Linear realizability of matroids

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23 May 2019

Submitted in partial fulfillment of a structured masters degree at AIMS South Africa

Abstract

The purpose of this paper is to check the representability of some paving matroids. To begin with, we check the representability of some known matroids which are the matroids arising from all Steiner triple systems $S(2,3,7)$, $S(2,3,9)$, $S(2,3,13)$ and some $S(2,3,15)$ and then, we aim to generalize the results of those examples. We see that matroids arising from certain Steiner triple systems $S(2,3,2^n-1)$ are representable and from certain Steiner triple systems $S(2,3,3n)$ with n odd and divisible by 5 are not representable.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Emmanuella Sandratra Rambeloson, 23 May 2019

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Introduction

A Matroid is a structure which generalizes the notion of dependence. It was found by Whitney in 1935 (Whitney, 1992). Dependence is a notion which is very well-known in several areas of mathematics, especially in linear algebra and graph theory. There are many analogous properties in those different areas, so Whitney captured those results and abstracted them. The theory of matroids is applicable also in some areas such as topology, network theory, combinatorial optimisation, coding theory and extensions of several algebraic structures. In order to know more applications of these areas see Bachem and Kern (1992), Recski (2013), or Wilson (1979). Problems involving matrices can be viewed as problems about matroids, which can make their solution easier. This makes interesting of a matroid which is representable because it corresponds to some matrices over some fields. The problem of representability of matroids in general remains open. There are many matroids that are neither known to be representable nor known to be non-representable. This is the case for the interesting class of paving matroids, and in particular for matroids arising from block designs. This project consists of three chapters. The first chapter will introduce the basic definitions and some examples of matroids where we need to solve our problems of representability. In Chapter 2 we will focus on checking the representability of some known matroids. In the final chapter, we will attempt to obtain stronger results. We see in Section 3.2 and in Section 3.3 that there are infinitely many Steiner triple systems $S(2,3,n)$ with representable matroid and infinitely many Steiner triple systems $S(2,3,n)$ with non-representable matroid.

1. Matroids

In this chapter, our principal goal is to understand some basic definitions of Matroid Theory which will help us in solving our problems in this paper.

1.1 Preliminaries

By abuse of notation we will write respectively the sets $\{a, b\}$, $\{a, b, c\}$ as $a\ b$ and $a\ b\ c$. The elementary result that we will use most frequently is this: a set of n vectors in n-dimensional space is linearly dependent if and only if the determinant of the $n \times n$ matrix whose columns are the given vectors is zero.

1.2 Independent sets

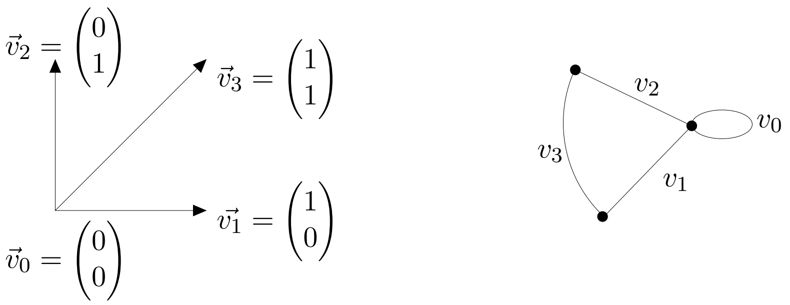
In this section, our aim is to describe the structure of a matroid and to see some examples. We will see also what is a basis, a circuit and the rank of matroid.

1.2.1 Definition. A matroid M is an ordered pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a set of subsets of E such that:

- (I_1) $\emptyset \in \mathcal{I}$.
- (I_2) If $I \in \mathcal{I}$ and $I' \subseteq I$ then $I' \in \mathcal{I}$.
- (I_3) If I and J are in $\mathcal{I}$ with $|I| < |J|$, then there exists $e \in J - I$ such that $I \cup e \in \mathcal{I}$.

We say that M is a matroid on E. We call the elements of $\mathcal{I}$ the independent sets of M and E the ground set of M. A subset of E which is not in $\mathcal{I}$ is called a dependent set.

1.2.2 Example.



$E = \{\vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ $\quad$ $E = \{v_0, v_1, v_2, v_3\}$

$I = \{\emptyset, \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_1\vec{v}_2, \vec{v}_1\vec{v}_3, \vec{v}_2\vec{v}_3\}$ $\quad$ $I = \{\emptyset, v_1, v_2, v_3, v_1v_2, v_1v_3, v_2v_3\}$

Figure 1.1: vector matroid and cycle matroid

In the first figure, E is a finite subset of the vector space $\mathbb{R}^2$ over the real numbers $\mathbb{R}$ and $\mathcal{I}$ is the set of linearly independent set of vectors in E. $M = (E, \mathcal{I})$ is a matroid which is called a vector matroid.

In the second figure, E is the set of edges from a graph G and $\mathcal{I}$ is the set of subsets of E which contain no cycle. This matroid is called a cycle matroid.

1.2.3 Definition. Two matroids $M_1 = (E_1, \mathcal{I}_1)$, $M_2 = (E_2, \mathcal{I}_2)$ are isomorphic if there is a bijection $\psi : E_1 \to E_2$ such that $I \in \mathcal{I}_1$ if and only if $\psi(I) \in \mathcal{I}_2$. We write $M_1 \simeq M_2$.

1.2.4 Example. The two matroids that we define in Example 1.2.2 are isomorphic. We can just consider a map v such that $v(\vec{v_i}) = v_i$ for $0 \leq i \leq 3$.

1.2.5 Definition. A maximal independent set of a matroid M is called a basis of M and a minimal dependent set is a circuit of M.

1.2.6 Example. In Example 1.2.2 the bases are v_1v_2 , v_1v_3 and v_2v_3 and the circuits are v_0 and $v_1v_2v_3$.

1.2.7 Lemma. If A and B are bases of a matroid M, then $|A| = |B|$.

Proof. Let A and B be two bases of a matroid M. Suppose that $|A| \neq |B|$. Then either $|A| < |B|$ or $|B| < |A|$. If $|A| < |B|$, then by (I_3) in Definition 1.2.1 there is $e \in B - A$ such that $A \cup e$ is an independent set. This is a contradiction to the maximality of A. By the same arguments, we show that $|A| \leq |B|$. Hence $|A| = |B|$. □

1.2.8 Definition. Let $M = (E, I)$ be a matroid and A be a subset of E. Consider the set I_A of all independent subsets of E that are contained in A. We define (A, I_A) as a submatroid of M.

1.2.9 Definition. The rank function of a matroid M is a function $f : 2^E \to \mathbb{N}$ defined by

$$f(A) = \max\{|X| : X \subseteq A, X \in \mathcal{I}\}.$$

The rank r of M is $f(E)$. This implies that r is also the size of a basis of M.

1.3 Paving matroids and Steiner systems $S(t, k, n)$

In this section, our main purpose is to see how from a Steiner system arises a matroid which is known as a paving matroid.

1.3.1 Definition. If a matroid M has no circuits of size less than its rank r, then we will call M a paving matroid.

1.3.2 Definition. A Steiner system $S(t, k, n)$ of order n is an ordered pair (S, D) where S is a set of n elements and D a set of k-subsets of S such that each t-subset of S is a subset of exactly one element of D. The elements of D will be called blocks.

A Steiner system $S(t, k, n)$ is trivial if there is an equality between the parameters t, k, n. In the following we assume that a Steiner system is non trivial which means $t < k < n$.

1.3.3 Proposition. Let $S(t, k, n) = (S, D)$ be a Steiner system of order n. Define a set $\mathcal{I}$ of subsets of S such that I is an element of $\mathcal{I}$ if either $|I| \leq t$ or $|I| = t + 1$ and I is not a subset of any block. Then $M = (S, \mathcal{I})$ is a paving matroid.

To prove the proposition, we show the conditions in Definition 1.2.1 are all satisfied.

Proof. The two conditions (I_1) and (I_2) in Definition 1.2.1 are obvious. Let I, J be two elements of $\mathcal{I}$ such that $|I| < |J|$.

If $|I| < t$ and $|J| \leq t+1$, then for any elements $u \in J - I$ we have $|I \cup u| \leq t$ which means that $I \cup u \in \mathcal{I}$.

Suppose $|I| = t$ and $|J| = t+1$. Let $u \in J - I$ and suppose that $I \cup u \notin \mathcal{I}$. As $|I \cup u| = t+1$, then it is a subset of a block K. Then for any $u \in J - I$, there is a block which contains $I \cup u$ as a subset. As $|I| = t$ then I is a subset of only one block. Then $I \cup (J - I)$ is a subset of the block K which means that J is a subset of one block. It is a contradiction of $J \in \mathcal{I}$. Then there is $u \in J - I$ such that $I \cup u \in \mathcal{I}$. So the third condition (I_3) holds. Therefore $\mathcal{I}$ is a set of independent sets of a matroid on S.

The rank of the matroid arising from $S(t, k, n)$ is $t+1$ and it has no circuits of size less than $t+1$. Thus it is a paving matroid. □

1.3.4 Definition. Two Steiner systems (S_1, D_1), (S_2, D_2) are isomorphic if there is a bijection map $\phi : S_1 \to S_2$ such that $B \in D_1$ if and only if $\phi(B) \in D_2$.

Given t, k, n, there might be many pairwise non-isomorphic Steiner systems $S(t, k, n)$.

1.3.5 Definition. A Steiner system $S(2, 3, n)$ is called a Steiner triple system of order n and its blocks are called triples.

1.3.6 Example. We will consider examples in the later section.

1.3.7 Lemma (Kirkman (1847))**.** A Steiner triple system of order n exists if and only if $n \cong 1$ or 3 (mod 6).

1.3.8 Lemma (Anderson (1989), p.102)**.** A Steiner triple system $S(2, 3, n)$ has $\frac{n(n-1)}{6}$ blocks.

1.4 Vector representation of matroids

In this section, we aim to understand the meaning of representability of matroid.

In the following, we let $\mathbb{F}$ be a field.

1.4.1 Proposition. Let U be a finite subset of a vector space over $\mathbb{F}$ and $\mathcal{I}$ be the set of linear independent subsets of U. Then $M = (U, \mathcal{I})$ is a matroid.

Proof. The two conditions (I_1) and (I_2) in the definition 1.2.1 are obvious. Let I, J be two linearly independent sets such that $|I| < |J|$. Write V_1 ,V_2 the vector space generated by the vectors in I and J respectively. Then $\dim(V_1) = |I| < \dim(V_2) = |J|$. Now suppose that for any $e \in J - I$, $I \cup e$ is not linearly independent. Then J is a subset of V_1 which means that $|J| = \dim(V_2) \leq \dim(V_1) = |I|$. Clearly this is a contradiction. Then there is $e \in J - I$ such that $I \cup e$ is a linearly independent set. □

1.4.2 Remark. We will call a matroid constructed this way a vector matroid over $\mathbb{F}$.

1.4.3 Definition. Let $\mathbb{F}$ be a field. A matroid M on a set E is $\mathbb{F}$-representable or $\mathbb{F}$-linear or representable over the field $\mathbb{F}$ if it is isomorphic to a vector matroid over $\mathbb{F}$.

