Gelfand theory for commutative Banach algebras

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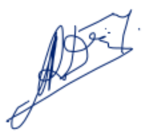
Abstract

In a commutative Banach algebra, characters (multiplicative linear functionals) play a very important role. Indeed, they determine completely the maximal ideals of the Banach algebra, as well as the spectrum of every element. Since the set of characters is a (weak*-) compact set, we can construct the Banach algebra of continuous complex-valued functions on this set. Moreover, some commutative Banach algebras can be identified with a subalgebra of the function algebra defined by their characters. In this project, we elaborate these theories and provide some applications.

Keywords: Commutative Banach algebra, Gelfand representation, characters.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



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Contents

Introduction

Commutative Banach algebras have some particular properties. One of these properties is the fact that they have many multiplicative linear functionals. We call these functionals *characters*. The Gelfand theory for commutative Banach algebras consists of using the set of characters to represent a Banach algebra. We shall mostly consider commutative Banach algebras in this project, except in the preliminary chapter and a few parts in the other chapters, where some results are still valid in the non-commutative setting.

In the first chapter, we recall some preliminary tools in Banach algebras, complex analysis and topology, namely some basic definitions, examples, as well as topological and algebraic properties of Banach algebras. We recall as well some fundamental theorems of functional analysis such as the Hahn-Banach Theorem, the Banach-Alaoglu Theorem, the Closed Graph Theorem, and the Holomorphic Functional Calculus.

The second chapter consists of the development of the Gelfand theory. The first two sections will be dedicated to defining "characters", and giving their relations with the spectrum and maximal ideals. We will explain how characters determine the maximal ideals (Theorem 2.2.1), and the spectrum (Corollary 2.2.2) in commutative Banach algebras. In Theorem 2.2.7, we determine the characters of $\mathcal{C}(K)$ (where K is a compact set). The set of characters of the disk algebra is given in Theorem 2.3.1. In Section 2.4, we introduce the concept of Gelfand representation. The main results in this section are the weak*-compactness of the set of characters (Theorem 2.4.2), and the fact that the Gelfand transformation is a continuous homomorphism (Theorem 2.4.5). In the last section, we give a direct application to function algebras (Theorem 2.5.4), and then some properties of the characters of finitely generated Banach algebras as we see in Theorem 2.6.5.

The third chapter focuses on further applications. In particular, we give four examples of applications of the above theory. The first of them is an application to Fourier series via the Wiener-Lévy Theorem (Corollary 3.1.2). We also use this theory to prove the automatic continuity of some (algebra) homomorphisms (Theorem 3.2.2). This continuity has some interesting consequences such as the equivalence of Banach algebra norms (Corollary 3.2.3), and also the continuity of involutions (Corollary 3.2.9), in the semisimple case. The third application is on the existence of a Banach algebra norm as we show in Theorem 3.3.2. In the last section, as shown in Corollary 3.4.3, we study the uniform continuity of the spectrum function.

1. Preliminaries

In this chapter, we recall the preliminary results that we will use later. We state them without proofs, since they can be found in most of the classical textbooks of functional analysis.

1.1 Banach algebras

Banach algebras are the main objects we manipulate in this project. Basic definitions and some examples are given in this section.

1.1.1 Definition (complex algebra)**.** A vector space A over the field $\mathbb{C}$ of complex numbers is called a *complex algebra* if there exists a multiplication $A \times A \to A,\ (x, y) \mapsto xy$, satisfying:

(1) $x(yz) = (xy)z$ (associativity),

(2) $(x + y)z = xz + yz,\ x(y + z) = zy + xz$ (left and right distributivity with respect to the addition),

(3) $\lambda(xy) = (\lambda x)y = x(\lambda y)$

for all $\lambda \in \mathbb{C}$ and $x, y, z \in A$.

1.1.2 Definition (Banach algebra)**.** A *Banach algebra* is a complex algebra A which is a Banach space with norm $||.||$, and has unit element $\mathbf{1}$ satisfying:

(1) $||xy|| \leq ||x||\ ||y||$

(2) $||\mathbf{1}|| = 1$

(3) $\mathbf{1}x = x\mathbf{1} = x$

for all $x, y \in A$.

1.1.3 Definition. We say that a Banach algebra (or in general a complex algebra) A is *commutative* if the multiplication is commutative, that is $xy = yx$ for all $x,\ y \in A$.

A trivial example of a commutative Banach algebra is the field $\mathbb{C}$ of complex numbers. More generally, we can construct a non-commutative Banach algebra by choosing an appropriate norm for the algebra of $n \times n$ matrices with complex entries, as shown in the following example.

1.1.4 Example. The set $\mathcal{M}_n(\mathbb{C})$ of $n \times n$ matrices with complex entries is a Banach algebra with the matrix addition and multiplication, and the norm $||.||$ defined, for all $M = (m_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_n(\mathbb{C})$, by

$$||M|| = \sup_{i \in \{1,\ldots,n\}} \sum_{j=1}^{n} |m_{ij}|.$$

1.1.5 Remark. We know that all norms are equivalent in finite dimensional normed spaces. However, there exist some norms which are not Banach algebra norms even in the finite dimensional case. For instance, $\mathcal{M}_n(\mathbb{C}),\ (n \geq 2)$ with the norm defined by

$$||M|| = \max_{1 \leq i,j \leq n} |m_{ij}|$$

is not a Banach algebra. Indeed, considering the element

$$M = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix},$$

we have that $||M|| = 1$, and then $||M||^2 = 1$. However,

$$M^2 = \begin{pmatrix} n & \cdots & n \\ \vdots & \ddots & \vdots \\ n & \cdots & n \end{pmatrix},$$

so $||M^2|| = n > 1 = ||M||^2$. Hence, Property (1) in Definition 1.1.2 is not satisfied.

We proceed to consider more interesting examples. Since matrices can be seen as linear operator on Banach spaces, Example 1.1.4 can be extended again, as given in the following.

1.1.6 Example. Let X be a Banach space. The set $\mathcal{L}(X)$ of bounded linear maps $X \to X$ is a Banach algebra under the pointwise addition, the composition of maps, with the operator norm. The unit element is the identity function. Generally, this Banach algebra is not commutative since composition is not commutative.

We take a closer look at the following Banach algebras in Chapter 2 and Chapter 3.

1.1.7 Example. Given a compact set K, the set $\mathcal{C}(K)$ of complex valued continuous functions over K is a commutative Banach algebra, under the pointwise addition and multiplication, with the supremum norm.

1.1.8 Example (Disk Algebra)**.** Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$, and $\overline{\mathbb{D}}$ its closure. The set $\mathcal{A}(\overline{\mathbb{D}})$ of continuous functions on $\overline{\mathbb{D}}$ which are analytic on $\mathbb{D}$ is a Banach algebra, as it is a closed subalgebra of $\mathcal{C}(\overline{\mathbb{D}})$. This is called *the Disk Algebra*.

1.1.9 Example (Wiener algebra)**.** Let W be the set of all continuous functions f with period 2π which have absolutely converging trigonometric series expansion $f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int} \quad (t \in [0, 2\pi])$. With pointwise addition and multiplication and norm $||f|| = \sum_{n=-\infty}^{\infty} |a_n|$, W is a commutative Banach algebra called the *Wiener algebra*.

As vector space, W is generated by $\{g^n,\ n \in \mathbb{Z}\}$, where $g(t) = e^{it}$ and $g^n(t) = (g(t))^n$ for all $t \in [0, 2\pi]$ and $n \in \mathbb{Z}$. We have also that $||g|| = ||g^{-1}|| = 1$. Indeed,

$$g(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$$

with $a_1 = 1$ and $a_n = 0$ for all $n \neq 1$, and

$$g^{-1}(t) = \sum_{n=-\infty}^{\infty} a'_n e^{int}$$

with $a'_{-1} = 1$ and $a'_n = 0$ for all $n \neq -1$.

1.1.10 Example. The algebra $\mathbb{C}[x_1, \ldots, x_n]$ of n-variate polynomials with complex coefficients is a complex algebra, but not a Banach algebra, since it does not have a Banach space norm.

