N}}$ *i.e.*

$$f_d(q) = \sum_{n\geqslant 0} p(S(\leqslant d),n)q^n,$$

- $f_m(z,q)$ *the generating function of* $(p(S,m,n))_{m,n\in\mathbb{N}}$ *i.e.*

$$f_m(z,q) = \sum_{m,n\geqslant 0} p(S,m,n)z^m q^n,$$

- $f_{d,m}(z,q)$ *the generating function of* $(p(S(\leqslant d),m,n))_{m,n\in\mathbb{N}}$ *i.e.*
$$f_{d,m}(z,q)=\sum_{m,n\geqslant 0}p(S(\leqslant d),m,n)z^m q^n.$$

Then for $|x|<1$*:*

$$f(q)=\prod_{n\in S}(1-q^n)^{-1} \tag{2.1.1}$$
$$f_d(q)=\prod_{n\in S}(1+q^n+q^{2n}+\cdots+q^{dn})=\prod_{n\in S}(1-q^{(d+1)n})(1-q^n)^{-1} \tag{2.1.2}$$
$$f_m(z,q)=\prod_{n\in S}(1-zq^n)^{-1} \tag{2.1.3}$$
$$f_{d,m}(z,q)=\prod_{n\in S}(1-z^{d+1}q^{(d+1)n})(1-zq^n)^{-1}. \tag{2.1.4}$$

Proof. Let us write $S=\{s_1,s_2,s_3,\ldots\}$.

The first product (2.1.1) gives us
$$\begin{aligned}\prod_{n\in S}(1-q^n)^{-1}&=\prod_{n\in S}(1+q^n+q^{2n}+\cdots)\\&=(1+q^{s_1}+q^{2s_1}+\cdots)(1+q^{s_2}+q^{2s_2}+\cdots)\cdots\\&=\sum_{a_1,a_2,\ldots\geqslant 0}q^{a_1s_1+a_2s_2+a_3s_3+\cdots}.\end{aligned}$$
We observe that the exponent of q is $N=a_1s_1+a_2s_2+a_3s_3+\cdots$ which can be viewed as the partition $(s_1^{a_1}s_2^{a_2}\ldots)$ of N lying in S.

Thus, given a positive integer N, the coefficient of q^N in the right-hand side polynomial is the number of way of writing N in terms of s_i's. Therefore,
$$\prod_{n\in S}(1-q^n)^{-1}=\sum_{n\geqslant 0}p(S,n)q^n.$$
For the Equation (2.1.2), instead of infinite geometric series, we have a finite one :
$$\begin{aligned}\prod_{n\in S}(1+q^n+q^{2n}+\cdots+q^{dn})&=(1+q^{s_1}+q^{2s_1}+\cdots+q^{ds_1})(1+q^{s_2}+q^{2s_2}+\cdots+q^{ds_2})\cdots\\&=\sum_{d\geqslant a_1,a_2,\ldots\geqslant 0}q^{a_1s_1+a_2s_2+a_3s_3+\cdots}.\end{aligned}$$
The coefficient of q^N, with $N=a_1s_1+a_2s_2+a_3s_3+\cdots$, is now the number of partition $(s_1^{a_1}s_2^{a_2}\ldots)$ of N $(a_i\leqslant d)$, i.e. $p(S(\leqslant d),N)$.

The equality holds since the two terms in left-hand side and right-hand side are convergent.

To prove (2.1.3) and (2.1.4), it is similar. The difference is that we play with 2 variables. □

2.1.4 Remark. We remark that when S is the set of all positive integers, we get the generating function of $p(n)$

$$\sum_{n\geqslant 0}p(n)q^n=\prod_{n=0}^{\infty}(1-q^n)^{-1} \tag{2.1.5}$$

which holds for the property that $p(0)=1$, the constant term.

From this theorem follows the first identity of Rogers-Ramanujan type.

2.1.5 Theorem (Euler)**.** *Let n be an integer, $\mathcal{O}$ the set of odd numbers and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. Then*

$$p(\mathcal{O}, n) = p(\mathbb{N}^*(\leqslant 1), n).$$

Proof. Theorem 2.1.3 gives us

$$\sum_{n\geqslant 0} p(\mathcal{O}, n)q^n = \prod_{n=1}^{\infty}(1-q^{2n-1})^{-1}.$$

Besides, we have

$$\begin{aligned}
\sum_{n\geqslant 0} p(\mathbb{N}^*(\leqslant 1), n)q^n &= \prod_{n=1}^{\infty}(1-q^{2n})(1-q^n)^{-1} \\
&= \frac{1-q^2}{1-q} \times \frac{1-q^4}{1-q^2} \times \frac{1-q^6}{1-q^3} \times \frac{1-q^8}{1-q^4} \times \cdots \\
&= \frac{1-q^2}{(1-q)(1-q^2)} \times \frac{1-q^4}{(1-q^3)(1-q^4)} \times \frac{1-q^6}{(1-q^5)(1-q^6)} \times \cdots \\
&= \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})^{-1}(1-q^{2n})^{-1} \\
&= \prod_{n=1}^{\infty}(1-q^{2n-1})^{-1}.
\end{aligned}$$

Thus, we have

$$\sum_{n\geqslant 0} p(\mathcal{O}, n)q^n = \sum_{n\geqslant 0} p(\mathbb{N}^*(\leqslant 1), n)q^n.$$

□

2.1.6 Remark. Let us make some remarks before proceeding.

- If we consider the product to be in the numerator and change the sign to plus in (2.1.1) i.e.

  $$\prod_n (1+q^n)$$

  then we only get the number of partition of n with distinct parts, $p(\mathbb{N}^*(\leqslant 1), n)$.

- Euler was also interested in studying the infinite product in Equation (2.1.5) itself. He expanded directly the inverse of the product and with some manipulation, as result, he got the Euler's pentagonal number theorem stating that

  $$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = \prod_{n=0}^{\infty}(1-q^n).$$

  After this, MacMahon derived a recurrence formulae of $p(n)$ which has up to now the best efficiency in terms of algorithm implementation (Dousse, 2015).

- After Euler, there was no big evolution in partition theory until Sylvester, in 1880, discovered new results (Dousse, 2015).

- There is a graphical point of view of partition theory, named Ferrers diagram. We will not address this result in this essay (Andrews, 1976).

2.2 q-Hypergeometric series results

In this section, we will be introducing some concepts that are useful to our work.

2.2.1 Definition. A series $\sum_k u_{n,k,q}$ is called basic hypergeometric of base q or q-hypergeometric if

$$\frac{u_{n+1,k,q}}{u_{n,k,q}} \quad \text{and} \quad \frac{u_{n,k+1,q}}{u_{n,k,q}}$$

are rational functions in terms of q^n and q^k.

2.2.2 Definition. Let a, q be real numbers and n be integer. Define:

- $(a)_n = (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$,
- $(a)_\infty = (a;q)_\infty := \lim_{n\to\infty}(a;q)_n$,
- $(a)_0 = 1$.

$(a;q)_n$ is called q-Pochhammer symbol or q-shifted factorial and we may observe that

$$(a)_n = \frac{(a)_\infty}{(aq^n)_\infty}.$$

This symbol is not to be confused with the ordinary rising factorial nor the sequence $(a_n)_{n\in\mathbb{N}}$, we always write explicitly the set of index in that case. Also, it shall be clear later that the semicolon inbetween is important.

2.2.3 Definition. We define the unilateral basic hypergeometric series as

$${}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ \\ b_1, b_2, \ldots, b_s \end{matrix} ; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1;q)_k(a_2;q)_k\cdots(a_r;q)_k}{(b_1;q)_k(b_2;q)_k\cdots(b_r;q)_k} z^k \left((-1)^k q^{k(k-1)/2}\right)^{1+s-r}.$$

2.2.4 Remark. The basic hypergeometric series

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ \\ b_1, b_2, \ldots, b_r \end{matrix} ; q, z\right]$$

is said to be:

- well-poised if
$$qa_1 = a_2b_1 = a_3b_2 = \cdots = a_{r+1}b_r$$
- very-well-poised if it is well-poised, $a_2 = q\sqrt{a_1}$ and $a_3 = -q\sqrt{a_1}$
- balanced if $b_1b_2\cdots b_r = qa_1a_2\cdots a_{r+1}$.

2.2.5 Theorem (Sills (2017))**.**

$${}_2\phi_1\left[\begin{matrix} a, q^{-n} \\ \\ c \end{matrix} ; q, \frac{cq^n}{a}\right] = \frac{(c/a;q)_n}{(c;q)_n}.$$

This is called the q-Chu-Vandermonde.

Proof. See (Sills, 2017) □

2.2.6 Theorem (Sills (2017)).

$$ {}_6\phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-n} \\ \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{n+1} \end{matrix} ; q, \frac{aq^{n+1}}{bc} \right] = \frac{(aq;q)_n (aq/bc;q)_n}{(aq/b;q)_n (aq/c;q)_n}. $$

This is called Jackson's ${}_6\phi_5$ sum.