1.4.4 Remark. If v is an isomorphism of a matroid M to a vector matroid in a vector space V over $\mathbb{F}$ then we will call v a representation of M.

1.4.5 Example. The cycle matroid in Example 1.2.2 is $\mathbb{R}$-linear.

1.4.6 Corollary. Let M be a matroid such that M is realizable over $\mathbb{F}$ and v a representation of M to a vector space V. Let U be a vector space over $\mathbb{F}$ such that there is a one to one linear transformation f that maps V to U. Then $f \circ v$ is a representation of M to the vector space U.

Proof. This is obvious because a one to one linear transformation preserves linear independence. □

1.4.7 Example. Let M be a rank r matroid such that M is $\mathbb{F}$-linear. Let v be a representation of M and A be an $r \times r$ invertible matrix such that all entries of A are elements of $\mathbb{F}$. Consider an isomorphism of the vector space $\mathbb{F}^r$ to itself define by

$$\begin{aligned} f :& \mathbb{F}^r \to \mathbb{F}^r \\ & v \mapsto Av \end{aligned}$$

An isomorphism of a vector space is also a one to one linear transformation, so by Proposition 1.4.6 $f \circ v$ is a representation of M to the vector space $\mathbb{F}^r$.

This example means that a multiplication of every column by an $r \times r$ fixed invertible matrix does not change anything about the matroid.

1.4.8 Proposition. Let M be a rank r matroid on a set E such that M is representable over $\mathbb{F}$, then there is a representation v of M to the r-dimensional vector space $\mathbb{F}^r$.

Proof. As the matroid M is representable over $\mathbb{F}$, then there is a representation g of M to a vector space V over $\mathbb{F}$, that is $g(E) = U$. Let r be the size of all maximal linearly independent sets in U. Then span(U) is an r-dimensional vector space which isomorphic to $\mathbb{F}^r$. Then by the proposition 1.4.6 there is a representation of M to the vector space $\mathbb{F}^r$. □

1.4.9 Corollary. Let M be a rank r matroid and representable over $\mathbb{F}$. Let B be a fixed basis of M, then there is a representation of M to the vector space $\mathbb{F}^r$ that maps B to another basis B_1 of $\mathbb{F}^r$.

Proof. We know that M is $\mathbb{F}$-linear of rank r, so by Proposition 1.4.8 there is a representation v of M to the vector space $\mathbb{F}^r$. Now fix one basis B of M then $v(B)$ is a basis of $\mathbb{F}^r$ and the linear transformation u that maps $v(B)$ to the basis B_1 of $\mathbb{F}^r$ is bijective. Therefore $u \circ v$ is a representation of M to the vector space $\mathbb{F}^r$ which maps the fix basis B to the basis B_1. □

1.4.10 Proposition. Let $S(t, k, n) = (S, D)$ be a Steiner system, where $t \geq 2$. Let M be the matroid arising from (S, D), where $S = \{a_1, \ldots, a_n\}$, such that M is representable over $\mathbb{F}$. Let v be a representation of M to the vector space $\mathbb{F}^r$ and $\alpha_1, \ldots, \alpha_n$ be n non-null elements of $\mathbb{F}$. Then M is isomorphic to the vector matroid $U = \{\alpha_1 v(a_1), \ldots, \alpha_n v(a_n)\}$.

Proof. Consider a map $f : M \to U$ defined by $f(a_i) = \alpha_i v(a_i)$ where $1 \leq i \leq n$. Obviously, f is an onto map. Now, let $a_i, a_j \in S$ such that $f(a_i) = f(a_j)$, then $\alpha_i v(a_i) = \alpha_j v(a_j)$ which means that $v(a_i)$ $v(a_j)$ is a linearly dependent set. Since a set of 2 elements is independent in the matroid M, $v(a_i) = v(a_j)$ and $a_i = a_j$ because v is a representation of M to the vector space $\mathbb{F}^r$. So f is a one to one map. Therefore f is a bijection and clearly it preserves independence. □

The proposition 1.4.10 means that for a matroid arising from a Steiner system $S(t, k, n)$ a multiplication of any column by a non zero scalar does not change anything about the matroid.

2. Linearisability of some interesting matroids

In this chapter, our goal is to check whether the Steiner triple system arising from $S(2,3,7)$, $S(2,3,9)$, $S(2,3,13)$ and some $S(2,3,15)$ are representable. We will say that a mapping v from a ground set of a Steiner triple system to a 3-dimensional vector space, *preserves triples* if $v(a\ b\ c)$ is linearly dependent for every triple $a\ b\ c$.

2.1 The Steiner triple system $S(2,3,7)$

The Steiner triple System $S(2,3,7) = (S, D)$ is uniquely determined. This is the finite projective plane $PG(2,2)$ whose points are the 1-dimensional subspaces of a 3-dimensional vector space over the field of order 2 and whose lines are the 2-dimensional subspaces of the same space. This Steiner triple system is called Fano plane. A Fano Plane, as depicted in Figure 2.1, has one block represented by a circle and the others by lines.

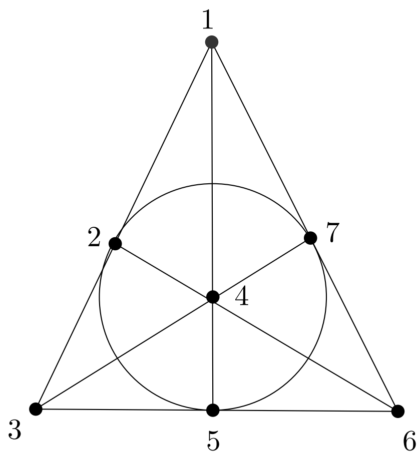


Figure 2.1: Fano Plane

2.1.1 Remark. We will write M_7 for the matroid arising from the Fano plane.

2.1.2 Lemma. Let $\mathbb{F}$ be a field and v be an injection from the ground set S of the matroid M_7 to the 3-dimensional vector space $\mathbb{F}^3$ over the field $\mathbb{F}$. Suppose that v preserves triples and the assignment $v(S)$ of the set S spans the 3-dimensional vector space $\mathbb{F}^3$. Then v is a representation of the matroid M_7 to the vector space $\mathbb{F}^3$.

Proof. To prove Lemma 2.1.2 we have to show that the assignment v of all bases of the matroid M_7 are bases of the vector space $\mathbb{F}^3$. Suppose that there is a basis $u_1\ u_2\ u_3$ of M_7 such that $v(u_1)\ v(u_2)\ v(u_3)$ is linearly dependent. As $u_1\ u_2\ u_3$ is not a triple then there are three other points u_4, u_5, u_6 in S such that $u_1\ u_2\ u_4$, $u_1\ u_3\ u_5$, $u_2\ u_3\ u_6$ are triples by definition of a Steiner system. Then these six vectors $v(u_1), v(u_2), v(u_3), v(u_4), v(u_5), v(u_6)$ are different because v is injective. The six vectors span a 2-dimensional subspace of $\mathbb{F}^3$. The assignment $v(u_7)$ of the one remaining element u_7 of S is also in that 2-dimensional vector space by the definition of a Steiner system. Then $v(S)$ spans a 2-dimensional subspace of $\mathbb{F}^3$ which is a contradiction of the fact that $v(S)$ spans $\mathbb{F}^3$. Thus v is a representation of M_7 to the vector space $\mathbb{F}^3$. □

The following result is already known (see Oxley (2011), Proposition 6.4.8).

2.1.3 Proposition. Let $\mathbb{F}$ be a field. The matroid M_7 is representable over $\mathbb{F}$ if and only if the characteristic of $\mathbb{F}$ is 2.

Proof. By Proposition 1.3.3 M_7 is a 3-rank matroid. Suppose that M_7 is representable over the field $\mathbb{F}$. Let v be a representation of M_7 to the vector space $\mathbb{F}^3$. From Figure 2.1, $1\ 2\ 4$ is a basis of the matroid then by Corollary 1.4.9 we can assume that $v(1) = (1\ 0\ 0)$, $v(2) = (0\ 1\ 0)$ and $v(4) = (0\ 0\ 1)$.

Let $v(3) = (a\ b\ c)$, $v(5) = (a_1\ b_1\ c_1)$ and $v(6) = (a_2\ b_2\ c_2)$ for some $a, b, c, a_1, b_1, c_1, a_2, b_2$ and c_2 in $\mathbb{F}$. Since $1\ 2\ 3$, $1\ 4\ 5$ and $2\ 4\ 6$ are circuits of M then $v(3)$ is a linear combination of $v(1)$ and $v(2)$, $v(5)$ is a linear combination of $v(1)$ and $v(4)$ and $v(6)$ a linear combination of $v(2)$ and $v(4)$.Then $c = b_1 = a_2 = 0$. We know also that $2\ 4\ 3$, $1\ 4\ 3$, $1\ 2\ 5$, $1\ 2\ 6$, $2\ 4\ 5$ and $1\ 4\ 6$ are bases of M so $v(2)\ v(4)\ v(3)$, $v(1)\ v(4)\ v(3)$, $v(1)\ v(2)\ v(5)$, $v(1)\ v(2)\ v(6)$, $v(2)\ v(4)\ v(5)$ and $v(1)\ v(4)\ v(6)$ are bases of $\mathbb{F}^3$ implying that a, b, a_1, c_1, b_2 and c_2 are non-null. In addition, by Proposition 1.4.10 a multiplication of a scalar does not change anything about the matroid so we can assume that $v(3) = (1\ s\ 0)$, $v(5) = (1\ 0\ t)$ and $v(6) = (0\ 1\ u)$ for some s, t, u which are non-null in $\mathbb{F}$.

Now consider the diagonal matrix diag$(1,\ \frac{1}{s},\ 1)$, then by the example 1.4.7 we can assume that $v(1) = (1\ 0\ 0)$, $v(2) = (0\ \frac{1}{s}\ 0)$, $v(3) = (1\ 1\ 0)$, $v(4) = (0\ 0\ 1)$, $v(5) = (1\ 0\ t)$ and $v(6) = (0\ \frac{1}{s}\ u)$. Since a multiplication by a scalar does not change anything, we can assume that $v(1) = (1\ 0\ 0)$, $v(2) = (0\ 1\ 0)$, $v(3) = (1\ 1\ 0)$, $v(4) = (0\ 0\ 1)$, $v(5) = (1\ 0\ t)$ and $v(6) = (0\ 1\ su)$.