1.1.11 Definition. Let A be a Banach algebra. An element $a \in A$ is said to be *invertible* if there exist $x \in A$ such that

$$xa = ax = \mathbf{1}.$$

This element is called the *inverse* of a, and we write $x = a^{-1}$. We denote the set of all invertible elements of A by A^{-1}. This set has interesting properties, both algebraic and topological:

1.1.12 Proposition ([8], Definitions 10.10, Theorem 10.12)**.** *Let A be a Banach algebra. Then*

- *A^{-1} is a multiplicative group,*
- *A^{-1} is an open subset of A.*

1.2 Ideals and homomorphisms

Ideals and homomorphisms are very important concepts in Banach algebras. We will see the relations between maximal ideals and some particular linear functionals (which are themselves homomorphisms) in Chapter 2.

1.2.1 Definition (Ideals)**.** Let A be a Banach algebra. A vector subspace I of A is called a:

- *left* ideal of A if, for any $x \in I$ and $a \in A$, $ax \in I$.
- *right* ideal of A if, for any $x \in I$ and $a \in A$, $xa \in I$.
- *two-sided ideal* if I is both a right ideal and a left ideal.
- *proper* (left, right, two-sided) ideal if it is a (left, right, two-sided) ideal and $I \neq A$.
- *maximal* (left, right, two-sided) ideal if it is a proper (left, right, two-sided) ideal which is not strictly contained in a proper (left, right, two-sided) ideal.

We note that the trivial (two-sided) ideals of a Banach algebra A are $\{0\}$ and A. If A is a commutative Banach algebra, then every ideal of A is two-sided. In particular, if A is isometrically isomorphic to $\mathbb{C}$ (or equivalently, as we will see in Theorem 1.2.12, every non-zero element of A is invertible), then the only ideals of A are $\{0\}$ and A itself. Let us now see a non-trivial example.

1.2.2 Example. Let X be a Banach space and $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. The set $\mathcal{K}(X)$ of compact operators on X is a two-sided ideal of $\mathcal{L}(X)$.

The following result shows an interesting use of ideals. We can construct a new Banach algebra from a given one and an ideal.

1.2.3 Proposition ([1], p.33)**.** *Let A be a (commutative) Banach algebra, and I a closed two-sided ideal of A. Then, the quotient algebra A/I is a (commutative) Banach algebra with the multiplication*

$$(x + I)(y + I) = xy + I$$

and the norm

$$||x + I|| = \inf_{u \in I} ||x - u||.$$

Krull's Theorem, derived from ring theory, is also valid for Banach algebras. Indeed Banach algebras are also rings.

1.2.4 Proposition (Krull)**.** ([1], Lemma 3.1.1). *Let A be a Banach algebra and I a left ideal (resp. right ideal) of A. Then I is contained in a maximal left ideal (resp. maximal right ideal) of A.*

The sets of all maximal left ideals and all maximal right ideals of a Banach algebra A are denoted, respectively, by $\mathrm{MLI}(A)$ and $\mathrm{MRI}(A)$. We have the following result.

1.2.5 Theorem ([1], Theorem 3.1.3)**.** *Let A be a Banach algebra. The following sets are identical:*

(i) $\bigcap \mathrm{MLI}(A)$,

(ii) $\bigcap \mathrm{MRI}(A)$,

(iii) $\{x \in A : \mathbf{1} - xy \in A^{-1}$ *for all* $y \in A\}$,

(iv) $\{x \in A : \mathbf{1} - yx \in A^{-1}$ *for all* $y \in A\}$.

A very important and particular kind of ideal is the Jacobson radical. This concept also comes from ring theory, and is still valid for Banach algebras.

1.2.6 Definition (Radical)**.** The set defined in the preceding theorem is a two-sided ideal of A, called the *(Jacobson) radical* of A, and denoted by $\mathrm{Rad}(A)$. If $\mathrm{Rad}(A) = \{0\}$, A is said to be *semisimple*.

A straight and forward way to construct a semisimple Banach algebra is to take the quotient algebra $A/\mathrm{Rad}(A)$ for a given Banach algebra A. We now give some other important semisimple Banach algebra.

1.2.7 Example.

(1) If X is a Banach space, then $\mathcal{L}(X)$ is a semisimple Banach algebra.

(2) If K is a compact set, then the Banach algebra $\mathcal{C}(K)$ is semisimple.

As in group theory, ring theory and vector spaces, we also have the concept of homomorphisms in Banach algebras. These are mappings which preserve algebraic structures.

1.2.8 Definition (Homomorphism)**.** Let A and B be two Banach algebras. A linear map $\phi : A \to B$ is a *homomorphism* if $\phi(xy) = \phi(x)\phi(y)$ for all $x,\ y \in A$ and $\phi(\mathbf{1}) = \mathbf{1}$. A homomorphism is called an *isomorphism* if it is bijective.

1.2.9 Definition (Kernel and range)**.** Let A and B be Banach algebras and $\phi : A \to B$ a homomorphism. The *kernel* and the *range* of ϕ are, respectively, defined and denoted by

$$\ker \phi =: \{x \in A : \phi(x) = 0\} \text{ and } \mathcal{R}(\phi) =: \{\phi(x) : x \in A\}.$$

Clearly, $\ker \phi$ is an ideal of A, and $\mathcal{R}(\phi)$ is a subalgebra of B.

1.2.10 Example. Let A be a Banach algebra and I an ideal of A. The mapping $\pi : A \to A/I$ defined by $\pi(x) = x + I$ is a homomorphism. Moreover, we have that $\ker \pi = I$ and $\mathcal{R}(\pi) = A/I$.

1.2.11 Example. The identity mapping on A is an isomorphism.

1.2.12 Theorem (Gelfand-Mazur)**.** ([1], Corollary 3.2.9). *Let A be a commutative Banach algebra in which all non zero elements are invertible. Then A is isometrically isomorphic to $\mathbb{C}$. That is, there exists an isomorphism $\phi : A \to \mathbb{C}$ such that $|\phi(x)| = ||x||$ for all $x \in A$.*

1.3 Spectrum

We now introduce some notions in spectral theory, namely the definition and basic properties of the spectrum, and then some examples illustrating them.

1.3.1 Definition. Let A be a Banach algebra, and $x \in A$. The *spectrum* of x is the set

$$\mathrm{Sp}(x, A) := \{\lambda \in \mathbb{C} : x - \lambda \mathbf{1} \notin A^{-1}\}.$$

If there is no risk of confusion, we simply denote the spectrum of x by $\mathrm{Sp}(x)$.

1.3.2 Example. The spectrum of an element of the Banach algebra $\mathcal{M}_n(\mathbb{C})$ is the set of its eigenvalues.

1.3.3 Example (Spectrum of the shift operators)**.** The right shift operator R and left shift operator L on l^2 are defined respectively as follows. For all $x = (x_1, x_2, x_3, \ldots) \in l^2$,

$$R(x) = (0, x_1, x_2, \ldots) \text{ and } L(x) = (x_2, x_3, \ldots).$$

The spectrum of each of these operators is the closed unit disk of $\mathbb{C}$.

The spectrum of an element in a Banach algebra has some nice properties, for instance as shown in the following theorem.

1.3.4 Theorem ([1], Theorem 3.2.8)**.** *Let A be a Banach algebra and $x \in A$. Then $\mathrm{Sp}(x)$ is a non-empty compact subset of $\mathbb{C}$.*

1.3.5 Definition (Spectral radius)**.** Let A be a Banach algebra and $x \in A$. The *spectral radius* of x is the positive real number $\rho(x)$ defined by

$$\rho(x) := \sup_{\lambda \in \mathrm{Sp}(x)} |\lambda|.$$

We will use the following result in the proof of Proposition 2.5.3.

1.3.6 Theorem ([1], Theorem 3.2.8)**.** *Let A be a Banach algebra and $x \in A$. Then*

$$\rho(x) = \lim_{n \to \infty} ||x^n||^{\frac{1}{n}}.$$

As a consequence of the preceding theorem, we have the following corollary.

1.3.7 Corollary. *Let A be a Banach algebra and $x \in A$. Then $\rho(x) \leq ||x||$.*

We will use the following proposition to prove the uniform continuity of the spectrum (§3.4).