Proof. See (Sills, 2017) □

These are not the only interesting results about the unilateral basic hypergeometric series. There exist many of them as shown in Sills (2017). Their proofs require some theory called the Wilf-Zeilberger (WZ) theory so we did not give the proofs.

2.2.7 Definition. Ramanujan's general theta function $f(a,b)$ is defined by

$$ f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} $$

for $|ab| < 1$. Special values of $f(a,b)$ are

$$ \Phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty} $$

and

$$ \Psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} $$

where $|q| < 1$, notations given by Ramanujan because they arise so often.

2.2.8 Remark. We have

- $f(-1,a) = 0$,
- $f(a,b) = f(b,a)$.
- It is called theta function despite the absence of theta in the formulae. It is due to its equivalence, after a change of variable, to the Jacobi's theta function

$$ \vartheta(z,w) = \sum_{n=-\infty}^{\infty} (-1)^n w^{n^2} e^{2inz} $$

for $|w| < 1$.

- We also remark that

$$ f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q)_\infty $$

which is the Euler's pentagonal number theorem.

2.2.9 Theorem (Cauchy)**.** *Let* $|q| < 1$ *and* $|t| < 1$. *We have*

$$
\begin{aligned}
{}_1\phi_0[a; q, t] = 1 + \sum_{n=1}^{\infty} \frac{(a)_n}{(q)_n} t^n &= \frac{(at)_\infty}{(t)_\infty} \\
&= \prod_{n=0}^{\infty} \frac{(1 - atq^n)}{(1 - tq^n)}. \qquad (2.2.1)
\end{aligned}
$$

Proof. Let

$$F(t) = \frac{(at)_\infty}{(t)_\infty} = \sum_{n=0}^{\infty} A_n t^n. \qquad (2.2.2)$$

We observe that A_n exist since we are dealing with an infinite product which is uniformly convergent in $|t| < 1$.

From the notation of $F(t)$ seen in (2.2.1), we have that

$$(1 - t)F(t) = (1 - at)F(tq).$$

Substituting (2.2.2) in this last equation, we have

$$(1 - t) \sum_{n=0}^{\infty} A_n t^n = (1 - at) \sum_{n=0}^{\infty} A_n t^n q^n$$

meaning

$$\sum_{n=0}^{\infty} A_n t^n - \sum_{n=1}^{\infty} A_{n-1} t^n = \sum_{n=0}^{\infty} A_n q^n t^n - \sum_{n=1}^{\infty} A_{n-1} a q^{n-1} t^n.$$

Comparing the coefficient of t^n in both side and noting that $A_0 = F(0) = 1$, we have

$$A_n - A_{n-1} = q^n A_n - a q^{n-1} A_{n-1}.$$

It follows that

$$A_n = \frac{(1 - aq^{n-1})}{(1 - q^n)} A_{n-1}$$

Iterating this relation, we obtain

$$A_n = \frac{(1 - aq^{n-1})(1 - aq^{n-2}) \cdots (1 - a)}{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q)} A_0 = \frac{(a)_n}{(q)_n}.$$

□

Many corollaries follow from this theorem, we are considering few of them.

2.2.10 Corollary (Euler)**.** Let $|t| < 1$ and $|q| < 1$. We have that:

$$
\begin{aligned}
1 + \sum_{n=1}^{\infty} \frac{t^n}{(q)_n} &= \frac{1}{(t)_\infty} \qquad (2.2.3) \\
&= \prod_{n=0}^{\infty} (1 - tq^n)^{-1}
\end{aligned}
$$

and

$$1+\sum_{n=1}^{\infty}\frac{t^n q^{\frac{1}{2}n(n-1)}}{(q)_n}=(-t;q)_\infty \tag{2.2.4}$$
$$=\prod_{n=0}^{\infty}(1+tq^n)$$

Proof. The first Equation (2.2.3) can be obtained by setting $a=0$ in Theorem 2.2.9.

For the Equation (2.2.4), we replace t by $-\frac{t}{a}$ and let $a\to\infty$. □

2.2.11 Theorem (Jacobi's triple product identity). *Let $z\neq 0$ and $|q|<1$. We have that*

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2}=(q^2;q^2)_\infty(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty \tag{2.2.5}$$
$$=\prod_{n=0}^{\infty}(1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1}).$$

Proof. We have for $|z|>|q|$, $|q|<1$:

$$(-zq;q^2)_\infty=\prod_{n=0}^{\infty}(1+zq^{2n+1})$$
$$=\sum_{m=0}^{\infty}\frac{z^m q^{m^2}}{(q^2;q^2)_m}\quad\text{by Equation (2.2.4)}$$
$$=\frac{1}{(q^2;q^2)_\infty}\sum_{m=0}^{\infty}z^m q^{m^2}(q^{2m+2};q^2)_\infty$$
$$=\frac{1}{(q^2;q^2)_\infty}\sum_{m=-\infty}^{\infty}z^m q^{m^2}(q^{2m+2};q^2)_\infty$$

since $(q^{2m+2};q^2)_\infty=0$ for $m<0$

$$=\frac{1}{(q^2;q^2)_\infty}\sum_{m=-\infty}^{\infty}z^m q^{m^2}\sum_{r=0}^{\infty}\frac{(-1)^r q^{r^2+2mr+r}}{(q^2;q^2)_r}$$
$$=\frac{1}{(q^2;q^2)_\infty}\sum_{r=0}^{\infty}\frac{(-1)^r z^{-r}q^r}{(q^2;q^2)_r}\sum_{m=-\infty}^{\infty}q^{(m+r)^2}z^{m+r}$$
$$=\frac{1}{(q^2;q^2)_\infty}\sum_{r=0}^{\infty}\frac{(-q/z)^r}{(q^2;q^2)_r}\sum_{m=-\infty}^{\infty}q^{m^2}z^m$$
$$=\frac{1}{(q^2;q^2)_\infty}\frac{1}{(-q/z;q^2)_\infty}\sum_{m=-\infty}^{\infty}q^{m^2}z^m.$$

We multiply by the denominator of the right-hand side and we get the result. □

This result holds in the set $|z|>|q|,|q|<1$. However, we can have all the other case using manipulation of the analytic property. This theorem is very practical in the sense that we derive later from it several important theorems in treating partition identities.

The Jacobi's triple product also admits several forms depending on what we want to emphasize. For instance, we can use Gaussian polynomial or view it in the general form of Ramanujan's theta function (Andrews, 1976). In that case the Jacobi's triple product becomes

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

2.2.12 Corollary. For $|q|<1$, for $0 \leqslant i \leqslant 2k+1$, we have:

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in} &= \sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in}(1-q^{(2n+1)i})\\ &= \prod_{n=0}^{\infty}(1-q^{(2k+1)(n+1)})(1-q^{(2k+1)n+i})(1-q^{(2k+1)(n+1)-i})\\ &= \prod_{\substack{n=1\\ n\equiv 0,\pm i (\bmod 2k+1)}}^{\infty}(1-q^n). \end{aligned} \tag{2.2.6}$$

Proof. We replace q by $q^{k+\frac{1}{2}}$ and take $z=-q^{k+\frac{1}{2}-i}$ in the Theorem 2.2.11. After, the left-hand side becomes:

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in}(1-q^{(2n+1)i})\\ &= \sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in} + \sum_{n=1}^{\infty}(-1)^n q^{(2k+1)n(n-1)/2+in}\\ &= \sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in} + \sum_{n=-1}^{-\infty}(-1)^n q^{(2k+1)n(n+1)/2-in}\\ &= \sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in}.\end{aligned}$$

□

3. On the Rogers-Ramanujan type identities

This chapter is the main focus of this project. We will prove here the two famous Rogers-Ramanujan identities which are given in the following theorems

3.0.1 Theorem (First Rogers-Ramanujan identity)**.**

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$
$$= \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})},$$

and

3.0.2 Theorem (Second Rogers-Ramanujan identity)**.**

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$$
$$= \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+2})(1-q^{5k+3})}.$$

We will first give the proof of the theorems themselves and after generalize the proof using Bailey's idea which was highlighted by other mathematicians. We will next give some of the identities' implications, especially we will consider them as generating functions of the cardinality of some partition's classes. We will further provide some examples of identities of the Rogers-Ramanujan type.

3.0.3 Remark. Some proofs of the results in this chapter were omitted. However, we have included them wherever they are important results, technically advantageous or new.

3.1 Analytic work

Let us first prove the Rogers-Ramanujan identities. The proof that we are about to introduce is the proof given by Ramanujan himself. It can be found in (Sills, 2017).