Now consider the diagonal matrix diag$(1,\ 1,\ \frac{1}{t})$, then we can assume that $v(1) = (1\ 0\ 0)$, $v(2) = (0\ 1\ 0)$, $v(3) = (1\ 1\ 0)$, $v(4) = (0\ 0\ \frac{1}{t})$, $v(5) = (1\ 0\ 1)$ and $v(6) = (0\ 1\ \frac{su}{t})$. Finally, we multiply the fourth vector by t to get $v(1) = (1\ 0\ 0)$, $v(2) = (0\ 1\ 0)$, $v(3) = (1\ 1\ 0)$, $v(4) = (0\ 0\ 1)$, $v(5) = (1\ 0\ 1)$ and $v(6) = (0\ 1\ \frac{su}{t})$.

Then without changing anything about the matroid we can assume that $v(1) = (1\ 0\ 0)$, $v(2) = (0\ 1\ 0)$, $v(3) = (1\ 1\ 0)$, $v(4) = (0\ 0\ 1)$, $v(5) = (1\ 0\ 1)$ and $v(6) = (0\ 1\ u)$ where u is non-null in $\mathbb{F}$.

Now write $v(7) = (a\ b\ c)$ for some a, b and c in $\mathbb{F}$. Since $1\ 2\ 7$, $1\ 4\ 7$ and $2\ 4\ 7$ are bases of M then none of a, b, c are null. By multiplying $v(7)$ by $\frac{1}{a}$ then we can assume that $v(7) = (1\ b\ c)$ for some b, c which are non-null in $\mathbb{F}$.

Now as $1\ 7\ 6$, $2\ 5\ 7$ and $3\ 4\ 7$ are circuits then $v(1)\ v(7)\ v(6)$, $v(2)\ v(5)\ v(7)$ and $v(3)\ v(4)\ v(7)$ are linearly dependent so:

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & b \\ 0 & u & c \end{vmatrix} = c - bu = 0\ , \quad \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & b \\ 1 & 0 & c \end{vmatrix} = c - 1 = 0\ , \quad \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & b \\ 0 & 1 & c \end{vmatrix} = -b + 1 = 0,$$

giving us $c = b = u = 1$.

Therefore, we have $v(1) = (1\ 0\ 0)$, $v(2) = (0\ 1\ 0)$, $v(3) = (1\ 1\ 0)$, $v(4) = (0\ 0\ 1)$, $v(5) = (1\ 0\ 1)$, $v(6) = (0\ 1\ 1)$ and $v(7) = (1\ 1\ 1)$.

Now using the circuit 3 5 6 then we get

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1 - 1 = -2 = 0.$$

Therefore $\mathbb{F}$ has characteristic 2.

Conversely, suppose that the characteristic of $\mathbb{F}$ is 2. Consider an assignment v of the elements of S to the vector space $\mathbb{F}^3$ define by $v(1) = (1\ 0\ 0)$, $v(2) = (0\ 1\ 0)$, $v(3) = (1\ 1\ 0)$, $v(4) = (0\ 0\ 1)$, $v(5) = (1\ 0\ 1)$, $v(6) = (0\ 1\ 1)$ and $v(7) = (1\ 1\ 1)$. We can see clearly that the assignment of all triples apart from $v(3)\ v(5)\ v(6)$ are linearly dependent but as we are in a field of characteristic 2 then $v(3)\ v(5)\ v(6)$ is also linearly dependent, v is an injection map, all 2-sets from $\{v(i)\}$ span a 2-dimensional subspace of $\mathbb{F}^3$ and as $v(1)\ v(2)\ v(4)$ is a basis of the vector space $\mathbb{F}^3$, then all hypothesis of Lemma 2.1.3 are satisfied. Then v is a representation of the matroid M_7 to the vector space $\mathbb{F}^3$.

Therefore the Fano plane is representable over $\mathbb{F}$.

□

2.2 The Steiner triple system $S(2,3,9)$

The Steiner triple system $S(2,3,9) = (S, D)$ is uniquely determined. This is the finite affine plane $AG(2,3)$ obtained from the projective plane $PG(2,3)$ by removing one line and all points in that line. As depicted in Figure 2.2, this Steiner triple system has four blocks represented by curves and the others by lines.

1 2 9	1 3 8	1 4 7
1 5 6	2 3 6	2 4 8
2 5 7	3 4 5	3 7 9
4 6 9	5 8 9	6 7 8

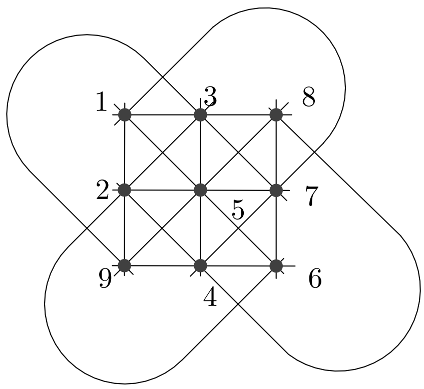


Figure 2.2: $S(2,3,9)$

2.2.1 Remark. We will write M_9 for the matroid arising from the Steiner triple system $S(2,3,9)$.

2.2.2 Lemma. Let $\mathbb{F}$ be a field and v be an injection map from the ground set S of the matroid M_9 to the 3-dimensional vector space $\mathbb{F}^3$ over the field $\mathbb{F}$. Suppose that v preserves triples and the assignment $v(S)$ of the set S spans the 3-dimensional vector space $\mathbb{F}^3$. Then v is a representation of the matroid M_9 to the vector space $\mathbb{F}^3$.

Proof. To prove Lemma 2.2.2 we have to show that the assignment v of all bases of the matroid M_9 are bases of the vector space $\mathbb{F}^3$. Suppose that there is a basis $u_1\ u_2\ u_3$ of M_9 such that its assignment $v(u_1)\ v(u_2)\ v(u_3)$ is a linearly dependent set of $\mathbb{F}^3$. As a 2-subset of the ground set S must be a subset of exactly one triple then there are three other points u_4, u_5, u_6 of S such that $v(u_1)\ v(u_2)\ v(u_4)$, $v(u_1)\ v(u_3)\ v(u_5)$, $v(u_2)\ v(u_3)\ v(u_6)$ are linearly dependent sets. Then the six $v(u_i)$ for $1 \leq i \leq 6$ are different because v is an injection, and span an r-dimensional space W, with $r \leq 2$. The three remaining vectors are also in W. Indeed, given any six of the nine points in our Steiner system, each of the other three points lies in a block with some two of the six. So $v(S)$ spans an r-dimensional subspace of $\mathbb{F}^3$ which is a contradiction of the fact that $v(S)$ spans $\mathbb{F}^3$.Therefore, v is a representation of the matroid M_9 to the vector space $\mathbb{F}^3$. □

For the following proposition, we can refer to Oxley (2011), p.653.

2.2.3 Proposition. The matroid M_9 is representable over a field $\mathbb{F}$ if and only if the polynomial x^2-x+1 has a root in $\mathbb{F}$.

Proof. Write M_9 the matroid arising from the Steiner triple $S(2,3,9)$.

Let $\mathbb{F}$ be a field such that the polynomial x^2-x+1 has a root s in $\mathbb{F}$. Then $s \neq 0$ and $s \neq 1$. Consider a map v from the ground set $\{1,\ldots,9\}$ to the 3-dimensional vector space $\mathbb{F}^3$ defined by $v(1) = (1\ 0\ 0)$, $v(2) = (0\ 1\ 0)$, $v(3) = (0\ 1\ s)$, $v(4) = (1-s\ \ 1\ \ 1)$, $v(5) = (1\ 0\ 1)$, $v(6) = (0\ 0\ 1)$, $v(7) = (1\ 1\ 1)$, $v(8) = (1\ 1\ s)$, $v(9) = (1\ s\ 0)$. Here we can observe clearly that v is an injection and the subspaces vector of $\mathbb{F}^3$ spanned by any 2-sets from $\{v(i)\}$ are 2-dimensional spaces.

Then we have an injection map v such that $v(S)$ span the vector space $\mathbb{F}^3$. To satisfy the hypothesis of Lemma 2.2.2 we have to show that each triple in the system is assigned a linearly dependent set. So $v(9) = v(1)+sv(2)$, $v(8) = v(1)+v(3)$, $v(4) = v(7)-sv(1)$, $v(6) = v(1)+v(5)$, $v(3) = v(2)+sv(6)$, $v(2) = \frac{1}{s}(v(4)+(s-1)v(8))$, $v(5) = v(2)+v(7)$, $v(5) = \frac{1}{1-s}(v(4)-v(3))$, $v(3) = s(v(5)-v(9))$, $v(6) = v(4)+(s-1)v(9)$, $v(6) = \frac{1}{1-s}(v(7)-v(8))$, $v(9) = s(v(8)-sv(5))$. Hence, all the hypothesis of Lemma 2.2.2 are satisfied. Thus, v is a representation of M_9 to the vector space $\mathbb{F}^3$. Therefore, M_9 is $\mathbb{F}$-representable.

Conversely, suppose that M_9 is representable over a field $\mathbb{F}$. Let v be a representation of M_9 to the vector space $\mathbb{F}^3$. As $1\ 2\ 3$ is a basis of the matroid then by Corollary 1.4.9 we can assume that $v(1) = (1\ 0\ 0)$, $v(2) = (0\ 1\ 0)$, $v(3) = (0\ 0\ 1)$. Now using the triples $1\ 2\ 9$, $1\ 3\ 8$, $2\ 3\ 6$ and using the same arguments that we did in the proof of Proposition 2.1.3, we can assume that $v(9) = (1\ 1\ 0)$, $v(8) = (1\ 0\ 1)$ and $v(6) = (0\ s\ 1)$ for some non-null s. Write $v(7) = (a\ b\ c)$ and $v(4) = (u\ v\ w)$. As $1\ 2\ i$, $1\ 3\ i$, $2\ 3\ i$ are bases for all remaining i then none of a, b, c, u, t, w are null. As multiplication by a scalar does not change anything about the matroid then we can assume that $v(7) = (1\ b\ c)$, $v(4) = (1\ t\ w)$. Now considering the triples $3\ 7\ 9$ and $6\ 7\ 8$, we get that $b = 1$ and $c = (s+1)s^{-1}$ respectively. So $v(7) = (1\ \ 1\ \ (s+1)s^{-1})$. Now using the triples $2\ 4\ 8$ and $4\ 6\ 9$, we see that $t = s+1$ and $w = 1$. So $v(4) = (1\ s+1\ \ 1)$. Now considering the triple $1\ 4\ 7$, we have $(s+1)^2s^{-1} = 1$, which means that $s^2+s+1 = 0$. Then the polynomial x^2+x+1 has a root in the field $\mathbb{F}$ then this polynomial is not irreducible over $\mathbb{F}$. So consider the ring isomorphism f over $\mathbb{F}[x]$ defined by $f(P(x)) = P(-x)$. We know that an isomorphism preserves irreducibility then $f(x^2+x+1) = x^2-x+1$ is not also irreducible. Therefore, x^2-x+1 has a root in $\mathbb{F}$. □

2.3 The Steiner triple systems $S(2,3,13)$

Up to isomorphism, there are two Steiner triple systems of order 13. They have twenty-two similar blocks. In the table below, all blocks that are neither underlined nor boxed belong to both systems. Those that are underlined belong to one system and those that are boxed belong to the other.