1.3.8 Proposition ([1], Theorem 3.2.10)**.** *Let A be a Banach algebra, $x \in A$ and $\alpha \in \mathbb{C} \setminus \mathrm{Sp}(a)$. Then*

$$\mathrm{d}(\alpha, \mathrm{Sp}(a)) = \frac{1}{\rho((\alpha \mathbf{1} - x)^{-1})},$$

where $\mathrm{d}(\alpha, \mathrm{Sp}(a)) := \sup_{\lambda \in \mathrm{Sp}(a)} |\lambda - \alpha|$.

If A is a Banach algebra and B a subalgebra of A, then $A \setminus A^{-1} \subseteq B \setminus B^{-1}$. As such we have the following lemma.

1.3.9 Lemma. *Let A be a Banach algebra, B a closed subalgebra of A and $x \in B$. Then* $\mathrm{Sp}(x, A) \subseteq \mathrm{Sp}(x, B)$.

More generally, we have the following results.

1.3.10 Proposition. *Let A be a Banach algebra and C a commutative subset of A. Then, there exists a subset M of A such that M is a maximal commutative subset containing C. Moreover, $\mathbf{1} \in M$ and M is a closed subalgebra of A.*

The maximal subset containing a commutative set is then itself a Banach algebra. In the case where the commutative subset is a singleton, we have the following corollary.

1.3.11 Corollary. *Let A be a Banach algebra, $x \in A$, and M a maximal commutative subset of A containing x. Then*

$$\mathrm{Sp}(x, A) = \mathrm{Sp}(x, M).$$

1.3.12 Definition (Quasi-nilpotent)**.** An element x of a Banach algebra A is called *quasinilpotent* if $\mathrm{Sp}(x) = \{0\}$, or equivalently $\rho(x) = 0$. We denote the set of all quasinilpotent elements of A by $\mathrm{QN}(A)$.

The following result follows from (iii) in Theorem 1.2.5, and the definition of $\mathrm{QN}(A)$.

1.3.13 Proposition. *Let A be a Banach algebra. Then* $\mathrm{Rad}(A) \subseteq \mathrm{QN}(A)$.

1.4 Holomorphic functions and functional calculus

The objective of this section is to recall some basic properties of holomorphic functions, and then to elaborate the holomorphic functional calculus.

1.4.1 Definition (Holomorphic function)**.** Let Ω be an open subset of $\mathbb{C}$ and $a \in \Omega$. A function $f : \Omega \to \mathbb{C}$ is said to be *holomorphic* at a if

$$f'(a) := \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists. The function f is said to be *holomorphic on* Ω if it is holomorphic at every point of Ω. If f is holomorphic in the entire complex plane, f is called an *entire function*.

For an open set $\Omega \subseteq \mathbb{C}$ we denote the algebra of holomorphic functions on Ω by $\mathcal{H}(\Omega)$. We can see for instance in Section 13 of [6] that every element of $\mathcal{H}(\Omega)$ is continuously differentiable, and then has a local power series expansion. This last property is the *analyticity* of holomorphic functions. The following result relates analytic and holomorphic functions.

1.4.2 Proposition ([8], p.78-79)**.** *Let A be a Banach algebra, $\Omega \subseteq \mathbb{C}$ an open set, and $f : \Omega \to A$ an analytic function. Then for every bounded linear functional l on A, $l \circ f \in \mathcal{H}(\Omega)$.*

One of the most important properties of holomorphic functions is the Cauchy integral formula.

1.4.3 Theorem (Cauchy integral formula)**.** ([6], §13.5) *Let Ω be an open subset of $\mathbb{C}$, $a \in \Omega$, $f \in \mathcal{H}(\Omega)$ and $\gamma \subset \Omega$ a contour surrounding a. Then*

$$f(a) = \frac{1}{2i\pi} \int_\gamma \frac{f(z)}{z - a} \mathrm{d}z.$$

More generally, for $n \in \mathbb{N}$

$$f^{(n)}(a) = \frac{n!}{2i\pi} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \mathrm{d}z.$$

where $f^{(0)} = f$ and $f^{(n)}$ $(n \geq 1)$ is the n-th derivative of f.

As a consequence of this theorem, we have Liouville's Theorem. We shall often use it in Chapter 3.

1.4.4 Corollary (Liouville's Theorem)**.** ([6], §13.3) *A bounded entire function is constant.*

Now, let us introduce the concepts of functional calculus. We consider a Banach algebra A, and an element $a \in A$.

Let $p(\lambda) = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_n\lambda^n \in \mathbb{C}[\lambda]$ a non-constant polynomial. We define the element $p(a) \in A$ as $p(a) = \alpha_0\mathbf{1} + \alpha_1 a + \cdots + \alpha_n a^n$.

1.4.5 Theorem ([1], Theorem 3.2.6)**.** *Let A be a Banach algebra, $x \in A$ and $p \in \mathbb{C}[\lambda]$ a non-constant polynomial. Then $\mathrm{Sp}(p(x)) = p(\mathrm{Sp}(x))$, where $p(\mathrm{Sp}(x)) = \{p(\mu) : \mu \in \mathrm{Sp}(x)\}$.*

We now consider a rational function $r(\lambda) = \dfrac{p(\lambda)}{q(\lambda)}$ such that p and q are coprimes and $q(\mu) \neq 0$ for all $\mu \in \mathrm{Sp}(a)$. By Theorem 1.4.5, $0 \notin \mathrm{Sp}(q(a))$, so $q(a)$ is invertible. Then we now define the element $r(a)$ of A as $r(a) = p(a)(q(a))^{-1}$.

More generally, let f be a complex-valued function which is analytic on an arbitrary open neighborhood Ω of $\mathrm{Sp}(a)$. Then we define the element $f(a)$ of A by

$$f(a) := \frac{1}{2i\pi} \int_\Gamma f(\lambda)(\lambda\mathbf{1} - a)^{-1} \mathrm{d}\lambda,$$

where Γ is a contour surrounding $\mathrm{Sp}(a)$ such that $\Gamma \subseteq \Omega \setminus \mathrm{Sp}(a)$. It is shown in [1], p. 42-43 that this integral is independent of the contour Γ. Then $f(a)$ is well defined. The following result shows that this definition is compatible with the case where f is a rational function as above.

1.4.6 Proposition ([1], Lemma 3.3.1)**.** *Let A be a Banach algebra and $a \in A$. Let r be a rational function and Γ a smooth contour surrounding $\mathrm{Sp}(a)$ such that r has no poles surrounded by Γ. Then*

$$r(a) = \frac{1}{2i\pi} \int_\Gamma r(\lambda)(\lambda\mathbf{1} - a)^{-1} \mathrm{d}\lambda.$$

Now we are ready to establish the holomorphic functional calculus.

1.4.7 Theorem (Holomorphic functional calculus)**.** ([1], Theorem 3.3.3). *Let A be a Banach algebra, $a \in A$, and Ω an open neighborhood of $\mathrm{Sp}(a)$. Then*

(i) The mapping $\phi : \mathcal{H}(\Omega) \to A$ defined by $\phi(f) = f(a)$ is a continuous algebra homomorphism.

(ii) $\mathrm{Sp}(f(a)) = \{f(\lambda) : \lambda \in \mathrm{Sp}(a)\}$.

1.5 The Hahn-Banach Theorem

In this section, we recall the Hahn-Banach Theorem, which is fundamental in functional analysis. A more extended form of this theorem can be found in Chapter 3 of [8], or §4.2 of [3] and almost all functional analysis books.

1.5.1 Theorem (Hahn-Banach)**.** ([1], Theorem 1.1.3) *Let X be a normed space and Y a non-trivial subspace of X. Let f be a linear functional on Y such that $|f(y)| \leq ||y||$ for all $y \in Y$. Then there exists a linear functional $\tilde{f}$ on X such that $\tilde{f}_{|_Y} = f$ and $|\tilde{f}(x)| \leq ||x||$ for all $x \in X$.*

The following result is a consequence of the Hahn-Banach Theorem.

1.5.2 Corollary (Point separation)**.** *Let X be a normed space and $x, y \in X$. If $l(x) = l(y)$ for all linear functionals l on X, then $x = y$.*

1.6 Product topology

In this section, we give a quick summary of the product topology, weak*-topology and the statement of the Closed Graph Theorem.