Proof. Let us define by

$$F(z) := F(z,q) = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q)_n}$$

and

$$G(z) := G(z,q) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m} q^{m(5m-1)/2}(1-zq^{2m})(z)_m}{(1-z)(q)_m}.$$

Using the techniques shown in the proof of Corollary 2.2.12, we can write $G(z)$ as

$$G(z) = \sum_{m=-\infty}^{\infty} \frac{(-1)^m z^{2m} q^{m(5m-1)/2}(z)_m}{(1-z)(q)_m}.$$

Now, let use write

$$H(z) := H(z,q) = \frac{G(z)}{(zq)_\infty}.$$

Then, with some computations, we may observe that

$$G(z) = (1-zq)G(zq) + zq(1-zq)(1-zq^2)G(zq^2)$$

implying that

$$H(z) = H(zq) + zqH(zq^2)$$

Similarly, we can observe (see later in 3.2.3) that

$$F(z) = F(zq) + zqF(zq^2).$$

We remark that $F(0) = H(0) = 1$ and they are power series which satisfy the same q-difference equation. Therefore, we have

$$F(z) = H(z) = \frac{G(z)}{(zq)_\infty}$$

Now, setting $z = 1$, we have

$$\begin{aligned}\frac{G(1)}{(q)_\infty} &= \frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{5/2n^2-1/2n} \\ &= \prod_{\substack{n=0 \\ n\equiv \pm 4 \mod 5}}^{\infty} \frac{1}{1-q^n} \quad \text{applying Jacobi's triple product identity } (k=2,\ i=2).\end{aligned}$$

Then we obtain Theorem 3.0.1.

Similarly, setting $z = q$, we have

$$\frac{G(q)}{(q^2)_\infty} = \prod_{\substack{n=0 \\ n\equiv \pm 3 \mod 5}}^{\infty} \frac{1}{1-q^n}$$

and we obtain Theorem 3.0.2. □

One may observe that this proof is a "verification proof". That is, the proof just checks whether the identities hold or not. Many mathematicians thought that the Rogers-Ramanujan identities were an isolated phenomenon. The source of the identities themselves was a mystery. Actually, we remark that most of the identities of the Rogers-Ramanujan type can be derived from a result found by Bailey. A methodical manipulation of this result is fruitful and we get a new identity of the Rogers-Ramanujan type.

3.1.1 Theorem. *Let*

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{3.1.1}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}.$$

Then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n.$$

It is called the Bailey's transform, a name introduced by Slater.

Proof. See (Sills, 2017), page 60. □

Having this theorem, Andrews introduced a new definition.

3.1.2 Definition. A pair of sequences $(\alpha_n(x,q), \beta_n(x,q))$ satisfying the Equation (3.1.1) with

$$u_n = \frac{1}{(q)_n} \quad \text{and} \quad v_n = \frac{1}{(xq)_n}$$

i.e.

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(xq)_{n+r}}$$

is called Bailey pair with respect to x.

We may notice that when one is specified, the other one is found so we do not need to always state the couple (α_n, β_n) together.

3.1.3 Lemma (Bailey's lemma)**.** Suppose that $(\alpha_n(x,q), \beta_n(x,q))$ is a Bailey pair.

Then $(\alpha'_n(x,q), \beta'_n(x,q))$ with

$$\alpha'_r(x,q) = \frac{(\rho_1)_r(\rho_2)_r}{\left(\frac{xq}{\rho_1}\right)_r \left(\frac{xq}{\rho_2}\right)_r} \left(\frac{xq}{\rho_1\rho_2}\right)^r \alpha_r(x,q),$$

and

$$\beta'_n(x,q) = \frac{1}{\left(\frac{xq}{\rho_1}\right)_n \left(\frac{xq}{\rho_2}\right)_n} \sum_{j=0}^{n} \frac{(\rho_1)_j(\rho_2)_j \left(\frac{xq}{\rho_1\rho_2}\right)_{n-j}}{(q)_{n-j}} \left(\frac{xq}{\rho_1\rho_2}\right)^j \beta_j(x,q)$$

form also a Bailey pair.

Proof. See (Warnaar, 2009) □

From this Lemma, an infinite sequence of Bailey pairs, called Bailey chains, can be created given one initial Bailey pair. We observe that this Lemma is equivalent to

$$\frac{1}{(xq/\rho_1)_n(xq/\rho_2)_n} \sum_{j=0}^{n} \frac{(\rho_1)_j(\rho_2)_j(xq/\rho_1\rho_2)_{n-j}}{(q)_{n-j}} \left(\frac{xq}{\rho_1\rho_2}\right)^j \beta_j(x,q)$$
$$= \sum_{r=0}^{n} \frac{(\rho_1)_r(\rho_2)_r}{(q)_{n-r}(xq)_{n+r}(xq/\rho_1)_r(xq/\rho_2)_r} \left(\frac{xq}{\rho_1\rho_2}\right)^r \alpha_r(x,q) \quad (3.1.2)$$

using the definition of Bailey pair.

3.1.4 Theorem. *Define*

$$\alpha_n^{(d,e,k)}(x^e,q^e) = \frac{(-1)^{n/d}x^{(k/d-1)n}q^{(k-d+1/2)n^2/d-n/2}(1-xq^{2n})}{(1-x)(q^d,q^d)_{n/d}}(x,q^d)_{n/d}[\![d|n]\!]$$

where

$$[\![P(n,d)]\!] = \begin{cases} 1 & \text{if } P(n,d) \text{ is true} \\ 0 & \text{if } P(n,d) \text{ is false} \end{cases} \quad \text{, the Iverson bracket}$$

and

$$\beta_n^{(d,e,k)}(x^e,q^e)$$
$$= \frac{1}{(q^e,q^e)_n(x^eq^e,q^e)_n}\sum_{r=0}^{\lfloor n/d\rfloor}\frac{(x,q^d)_r(1-xq^{2dr})(q^{-en},q^e)_{dr}}{(q^d,q^d)_r(1-x)(x^eq^{e(n+1)},q^e)_{dr}}(-1)^{(d+1)r}x^{(k-d)r}q^{Q(d,e,k,n,r)}$$

where

$$Q(d,e,k,n,r) = (2k-2d-ed+1)\frac{d}{2}r^2 + (e-1)\frac{d}{2}r + endr.$$

Then $(\alpha_n^{(d,e,k)}, \beta_n^{(d,e,k)})$ *form a Bailey pair. It is called "standard multiparameter Bailey pair"(SMBP).*

Proof. See (Sills, 2017), page 61. □

3.1.5 Corollary. For $|q| < 1$, we have

$$\sum_{n\geqslant 0} x^{en}q^{en^2}\beta_n^{(d,e,k)}(x^e,q^e) = \frac{1}{(x^eq^e;q^e)_\infty}\sum_{n\geqslant 0} x^{en}q^{en^2}\alpha_n^{(d,e,k)}(x^e,q^e).$$

Proof. Applying to $(\alpha_n^{(d,e,k)}, \beta_n^{(d,e,k)})$ the Equation (3.1.2), letting $\rho_1,\rho_2 \to \infty$ and replacing x by x^e and q by q^e we have:

- $(x^eq^e/\rho_1)_l$ and $(x^eq^e/\rho_2)_l \to 1$ for $l = n, j, r$,
- $(x^eq^e/\rho_1\rho_2)_{n-j} \to 1$.

Now we have, for $l = j, r$,

$$(\rho_1)_l(\rho_2)_l\left(\frac{x^eq^e}{\rho_1\rho_2}\right)^l = q^{el(l-1)}x^{el}q^{el}$$
$$= x^{el}q^{el^2}.$$

Finally, letting $n \to \infty$, we have

- $(q^e)_{n-j}$ and $(q^e)_{n-r} \to (q^e)_\infty$ and after we simplify (they converge to the same limit)
- $(x^eq^e;q^e)_{n+r} \to (x^eq^e;q^e)_\infty$.

Then combining all of these, we obtain the corollary. □

For our purpose, we will use the SMBP defined above to be able to get all the Rogers-Ramanujan identities family.