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	2 4 6	2 5 7	2 8 10	2 9 12	2 11 13
3 4 8	3 5 12	3 7 11	4 7 9	4 10 13	4 11 12	5 8 11	6 8 12	6 9 11	7 8 13	7 10 12
<u>3 6 10</u>	<u>3 9 13</u>	<u>5 6 13</u>	<u>5 9 10</u>				[3 6 13]	[3 9 10]	[5 6 10]	[5 9 13]

Table 2.1: Steiner triple systems of order 13.

2.3.1 Proposition. The matroids arising from the Steiner triple systems of order 13 are not representable.

Proof. To prove the Proposition 2.3.1 we just need some of the blocks which are neither underlined nor boxed.

Suppose that either of the two matroids is representable. Then there is a field $\mathbb{F}$ and a representation v of the matroid to the vector space $\mathbb{F}^3$.

Let us pick one basis of the matroid which is $1\ 2\ 4$. As usual, we can assume that $v(1) = (1\ 0\ 0)$, $v(2) = (0\ 1\ 0)$ and $v(4) = (0\ 0\ 1)$. As we did for the proof of Proposition 2.1.3 and by using the triples $1\ 2\ 3$, $1\ 4\ 5$, $2\ 4\ 6$ then without changing anything about the matroid we can assume that $v(3) = (1\ 1\ 0)$, $v(5) = (1\ 0\ 1)$ and $v(6) = (0\ s\ 1)$ for some non null s. As $1\ 2\ i$, $1\ 4\ i$ and $2\ 4\ i$ are bases for all remaining points i then we can assume that $v(i) = (1\ \ a_i\ \ b_i)$ with none of a_i, b_i null. Now pick the triple $3\ 4\ 8$, as v is a representation then $v(3)\ v(4)\ v(8)$ is a linearly dependent set then we have $v(8) = (1\ \ 1\ \ b_8)$. Now pick the triple $2\ 5\ 7$ then we get $b_7 = 1$ and from the triple $1\ 6\ 7$ we get $a_7 = s$ and $v(7) = (1\ \ s\ \ 1)$. Now the triple $4\ 7\ 9$ gives us $a_9 = s$ and $1\ 8\ 9$ gives that $b_9 = sb_8$ and $v(9) = (1\ \ s\ \ sb_8)$. Finally, we need the vector $v(12)$. Using the triple $2\ 9\ 12$ we get $b_{12} = sb_8$, and using $3\ 5\ 12$ we get $a_{12} = 1 - sb_8$. So, $v(12) = (1\ \ 1 - sb_8\ \ sb_8)$. Using the triple $6\ 8\ 12$, we calculate that $s^2 b_8 = 0$. This contradicts the fact that s and b_8 are non-null.

□

2.4 The Steiner triple systems $S(2,3,15)$

There are 80 Steiner triple systems of order 15 up to isomorphism listed in Mathon et al. (1982). Inspection shows that 23 of them contain the Fano plane 2.1. The others on the other hand contain the Fano plane 2.1 minus one triple or the Fano plane 2.1 minus two triples. In this section, we will consider the representability of all systems containing the Fano plane 2.1. Therefore, every field we will consider in this section has characteristic 2. We follow the numbering used by Mathon et al. (1982) for the systems, which is $\mathrm{NO}.i$ $(1 \leq i \leq 80)$. We denote the matroid arising from the system $\mathrm{NO}.i$ by $\mathrm{MO}.i$.

Firstly, let us consider the Steiner system $\mathrm{NO}.1$ which is the finite projective $PG(3,2) = (S, D)$ whose points are all 1-dimensional subspace of a 4-dimensional vector space over the field of order 2 and whose lines are the 2-dimensional subspace over the same space. So all blocks are the lines. This system contains 15 Fano planes. All triples are constructed from the triples of the Fano plane 2.1 and all triples are in some of those 15 Fano planes.

2.4.1 Lemma. Let $\mathbb{F}$ be a field and v be an injection map from the ground set S of the matroid MO.1 to the 3-dimensional vector space $\mathbb{F}^3$ over the field $\mathbb{F}$. Suppose that v preserves triples and $v(X)$ spans $\mathbb{F}^3$ for each Fano plane X in $MO.1$. Then v is a representation of the matroid MO.1 to the vector space $\mathbb{F}^3$.

Proof. To prove Lemma 2.4.1 we have to show that the assignment v of all bases of the matroid MO.1 are bases of the vector space $\mathbb{F}^3$. Suppose that there is a basis $u_1\ u_2\ u_3$ of the matroid such that its assignment $v(u_1)\ v(u_2)\ v(u_3)$ is a dependent set of $\mathbb{F}^3$. Then by definition of Steiner system there are three other points $u_4\ u_5\ u_6$ such that $v(u_1)\ v(u_2)\ v(u_4)$, $v(u_1)\ v(u_3)\ v(u_5)$, $v(u_2)\ v(u_3)\ v(u_6)$ are linearly dependent. Then we have six vectors, all distinct since v is an injection. These vectors span an r-dimensional subspace W, with $r \leq 2$. Now pick another point u_7 such that u_7 with some two points of u_i for $1 \leq i \leq 6$ is a triple. Then $v(u_7)$ is also in W. By hypothesis, $\{u_1, ...u_7\}$ cannot be a Fano plane, and there is another point u_8 not in $\{u_1, ..., u_6\}$ such that $u_i\ u_j\ u_8$ is a triple for some i, j in $\{1, ..., 6\}$. Then $v(u_8)$ is also in W. Indeed, given any eight of the fifteen points in our Steiner system, each of the other seven points lies in a block with some two of the eight because each point u_i is in seven triples. So the remaining $v(u_i)$ are also in W, which is a contradiction because v is an injection and $v(X)$ spans $\mathbb{F}^3$ for each Fano plane X. Therefore, v is a representation of MO.1 to the vector space $\mathbb{F}^3$. □

2.4.2 Proposition. The matroid MO.1 is representable.

Proof. See Theorem 3.2.2 and Theorem 3.2.3. □

Now, let us consider the Steiner $S(2,3,15)$ NO.2 and NO.4 since the triples containing 1,2,3,4,6 are the same in both of them and they contain also 5 10 15 as a triple.

2.4.3 Proposition. The matroids MO.2 and MO.4 are not representable.

Proof. In this proof, we need the triples in the Fano plane 2.1, the remaining triples containing $1,2,4,6$ and the triples 3 8 11, 5 10 15. They are

1 8 9	1 10 11	1 12 13
1 14 15	2 8 10	2 9 11
2 12 14	2 13 15	3 8 11
4 8 12	4 9 13	4 10 14
4 11 15	5 10 15	6 8 15
6 9 14	6 10 13	6 11 12

Suppose either of the two matroids MO.2 and MO.4 is representable over a field $\mathbb{F}$ of characteristic 2. Let v be a representation of this matroid to the 3-dimensional vector space $\mathbb{F}^3$. Let us pick the basis 1 2 4 of this matroid. By using the proof of Proposition 2.1.3, we can assume that

$$v(1) = (1\ 0\ 0), \quad v(2) = (0\ 1\ 0), \quad v(4) = (0\ 0\ 1),$$
$$v(5) = (1\ 0\ 1), \quad v(6) = (0\ 1\ 1), \quad v(7) = (1\ 1\ 1).$$

For all remaining points i, we can assume that $v(i) = (1\ a_i\ b_i)$ for some a_i, b_i non-null, because 1 2 i, 1 4 i and 2 4 i are bases of the matroid. Now using all triples which contain 2, we see $b_8 = b_{10}$, $b_9 = b_{11}$,

$b_{12} = b_{14}$ and $b_{13} = b_{15}$. Also, using all triples which contain 4, we see $a_8 = a_{12}$, $a_9 = a_{13}$, $a_{10} = a_{14}$ and $a_{11} = a_{15}$. Then we can assume that for some non-null a, b, c, u, e, g, p, r, we have

$$v(8) = (1\ a\ b)\,, \qquad v(9) = (1\ c\ u)\,, \qquad v(10) = (1\ e\ b)\,,$$
$$v(11) = (1\ g\ u)\,, \qquad v(12) = (1\ a\ p)\,, \qquad v(13) = (1\ c\ r)\,,$$
$$v(14) = (1\ e\ p)\,, \qquad v(15) = (1\ g\ r)\,.$$

Now using the triple 1 8 9, we see that $u = \frac{cb}{a}$. Now $v(11) = \left(1\ \ g\ \ \frac{cb}{a}\right)$ and using the triple 1 10 11, we see that $c = \frac{ga}{e}$. Now $v(13) = (1\ \frac{ag}{e}\ r)$ and using the triple 1 12 13, we see that $r = \frac{gp}{e}$. Now $v(11) = \left(1\ g\ \frac{bg}{e}\right)$ and using the triple 3 8 11, we see that $a = \frac{(e+1)g+e}{g}$. From the triple 5 10 15, we see that $e = -(b+1)g - gp = (b+1+p)g$. So $b+1+p \neq 0$ otherwise $e = 0$. Now $v(11) = \left(1\ \ g\ \ \frac{b}{b+p+1}\right)$ and $v(13) = \left(1\ \ \frac{bg+b+gp+g+p}{b+p+1}\ \ \frac{p}{b+p+1}\right)$. Then $v(6)\ v(11)\ v(13)$ is linearly dependent which contradicts the fact that 6 11 13 is not a block.

□

Next, we consider the representability of the Steiner $S(2,3,15)$ NO.3.

2.4.4 Proposition. The matroid MO.3 is not representable.

Proof. In this proof, we need the triples in the Fano plane 2.1, the remaining triples containing $1, 2, 4$ and also the triples 3 8 11, 5 8 14 and 5 10 12. They are

1 8 9	1 10 11	1 12 13
1 14 15	2 8 10	2 9 11
2 12 14	2 13 15	3 8 11
4 8 12	4 9 13	4 10 14
4 11 15	5 8 14	5 10 12

Suppose v is a representation of MO.3 to $\mathbb{F}^3$ for some field $\mathbb{F}$ of characteristic 2. For $i = 1, 2, 3, 4$ the triples containing i in the Steiner system NO.3 are the same as those containing i in NO.2.