1.6.1 Weak*-topology .

Let $(X_1, \tau_1), \ldots, (X_n, \tau_n)$ be n topological spaces ($n \geq 2$). The product topology τ on the Cartesian product $Y = X_1 \times \cdots \times X_n$ is constructed as follows: The set $\{U_1 \times \cdots \times U_n : U_i \in \tau_i,\ i = 1, \ldots, n\}$ is a base for τ. We now remark that Y is the set of all functions $x : \{1, \ldots, n\} \to X_k,\ k \mapsto x_k$, and τ is the smallest topology on Y, making the projections $p_i : Y \to X_i,\ x \mapsto x_i$ continuous.

More generally, we have the following definition.

1.6.2 Definition (Product topology)**.** Consider a family of topological spaces $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in I}$.

(i) The *Cartesian product* $\mathcal{Y} = \prod_{\alpha \in I} X_\alpha$ is defined to be the set of all functions $x : I \to X_\alpha,\ \alpha \mapsto x_\alpha$.

(ii) The *product topology* is the smallest topology of $\mathcal{Y}$ making the projections $p_\alpha : \mathcal{Y} \to X_\alpha,\ x \mapsto x_\alpha$ continuous.

Now we consider the case where $I = \mathbb{C}$ and X_α is a normed space X for each $\alpha \in I$. In this case, we denote X^I the Cartesian product $\prod_{\alpha \in I} X$.

1.6.3 Definition (Weak*-topology)**.** Let X be a normed space over $\mathbb{C}$. Let X' be the Banach space of bounded linear functionals on X. The weak*-topology of X' is the restriction of the product topology of $\mathbb{C}^X$ to X'.

By construction of the weak*-topology, for every $x \in X$, the projection $p_x : X' \to \mathbb{C},\ f \mapsto f(x)$ is continuous. So, for every open set $U \subseteq \mathbb{C}$, $p_x^{-1}(U)$ is an open set of the weak topology.

1.6.4 Theorem (Banach-Alaoglu)**.** ([4], Theorem 5.41). *Let X be a normed vector space over $\mathbb{R}$ or $\mathbb{C}$. Then, the set $\{f \in X' : ||f|| \leq 1\}$ is weak*-compact in X'.*

1.6.5 The Closed Graph Theorem.

1.6.6 Definition (Graph of a function)**.** Let X and Y be two topological spaces and $f : A \to B$ a function. The *graph* of f is the subset of $X \times Y$ defined by $\mathcal{G}(f) := \{(x, y) \in X \times Y : y = f(x)\}$.

1.6.7 Theorem (Closed Graph Theorem)**.** ([8], p.49-50). *Let X and Y be normed spaces and $f : X \to Y$ a linear map. Then f is continuous if and only if $\mathcal{G}(f)$ is closed in the product topology of $X \times Y$.*

Since a linear operator is continuous if and only if it is continuous at 0, we have the following corollary.

1.6.8 Corollary. *Let X and Y be normed spaces and $T : X \to Y$ a linear map. Then the following assertions are equivalent:*

(i) T is continuous.

(ii) For every sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ converging to 0, if the sequence $(Tx_n)_{n\in\mathbb{N}} \subseteq Y$ converges to a limit $l \in Y$, then $l = 0$.

2. The Gelfand theory for commutative Banach Algebras

Throughout this chapter, we will be working in Banach algebras with unit element $\mathbf{1}$. Our objective is to develop in detail the Gelfand theory for commutative Banach algebras. Most of the time, we follow [1].

2.1 Characters of Banach algebras

In this section, we introduce the concept of *characters* on Banach algebras. As homomorphisms, they have important algebraic as well as topological properties.

2.1.1 Definition. Let A be a Banach algebra, and χ a linear functional on A. We call χ a *character* of A if it is multiplicative and not identically zero, that is

$$\begin{cases} \chi(xy) = \chi(x)\chi(y), & \text{for any } x, y \in A \\ \chi \not\equiv 0. \end{cases}$$

We denote by $\mathfrak{M}(A)$ the set of all characters of A.

The set $\mathfrak{M}(A)$ has the following properties.

2.1.2 Proposition. *Let A be a Banach algebra, and $\chi \in \mathfrak{M}(A)$. Then $\chi(\mathbf{1}) = 1$.*

Proof. Let $\chi \in \mathfrak{M}(A)$, and $x \in A$ such that $\chi(x) \neq 0$. Since χ is multiplicative, $\chi(x) = \chi(x.\mathbf{1}) = \chi(x)\chi(\mathbf{1})$. So $\chi(\mathbf{1}) = 1$. □

2.1.3 Example. The only element of $\mathfrak{M}(\mathbb{C})$ is the identity function. Indeed, the only linear functionals on $\mathbb{C}$ are scalar multiplications. Then, if $f \in \mathfrak{M}(C)$, we must have that $f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$ and all $z \in \mathbb{C}$, and $f(1) = 1$. So, $\lambda = 1$.

2.1.4 Proposition. *Let A be a Banach algebra, and $\chi_1, \chi_2 \in \mathfrak{M}(A)$. If $\ker \chi_1 = \ker \chi_2$, then $\chi_1 = \chi_2$.*

Proof. Let $\chi_1, \chi_2 \in \mathfrak{M}(A)$, such that $\ker \chi_1 = \ker \chi_2$. Let $x \in A$. By linearity and the multiplicative property of characters,

$$\chi_1(x) - \chi_2(x) = \chi_1(x) - \chi_2(x)\chi_1(\mathbf{1}) = \chi_1(x - \chi_2(x)\mathbf{1}),$$

but we remark that

$$\chi_2(x - \chi_2(x)\mathbf{1}) = \chi_2(x) - \chi_2(x)\chi_2(\mathbf{1}) = \chi_2(x) - \chi_2(x) = 0.$$

So $x - \chi_2(x)\mathbf{1} \in \ker \chi_2$. Since $\ker \chi_2 = \ker \chi_1$ (by hypothesis), it follows that $x - \chi_2(x)\mathbf{1} \in \ker \chi_1$. Therefore $\chi_1(x) - \chi_2(x) = \chi_1(x - \chi_2(x)\mathbf{1}) = 0$, and this implies that $\chi_1(x) = \chi_2(x)$. Since x was chosen arbitrarily, we conclude that $\chi_1 = \chi_2$. □

2.1.5 Proposition. *Let A be a Banach algebra, $\chi \in \mathfrak{M}(A)$, and $x \in A$. Then*

(i) $\chi(x) \in \mathrm{Sp}(x)$.

(ii) χ *is continuous of norm* 1.

Proof. Let A, χ and x be as stated.

(i) Suppose that $\chi(x) \notin \mathrm{Sp}(x)$. Then, by definition of the spectrum, $x - \chi(x)\mathbf{1}$ is invertible in A. So $(x - \chi(x)\mathbf{1})z = \mathbf{1}$ and $z(x - \chi(x)\mathbf{1}) = \mathbf{1}$ for some $z \in A$. Applying χ to both sides of the first equation, we get

$$\chi((x - \chi(x)\mathbf{1})z) = (\chi(x) - \chi(x)\chi(\mathbf{1}))\chi(z) = (\chi(x) - \chi(x))\chi(z) = 0,$$

which is a contradiction since $\chi(\mathbf{1}) = 1$. Thus, $x - \chi(x)\mathbf{1}$ is not invertible in A, which means that $\chi(x) \in \mathrm{Sp}(x)$.

(ii) Let $\rho(x)$ be the spectral radius of x. Since $\chi(x) \in \mathrm{Sp}(x)$ (according to (i)), $|\chi(x)| \leq \rho(x)$. But we know also that $\rho(x) \leq ||x||$ for any x in a Banach algebra. So, $|\chi(x)| \leq ||x||$ for any $x \in A$ which implies that χ is continuous, and $||\chi|| \leq 1$. Now, taking $x = \mathbf{1}$, we have that $|\chi(\mathbf{1})| = 1$, and we conclude that $||\chi|| = 1$.

□

2.2 Characters and spectrum

In this and all the subsequent sections, we will be working on commutative Banach algebras. Therefore, there is no need to distinguish left, right and two-sided ideals. We use only the concept of ideals instead.