3.1.6 Example. For instance,

$$(\alpha_n^{(1,1,1)}(x,q), \beta_n^{(1,1,1)}(x,q)) = \left(\frac{(-1)^n q^{n(n-1)/2}(1-xq^{2n})(x)_n}{(1-x)(q)_n}, \delta_{n,0} \right)$$

is called the unit Bailey pair where $\delta_{n,0}$ is the Kronecker delta function. Indeed, setting $a = b = x$ and $c = q$ in Theorem 2.2.6, we have

$$\begin{aligned}
\frac{1}{(q)_n(xq)_n} {}_6\phi_5 \left[\begin{matrix} x, q\sqrt{x}, -q\sqrt{x}, x, q, q^{-n} \\ \sqrt{x}, -\sqrt{x}, x, q, xq^{n+1} \end{matrix} ; q, q^n \right] &= \frac{1}{(q)_n(xq)_n} \sum_{r=0}^{\infty} \frac{(x)_r(q\sqrt{x})_r(-q\sqrt{x})_r(q^{-n})_r(q)_r(x)_r}{(q)_r(\sqrt{x})_r(-\sqrt{x})_r(xq^{n+1})_r(x)_r(q)_r} q^{nr} \\
&= \frac{1}{(q)_n(xq)_n} \sum_{r=0}^{\infty} \frac{(x)_r(xq^2;q^2)_r(q^{-n})_r}{(q)_r(x;q^2)_r(xq^{n+1})_r} q^{nr} \\
&= \frac{1}{(q)_n(xq)_n} \sum_{r=0}^{\infty} \frac{(x)_r(1-xq^{2r})(q^{-n})_r}{(q)_r(1-x)(xq^{n+1})_r} q^{nr} \\
&= \frac{1}{(q)_n(xq)_n} \sum_{r=0}^{n} \frac{(x)_r(1-xq^{2r})(q^{-n})_r}{(q)_r(1-x)(xq^{n+1})_r} q^{nr}
\end{aligned}$$

since $(q^{-n})_r = 0$ for $r > n$

$$\begin{aligned}
&= \beta_n^{(1,1,1)}(x,q) \\
&= \frac{(1)_n}{(x)_n(q)_n^2} \quad \text{from the Theorem 2.2.6.}
\end{aligned}$$

Since $(1)_n = 0$ if $n \neq 0$ and $(1)_n = (x)_n = (q)_n = 1$ if $n = 0$ by definition, we have the result.

For $\alpha_n^{(1,1,1)}(x,q)$, we just set $(d,e,k) = (1,1,1)$ in the definition of $\alpha_n^{(d,e,k)}(x^e, q^e)$.

3.1.7 Remark. There are also two other types of multiparameter Bailey pair: Eulerian multiparameter Bailey pair (EMBP) and Jackson-Slater multiparameter Bailey pair (JSMBP).

Now, similarly to the power series H introduced by Ramanujan in the Proof 3.1, let us introduce a more general definition.

3.1.8 Definition (Andrews (1976))**.** Let $|x| < |q|^{-1}$ and $|q| < 1$. Define the following:

$$H_{k,i}(a,x,q) = \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2+n-in} a^n (1-x^i q^{2ni})(axq^{n+1})_\infty (a^{-1})_n}{(q)_n (xq^n)_\infty}. \tag{3.1.3}$$

3.1.9 Remark. We note that $H_{k,i}(a,x,q)$ is well-defined for $a = 0$. Indeed, we have:

$$a^n(a^{-1})_n = (a-1)(a-q)\cdots(a-q^{n-1}),$$

and the value at $a = 0$ is $(-1)^n q^{n(n-1)/2}$.

3.1.10 Theorem (Sills (2017))**.** *Let d, n and k be positive integers. We have that*

$$\sum_{n=0}^{\infty} x^{en} q^{en^2} \beta_n^{(d,e,k)}(x^e, q^e) = \frac{(xq^d, q^d)_\infty}{(x^e q^e, q^e)_\infty} H_{d(e-1)+k,1}(0, x, q^d).$$

Proof. We see from Remark 3.1.5 that

$$\begin{aligned}
&\sum_{n=0}^{\infty} x^{en} q^{en^2} \beta_n^{(d,e,k)}(x^e, q^e) \\
&= \frac{1}{(x^e q^e; q^e)_\infty} \sum_{n\geqslant 0} x^{en} q^{en^2} \alpha_n^{(d,e,k)}(x^e, q^e) \\
&= \frac{1}{(x^e q^e; q^e)_\infty} \sum_{n\geqslant 0} x^{en} q^{en^2} \frac{(-1)^{n/d} x^{(k/d-1)n} q^{(k-d+1/2)n^2/d-n/2}(1-xq^{2n})}{(1-x)(q^d,q^d)_{n/d}} (x,q^d)_{n/d} [\![d|n]\!] \\
&= \frac{1}{(x^e q^e; q^e)_\infty (1-x)} \sum_{j\geqslant 0} x^{edj} q^{ed^2 j^2} \frac{(-1)^j x^{(k/d-1)dj} q^{(k-d+1/2)dj^2 - dj/2}(1-xq^{2dj})}{(q^d,q^d)_j} (x,q^d)_j \\
&= \frac{(xq^d;q^d)_\infty}{(x^e q^e; q^e)_\infty (x;q^d)_\infty} \sum_{j\geqslant 0} \frac{(-1)^j x^{j(d(e-1)+k)} q^{j^2 d(d(e-1)+k)+dj(j-1)/2}(1-xq^{2jd})(x;q^d)_j}{(q^d;q^d)_j} \\
&= \frac{(xq^d;q^d)_\infty}{(x^e q^e; q^e)_\infty} \frac{(0;q^d)_\infty}{(x;q^d)_\infty} \sum_{j\geqslant 0} \frac{x^{j(d(e-1)+k)} q^{j^2 d(d(e-1)+k)} (-1)^j q^{dj(j-1)/2}(1-xq^{2jd})(x;q^d)_j}{(q^d;q^d)_j} \\
&= \frac{(xq^d;q^d)_\infty}{(x^e q^e; q^e)_\infty} H_{d(e-1)+k,1}(0,x,q^d).
\end{aligned}$$

□

Seeking for the value of (d, e, k) suitable in which case we can have a product representation of the series $H_{d(e-1)+k,1}$ is the issue faced in order to create an identity. The product representation is done in such a way that after setting $x = 1$ and applying Jacobi's triple product, we get the result. Additionally to the finding of an infinite product, the β_n should also be nice in some unquantifiable aesthetic sense. That is, the right-hand side should be simplified as much as possible.

3.1.11 Example. This example (and other proof of the first Rogers-Ramanujan identities) can be seen in (Sills, 2017). We obtain, after some computation,

$$\begin{aligned}
\beta_n^{(1,1,2)} &= \frac{1}{(q)_n (xq)_n} \sum_{r=0}^{n} \frac{(x)_r (1-xq^{2r})(q^{-n})_r}{(q)_r (1-x)(xq^{n+1})_r} x^r q^{r^2+nr} \\
&= \frac{1}{(q)_n (xq)_n} \lim_{b\to\infty} {}_6\phi_5 \left[\begin{matrix} x, q\sqrt{x}, -q\sqrt{x}, q^{-n}, b, b \\ \sqrt{x}, -\sqrt{x}, xq^{n+1}, xq/b, xq/b \end{matrix} ; q, \frac{xq^{n+1}}{b^2} \right] \\
&= \frac{1}{(q)_n (xq)_n} \lim_{b\to\infty} \frac{(xq)_\infty (xq/b^2)_n}{(xq/b)_n^2} = \frac{1}{(q)_n} \quad \text{by Theorem 2.2.6.}
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^n q^{n^2}}{(q)_n} = H_{2,1}(0,x,q) &= \frac{1}{(x)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} q^{5/2n^2-1/2n}(1-xq^{2n})(x)_n}{(q)_n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} q^{5/2n^2-1/2n}(1-xq^{2n})}{(q)_n (xq^{n+1})_\infty}.
\end{aligned}$$

Setting $x = 1$ in the right-hand side, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n q^{5/2n^2-1/2n}(1-q^2n)}{(1-q^n)(q)_\infty} &= \frac{1}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{5/2n^2-1/2n}(1+q^n) \\ &= \frac{1}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{5/2n^2-1/2n} + \frac{1}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{5/2n^2+1/2n} \\ &= \frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{5/2n^2+1/2n} \\ &= \prod_{\substack{n=0 \\ n\equiv\pm 4 \mod 5}}^{\infty} \frac{1}{1-q^n} \quad \text{applying Jacobi's triple product identity } (k=2,\ i=2).\end{aligned}$$

Therefore, we get the first Rogers-Ramanujan identity. With this same triplet, we can derive the first Ramanujan-Slater mod 8 identity which is

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} &= \frac{(q^3;q^8)_\infty(q^5;q^8)_\infty(q^8;q^8)_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty} \\ &= \frac{f(-q^3,-q^5)}{\Psi(-q)}\end{aligned} \tag{3.1.4}$$

using the notation seen in the Ramanujan's theta function.

Now, analogous to what we did in Proof 3.1, let use define the following q-difference equation.

3.1.12 Definition. Let

$$J_{k,i}(a,x,q) = H_{k,i}(a,xq,q) - xqaH_{k,i-1}(a,xq,q). \tag{3.1.5}$$

Let us look at few properties of this functions.