We can therefore assume that for some non-null e, g, b, p

$$v(1) = (1\ 0\ 0)\,, \qquad v(8) = \left(1\ \ \frac{eg+e+g}{g}\ \ b\right), \qquad v(9) = \left(1\ \ \frac{eg+e+g}{e}\ \ \frac{bg}{e}\right),$$
$$v(2) = (0\ 1\ 0)\,, \qquad v(10) = (1\ e\ b)\,, \qquad v(11) = \left(1\ g\ \frac{bg}{e}\right),$$
$$v(3) = (1\ 1\ 0)\,, \qquad v(12) = \left(1\ \ \frac{eg+e+g}{g}\ \ p\right), \qquad v(13) = \left(1\ \ \frac{eg+e+g}{e}\ \ \frac{gp}{e}\right),$$
$$v(4) = (0\ 0\ 1)\,, \qquad v(14) = (1\ \ e\ \ p)\,, \qquad v(15) = \left(1\ \ g\ \ \frac{gp}{e}\right),$$
$$v(5) = (1\ 0\ 1)\,, \qquad v(6) = (0\ 1\ 1)\,, \qquad v(7) = (1\ 1\ 1)\,.$$

Using the triple 5 8 14, we see that $(bg + (g+1)p + 1)e = -(gp + g)$. If $bg + (g+1)p + 1 = 0$ then $g(p+1) = 0$ and $p = 1$. However, if $p = 1$ then $v(2)\ v(5)\ v(12)$ is linearly dependent, which contradicts the fact that 2 5 12 is not a block. Therefore, $e = \frac{gp+g}{bg+(g+1)p+1}$. Now using the triple 5 10 12, we see that $b = p$. Now $v(8) = v(12)$, which is a contradiction.

□

Now, we consider the Steiner $S(2,3,15)$ NO.5 and NO.6 since the triples containing 1,2 and 4 are the same in both of them.

2.4.5 Proposition. The matroids MO.5 and MO.6 are not representable.

Proof. In this proof, we need the triples in the Fano plane 2.1 and also the remaining triples containing $1, 2, 4$. These are

1 8 9	1 10 11	1 12 13
1 14 15	2 8 10	2 9 11
2 12 14	2 13 15	4 8 12
4 9 13	4 10 15	4 11 14

Suppose either of the two matroids MO.5 and MO.6 is representable over a field $\mathbb{F}$ of characteristic 2. Let v be a representation of this matroid to the vector space $\mathbb{F}^3$. Now using all triples in the Fano plane 2.1 and those containing 2 and 4. We can assume that for some non-null a, b, c, u, e, g, p, r, we have

$$
\begin{aligned}
&v(1) = (1\ 0\ 0)\,, && v(5) = (1\ 0\ 1)\,, && v(9) = (1\ c\ u)\,, && v(13) = (1\ c\ r)\,,\\
&v(2) = (0\ 1\ 0)\,, && v(6) = (0\ 1\ 1)\,, && v(10) = (1\ e\ b)\,, && v(14) = (1\ g\ p)\,,\\
&v(3) = (1\ 1\ 0)\,, && v(7) = (1\ 1\ 1)\,, && v(11) = (1\ g\ u)\,, && v(15) = (1\ e\ r)\,,\\
&v(4) = (0\ 0\ 1)\,, && v(8) = (1\ a\ b)\,, && v(12) = (1\ a\ p)\,.
\end{aligned}
$$

Now using the triple 1 8 9, we see that $ua = cb$. From the triple 1 10 11, we see that $ue = bg$. Now considering the triples 1 12 13 and 1 14 15, we get that $ar = cp$ and $ep = gr$ respectively. Then $u = \frac{cb}{a} = \frac{bg}{e}$, which means that $\frac{c}{a} = \frac{g}{e}$. We see also that $r = \frac{cp}{a} = \frac{ep}{g}$ then $\frac{c}{a} = \frac{e}{g}$. So, $\frac{e}{g} = \frac{g}{e}$. Since we are in a field of characteristic 2 we get $e = g$. However, if $e = g$ then $v(4)\ v(14)\ v(15)$ is linearly dependent, which contradicts the fact that 4 14 15 is not a triple. □

Let us consider the Steiner $S(2,3,15)$ NO.7.

2.4.6 Proposition. The matroid MO.7 is not representable.

Proof. In this proof, we need the triples in the Fano plane 2.1 and also the remaining triples containing $1, 2, 4$. This includes

1 8 9	1 10 11	1 12 13
1 14 15	2 8 10	2 9 11
2 12 14	2 13 15	4 8 12
4 9 14	4 10 15	4 11 13

Suppose v is a representation of MO.7 to $\mathbb{F}^3$ for some field $\mathbb{F}$ of characteristic 2. Using all triples in the Fano plane 2.1 and all triples which contain 2 and 4. We can assume that for some non-null

a, b, c, u, e, g, p, r:

$$\begin{array}{llll} v(1) = (1\ 0\ 0)\,, & v(5) = (1\ 0\ 1)\,, & v(9) = (1\ c\ u)\,, & v(13) = (1\ g\ r)\,, \\ v(2) = (0\ 1\ 0)\,, & v(6) = (0\ 1\ 1)\,, & v(10) = (1\ e\ b)\,, & v(14) = (1\ c\ p)\,, \\ v(3) = (1\ 1\ 0)\,, & v(7) = (1\ 1\ 1)\,, & v(11) = (1\ g\ u)\,, & v(15) = (1\ e\ r)\,, \\ v(4) = (0\ 0\ 1)\,, & v(8) = (1\ a\ b)\,, & v(12) = (1\ a\ p)\,. & \end{array}$$

Now considering the triples $1\ 8\ 9$, $1\ 10\ 11$, $1\ 12\ 13$ and $1\ 14\ 15$, we get that $ua = cb$, $ue = bg$, $ar = gp$ and $ue = bg$ respectively. Then we have $u = \frac{cb}{a} = \frac{bg}{e}$, which means that $\frac{c}{a} = \frac{g}{e}$. We also have $r = \frac{gp}{a} = \frac{ep}{c}$, then $\frac{g}{a} = \frac{e}{c}$. So $c = \frac{ga}{e} = \frac{ea}{g}$ then $\frac{g}{e} = \frac{e}{g}$ and we have $g = e$. Then $v(4)\ v(13)\ v(15)$ is linearly dependent, which contradicts the fact that $4\ 13\ 15$ is not a block. □

We will consider the Steiner $S(2,3,15)$ NO.8, NO.9 and NO.10 since the triples containing 1,2 and 4 are the same in three of them.

2.4.7 Proposition. The matroids MO.8, MO.9 and MO.10 are not representable.

Proof. We make use of the triples in the Fano plane 2.1 and also the remaining triples containing $1, 2, 4$. That is

1 8 9	1 10 11	1 12 13
1 14 15	2 8 10	2 9 11
2 12 14	2 13 15	4 8 13
4 9 10	4 11 14	4 12 15

Suppose either of the three matroids MO.8, MO.9 and MO.10 is representable over a field $\mathbb{F}$ of characteristic 2. Let v be a representation of this matroid to the vector space $\mathbb{F}^3$. Using all triples of the Fano plane 2.1 and also all triples which contain 2 and 4. We can assume that for some non-null a, b, c, u, g, s, p, r. We have

$$\begin{array}{llll} v(1) = (1\ 0\ 0)\,, & v(5) = (1\ 0\ 1)\,, & v(9) = (1\ c\ u)\,, & v(13) = (1\ a\ r)\,, \\ v(2) = (0\ 1\ 0)\,, & v(6) = (0\ 1\ 1)\,, & v(10) = (1\ c\ b)\,, & v(14) = (1\ g\ p)\,, \\ v(3) = (1\ 1\ 0)\,, & v(7) = (1\ 1\ 1)\,, & v(11) = (1\ g\ u)\,, & v(15) = (1\ s\ r)\,, \\ v(4) = (0\ 0\ 1)\,, & v(8) = (1\ a\ b)\,, & v(12) = (1\ s\ p)\,. & \end{array}$$

Now considering the triples $1\ 8\ 9$, $1\ 10\ 11$, $1\ 12\ 13$ and $1\ 14\ 15$, we get $ua = cb$, $uc = bg$, $ap = sr$ and $gr = sp$ respectively. Then we have $u = \frac{cb}{a} = \frac{bg}{c}$, which means that $\frac{c}{a} = \frac{g}{c}$. We have also $r = \frac{ap}{s} = \frac{sp}{g}$ then $\frac{a}{s} = \frac{s}{g}$ and then $c = s$. However, if $c = s$ then $v(4)\ v(10)\ v(12)$ is linearly dependent, which contradicts the fact that $4\ 10\ 12$ is not a triple. □

We will consider the Steiner $S(2,3,15)$ NO.11 and NO.12 since the triples containing 1,2,4 are the same in both of them.

2.4.8 Proposition. The matroids MO.11 and MO.12 are not representable.

Proof. To prove this proposition, we need the triples in the Fano plane 2.1 and also the following triples:

1 8 9	1 10 11	1 12 13
1 14 15	2 8 10	2 9 11
2 12 14	2 13 15	4 8 13
4 9 14	4 10 12	4 11 15

Suppose either of the two matroids MO.11 and MO.12 is representable over a field $\mathbb{F}$ of characteristic 2. Let v be a representation of this matroid to the vector space $\mathbb{F}^3$. Using all triples in the Fano plane 2.1 and those triples containing 2 and 4, we see that we can assume that there are some non-null a, b, c, u, e, g, p, r such that:

$$\begin{aligned} &v(1) = (1\ 0\ 0)\,, && v(5) = (1\ 0\ 1)\,, && v(9) = (1\ c\ u)\,, && v(13) = (1\ a\ r)\,,\\ &v(2) = (0\ 1\ 0)\,, && v(6) = (0\ 1\ 1)\,, && v(10) = (1\ e\ b)\,, && v(14) = (1\ c\ p)\,,\\ &v(3) = (1\ 1\ 0)\,, && v(7) = (1\ 1\ 1)\,, && v(11) = (1\ g\ u)\,, && v(15) = (1\ g\ r)\,.\\ &v(4) = (0\ 0\ 1)\,, && v(8) = (1\ a\ b)\,, && v(12) = (1\ e\ p)\,, && \end{aligned}$$

Now considering the triples 1 8 9, 1 10 11, 1 12 13 and 1 14 15, we get that $ua = cb$, $ue = bg$, $ap = er$ and $cr = gp$. So $u = \frac{cb}{a} = \frac{bg}{e}$ then $\frac{c}{a} = \frac{g}{e}$. We have also $r = \frac{ap}{e} = \frac{gp}{c}$ then $\frac{a}{e} = \frac{g}{c}$. Then $\frac{e}{a} = \frac{g}{c} = \frac{a}{e}$, which means that $e = a$. However, if $e = a$ then $v(8) = v(10)$ which is a contradiction because v is a representation. □

We consider the two Steiner $S(2,3,15)$ NO.13 and NO.14 since the triples containing 1,2,3 and 4 are the same in both systems.