2.2.1 Theorem (Gelfand)**.** ([1], Theorem 4.1.2 (i)). *Let* A *be a commutative Banach algebra and* $\mathcal{M}(A)$ *the set of all maximal ideals of* A*. Then*

$$\begin{array}{rccc} \varphi: & \mathfrak{M}(A) & \longrightarrow & \mathcal{M}(A) \\ & \chi & \longmapsto & \ker\chi \end{array}$$

is a bijection.

Proof. Let $\chi \in \mathfrak{M}(A)$. Since χ is continuous, and $\{0\}$ is a closed subset of $\mathbb{C}$, $\ker\chi = \chi^{-1}(\{0\})$ is a closed subset of A. By linearity and the multiplicative property of χ, it is not difficult to prove that $\ker\chi$ is an ideal of A. By the First Isomorphism Theorem, $A/\ker\chi$ is isomorphic (as vector space) to the range of χ, which is a 1-dimensional vector space over $\mathbb{C}$. Hence, $A/\ker\chi$ is also isomorphic to $\mathbb{C}$ as a ring, which means that any non-zero element of $A/\ker\chi$ is invertible. So $\ker\chi$ is a maximal ideal. Therefore, the map φ is well defined.

The fact that φ is injective follows from 2.1.4.

Now, let M be a maximal ideal of A and $x \in A \setminus M$. Consider the set $I = \{ax + m : a \in A, m \in M\}$. Clearly, I is an ideal of A properly containing the maximal ideal M. Therefore $I = A$, and then $x + M$ is invertible in A/M. Thus A/M is a division algebra. Hence, by the Gelfand-Mazur Theorem (Theorem 1.2.12), A/M is isometrically isomorphic to $\mathbb{C}$. Let $i : A/M \to \mathbb{C}$ be that isomorphism, and consider the homomorphism $\pi : A \to A/M$, $x \mapsto x + M$. The map $\chi_0 = i \circ \pi$ is a character of A whose kernel is the maximal ideal M. Indeed, the linearity of χ_0 is obvious, and for $x, y \in A$, we have

$$\chi_0(xy) = i(xy + M) = i((x + M)(y + M)) = i(x + M)i(y + M) = (i \circ \pi(x))(i \circ \pi(y)).$$

Moreover $\chi_0(x) = 0$ if and only if $i(x + M) = 0$ if and only if $x + M = M$ if and only if $x \in M$. So $\ker\chi_0 = M$. Thus φ is surjective. □

Knowing the characters of a Banach algebra, we can determine the spectrum of every element, as shown by the following result.

2.2.2 Corollary ([1], Theorem 4.1.2-ii)**.** *Let A be a commutative Banach algebra. For every $x \in A$, we have*

$$\mathrm{Sp}(x) = \{\chi(x) : \chi \in \mathfrak{M}(A)\}.$$

Proof. Let $x \in A$ where A is a commutative Banach algebra. It follows from 2.1.5-(i) that

$$\{\chi(x) : \chi \in \mathfrak{M}(A)\} \subseteq \mathrm{Sp}(x).$$

Conversely, let $\lambda \in \mathrm{Sp}(x)$. Then, by definition of the spectrum, $x - \lambda\mathbf{1}$ is not invertible in A. So, the ideal I generated by $x - \lambda\mathbf{1}$ must be contained in certain maximal ideal J of A. But according to Theorem 2.2.1, $J = \ker \chi_0$ for some $\chi_0 \in \mathfrak{M}(A)$. So $x - \lambda\mathbf{1} \in \ker \chi_0$, which means that $\chi_0(x - \lambda\mathbf{1}) = 0$. Since χ_0 is linear and multiplicative, we have

$$\chi_0(x - \lambda\mathbf{1}) = \chi_0(x) - \lambda\chi_0(\mathbf{1}) = \chi_0(x) - \lambda = 0.$$

So $\lambda = \chi_0(x) \in \{\chi(x) : \chi \in \mathfrak{M}(A)\}$. Then

$$\mathrm{Sp}(x) \subseteq \{\chi(x) : \chi \in \mathfrak{M}(A)\}.$$

□

2.2.3 Corollary. *Let A be a commutative Banach algebra. Then*

(i) $\mathfrak{M}(A) \neq \emptyset$,

(ii) $\mathrm{Rad}(A) = \bigcap_{\chi \in \mathfrak{M}(A)} \ker \chi = \mathrm{QN}(A)$.

Proof.

(i) It is immediate from Corollary 2.2.2 since the spectrum of each element of A is not empty.

(ii) Since $\mathrm{Rad}(A)$ is the intersection of all maximal ideals of A, by Theorem 2.2.1, we have

$$\begin{aligned}\mathrm{Rad}(A) &= \bigcap_{\chi \in \mathfrak{M}(A)} \ker \chi \\ &= \{x \in A : \chi(x) = 0, \text{ for all } \chi \in \mathfrak{M}(A)\} \\ &= \{x \in A : \mathrm{Sp}(x) = \{0\}\} \text{ (by Corollary 2.2.2)} \\ &= \mathrm{QN}(A).\end{aligned}$$

□

2.2.4 Remark. The assertion (i) in the preceding corollary is not always true if the Banach algebra is not commutative. For instance if A is the Banach algebra $\mathcal{M}_n(\mathbb{C})$, $(n \geq 2)$, then $\mathfrak{M}(A) = \emptyset$, but we know that the spectrum is not empty.

2.2.5 Corollary. *Let A be a Banach algebra and $x,\ y \in A$ such that $xy = yx$. Then*

$$\mathrm{Sp}(x + y) \subseteq \mathrm{Sp}(x) + \mathrm{Sp}(y) \text{ and } \mathrm{Sp}(xy) \subseteq \mathrm{Sp}(x)\mathrm{Sp}(y),$$

where $\mathrm{Sp}(x) + \mathrm{Sp}(y) = \{\lambda + \mu : \lambda \in \mathrm{Sp}(x) \text{ and } \mu \in \mathrm{Sp}(y)\}$ *and* $\mathrm{Sp}(x)\mathrm{Sp}(y) = \{\lambda\mu : \lambda \in \mathrm{Sp}(x) \text{ and } \mu \in \mathrm{Sp}(y)\}$.

Proof. (i) We first assume that A is a commutative Banach algebra M. Let $x, y \in M$. By Corollary 2.2.2, we have

$$\begin{aligned}\mathrm{Sp}(x+y, M) &= \{\chi(x+y) : \chi \in \mathfrak{M}(M)\}\\ &= \{\chi(x)+\chi(y) : \chi \in \mathfrak{M}(M)\}\\ &\subseteq \{\chi_1(x)+\chi_2(y) : \chi_1, \chi_2 \in \mathfrak{M}(M)\}\\ &= \mathrm{Sp}(x, M) + \mathrm{Sp}(y, M)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Sp}(xy, M) &= \{\chi(xy) : \chi \in \mathfrak{M}(M)\}\\ &= \{\chi(x)\chi(y) : \chi \in \mathfrak{M}(M)\}\\ &\subseteq \{\chi_1(x)\chi_2(y) : \chi_1, \chi_2 \in \mathfrak{M}(M)\}\\ &= \mathrm{Sp}(x, M)\mathrm{Sp}(y, M).\end{aligned}$$

So $\mathrm{Sp}(x+y, M) \subseteq \mathrm{Sp}(x, M) + \mathrm{Sp}(y, M)$ and $\mathrm{Sp}(xy, M) \subseteq \mathrm{Sp}(x, M)\mathrm{Sp}(y, M)$.

(ii) Now, let $x,\ y \in A$ such that $xy = yx$. Let M be a maximal commutative subset containing x and y. Then M itself is a commutative Banach algebra as a closed commutative subalgebra of A. Moreover, M contains x and y, and we have that

$$\mathrm{Sp}(x+y, A) = \mathrm{Sp}(x+y, M) \text{ and } \mathrm{Sp}(xy, A) = \mathrm{Sp}(xy, M).$$

Thus, we can conclude using (i).

□

Now we consider the Banach algebra $\mathcal{C}(K)$ for a compact set K.

2.2.6 Lemma. *Let K be a compact set, $A = \mathcal{C}(K)$, and $x \in K$. Then*

$$\chi_x : \begin{array}{ccc} A & \longrightarrow & \mathbb{C} \\ f & \longmapsto & f(x) \end{array}$$

is a character of A.