3.1.13 Lemma. We have:

$$H_{k,i}(a,x,q) - H_{k,i-1}(a,x,q) = x^{i-1}J_{k,k-i+1}(a,x,q). \tag{3.1.6}$$

Proof. See (Andrews, 1976). □

3.1.14 Lemma. We also have

$$J_{k,i}(a,x,q) - J_{k,i-1}(a,x,q) = (xq)^{i-1}(J_{k,k-i+1}(a,xq,q) - aJ_{k,k-i+2}(a,xq,q)). \tag{3.1.7}$$

Proof. See (Andrews, 1976) □

3.1.15 Lemma. Let $1 \leqslant i \leqslant k$ and $|q| < 1$. We have:

$$J_{k,i}(0,1,q) = \prod_{\substack{n=1 \\ n\not\equiv 0,\ \pm i \mod 2k+1}}^{\infty} (1-q^n)^{-1}. \tag{3.1.8}$$

Proof. By the Equation (3.1.5), we have that

$$\begin{aligned} J_{k,i}(0,1,q) &= H_{k,i}(0,q,q) \\ &= (q)_\infty^{-1} \sum_{n=0}^{\infty} q^{kn^2+(k-i+1)n}(-1)^n q^{n(n-1)/2}(1-q^{(2n+1)i}) \\ &= (q)_\infty^{-1} \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2-in}(1-q^{(2n+1)i}) \\ &= \prod_{\substack{n=1 \\ n\not\equiv 0,\pm i (\mod 2k+1)}}^{\infty} (1-q^n)^{-1} \quad \text{by Corollary 2.2.12.} \end{aligned}$$

□

3.1.16 Lemma. Let $1 \leqslant i \leqslant k$ and $|q| < 1$. We have:

$$J_{k,i}(-q^{-1},1,q^2) = \prod_{\substack{n=1 \\ n\not\equiv 2 \mod 4 \\ n\not\equiv 0,\ \pm(2i-1) \mod 4k}}^{\infty} (1-q^n)^{-1}. \tag{3.1.9}$$

Proof.

$$\begin{aligned} J_{k,i}(-q^{-1},1,q^2) &= H_{k,i}(-q^{-1},1,q^2) + qH_{k,i-1}(-q^{-1},1,q^2) \\ &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\Big(\sum_{n=0}^{\infty}(-1)^n q^{2kn+2kn^2-(2i-1)n}\frac{(1-q^{2i+4ni})}{(1+q^{2n+1})} \\ &+ q\sum_{n=0}^{\infty}(-1)^n q^{2kn+2kn^2-(2i-3)n}\frac{(1-q^{2i-2+4ni-4n})}{(1+q^{2n+1})}\Big) \\ &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{2kn+2kn^2+n}\frac{(q^{-2in}-q^{2i+2ni}+q^{1-2in+2n}-q^{-1+2i+2ni-2n})}{(1+q^{2n+1})} \\ &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{2kn^2+(2k+1-2i)n}(1-q^{(2n+1)(2i-1)}) \\ &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{2kn^2+(2k-2i+1)n} \end{aligned}$$

using the same techniques as in proof of Corollary 2.2.12,

$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\prod_{n=0}^{\infty}(1-q^{4kn+4k})(1-q^{4kn+2i-1})(1-q^{4kn+4k-2i+1})$$

using Corollary 2.2.12 by replacing $2k+1$ with $4k$ and i with $2i-1$,

$$= \frac{1}{(q,q^4)_\infty(q^3,q^4)_\infty(q^4,q^4)_\infty}\prod_{n=0}^{\infty}(1-q^{4k(n+1)})(1-q^{4kn+2i-1})(1-q^{4k(n+1)-2i+1})$$

using the value of Ramanujan's theta function $f(-q,-q^3)$

$$= \prod_{\substack{n=1 \\ n\not\equiv 2 \mod 4 \\ n\not\equiv 0,\pm(2i-1) \mod 4k}}^{\infty} (1-q^n)^{-1}.$$

□

The following theorem is the generalization of the Rogers-Ramanujan identities due to Andrews-Gordon.

3.1.17 Theorem (Andrews (1976)). *For* $1 \leqslant i \leqslant k$, $k \geqslant 2$, $|q| < 1$

$$\sum_{n_1,n_2,\dots,n_{k-1}\geqslant 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} = \prod_{\substack{n=1 \\ n\not\equiv 0,\pm i \pmod{2k+1}}}^{\infty} (1-q^n)^{-1}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

Proof. Let us first prove that

$$J_{k,i}(0,x,q) = \sum_{n\geqslant 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n} J_{k-1,i}(0,xq^{2n},q). \tag{3.1.10}$$

Suppose that

$$R_{k,i}(x,q) = \sum_{n\geqslant 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n} J_{k-1,i}(0,xq^{2n},q).$$

Then we have, for $1 \leqslant i \leqslant k$,

$$R_{k,i}(0,q) = R_{k,i}(x,0) = 1 \tag{3.1.11}$$

and

$$R_{k,0}(x,q) = 0. \tag{3.1.12}$$

Furthermore, we have

$$\begin{aligned}
&R_{k,i}(x,q) - R_{k,i-1}(x,q) \\
&= \sum_{n\geqslant 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n}(J_{k-1,i}(0,xq^{2n},q) - q^n J_{k-1,i-1}(0,xq^{2n},q)) \\
&= \sum_{n\geqslant 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n}(J_{k-1,i-1}(0,xq^{2n},q) + (xq^{2n+1})^{i-1}J_{k-1,k-i}(0,xq^{2n+1},q) \\
&- q^n J_{k-1,i-1}(0,xq^{2n},q)) \\
&= \sum_{n\geqslant 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n}(1-q^n)J_{k-1,i-1}(0,xq^{2n},q) \\
&+ (xq)^{i-1}\sum_{n\geqslant 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k-i-2)n}}{(q)_n} J_{k-1,k-i}(0,xq^{2n+1},q) \\
&= x^{k-1}q^{2k-i-1}\sum_{n\geqslant 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(3k-i-2)n}}{(q)_n} J_{k-1,i-1}(0,xq^{2n+2},q)
\end{aligned}$$

$$
\begin{aligned}
&+ (xq)^{i-1} \sum_{n\geqslant 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(k-i-2)n}}{(q)_n} (J_{k-1,k-i+1}(0, xq^{2n+1}, q) \\
&- (xq^{2n+2})^{k-i} J_{k-1,i-1}(0, xq^{2n+2}, q)) \\
&= x^{k-1} q^{2k-i-1} \sum_{n\geqslant 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(3k-i-2)n}}{(q)_n} J_{k-1,i-1}(0, xq^{2n+2}, q) \\
&+ (xq)^{i-1} \sum_{n\geqslant 0} (xq)^{(k-1)n} q^{(k-1)n^2+(k-(k-i+1))n} J_{k-1,k-i+1}(0, xq^{2n+1}, q) \\
&- x^{k-1} q^{2k-i-1} \sum_{n\geqslant 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(3k-i-2)n}}{(q)_n} J_{k-1,i-1}(0, xq^{2n+2}, q) \\
&= (xq)^{i-1} R_{k,k-i+1}(xq, q). \qquad (3.1.13)
\end{aligned}
$$

Since the coefficients of $J_{k,i}(0, x, q)$ were uniquely defined (see later in Remark 3.3.5) and $R_{k,i}(x, q)$ satisfies (3.1.11), (3.1.12) and (3.1.13), which implies that its coefficients satisfy also the same properties as seen in 3.3.5, for $0 \leqslant i \leqslant k$, $R_{k,i}(x, q) = J_{k,i}(0, x, q)$. Now, noticing that $J_{k,k+1}(0, x, q) = J_{k,k}(0, x, q)$ (setting $i = k + 1$ in Lemma 3.1.14 and observing that $J_{k,0}(0, xq, q) = H_{k,0}(0, xq^2, q) = 0$) and $J_{1,1}(0, x, q) = 1$ (by 3.1.14 again, $J_{1,1}(0, x, q) = J_{1,1}(0, xq, q) = \cdots = J_{1,1}(0, xq^n, q) \to J_{1,1}(0, 0, q) = 1$), by induction on k in the Equation (3.1.10), we obtain

$$
J_{k,i}(0, x, q) = \sum_{n_1, n_2, \ldots, n_{k-1} \geqslant 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_{k-1}}}.
$$

Now setting $x = 1$ and using Lemma 3.1.15, we have Theorem 3.1.17. □

This proof, given in Andrews (1976) can also be obtained iterating the Unit Bailey pair in Bailey's lemma. However, it will give an equivalent version of Theorem 3.1.17 as seen in Sills (2017). Setting $k = 2$, $i = 2$ and $k = 2$, $i = 1$, as special cases, we derive respectively the first (Theorem 3.0.1) and the second (Theorem 3.0.2) Rogers-Ramanujan identity.

3.2 Application to Ramanujan's continued fractions

In this section, we use the Rogers-Ramanujan identities seen above to show a well-known result. We cannot pass through the identities without stating the famous Ramanujan's continued fraction.