2.4.9 Proposition. The matroids MO.13 and MO.14 are not representable.

Proof. In this proof, we need the triples in the Fano plane 2.1 and also the remaining triples containing $1, 2, 3, 4$. This includes

1 8 9	1 10 11	1 12 13
1 14 15	2 8 10	2 9 11
2 12 14	2 13 15	3 8 11
3 9 12	3 10 15	3 13 14
4 8 13	4 9 15	4 10 12
	4 11 14	

Suppose either of the two matroids MO.13 and MO.14 is representable over a field $\mathbb{F}$ of characteristic 2. Let v be a representation of this matroid to the vector space $\mathbb{F}^3$.

Using all triples in the Fano plane 2.1 and those which contain $2, 4$, we can assume that for some non-null a, b, c, u, e, g, p, r:

$$\begin{aligned} &v(1) = (1\ 0\ 0)\,, && v(4) = (0\ 0\ 1)\,, && v(7) = (1\ 1\ 1)\,, && v(10) = (1\ e\ b)\,, && v(13) = (1\ a\ r)\,,\\ &v(2) = (0\ 1\ 0)\,, && v(5) = (1\ 0\ 1)\,, && v(8) = (1\ a\ b)\,, && v(11) = (1\ g\ u)\,, && v(14) = (1\ g\ p)\,,\\ &v(3) = (1\ 1\ 0)\,, && v(6) = (0\ 1\ 1)\,, && v(9) = (1\ c\ u)\,, && v(12) = (1\ e\ p)\,, && v(15) = (1\ c\ r)\,. \end{aligned}$$

Now considering the triples 1 8 9, 1 10 11, 1 12 13 and 1 14 15, we get that $ua = cb$, $ue = bg$, $ap = er$ and $cp = gr$. Then $p = \frac{er}{a}$, $u = \frac{bg}{e}$ and $c = \frac{ua}{b} = \frac{bga}{be} = \frac{ag}{e}$. Considering the triple 3 13 14, where $v(14) = (1\ g\ \frac{er}{a})$, we have $g = \dfrac{a+e+ae}{a}$. So $v(3)\ v(12)\ v(15)$ is linearly dependent, where $v(12) = (1\ e\ \frac{er}{a})$ and $v(15) = (1\ \ \dfrac{a+e+ae}{e}\ \ r)$, which contradicts the fact that 3 12 15 is not a triple. □

Let us now focus on the Steiner $S(2,3,15)$ NO.15.

2.4.10 Proposition. The matroid MO.15 is not representable.

Proof. Let us consider the triples in the Fano plane 2.1 and those triples containing $1, 2, 4$, namely:

1 8 9	1 10 11	1 12 13
1 14 15	2 8 10	2 9 11
2 12 14	2 13 15	4 8 15
4 9 10	4 11 12	4 13 14

Suppose that the matroid MO.15 is representable over a field of characteristic 2. Using all triples in the Fano plane 2.1 and those containing 2 and 4, we can assume that for some non-null a, b, c, g, q, p, r, u:

$$\begin{array}{lllll} v(1) = (1\ 0,\ 0)\,, & v(4) = (0\ 0\ 1)\,, & v(7) = (1\ 1\ 1)\,, & v(10) = (1\ c\ b)\,, & v(13) = (1\ q\ r)\,, \\ v(2) = (0\ 1\ 0)\,, & v(5) = (1\ 0\ 1) & v(8) = (1\ a\ b)\,, & v(11) = (1\ g\ u)\,, & v(14) = (1\ q\ p)\,, \\ v(3) = (1\ 1\ 0) & v(6) = (0\ 1\ 1)\,, & v(9) = (1\ c\ u)\,, & v(12) = (1\ g\ p)\,, & v(15) = (1\ a\ r)\,. \end{array}$$

Now using the triple 1 8 9, we see $ua = cb$. From the triple 1 10 11, we see that $uc = bg$. Now using the triple 1 12 13, we see that $gr = pq$ and from the triple 1 14 15, we see that $ap = qr$. Then we have $u = \frac{cb}{a} = \frac{bg}{c}$, which means $\frac{c}{a} = \frac{g}{c}$. We have also $r = \frac{pq}{g} = \frac{ap}{q}$ then $\frac{q}{g} = \frac{a}{q}$. Then $c^2 = ag = q^2$, which means that $c = q$. Then $v(4)\ v(9)\ v(13)$ is linearly dependent, which contradicts the fact that 4 9 13 is not a block. □

Let us consider the Steiner $S(2,3,15)$ NO.16 and NO.17 since for $i = 1, 2, 4$ the triples containing i in the Steiner system NO.16 are the same as those containing i in NO.17 and also both of them contain 3 10 14, 3 11 15 and 6 8 14 as triples.

2.4.11 Proposition. The matroids MO.16 and MO.17 are not representable.

Proof. In this proof, we need the triples in the Fano plane 2.1, the remaining triples containing $1, 2, 4$, the triples 3 10 14, 3 11 15 and 6 8 1.

1 8 9	1 10 11	1 12 13
1 14 15	2 8 10	2 9 11
2 12 14	2 13 15	3 10 14
3 11 15	4 8 15	4 9 14
4 10 13	4 11 12	6 8 14

Suppose either of the two matroids MO.16 and MO.17 is representable over a field $\mathbb{F}$ of characteristic 2. Let v be a representation of this matroid to the vector space $\mathbb{F}^3$. Using the triples in the Fano plane 2.1 and which contain 2 and 4. We can assume that for some non-null a, b, c, u, e, g, h, f:

$$\begin{array}{lllll} v(1)=(1\ 0\ 0)\,, & v(4)=(0\ 0\ 1)\,, & v(7)=(1\ 1\ 1)\,, & v(10)=(1\ c\ b)\,, & v(13)=(1\ c\ u)\,, \\ v(2)=(0\ 1\ 0)\,, & v(5)=(1\ 0\ 1)\,, & v(8)=(1\ a\ b)\,, & v(11)=(1\ f\ g)\,, & v(14)=(1\ h\ e)\,, \\ v(3)=(1\ 1\ 0)\,, & v(6)=(0\ 1\ 1)\,, & v(9)=(1\ h\ g)\,, & v(12)=(1\ f\ e)\,, & v(15)=(1\ a\ u)\,. \end{array}$$

Consider all the triples 1 8 9, 1 10 11, 1 12 13 and 1 14 15, we see that $g=\frac{bh}{a}$, $f=\frac{ch}{a}$ and $u=\frac{ae}{h}$. Now using the triple 3 10 14, we see $c=\dfrac{bh+b+e}{e}$. Now pick the triple 6 8 14, we see $a=b+e+h$ and finally the triple 3 11 15 where $v(11)=\begin{pmatrix}1 & \frac{bh^2+bh+eh}{be+e^2+eh} & \frac{bh}{b+e+h}\end{pmatrix}$ and $v(15)=\begin{pmatrix}1 & b+e+h & \frac{be+e^2+eh}{h}\end{pmatrix}$, we see $b^2e+b^2h+e^3+e^2h=0$, which means $b^2(e+h)=e^2(e+h)$. If $e=h$ then $v(1)$ $v(6)$ $v(14)$ is linearly dependent, which contradicts the fact that 1 6 14 is not a block. So $b=e$, then we have $a=h$ and then $v(8)=v(14)$ which is also a contradiction. Then v does not exist. Hence, the matroids MO.16 and MO.17 are not representable. □

We will consider the Steiner $S(2,3,15)$ NO.18 and NO.20 since the triples containing 1,2 and 4 are the same in both Steiner systems.

2.4.12 Proposition. The matroids MO.18 and MO.20 are not representable.

Proof. In this proof, we need the triples in the Fano plane 2.1 and the remaining triples containing $1,2,4$. These are

1 8 9	1 10 11	1 12 13
1 14 15	2 8 10	2 9 11
2 12 14	2 13 15	4 8 15
4 9 10	4 11 12	4 13 14

Suppose either of the two matroids MO.18 and MO.20 is representable over a field $\mathbb{F}$ of characteristic 2. Let v be a representation of this matroid to the vector space $\mathbb{F}^3$. Using all triples in the Fano plane 2.1 and those triples containing 2 and 4, we see that we can assume that there are some non-null a,b,c,h,e,f,g,u such that

$$\begin{array}{lllll} v(1)=(1\ 0\ 0)\,, & v(4)=(0\ 0\ 1)\,, & v(7)=(1\ 1\ 1)\,, & v(10)=(1\ c\ b)\,, & v(13)=(1\ f\ u)\,, \\ v(2)=(0\ 1\ 0)\,, & v(5)=(1\ 0\ 1)\,, & v(8)=(1\ a\ b)\,, & v(11)=(1\ h\ e)\,, & v(14)=(1\ f\ g)\,, \\ v(3)=(1\ 1\ 0)\,, & v(6)=(0\ 1\ 1)\,, & v(9)=(1\ c\ e)\,, & v(12)=(1\ h\ g)\,, & v(15)=(1\ a\ u)\,. \end{array}$$

Now consider the triples 1 8 9, 1 10 11,1 12 13 and 1 14 15, we see that $ae=cb$, $ce=hb$, $hu=fg$, $fu=ag$. So $e=\frac{cb}{a}=\frac{hb}{c}$ and $\frac{c}{a}=\frac{h}{c}$. Also $u=\frac{fg}{h}=\frac{ag}{f}$ and $\frac{f}{h}=\frac{a}{f}$. Then $f^2=c^2$ which means $f=c$. So $v(4)$ $v(9)$ $v(14)$ is linearly dependent, which contradicts the fact that 4 9 14 is not a block. Therefore, the matroids MO.18 and MO.20 are not representable. □

We will consider the Steiner $S(2,3,15)$ NO.19.

2.4.13 Proposition. The matroid MO.19 is not representable.

Proof. Let us consider the following triples and those the Fano plane 2.1:

1 8 9	1 10 11	1 12 13
1 14 15	2 8 10	2 9 11
2 12 14	2 13 15	4 8 15
4 9 12	4 10 14	4 11 13

Suppose that the matroid MO.19 is representable over a field $\mathbb{F}$ of characteristic 2. Let v be a representation of the matroid to the vector space $\mathbb{F}^3$. Using all triples in the Fano plane 2.1 and those triples containing 2 and 4, we see that we can assume that there are some non-null a, b, c, u, e, g, p, r such that

$$\begin{array}{lllll} v(1) = (1\ 0\ 0)\,, & v(4) = (0\ 0\ 1)\,, & v(7) = (1\ 1\ 1)\,, & v(10) = (1\ e\ b)\,, & v(13) = (1\ g\ r)\,, \\ v(2) = (0\ 1\ 0)\,, & v(5) = (1\ 0\ 1)\,, & v(8) = (1\ a\ b)\,, & v(11) = (1\ g\ u)\,, & v(14) = (1\ e\ p)\,, \\ v(3) = (1\ 1\ 0)\,, & v(6) = (0\ 1\ 1)\,, & v(9) = (1\ c\ u) & v(12), = (1\ c\ p)\,, & v(15) = (1\ a\ r)\,. \end{array}$$

Now using the triples 1 8 9 and 1 10 11, we see that $u = \frac{cb}{a} = \frac{bg}{e}$ then $\frac{c}{a} = \frac{g}{e}$. From the triples 1 12 13 and 1 14 15, we see $r = \frac{gp}{c} = \frac{ap}{e}$ then $\frac{g}{c} = \frac{a}{e}$. So $\frac{a}{e} = \frac{e}{a}$ then $a = e$ and then $v(8) = v(10)$ which is a contradiction. Hence the matroid MO.19 is not representable.