Proof. It follows from the construction of addition and multiplication on A. □

Note that Lemma 2.2.6 just tells us that the χ_x $(x \in K)$ are characters of A $(A = \mathcal{C}(K))$. The following theorem determines entirely $\mathfrak{M}(A)$.

2.2.7 Theorem ([1], Theorem 4.1.3)**.** *Let K be a compact set and $A = \mathcal{C}(K)$. Then, for every $\chi \in \mathfrak{M}(A)$, there exists $x \in K$ such that $\chi = \chi_x$.*

Proof. Since, for every $x \in K$, χ_x is a character of A (by Lemma 2.2.6), $\ker \chi_x = \{f \in A : f(x) = 0\}$ is a maximal ideal of A for any $x \in K$.

By the bijective correspondence between maximal ideals and characters of A (Theorem 2.2.1), Theorem 2.2.7 is equivalent to the following assertion: every maximal ideal of A is a $\ker \chi_x$ for some $x \in K$, or equivalently, that $\ker \chi_x (x \in K)$ are the only maximal ideals of A.

Now we are going to prove that the kernels $\ker \chi_x$ $(x \in K)$ are the only maximal ideals of A. To do so, we consider an arbitrary proper ideal J of A. By Proposition 1.2.4, J must be contained in some maximal ideal of A. We prove (by contradiction) that J must be contained in certain $\ker \chi_x$.

Let J be a proper ideal of A. Suppose that J is not contained in $\ker \chi_x$ for any $x \in K$. Then, for each $x \in K$, there exists $f_x \in J$ such that $f_x \notin \ker \chi_x$, that is, $\chi_x(f_x) = f_x(x) \neq 0$. Since f_x is a continuous function, there exists an open neighborhood O_x of x wherein f_x does not have any zero. Then $\{O_x : x \in K\}$ is an open covering of K. Therefore, by compactness of K, there exist $x_1, \ldots, x_n \in K$ corresponding to $f_1 := f_{x_1}, \ldots, f_n := f_{x_n}$ such that $\{O_{x_1}, \ldots, O_{x_n}\}$ is a covering of K. But we remark that, given $y \in K$, there exists $i \in \{1, \ldots, n\}$ such that $y \in O_{x_i}$, so $f_i(y) \neq 0$ and $f_i(y)\overline{f_i(y)} = |f_i(y)|^2 > 0$. Now, consider $g := f_1\overline{f_1} + \cdots + f_n\overline{f_n}$. Since the f_j's belong to the ideal J, and $\overline{f_j}$'s belong to A, we have that $g \in J$. Therefore $g(y) > 0$ for any $y \in K$. Hence g is an element of J which is invertible in A. So $J = A$, which is a contradiction since J is a proper ideal.

Thus J must be a subset of a $\ker \chi_x$ for some $x \in K$. □

2.3 The disk algebra

There is also an important application of the preceding results in the disk algebra $\mathcal{A}(\overline{\mathbb{D}})$ described in Example 1.1.8. In the following theorem, we determine the characters of $\mathcal{A}(\overline{\mathbb{D}})$.

2.3.1 Theorem ([1], Theorem 4.1.6)**.** *Let $\mathcal{A}(\overline{\mathbb{D}})$ be the commutative Banach algebra of continuous functions on $\overline{\mathbb{D}}$ which are holomorphic on $\mathbb{D}$. Then the mapping $\phi : \overline{\mathbb{D}} \to \mathfrak{M}(\mathcal{A}(\overline{\mathbb{D}}))$ defined by $\phi(x) = \chi_x$ (where $\chi_x(f) = f(x)$ for all $f \in \mathcal{A}(\overline{\mathbb{D}})$) for all $x \in \overline{\mathbb{D}}$, is a bijection.*

Proof. Let $x,\ y \in \overline{\mathbb{D}}$ such that $\chi_x = \chi_y$. Then, $f(x) = f(y)$ for all $f \in \mathcal{A}(\overline{\mathbb{D}})$. In particular, $x = y$ by taking I such that $I(z) = z$ for all $z \in \overline{\mathbb{D}}$. Therefore, ϕ is injective.

Now, let $\chi \in \mathfrak{M}(\mathcal{A}(\overline{\mathbb{D}}))$, and I the identity function on $\mathbb{C}$. It is clear that $I \in \mathcal{A}(\overline{\mathbb{D}})$. Moreover, $x := \chi(I) \in \overline{\mathbb{D}}$. Indeed, if $x \notin \overline{\mathbb{D}}$, the function $g : \overline{\mathbb{D}} \to \mathbb{C}$ defined by $g(z) = \dfrac{1}{z - x}$ belongs to $\mathcal{A}(\overline{\mathbb{D}})$. So, $\chi((I - x\mathbf{1})g) = \chi(\mathbf{1}) = 1$ on the one hand, and $\chi((I - x\mathbf{1})g) = \chi(I - x\mathbf{1})\chi(g) = 0$ on the other hand; that is a contradiction.

Let $f \in \mathcal{A}(\overline{\mathbb{D}})$ and
$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \ |z| < 1$$
the series expansion of f at 0. We note that $f = \sum_{n=0}^{\infty} a_n I^n$. So if $x \in \mathbb{D}$, we have that
$$\chi(f) = \sum_{n=0}^{\infty} a_n \chi(I)^n = \sum_{n=0}^{\infty} a_n x^n = f(x).$$
Since f is continuous on $\overline{\mathbb{D}}$, this equality holds even if x is in the boundary of $\overline{\mathbb{D}}$. So, $\chi = \chi_x = \phi(x)$. □

2.3.2 Corollary. *Let $G = \{f_1, \ldots, f_n\} \subseteq \mathcal{A}(\overline{\mathbb{D}})$ such that, for every $x \in \overline{\mathbb{D}}$, $f(x) \neq 0$ for some $f \in G$, and I the ideal generated by G. Then $I = \mathcal{A}(\overline{\mathbb{D}})$.*

Proof. Suppose that I is a proper ideal of $\mathcal{A}(\overline{\mathbb{D}})$. Then, by Proposition 1.2.4, there exists a maximal ideal M of $\mathcal{A}(\overline{\mathbb{D}})$ such that $I \subseteq M$. By Theorem 2.2.1, $M = \ker \chi$ for some $\chi \in \mathfrak{M}(\mathcal{A}(\overline{\mathbb{D}}))$. But, by Theorem 2.3.1, $\chi = \chi_x$ for some $x \in \overline{\mathbb{D}}$. So $\chi_x(f) = f(x) = 0$ for all $f \in G$, and that contradicts the properties of the elements of G. So $I = \mathcal{A}(\overline{\mathbb{D}})$. □

2.4 Gelfand representation

In this section, we present the concept of Gelfand representation for commutative Banach algebras. We start by defining and giving some properties of the Gelfand topology.

2.4.1 Definition. Let A be a commutative Banach algebra, and A' the Banach algebra of all linear continuous functionals on A. The *Gelfand topology* is the restriction of the weak*-topology of A' to $\mathfrak{M}(A)$.

The essence of this topology is the compactness, as we see in the following theorem.

2.4.2 Theorem (Gelfand)**.** ([1], Theorem 4.1.8). *Let A be a commutative Banach algebra. Then $\mathfrak{M}(A)$ is compact with respect to the Gelfand topology.*

Proof. According to the Banach-Alaoglu theorem, the set $U := \{f : ||f|| \leq 1, \ f \in A'\}$ is weak*-compact. Since $\mathfrak{M}(A)$ consists of bounded linear functionals of norm 1, we have that $\mathfrak{M}(A) \subseteq U$. Thus, it is sufficient to show that $\mathfrak{M}(A)$ is weak*-closed. To do so, we consider an element $\mathfrak{X}$ in the the weak*-closure of $\mathfrak{M}(A)$, and prove that $\mathfrak{X} \in \mathfrak{M}(A)$ (i.e $\mathfrak{X}$ is multiplicative and $\mathfrak{X}(\mathbf{1}) = 1$).