3.2.1 Definition. Let us consider a function $F(x)$ which is analytic in 0 with $F(0) = 1$ and which satisfies a linear second-order q-difference equation

$$
F(x) = F(xq) + xqF(xq^2). \qquad (3.2.1)
$$

3.2.2 Proposition. If we denote by $c(x, q) = \dfrac{F(x)}{F(xq)}$ then we have

$$
\begin{aligned}
c(x, q) &= 1 + \frac{xq}{c(xq, q)} \\
&= 1 + \cfrac{xq}{1 + \cfrac{xq^2}{c(xq^2, q)}} \\
&= \ldots
\end{aligned}
$$

Proof. It follows from the definition of $c(x, q)$ itself and Equation (3.2.1). □

3.2.3 Proposition. The function $F(x)$ defined above is given by

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n q^{n^2}}{(q)_n}.$$

Proof. Suppose that

$$F(x) = \sum_{n=0}^{\infty} A_n x^n.$$

Substituting this series into Equation (3.2.1) and comparing the coefficients of x^n in both sides, we get

$$A_n(q) = q^n A_n(q) + q^{2n-1} A_{n-1}(q).$$

It follows that

$$A_n(q) = \frac{q^{2n-1}}{1-q^n} A_{n-1}(q)$$

which, in iterating, gives us

$$A_n(q) = \frac{q^{1+3+\cdots+(2n-1)}}{(q)_n} A_0(q) = \frac{q^{n^2}}{(q)_n}.$$

Hence

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n q^{n^2}}{(q)_n}.$$

□

3.2.4 Remark. We notice that $F(1)$ gives the series in the left-hand side in the first Rogers-Ramanujan identities while $F(q)$ gives the second one.

We also remark that this is not the same as Andrews' x-generalization, different result in partition identity.

3.2.5 Proposition. We have

$$\cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{\vdots}}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right) e^{2\pi/5} \tag{3.2.2}$$

and

$$1-\cfrac{e^{-\pi}}{1+\cfrac{e^{-2\pi}}{1-\cfrac{e^{-3\pi}}{\vdots}}} = \left(\sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}\right) e^{\pi/5}. \tag{3.2.3}$$

Proof. We remark that $c(1,q) = \dfrac{F(1)}{F(q)}$ and the left-hand side of Equation (3.2.2) and (3.2.3) are respectively $c(1,e^{-2\pi})$ and $c(1,-e^{-\pi})$. The evaluation on the right sides requires the theory of modular functions, which is beyond the scope of this essay, but a nice presentation on this can be found in (Berndt et al., 1999). □

3.3 Combinatorial interpretation

The Rogers-Ramanujan identities can also be interpreted as partition identities. Let us see their interpretations in the theory of partition and we will after state Gordon's generalization of the results.

3.3.1 Theorem (First Rogers-Ramanujan identity)**.** *The number of partitions of n into non-consecutive parts equals the number of partitions of n into parts $\equiv \pm 1 \mod 5$.*

Proof. From Equation 2.1.1, we see that the right-hand side of the identity in Theorem 3.0.1 is the generating function of the number of partitions of n into parts $\equiv \pm 1 \mod 5$.

Now, let $r(n)$ be the number of partitions defined in the left-hand side of 3.3.1. Let l be an integer and consider a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ with $m \leqslant l$. Then we can always consider λ as a l-tuple by completing the $l-m$ missing piece with 0. The generating function of the number of such l-tuples is given by

$$\prod_{k=0}^{l} \frac{1}{1-q^{k+1}}$$

using the observation of Euler seen in 2.0.1.

Adding $2l-1$, $2l-3, \dots$, 3,1 to λ_1, $\lambda_2, \dots$, λ_l respectively, we obtain a partition of length l enumerated by $r(n)$. Using again Euler's observation, we see that the generating function of the number of such partitions is

$$\frac{q^{(2l-1)+(2l-3)+\cdots+1}}{(q)_l} = \frac{q^{l^2}}{(q)_l}.$$

Summing over all possible lengths l, we have

$$\sum_{l=0}^{\infty} \frac{q^{l^2}}{(q)_l} = \sum_{n=0}^{\infty} r(n) q^n.$$

Using the first Rogers-Ramanujan identity, we have the partition identity. □

3.3.2 Theorem (Second Rogers-Ramanujan identity)**.** *The number of partitions of n into non-consecutive parts greater than 1 equals the number of partitions of n into parts $\equiv \pm 2 \mod 5$.*

Proof. Similarly to the proof of the first one, the right-hand side of Theorem 3.0.2 define the number of partitions of n with parts $\equiv \pm 3 \mod 5$.

Now, for the left-hand side, we proceed exactly as above but now, the generating function is

$$\frac{q^{2l+(2l-2)+\cdots+2}}{(q)_l} = \frac{q^{l^2+l}}{(q)_l}.$$

We add multiples of 2 to the parts to assure that they are greater than 1 and they mutually differ by at least 2. Referring to the second Rogers-Ramanujan identity, we obtain the result. □

Gordon generalized these results as the following.

3.3.3 Definition (Sills (2017)). A partition $\lambda = (\lambda_1, \dots, \lambda_l)$ is said to be a $(k-i)-$Gordon partition if it satisfies the properties:

- at most $i-1$ of the parts are equal to 1,
- $\lambda_j - \lambda_{j+k-1} \geqslant 2$ for $j = 1, 2, \dots, l-k-1$.

3.3.4 Proposition (Andrews (1976)). Let us write $G_{k,i}(m,n)$ the number of $(k-i)-$Gordon partition with exactly m parts. We have for $1 \leqslant i \leqslant k$:

$$G_{k,i}(m,n) = \begin{cases} 1 & \text{if } m = n = 0 \\ 0 & \text{if } m \leqslant 0 \text{ or } n \leqslant 0 \text{ but } (m,n) \neq 0 \end{cases} \tag{3.3.1}$$

$$G_{k,0}(m,n) = 0 \tag{3.3.2}$$

$$G_{k,i}(m,n) - G_{k,i-1}(m,n) = G_{k,k-i+1}(m-i+1, n-m). \tag{3.3.3}$$

Proof. From the fact that the empty-tuple is the only partition of 0, the parts are positive number and $p(n) = 0$ if $n \leqslant 0$, we get the Equations (3.3.1) and (3.3.2)

Now, the left-hand side of the Equation (3.3.3) is the number of $(k-i)-$Gordon partition with m parts having exactly $i-1$ appearances of 1. Let λ be one of this partition.

If we subtract 1 from each parts of λ, we get a new partition $\mu = (\mu_1, \dots, \mu_{m-i+1})$ with length $m-i+1$. The $i-1$ one's of λ have been reduced to 0's and we don't consider them as parts of the partition.

We notice that μ is a partition of $n-m$ and its parts satisfy $\mu_j - \mu_{j+k-1} \geqslant 2$. Also, since 1 appears $i-1$ times in λ and the apparition of 1 and 2 together is at most $k-1$ times due to the difference condition, in μ, 1 appears at most $k-i+1$ times. Thus we made a bijection between the partitions enumerated in the left-hand side with those in the right-hand side. □

3.3.5 Remark. We observe that from the recurrence relation and the initial condition given in Proposition 3.3.4, $(G_{k,i}(m,n))_{n,i}$ is uniquely defined via an induction on i and n. The Equation (3.3.1) assure the case $n \leqslant 0$, $m \leqslant 0$ and $i > 0$ while Equation (3.3.2) plays this role for all n when $i = 0$. The Equation (3.3.3) is the recurrence relation.

3.3.6 Theorem (Gordon). *Let $\mathcal{G}_{k,i}$ the set of $(k-i)-$Gordon partition and let $\mathcal{A}_{k,i}$ the set of partitions such that the parts $\not\equiv 0, \pm i \mod (2k+1)$.*

For an integer n, we have:

$$p(\mathcal{A}_{k,i}, n) = p(\mathcal{G}_{k,i}, n)$$

Proof. Let us define explicitly

$$J_{k,i}(0,x,q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{k,i}(m,n) x^m q^n$$

We have

$$J_{k,i}(0,0,q) = J_{k,i}(0,x,0) = 1$$

which implies that for $1 \leqslant i \leqslant k$, we have

$$c_{k,i}(m,n) = \begin{cases} 1 & \text{if } m = n = 0 \\ 0 & \text{if } m \leqslant 0 \text{ or } n \leqslant 0 \text{ but } (m,n) \neq 0 \end{cases}.$$

We also have

$$J_{k,0}(0,x,q) = H_{k,0}(0,xq,q) = 0$$

which gives us $c_{k,0}(m,n) = 0$.

Finally, setting $a = 0$ in Equation (3.1.7) and comparing the coefficients of $x^m q^n$, it follows that

$$c_{k,i}(m,n) - c_{k,i-1}(m,n) = c_{k,k-i+1}(m-i+1, n-m).$$

Hence the sequence $(c_{k,i}(m,n))_{i,n}$ satisfies the same properties as the $(G_{k,i}(m,n))_{i,n}$ which we said early to be uniquely defined. Therefore, they have to be equal for all m, n and $0 \leqslant i \leqslant k$.