□

Now, we will consider the Steiner $S(2,3,15)$ NO.21 and NO.22 since the triples containing 1,2 and 4 are the same in both of them.

2.4.14 Proposition. The matroids MO.21 and MO.22 are not representable.

Proof. In this proof, we need the triples in the Fano plane 2.1 and also these triples:

1 8 9	1 10 11	1 12 13
1 14 15	2 8 10	2 9 12
2 11 14	2 13 15	4 8 12
4 9 14	4 10 13	4 11 15

Suppose either of the two matroids MO.21 and MO.22 is representable over a field $\mathbb{F}$ of characteristic 2. Let v be a representation of this matroid to the vector space $\mathbb{F}^3$. Using all triples in the Fano plane 2.1 and those triples containing 2 and 4, we see that we can assume that there are some non-null a, b, c, e, f, g, h, u such that

$$\begin{array}{lllll} v(1) = (1\ 0\ 0)\,, & v(4) = (0\ 0\ 1)\,, & v(7) = (1\ 1\ 1)\,, & v(10) = (1\ c\ b)\,, & v(13) = (1\ c\ e)\,, \\ v(2) = (0\ 1\ 0)\,, & v(5) = (1\ 0\ 1)\,, & v(8) = (1\ a\ b)\,, & v(11) = (1\ h\ g)\,, & v(14) = (1\ f\ g)\,, \\ v(3) = (1\ 1\ 0)\,, & v(6) = (0\ 1\ 1)\,, & v(9) = (1\ f\ u)\,, & v(12) = (1\ a\ u)\,, & v(15) = (1\ h\ e)\,. \end{array}$$

Considering the triples 1 8 9, 1 10 11, 1 12 13 and 1 14 15, we see that $au = fb$, $cg = hb$, $ae = cu$, $fe = hg$. Then $b = \dfrac{au}{f}$ and $g = \dfrac{bh}{c} = \dfrac{auh}{fc}$. We also have that $e = \dfrac{cu}{a} = \dfrac{hg}{f} = \dfrac{h^2au}{f^2c}$. Then $(cf)^2 = (ha)^2$ and then $cf = ah$. So $c = \dfrac{ah}{f}$, $b = \dfrac{au}{f}$, $g = \dfrac{auh}{fc}$. Then we have $v(8) = \left(1\ a\ \dfrac{ua}{f}\right)$ and $v(14) = (1\ f\ u)$. Then $v(1)\ v(8)\ v(14)$ is linearly dependent, which contradicts the fact that

$1\ 8\ 14$ is not a triple. Then, v does not exist. Therefore, the matroids $\mathrm{MO.21}$ and $\mathrm{MO.22}$ are not representable. $\square$

Finally, let us consider the Steiner $S(2,3,15)$ $\mathrm{NO.61}$.

2.4.15 Proposition. The matroid $\mathrm{MO.61}$ is not representable.

Proof. In this proof, we need the triples in the Fano plane 2.1 and also these triples:

1 8 9	1 10 11	1 12 13
1 14 15	2 8 10	2 9 12
2 11 14	2 13 15	4 8 12
4 9 11	4 10 15	4 13 14

Suppose that the matroid is representable over a field $\mathbb{F}$ of characteristic 2. Let v be a representation of $\mathbb{F}$ to the vector space $\mathbb{F}^3$. Using all triples in the Fano plane 2.1 and those triples containing 2 and 4, we see that we can assume that there are some non-null a, b, f, u, c, b, g, e such that

$$\begin{array}{lllll} v(1) = (1\ 0\ 0)\,, & v(4) = (0\ 0\ 1)\,, & v(7) = (1\ 1\ 1)\,, & v(10) = (1\ c\ b)\,, & v(13) = (1\ h\ e)\,, \\ v(2) = (0\ 1\ 0)\,, & v(5) = (1\ 0\ 1)\,, & v(8) = (1\ a\ b)\,, & v(11) = (1\ f\ g)\,, & v(14) = (1\ h\ g)\,, \\ v(3) = (1\ 1\ 0)\,, & v(6) = (0\ 1\ 1)\,, & v(9) = (1\ f\ u)\,, & v(12) = (1\ a\ u)\,, & v(15) = (1\ c\ e)\,. \end{array}$$

Now using the triples $1\ 8\ 9$, $1\ 10\ 11$, $1\ 12\ 13$ and $1\ 14\ 15$, we see that $au = bf,\ cg = bf,\ hu = ae,\ eh = cg$. Then we have $e = \dfrac{cg}{h} = \dfrac{hu}{a}$ and $cg = bf = ua$. So $e = \dfrac{cg}{h} = \dfrac{au}{h} = \dfrac{hu}{a}$ and $\dfrac{a}{h} = \dfrac{h}{a}$, which means that $a = h$. So $v(8) = (1\ h\ b)$ and $v(13) = (1\ h\ e)$. It follows that $v(4)\ v(8)\ v(13)$ is linearly dependent, which contradicts the fact that $4\ 8\ 13$ is not a block. $\square$

3. Some Generalisations

In this Chapter, we will obtain stronger results which help us to know about the representability of infinitely many paving matroids.

3.1 Construction of some Steiner triple systems

Our aim in this section is to generalise some construction of Steiner triple systems which contain a Fano plane and see also one other construction of some Steiner triple systems.

3.1.1 A construction of some Steiner triple system $S(2,3,2^n-1)$. Let n be an integer such that $n \geq 3$. One construction of a Steiner triple system $S(2,3,2^n-1)$ is using all triples in a system $S(2,3,2^{n-1}-1)$. Let $S(2,3,2^{n-1}-1)$ be a Steiner triple system of order $2^{n-1}-1$ which we call the "small system". We label all 2^n-1 points of the system $S(2,3,2^n-1)$ which we aim to construct by $a_1,\dots,a_{2^{n-1}-1},u,f_1,\dots,f_{2^{n-1}-1}$ where the first $2^{n-1}-1$ is the label of the points in the small system. We assume that all triples in the small system are also triples in the system $S(2,3,2^n-1)$. The triples in our system $S(2,3,2^n-1)$ include all $a_i\ a_j\ a_k$ which are triples in the small system, all $a_i\ f_j\ f_k$ such that $a_i\ a_j\ a_k$ is a triple in the small system, and all $a_i\ u\ f_i$.

3.1.2 Proposition. Every Steiner system whose points are the non-zero vectors in some finite dimensional vector space over $\mathbb{F}_2$ and whose triples consists of points lying in a given 2-space can be constructed this way

Proof. Let us prove this by induction. Say we are given such a system coming from $\mathbb{F}_2^{n-1}$, having labelled its points $a_1,\dots,a_{2^{n-1}-1}$. Each such point x is a vector with $n-1$ entries. Give the same label a_i to the vector w obtained from x by appending an n-th entry 0. Now give the label u to the vector $(0\ 0\ \dots\ 0\ 1)$ in $\mathbb{F}_2^n$, and give the label f_i to the vector $w+u$ for each w as above. □

3.1.3 Bose's Construction for a Steiner triple $S(2,3,6k+3)$. The ground set of one Steiner triple system $S(2,3,6k+3)$ is the group $\mathbb{Z}/(2k+1)\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$ of order $6k+3$. So all triples are:

$$\{(u,0),\ (u,1),\ (u,2)\} \text{ for all } u\in\mathbb{Z}/(2k+1)\mathbb{Z}$$

and also

$$\{(u,i),\ (v,i),\ (\frac{u+v}{2},i+1)\} \text{ for } u,v\in\mathbb{Z}/(2k+1)\mathbb{Z}\text{ , with } u\neq v,\ i\in\mathbb{Z}/3\mathbb{Z}$$

.

3.2 Realisability of some $S(2,3,2^n-1)$

In this section, our aim is to see the generalisation of representability of some Steiner triple systems.

3.2.1 Remark. Let n be an integer such that $n\geq 3$. We will write M_{2^n-1} the matroid arising from a Steiner triple system $S(2,3,2^n-1)$ constructed by the way in 3.1.1 such that its subsystems $S(2,3,2^k-1)$ where $3\leq k<n$ is constructed by the same way.

3.2.2 Theorem. *Let n be an integer such that $n \geq 4$. Write $E = \{a_1, \ldots, a_{2^{n-1}-1}\}$ for the ground set of the matroid $M_{2^{n-1}-1}$. Suppose that $M_{2^{n-1}-1}$ is representable over a finite field $\mathbb{F}$ of characteristic 2 and let v be its representation to the vector space $\mathbb{F}^3$. Suppose that there exists a non-zero vector p in $\mathbb{F}^3$ such that $p\ v(a_i)\ v(a_j)$ is a basis of $\mathbb{F}^3$ for all i, j. Let g be a map from the ground set $E' = \{a_1, \ldots, a_{2^{n-1}-1}, u, f_1, \ldots, f_{2^{n-1}-1}\}$ of the matroid M_{2^n-1} to the vector space $\mathbb{F}^3$ such that $g(a_i) = v(a_i)$, $g(u) = p$, $g(f_i) = g(u) + t_i v(a_i)$ for all i, for some non-null t_i and g preserves triples. Then the matroid M_{2^n-1} is representable.*