Let $f \in A'$ such that f belongs to the weak*-closure of $\mathfrak{M}(A)$ and $x, y \in A$. Let $\epsilon > 0$. For each $z \in Z = \{\mathbf{1}, x, y, xy\}$, since the projection p_z is weak*-continuous, the set $p_z^{-1}(D(f(z), \epsilon)) = \{g \in A' : |g(z) - f(z)| < \epsilon\}$ is a weak*-open neighborhood of f in A'. Then the set $U := \bigcap_{z \in Z} p_z^{-1}(D(f(z), \epsilon)$ is also a weak*-open neighborhood of f in A'. Therefore there exists an element $\chi \in \mathfrak{M}(A)$ such that $\chi \in U$. So, we have

$$|1 - f(\mathbf{1})| = |\chi(\mathbf{1}) - f(\mathbf{1})| < \epsilon.$$

Now we note that

$$f(xy) - f(x)f(y) = f(xy) - \chi(xy) + (\chi(y) - f(y))\chi(x) + (\chi(x) - f(x))f(y),$$

and therefore,

$$\begin{aligned} |f(xy) - f(x)f(y)| &= |f(xy) - \chi(xy) + (\chi(y) - f(y))\chi(x) + (\chi(x) - f(x))f(y)| \\ &\leq |f(xy) - \chi(xy)| + |\chi(y) - f(y)| \ |\chi(x)| + |\chi(x) - f(x)| \ |f(y)| \\ &< \epsilon(1 + |\chi(x)| + |f(y)|) \end{aligned}$$

Then, for any $\epsilon > 0$ and $x, y \in A$, we have that

$$\begin{cases} |1 - f(\mathbf{1})| < \epsilon \\ |f(xy) - f(x)f(y)| < \epsilon(1 + |\chi(x)| + |f(y)|), \end{cases}$$

which implies that $|1 - f(\mathbf{1})| = 0$ and $|f(xy) - f(x)f(y)| = 0$. Thus, $f(\mathbf{1}) = 1$ and $f(xy) = f(x)f(y)$, for any $x, y \in A$, that is $f \in \mathfrak{M}(A)$. □

Now, since $\mathfrak{M}(A)$ is a compact set in some topology, we can consider the commutative Banach algebra $\mathcal{C}(\mathfrak{M}(A))$, with pointwise addition and multiplication, and the supremum norm.

2.4.3 Definition (Gelfand transform)**.** Let A be a commutative Banach algebra, and $x \in A$. The *Gelfand transform* of x is the mapping

$$\begin{aligned} \hat{x} : \mathfrak{M}(A) &\longrightarrow \mathbb{C} \\ \chi &\longmapsto \chi(x) \end{aligned}$$

2.4.4 Proposition ([1], Theorem 4.1.8)**.** *Let A be a commutative Banach algebra and $x \in A$. Then*

$$\mathcal{R}(\hat{x}) = \mathrm{Sp}(x), \text{ and } ||\hat{x}|| = \rho(x).$$

Proof. The first part is straightforward using Corollary 2.2.2. For the second part we have, for every $x \in A$,

$$||\hat{x}|| = \sup_{\chi \in \mathfrak{M}(A)} |\hat{x}(\chi)| = \sup_{\chi \in \mathfrak{M}(A)} |\chi(x)| = \sup_{\lambda \in \mathrm{Sp}(x)} |\lambda| = \rho(x).$$

□

2.4.5 Theorem ([1], Theorem 4.1.8)**.** *Let A be a commutative Banach algebra and $x \in A$. Let $\varphi : A \to \mathcal{C}(\mathfrak{M}(A))$ such that $\varphi(x) = \hat{x}$. Then*

(i) φ is a continuous homomorphism.

(ii) $\ker \varphi = \mathrm{Rad}(A)$.

Proof.

(i) The fact that φ is a homomorphism is obvious. Moreover, by Proposition 2.4.4, for each $x \in A$, we have that $||\varphi(x)|| = ||\hat{x}|| = \rho(x)$. But $\rho(x) \leq ||x||$, so $||\varphi(x)|| \leq ||x||$ for all $x \in A$. Hence φ is continuous with norm ≤ 1.

(ii) Since the $\ker \chi$ $(\chi \in \mathfrak{M}(A))$ are the maximal ideals of A, we have that

$$\begin{aligned} \ker \phi &= \{x \in A : \phi(x) = 0\} \\ &= \{x \in A : \hat{x}(\chi) = 0, \quad \text{for all } \chi \in \mathfrak{M}(A)\} \\ &= \{x \in A : \chi(x) = 0, \quad \text{for all } \chi \in \mathfrak{M}(A)\} \\ &= \bigcap_{\chi \in \mathfrak{M}(A)} \ker \chi \\ &= \mathrm{Rad}(A). \end{aligned}$$

□

2.4.6 Corollary. *Let A be a semisimple Banach algebra. Then A is isomorphic to the subalgebra $\hat{A} := \{\hat{x} : x \in A\}$ of $\mathcal{C}(\mathfrak{M}(A))$.*

Proof. Let $\phi : A \to \hat{A}$ such that $\phi(x) = \hat{x}$ for all $x \in A$. It is clear that ϕ is a surjective homomorphism. Since A is semisimple, $\mathrm{Rad}(A) = \{0\}$ and then, by Theorem 2.4.5 (ii), $\ker \phi = \{0\}$, which means that ϕ is injective. Then ϕ is an isomorphism. □

Since it is given by a norm of an element of a Banach algebra, in the commutative case, the spectral radius satisfies the triangular inequalities and the submultiplicative property.

2.4.7 Corollary. *Let A be a commutative Banach algebra and $x,\ y \in A$. We have*

$$\rho(x+y) \leq \rho(x) + \rho(y) \text{ and } \rho(xy) \leq \rho(x)\rho(y).$$

Proof. By Proposition 2.4.4, we have

$$\rho(x+y) = ||\widehat{(x+y)}|| = ||\hat{x} + \hat{y}|| \leq ||\hat{x}|| + ||\hat{y}|| = \rho(x) + \rho(y),$$

and

$$\rho(xy) = ||\widehat{(xy)}|| = ||\hat{x}\hat{y}|| \leq ||\hat{x}||\ ||\hat{y}|| = \rho(x)\rho(y).$$

□

2.5 Function algebras

In §2.2, we saw some particular properties of the characters of the Banach algebra of continuous functions on a compact set. We will see in this section that some commutative Banach algebras can be identified with subalgebras of such Banach algebras.

2.5.1 Definition. Let A be a commutative Banach algebra, and K a compact set. A is called a *function algebra* on K if there exists a closed subalgebra B of $\mathcal{C}(K)$ and a map $\phi : A \to B$ such that:

(i) ϕ is an isometric isomorphism,

(ii) B contains all constant functions on K,

(iii) B separates the points of K.

2.5.2 Lemma. *Let A be a commutative Banach algebra and $\varphi : A \to \mathcal{C}(\mathfrak{M}(A))$ defined by $\varphi(x) = \hat{x}$. Then $\mathcal{R}(\varphi)$ separates the points of $\mathfrak{M}(A)$ and contains the constant functions on $\mathfrak{M}(A)$.*

Proof.

- Let $\chi_1,\ \chi_2 \in \mathfrak{M}(A)$ such that $\chi_1 \neq \chi_2$. Then, there exists $x \in A$ such that $\chi_1(x) \neq \chi_2(x)$. So $\hat{x}(\chi_1) \neq \hat{x}(\chi_2)$.

- Let $f \in \mathcal{C}(\mathfrak{M}(A))$ be the constant function $c \in \mathbb{C}$, and $x = c\mathbf{1} \in A$. For each $\chi \in \mathfrak{M}(A)$, we have

$$\hat{x}(\chi) = \chi(x) = \chi(c\mathbf{1}) = c\chi(\mathbf{1}) = c.$$

So $\hat{x} = f$, and then, $f \in \mathcal{R}(\varphi)$.

□

2.5.3 Proposition ([1], Proof of Theorem 4.1.13)**.** *Let A be a Banach algebra such that $||x^2|| = ||x||^2$ for every $x \in A$. Then $\rho(x) = ||x||$ for all $x \in A$. Consequently, A is commutative and semisimple.*

Proof.