Now, we have

$$p(\mathcal{G}_{k,i},n) = \sum_{m \leqslant 0} G_{k,i}(m,n)$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} p(\mathcal{G}_{k,i},n)q^n &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} G_{k,i}(m,n)q^n \\
&= J_{k,i}(0,1,q) \\
&= \prod_{\substack{n=1 \\ n \not\equiv 0,\ \pm i \mod 2k+1}}^{\infty} (1-q^n)^{-1} \quad \text{by Equation (3.1.8)} \\
&= \sum_{n=0}^{\infty} p(\mathcal{A}_{k,i},n)q^n \quad \text{by Equation (2.1.1)}.
\end{aligned}$$

By comparing the coefficient in both sides, we obtain the result. □

Setting $k = i = 2$, or $k = i + 1 = 2$, this theorem generates as corollaries the two identities of Rogers-Ramanujan in their combinatorial version.

3.4 More identities of Rogers-Ramanujan type

We are now able to generate more identities of Rogers-Ramanujan type from the notion of Bailey pair. We will see in this chapter some of them. A complete list can be found in Sills (2007) and Sills (2017).

First, let us use again Theorem 3.1.4 and insert them in Theorem 3.1.3. We refrain to let directly $\rho_1, \rho_2 \to \infty$ as we did before.

3.4.1 Theorem (Sills (2007)). *We have*

$$\begin{aligned}
&\frac{1}{(\frac{x^e q^e}{\rho_1^e};q^e)_n(\frac{x^e q^e}{\rho_1^e};q^e)_n} \sum_{j \geqslant 0} \frac{(\rho_1^e;q^e)_j(\rho_2^e;q^e)_j((\frac{xq}{\rho_1\rho_2})^e;q^e)_{n-j}}{(q^e;q^e)_{n-j}} \left(\frac{xq}{\rho_1\rho_2}\right)^{ej} \beta_j^{(d,e,k)}(x^e,q^e) \\
&= \sum_{r=0}^{\lfloor n/d \rfloor} \frac{(\rho_1^e;q^e)_{dr}(\rho_2^e;q^e)_{dr}}{((\frac{xq}{\rho_1})^e;q^e)_{dr}((\frac{xq}{\rho_2})^e;q^e)_{dr}(q^e;q^e)_{n-dr}(x^e q^e;q^e)_{n+dr}} \left(\frac{xq}{\rho_1\rho_2}\right)^{der} \alpha_{dr}^{(d,e,k)}(x^e,q^e). \qquad (3.4.1)
\end{aligned}$$

To make our notation more simpler, we will keep all the previous notations and let us define a new one, which is the general case of Definition 2.2.2.

3.4.2 Definition. Let $a_1, a_2, \ldots, a_n$ and q be reals. We have

$$(a_1, a_2, \ldots, a_n; q)_s = (a_1; q)_s (a_2; q)_s \cdots (a_n; q)_s$$

a notation which justify the semicolon in Definition 2.2.2.

We have already seen the case $(d, e, k) = (1, 1, 1)$ which is the unit Bailey pair and $(d, e, k) = (1, 1, 2)$ related to Rogers-Ramanujan identities. Let us consider other examples.

3.4.3 Case $(d, e, k) = (1, 2, 3)$**.** Setting $\rho_1 \to \infty$, $\rho_2 = -\sqrt{q}$, $x = 1$ in Equation (3.4.1), we have

$$\sum_{n,r \geqslant 0} \frac{q^{n^2+2nr+2r^2}(-q;q^2)_{n+r}}{(-q)_{n+r}(-q)_{n+2r}(q)_n(q)_r} = \frac{(q^3, q^4, q^7; q^7)_\infty (-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

3.4.4 Case $(d, e, k) = (2, 2, 3)$**.** Setting $\rho_1, \rho_2 \to \infty$, $x = 1$ in Equation (3.4.1), we have

$$\sum_{n,r \geqslant 0} \frac{q^{n^2+3nr+2r^2}(q)_{n+r}}{(q)_{2n+2r}(q)_r(q)_n} = \frac{(q^5, q^6, q^{11}; q^{11})_\infty}{(q)_\infty}.$$

This identity is due to S.O Warnaar.

3.4.5 Case $(d, e, k) = (2, 2, 5)$**.** Setting $\rho_1, \rho_2 \to \infty$, $x = q^2$ in Equation (3.4.1), we have

$$\sum_{n,r \geqslant 0} \frac{q^{2n^2+4nr+3r^2+4n+6r}}{(-q)_{2n+2r+2}(-q;q^2)_{r+1}(q)_r(q^2;q^2)_n} = \frac{(q^2, q^{28}, q^{30}; q^{30})_\infty}{(q^2;q^2)_\infty}.$$

3.4.6 Case $(d, e, k) = (3, 2, 7)$**.** Setting $\rho_1, \rho_2 \to \infty$, $x = q^3$ in Equation (3.4.1), we have

$$\sum_{n,r \geqslant 0} \frac{q^{2n^2+4nr+3r^2+6n+9r}(q^3;q^3)_r}{(-q)_{2n+2r+3}(q)_{2r+2}(q)_r(q^2;q^2)_n} = \frac{(q^3, q^{60}, q^{63}; q^{63})_\infty}{(q^2;q^2)_\infty}.$$

There were no criteria on the choice of (d, e, k) presented in this essay, it was just a choice among others. Many of them don't have combinatorial interpretation or any application. These reasons, together with the fact that they were independently discovered, make the Rogers-Ramanujan identities unique and more interesting.

4. Further results in partition theory

We have seen many partition theory results in the previous chapters. In this chapter, we will look at a result of Alladi and Schur (Andrews, 2019) and its generalization (Rapudi, 2019). These results were proven using analytic methods. Here, we present a bijective proofs for some special cases.

4.1 Analytic proof

We will show analytically some partition theory results in this section.

First, let us recall Euler's partition identity seen in Theorem 2.1.5 in its analytic version.

4.1.1 Theorem. *We have*

$$\prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}} \tag{4.1.1}$$

equivalent to

$$(-q)_\infty = \frac{1}{(q;q^2)_\infty}$$

using q-Pochhammer symbol.

Glaisher generalized this theorem extending the notion of distinctness in the left-hand side. Instead of considering the parts as distinct each other, he made a restriction on the multiplicity of the parts.

4.1.2 Theorem (Glaisher)**.** *Let k be a positive integer and let n be an integer. We denote by N_k the set of positive integers not divisible by k. Then we have*

$$p(N_k, n) = p(\mathbb{N}^*(\leqslant k-1), n).$$

We may observe that Euler's partition identity is the case $k = 2$.

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} p(\mathbb{N}^*(\leqslant k-1), n)q^n &= \prod_{n=1}^{\infty}\frac{1-q^{kn}}{1-q^n} \quad \text{from Equation (2.1.2)}\\ &= \prod_{\substack{n=1\\ k\nmid n}}^{\infty}\frac{1}{1-q^n}\\ &= \sum_{n=0}^{\infty} p(N_k, n)q^n.\end{aligned}$$

□

Now, if the odd parts in Theorem 2.1.5 are allowed to appear at most twice, we have the following

4.1.3 Theorem (Alladi-Schur (Andrews, 2019))**.** *Let $o_3(n)$ be the number of partitions of n into odd parts which occur at most 2 times. Let $d_3(n)$ be the number of partitions of n into distinct parts non-multiples of 3. We have*

$$o_3(n) = d_3(n)$$

This theorem is attributed to Alladi-Schur because Schur gave, in 1926, a theorem stating equality between $d_3(n)$ and the cardinalities of 2 other partition sets. After, Alladi showed that there is one more equality which is stated above.

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} o_3(n)q^n &= \prod_{n=1}^{\infty}(1+q^{2n-1}+q^{2(2n-1)}) \quad \text{from Equation (2.1.2)}\\ &= \prod_{n=1}^{\infty}\frac{1+q^n}{1+q^{3n}} \quad \text{from Equation (4.1.1)}\\ &= \prod_{n=1}^{\infty}\frac{(1+q^{3n})(1+q^{3n-1})(1+q^{3n-2})}{1+q^{3n}}\\ &= \prod_{n=1}^{\infty}(1+q^{3n-1})(1+q^{3n-2})\\ &= \sum_{n=0}^{\infty} d_3(n)q^n.\end{aligned}$$

□

This theorem can be generalized as follows.