Proof. Let α be a non-null scalar. Consider a map g_α from the ground set E' to the vector space $\mathbb{F}^3$ defined by $g_\alpha(a_i) = g(a_i)$, $g_\alpha(u) = g(u)$ and $g_\alpha(f_i) = \alpha g(u) + t_i v(a_i)$. We aim to show that g_α preserves also triples. Obviously, g_α preserves all triples in the small system $M_{2^{n-1}-1}$. So let us prove that all remaining triples are also linearly dependent. All remaining triples are of the form $u\ a_i\ f_i$ and $f_i\ f_j\ a_k$. By construction $g_\alpha(u)\ g_\alpha(a_i)\ g_\alpha(f_i)$ are linearly dependent. Let then a_i, f_j, f_k be three points such that $a_i\ f_j\ f_k$ is a triple. Then there are a_j, a_k in the small system such that $a_i\ a_j\ a_k$, $u\ a_j\ f_j$, $u\ a_k\ f_k$ are triples. As g preserves triples and there are two non-null scalars β, γ such that $g(a_i) = \gamma g(f_j) + \beta g(f_k)$, which means that $v(a_i) = (\gamma + \beta) g(u) + \gamma t_j v(a_j) + \beta t_k v(a_k)$ then $(\gamma + \beta) g(u) = v(a_i) + \gamma t_j v(a_j) + \beta t_k v(a_k)$ (we are in a field of characteristic 2). Then $(\alpha + \beta) g(u)$ is in the 2-dimensional subspace spanned by $v(a_i)\ v(a_j)\ v(a_k)$ and $\gamma = \beta$ because $g(u)\ v(a_i)\ v(a_j)$ is a basis of the vector space $\mathbb{F}^3$. Now $g(a_i) = \gamma(g(f_j) + g(f_k))$. So we have also $g_\alpha(a_i) = \gamma(g_\alpha(f_j) + g_\alpha(f_k))$ and $g_\alpha(a_i)\ g_\alpha(f_j)\ g_\alpha(f_k)$ is linearly dependent. Then g_α preserves triples for all non-zero scalar α. Let α be a non-null scalar. Suppose that for all g_α there exists a_i, a_j, f_k such that $g_\alpha(a_i)\ g_\alpha(a_j)\ g_\alpha(f_k)$ is linearly dependent. Then there exist two non-zero scalars α_i, α_j such that $g_\alpha(f_k) = \alpha_i g_\alpha(a_i) + \alpha_j g_\alpha(a_j)$ otherwise, $g_\alpha(u)$ is in the vector space spanned by a 2-dimensional vectors in the small system. Then for every non-null scalar α there exists a basis of the small system $v(a_k)\ v(a_i)\ v(a_j)$ such that $\alpha g(u) = t_k v(a_k) + \alpha_i v(a_i) + \alpha_j v(a_j)$. Suppose that the field $\mathbb{F}$ has size more than the number of bases of the small system (if not we can choose another field which has that size), then there must be a t_k, two different scalars α, β and also four other scalars $\alpha_i, \alpha_i, \alpha_i^1, \alpha_i^1$ such that $\alpha g(u) = t_k v(a_k) + \alpha_i v(a_i) + \alpha_j v(a_j)$ and $\beta g(u) = t_k v(a_k) + \alpha_i^1 v(a_i) + \alpha_j^1 v(a_j)$ then $(\alpha + \beta) g(u) = (\alpha_i + \alpha_i^1) v(a_i) + (\alpha_j + \alpha_j^1) v(a_j)$ then $\alpha = \beta$. This is a contradiction. So there is a non-null scalar α such that g_α preserves triples and also the assignment g_α of all bases of the form $a_i\ a_j\ f_k$ are bases of the vector space $\mathbb{F}^3$. Let α be such a scalar. Our aim is now to prove that $g_\alpha(B)$ is linearly independent for every basis B of our matroid. The remaining bases of the system are of the form $f_i\ f_j\ f_k$ for all i, j, k and also of the form $a_i\ f_j\ f_k$ which are not triples. Suppose that there is a basis of the form $a_i\ f_j\ f_k$ in the system such that $g_\alpha(a_i)\ g_\alpha(f_j)\ g_\alpha(f_k)$ is linearly dependent. As there must be a_t with $a_t \neq a_i$ such that $a_t\ f_j\ f_k$ is a triple then the 2-dimensional subspaces spanned by $g_\alpha(a_i)\ g_\alpha(a_t)$, $g_\alpha(f_j)\ g_\alpha(f_k)$ are equal. Then $g_\alpha(a_i)\ g_\alpha(a_t)\ g_\alpha(f_j)$ is linearly dependent which is a contradiction. Then the assignment of the bases of the form $a_i\ f_j\ f_k$ are bases. Now suppose that there is a basis of the form $f_i\ f_j\ f_k$ such that $g_\alpha(f_i)\ g_\alpha(f_j)\ g_\alpha(f_k)$ is linearly dependent. As there must be a point a_t such that $a_t\ f_i\ f_j$ is a triple then $g_\alpha(a_t)\ g_\alpha(f_i)\ g_\alpha(f_j)$ is linearly dependent and also $g_\alpha(a_t)\ g_\alpha(f_i)\ g_\alpha(f_k)$ is linearly dependent which is a contradiction because $g_\alpha(a_t)\ g_\alpha(f_i)\ g_\alpha(f_k)$ is basis. Then the assignments g_α of all bases of the system are bases. Hence, g_α is a representation of the matroid M_{2^n-1} to the vector space $\mathbb{F}^3$. Then the matroid M_{2^n-1} is representable. □

3.2.3 Theorem. *Let n be an integer such that $n \geq 3$. There exists a finite field $\mathbb{F}$ of characteristic 2 and a map v from the ground set of M_{2^n-1} to the 3- dimensional vector space $\mathbb{F}^3$ such that v preserves independence and for all triples $a_i\ a_j\ a_t$ of the matroid M_{2^n-1} we have $v(a_i) + v(a_j) + v(a_t) = 0$.*

Proof. Let us prove this by induction. For $n = 3$, M_{2^n-1} is a Fano plane. In our proof of the Proposition 2.1.3 we can see clearly that for all triples $a_i\ a_j\ a_k$ of the Fano plane the representation v

satisfies $v(a_i) + v(a_j) + v(a_k) = 0$.

Suppose that for all $3 \geq p \leq n-1$ we have the desired result and let us prove it for $p = n$. As above, we write $a_1, \dots, a_{2^{n-1}-1}, u, f_1, \dots, f_{2^{n-1}-1}$ for the elements of M_{2^n-1} where the first $2^{n-1}-1$ points a_i are the elements of $M_{2^{n-1}-1}$. Consider a map g defined by $g(a_i) = v(a_i)$ for all a_i and $g(f_i) = g(u) + g(a_i)$ where $g(u)$ is in $\mathbb{E}^3$ and $\mathbb{E}$ is some extension field of $\mathbb{F}$. The remaining triples of M_{2^n-1} are either of the form $a_i\ f_j\ f_k$ with $a_i\ a_j\ a_k$ a triple in our $S(2,3,2^{n-1}-1)$ or of either $a_i\ u\ f_i$. Obviously, $g(a_i) + g(u) + g(f_i) = 0$. So, it remains to consider $g(a_i)\ g(a_j)\ g(a_k)$ with $a_i\ a_j\ a_k$ a triple. We aim to show that $g(f_i) + g(f_j) + g(a_k) = 0$. By definition of the system there exists a_i, a_j in the small system such that $a_i\ a_j\ a_k$, $u\ f_i\ a_i$ and $u\ f_j\ a_j$ are triples. Then $g(a_i) + g(a_j) + g(a_k) = 0$, $g(f_i) = g(u) + g(a_i)$ and $g(f_j) = g(u) + g(a_j)$. Then $g(f_j) + g(f_i) + g(a_k) = 0$ because we are in a field of characteristic 2. Therefore, g preserves triples. Now because we have a map g which preserves triples then by Theorem 2.2.2 there exists a map g' which preserves independence and for all triples $u_i\ u_j\ u_k$ of the system we have $g'(u_i) + g'(u_j) + g'(u_k) = 0$. □

3.2.4 Corollary. Let $n \geq 3$. The matroids M_{2^n-1} are representable.

Proof. The proof is obvious by Theorem 3.2.3. □

3.2.5 Corollary. Let $n \geq 4$ and $S(2,3,2^n-1)$ be a Steiner triple system of order 2^n-1 which contain one of the systems $S(2,3,15)$ in Chapter 3 apart from the finite projective $PG(3,2)$. Then the matroid arising from $S(2,3,2^n-1)$ is not representable.

Proof. The proof is obvious. □

3.3 Non-realizability of some $S(2,3,6k+3)$

As I introduced in Chapter 3, there are 80 Steiner triples systems $S(2,3,15)$. So, 23 of them contain the Fano plane 2.1. Some contain the Fano plane 2.1 minus one triple and the others contain the same Fano plane minus two triples. As a result, all of those which contain the Fano plane 2.1 minus one are not representable and some of the remaining are not representable. Also, the system $S(2,3,15)$ constructed by Bose's construction 3.1.3 is not representable.

3.3.1 Corollary. Let n be an integer such that n odd and $5 \mid n$. The Steiner triple system $S(2,3,3n)$ constructed by Bose's construction in 3.1.3 is not representable.

Proof. Given n odd and divisible by 5, we consider the Steiner system $S(2,3,3n)$ whose points are the elements of $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and whose triples are:

- All $\{(x,0),(x,1),(x,2)\}$ where $x \in \mathbb{Z}/n\mathbb{Z}$
- All $\{(x,i),(y,i),((x+y)/2, i+1)\}$ where $x, y \in \mathbb{Z}/n\mathbb{Z}$ and $i \in \mathbb{Z}/3\mathbb{Z}$.

Since $5 \mid n$, there is a subgroup H of order 5 of $\mathbb{Z}/n\mathbb{Z}$. Then H is a cyclic group. So there is $x \in \mathbb{Z}/n\mathbb{Z}$ such that $o(x) = 5$ and $x \in H$. Since n is odd, 2 is invertible in $\mathbb{Z}/n\mathbb{Z}$. Let $1 \leq u, v \leq 5$ such that $u \neq v$. We have $\frac{ux+vx}{2} = (\frac{u+v}{2})x$. Then $\frac{ux+vx}{2} \in H$. So the system $S(2,3,3n)$ with n odd and $5 \mid n$ constructed by Bose's construction 3.1.3 contains a copy of the Steiner system $S(2,3,15)$ constructed in the same way by identifying the elements of $S(2,3,15)$ by the elements of $H \times \mathbb{Z}/3\mathbb{Z}$. We know that the system $S(2,3,15)$ constructed by Bose's construction is not representable. Hence, the matroid is not representable. □

Acknowledgements

Foremost, I would first like to thank my God for giving me strength, health, and guidance when I felt lost during the preparation of this paper.

I would like to express my special thanks to my supervisor Professor John Shareshian for his constant guidance, motivation and support. I indeed appreciate his guidance and all of his supports. He has been a great mentor and I could not imagine having a better supervisor.

A very profound thanks goes to the AIMS Staff for all of the supports, particularly, to Professor J. W. Sanders, Jan Groenewald, Noluvuyo Hobana, and also to my tutor Kenneth Dadedzi.

A special thanks goes to Dr. Gerard Razafinimanantsoa, his teaching really helps me for many things and my best friend who shared his knowledge during my undergraduate study in Madagasikara.

Finally, I would like to express my appreciation to my parents Rambeloson Patrick and Rahariseheno Verohanta, my siblings and my friends for their supports, encouragements and for always keeping me in their prayers. Thank you.

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