4.1.4 Theorem (Rapudi (2019))**.** *Let $k \geqslant 2$. Let $o_k(n)$ be the number of partitions of n into odd parts which occur at most $k-1$ times. Let $d_k(n)$ be the number of partitions of n into distinct parts non-multiples of k. We have*

$$o_k(n) = d_k(n).$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} o_k(n)q^n &= \prod_{n=1}^{\infty}(1+q^{2n-1}+q^{2(2n-1)}+\cdots+q^{(k-1)(2n-1)}) \quad \text{from Equation (2.1.2)}\\ &= \prod_{n=1}^{\infty}\frac{1+q^n}{1+q^{kn}} \quad \text{from Equation (4.1.1)}\\ &= \prod_{n=1}^{\infty}\frac{(1+q^{kn})(1+q^{kn-1})(1+q^{kn-2})\cdots(1+q^{kn-k+1})}{1+q^{kn}}\\ &= \prod_{n=1}^{\infty}(1+q^{kn-1})(1+q^{kn-2})\cdots(1+q^{kn-k+1})\\ &= \sum_{n=0}^{\infty} d_k(n)q^n.\end{aligned}$$

□

4.2 Bijective proof

Euler proved his partition identity in the 1840's using generating function. Glaisher, in 1883, gave a bijective proof of this theorem, and after of his theorem. However no combinatorial proof of Alladi-Schur was given up to now. In this section, we will give a bijective proof of Alladi-Schur partition identity. Analytic proof is shorter and simpler, however having a bijective proof is always nice in the sense that we can see the links between the partitions in the two sides.

Throughout this section, we will use the notation seen in the second bullet, Remark 1.1.3 to represent a partition.

4.2.1 Definition. We define the map G_k as follows

$$\lambda = (\lambda_1^{f_1}, \lambda_2^{f_2}, \ldots, \lambda_m^{f_m}) \mapsto \bigcup_{i=1}^{m} (a_i)^{k^{j_i} f_i}$$

where $\lambda_i = k^{j_i} a_i$ and $\gcd(k, a_i) = 1$. We notice that we are allowed to write λ_i in this from the fundamental theorem of arithmetic.

4.2.2 Remark. In other words, G_k is a map which takes the highest power of k in every part of the partition and convert it into multiplicity. We also remark that G_k preserves the size, that is λ and $\bigcup\limits_{i=1}^{m} (a_i)^{k^{j_i} f_i}$ partition the same number.

$\bigcup\limits_{i=1}^{m} (a_i)^{k^{j_i} f_i}$ is also an unconventional way to write partition but it makes simpler our understanding of the results. Indeed, using this notation, we can have a partition like $(1^2, 1^3, 3, 4)$. However, we will accept that by this partition, we mean $(1^5, 3, 4)$.

This map is defined onto the set of partitions whose parts are not divisible by k.

4.2.3 Definition. Let $\mu = (\mu_1^{h_1}, \ldots, \mu_l^{h_l})$ be a partition of n. Let us write

$$h_i = \sum_{j=0}^{m_i} a_{i,j} k^j \quad \text{where } a_{i,j} \in \{0, 1, 2, \ldots, k-1\}$$

in base k. Then we can define

$$G_k^*(\mu) = \bigcup_{i=1}^{l} \bigcup_{j=0}^{m_i} (k^j \mu_i)^{a_{i,j}}$$

4.2.4 Remark. As above, this definition can be interpreted as a map which takes the power of k in the parts multiplicity and send them back to the parts. G_k^* also preserves size and is defined onto the set of partitions whose parts occur at most $k-1$ times.

The following proof is the bijective proof of the Glaisher's theorem given in 4.1.2.

4.2.5 Glaisher's bijection. Let k be a positive integer. G_k restricted to the set of partitions whose parts appears at most $k-1$ times is a bijection onto the set of partitions whose parts are not divisible by k. Its inverse is G_k^* restricted to the set of partitions whose parts are not divisible by k. Indeed, let $\lambda = (\lambda_1^{t_1}, \lambda_2^{t_2}, \ldots, \lambda_l^{t_l})$ be a partition of an integer n with parts not divisible by k. Applying G_k^* to λ, we obtain $\bigcup_{i=1}^{l} \bigcup_{j=0}^{m_i} (k^j \lambda_i)^{a_{i,j}}$, partition of n with parts appearing at most $k-1$ where $t_i = \sum_{j=1}^{m_i} a_{i,j} k^j$. Conversely, let $\mu = (\mu_1^{f_1}, \mu_2^{f_2}, \ldots, \mu_g^{f_g})$ be a partition of n with parts appearing at most $k-1$ (i.e. $1 \leqslant f_i \leqslant k-1$). Applying G_k to μ, we obtain $\lambda = \bigcup_{i=1}^{g} (a_i)^{k^{j_i} f_i}$, partition of n with parts not divisble by k where $\mu_i = k^{j_i} a_i$. This table shows few examples for $k = 3$ and $n = 21$

λ	μ
$(1^2, 5^2, 9)$	$(1^{11}, 5^2)$
$(3, 4^2, 5^2)$	$(1^3, 4^2, 5^2)$

Table 4.1: Examples of partitions under Glaisher's bijection

Inspired by this, let us also give a bijective proof of Theorem 4.1.4. We are going to prove for k odd.

4.2.6 Generalized Alladi-Schur: bijective proof. Let k be an odd integer. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n into distinct parts non-multiple of k. Then applying G_2 to λ, we get a partition $\mu = \bigcup_{i=1}^{m} (\mu_i)^{2^{t_i}} = \bigcup_{i=1}^{m'} (\mu_i)^{\alpha_i}$(we add the multiplicity for same μ_i, $m' \leqslant m$) where $\lambda_i = 2^{t_i}\mu_i$ and μ_i odd. Now, applying G_k^* to μ, we get a partition $\nu = \bigcup_{i=1}^{m'} \bigcup_{j=1}^{m_i} (k^j \mu_i)^{a_{i,j}}$ where $\alpha_i = \sum_{j=1}^{m_i} a_{i,j} k^j$, $0 \leqslant a_{i,j} \leqslant k-1$. Since μ_i and k are odd, ν is a partition of n with odd parts occurring at most $k-1$ times.

Conversely, let $\nu = (\nu_1^{f_1}, \ldots, \nu_g^{f_g})$ be a partition of n into odd parts appearing at most $k-1$ times, i.e. ν_i is odd and $1 \leqslant f_i \leqslant k-1$ for all $1 \leqslant i \leqslant g$. Applying G_k to ν, we obtain

$$\mu = \bigcup_{i=1}^{g} (\mu_i)^{k^{j_i} f_i} = \bigcup_{i=1}^{g'} (\mu_i)^{\beta_i}$$

(we add the multiplicity for same μ_i, $g' \leqslant g$) where $\nu_i = k^{j_i}\mu_i$ and $\gcd(\mu_i, k) = 1$ for all i. Applying G_2^* to μ, we obtain

$$\lambda = \bigcup_{i=1}^{m} \bigcup_{j=1}^{l_i} (2^j \mu_i)^{b_{i,j}}$$

where $\beta_i = \sum_{j=1}^{l_i} b_{i,j} 2^j$, $0 \leqslant b_{i,j} \leqslant 1$. This partition is a partition of n whose parts are distinct since the $b_{i,j} = 0$ or 1. The parts are also non-multiples of k since k is odd and $\gcd(\mu_i, k) = 1$.

For $k = 3$ which is the theorem of Alladi-Schur. $G_3^* \circ G_2$ is the bijection needed and $G_2^* \circ G_3$ is its inverse. For instance, this table shows some partitions of $n = 25$ under the bijection

$\lambda / G_2^*(\mu)$	$G_2(\lambda)/G_3(\mu)$	$G_3^*(\lambda)/\mu$
$(1, 2, 22)$	$(1^3, 11^2)$	$(3, 11^2)$
$(5, 7, 13)$	$(5, 7, 13)$	$(5, 7, 13)$
$(1, 2^2, 20)$	$(1^5, 5^4)$	$(1^2, 3, 5, 15)$
$(1, 8, 16)$	$(1^2, 5)$	$(3^2, 9^2)$

Table 4.2: Some example of partitions under the bijection

4.2.7 Remark. What we have done so far is for the case k odd. However, to prove for a general even k (i.e. different from 2^j, case done by Rapudi (2019)), maybe it is possible but requires lot of manipulations of the above defined maps. The difficulty is to find the bijection.

5. Conclusion

The Rogers-Ramanujan identities will never stop to impress us. Hardy even said that: “it would be difficult to find more beautiful formulae than the 'Rogers-Ramanujan's identities'”. During the last century, they were extensively studied and many of their applications in lot of fields in mathematics were found included knot theory, representation theory, statistical mechanics, non-parametric statistics, continued fractions and many others. Most part of this work was dedicated to the impact of these identities in partition theory, results which were combinatorial. Besides, some mathematicians attempted to find bijective proof of partition identities and succeeded in some case but failed in some others. We have presented special case where it can be done. The Rogers-Ramanujan was already a source of inspiration for a lot of mathematicians, we hope that this will remain an active field for future generations and who knows some new consequences for the upcoming science.

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