- By induction, we have that $||x^{2^n}|| = ||x||^{2^n}$ for all positive integers n. So, using the spectral radius formula (Theorem 1.3.6), we have that

$$\rho(x) = \lim_{n\to\infty} ||x^n||^{\frac{1}{n}} = \lim_{n\to\infty} ||x^{2^n}||^{\frac{1}{2^n}} = ||x||. \qquad (*)$$

- Let $x, y \in A$ and $h : \mathbb{C} \to A$ the function defined by $h(\lambda) = e^{\lambda y} x e^{-\lambda y}$ for all $\lambda \in \mathbb{C}$. Let $l \in A'$ be a bounded linear functional. Since h is an analytic function, by Proposition 1.4.2, $l \circ h$ is an entire function. By ($*$), we have that

$$|l \circ h(\lambda)| \leq ||l||\ ||h(\lambda)|| = ||l||\rho(e^{\lambda y} x e^{-\lambda y}) = ||l||\rho(x) = ||l||\ ||x||,$$

for all $\lambda \in \mathbb{C}$. By Liouville's Theorem (Theorem 1.4.4), $l \circ h$ is a constant function, say $l \circ h(\lambda) = l \circ h(0) = l(x)$, for all $\lambda \in \mathbb{C}$. Since l was taken arbitrarily, by Corollary 1.5.2, we have that $h(\lambda) = x$ for all $\lambda \in \mathbb{C}$.

Moreover

$$\begin{aligned} h(\lambda) &= e^{\lambda y} x e^{-\lambda y} \\ &= \left(\mathbf{1} + \lambda y + \mathrm{O}(\lambda^2)\right) x \left(\mathbf{1} - \lambda y + \mathrm{O}(\lambda^2)\right) \\ &= x + \lambda(yx - xy) + \mathrm{O}(\lambda^2). \end{aligned}$$

Since this is true for all $\lambda \in \mathbb{C}$, we have that $yx - xy = 0$.

Thus A is commutative.

- By Corollary 2.2.3 -(ii), $x \in \mathrm{Rad}(A)$ if and only if $x \in \mathrm{QN}(A)$, that is $\rho(x) = 0$. But as we see in ($*$), $\rho(x) = ||x||$. So $\rho(x) = 0$ if and only if $x = 0$. Hence $\mathrm{Rad}(A) = \{0\}$, which means that A is semisimple.

□

We now give the necessary and sufficient conditions for a Banach algebra to be a function algebra.

2.5.4 Theorem ([1], Theorem 4.1.13)**.** *Let A be Banach algebra. The following assertions are equivalent:*

(i) A is a function algebra.

(ii) For all $x \in A$, $||x^2|| = ||x||^2$.

Proof. Suppose that A is a function algebra on a compact set K. Let ϕ and B be as in Definition 2.5.1, $x \in A$ and $f = \phi(x)$. Since ϕ is isometric, $||x|| = ||f|| = \sup_{z \in K} |f(z)|$. So $||x||^2 = \left(\sup_{z \in K} |f(z)|\right)^2 = \sup_{z \in K} |f(z)|^2 = \sup_{z \in K} |f^2(z)| = ||x^2||$.

Conversely, suppose that $||x^2|| = ||x||^2$ for all $x \in A$. By Proposition 2.5.3, A is commutative. Consider the homomorphism $\varphi : A \to \mathcal{C}(\mathfrak{M}(A))$ defined by $\varphi(x) = \hat{x}$ for all $x \in A$. By Lemma 2.5.2, $\hat{A} := \mathcal{R}(\varphi)$ separates the points of $\mathfrak{M}(A)$ and contains the constants. Since A is semisimple (Proposition 2.5.3), φ is an isomorphism $A \to \hat{A}$ (Corollary 2.4.6).

It follows from Proposition 2.4.4 and Proposition 2.5.3 that $||\hat{x}|| = ||x||$. So $||\varphi(x)|| = ||x||$ for all $x \in A$, which means that φ is isometric. □

2.6 Finitely generated Banach algebras

We know that the set $\mathbb{C}[x_1, \ldots, x_n]$ of n-variate polynomials with complex coefficients does not have a Banach algebra norm (Example 1.1.10). However, we can use it to construct certain types of Banach algebras as we see in the following definition.

2.6.1 Definition. Let A be a Banach algebra. We say that A is *(topologically) generated* by n $(n \geq 1)$ element(s) if there exist $a_1, \ldots, a_n \in A$ such that for every $x \in A$, there exists a sequence $(P_k)_{k \in \mathbb{N}}$ of n-variate polynomials such that

$$x = \lim_{k \to \infty} P_k(a_1, \ldots, a_n).$$

The elements $a_1, \ldots, a_n$ are called *(topological) generators* of A.

As an immediate consequence of this definition, we have the following result.

2.6.2 Proposition. *Let A be a Banach algebra which is generated by n elements $a_1, \ldots, a_n$. Then the set*

$$\mathbb{C}[a_1, \ldots, a_n] := \{P(a_1, \ldots, a_n) : P \text{ an } n\text{-variate polynomial}\}$$

is a dense subalgebra of A.

Proof. Let A and $a_1, \ldots, a_n$ be as stated.

It is obvious that $\mathbb{C}[a_1, \ldots, a_n]$ is a subalgebra of A. Let U be a non-empty open subset of A, and $x \in U$. Since A is generated by $a_1, \ldots, a_n$, there exists a sequence $(P_k)_{k\in\mathbb{N}}$ of $n-$variate polynomials such that $(P_k(a_1, \ldots, a_n))_{k\in\mathbb{N}}$ converges to x. Then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $P_k(a_1, \ldots, a_n) \in U$. Hence U contains at least one element of $\mathbb{C}[a_1, \ldots, a_n]$, namely P_{k_0}. Thus $\mathbb{C}[a_1, \ldots, a_n]$ is dense in A. □

2.6.3 Remark. Due to this proposition, any continuous function on A into a Hausdorff space is uniquely determined by its restriction to $\mathbb{C}[a_1, \ldots, a_n]$. In particular, any character of A is uniquely determined by its restriction to $\{a_1, \ldots, a_n\}$.

2.6.4 Definition. Let K be a compact subset of $\mathbb{C}$, and

$$\hat{K} := \{z \in \mathbb{C}^n : |p(z)| \leq \max_{x\in K} |p(x)|, \text{ for all } p \in \mathbb{C}[x_1, \ldots, x_n]\}.$$

(i) $\hat{K}$ is called the *polynomial convex hull* of K.

(ii) K is said to be *polynomially convex* if $K = \hat{K}$.

Obviously, the inclusion $K \subseteq \hat{K}$ is always true. If $n = 1$, $\hat{K}$ is called the *connected hull* of K.

2.6.5 Theorem ([1], Theorem 4.1.14)**.** *Let A be a commutative Banach algebra which is topologically generated by n elements $a_1, \ldots, a_n$, and $\psi : \mathfrak{M}(A) \to \mathbb{C}^n$ the function defined by $\psi(\chi) = (\chi(a_1), \ldots, \chi(a_n))$. Then ψ is a homeomorphism $\mathfrak{M}(A) \to \mathcal{R}(\psi)$. Moreover, $\mathcal{R}(\psi)$ is polynomially convex.*

Proof. By definition of the Gelfand topology, the projections $\chi \mapsto \chi(x_i)$ $(i = 1, \ldots, n)$ are continuous. So ψ is continuous. Since $\mathfrak{M}(A)$ is compact for the Gelfand topology, then so $\psi(\mathfrak{M}(A))$ is for the usual topology of $\mathbb{C}^n$. Moreover, $\psi(\mathfrak{M}(A))$ is Hausdorff, so that ψ is a homeomorphism if and only if it is a bijection. The surjectivity is obvious. Now, let $\chi_1,\ \chi_2 \in \mathfrak{M}(A)$ such that $\psi(\chi_1) = \psi(\chi_2)$. So $\chi_1(a_i) = \chi_2(a_i)$ for all $i = 1, \ldots, n$, and then, by Remark 2.6.3, $\chi_1 = \chi_2$.

Now let us prove that $K := \psi(\mathfrak{M}(A))$ is polynomially convex. Let $x \in \hat{K}$ and p an n-variate polynomial with coefficients in $\mathbb{C}$. Then

$$|p(x)| \leq \max_{z\in K} |p(z)| = \max_{\chi\in\mathfrak{M}(A)} |p(\chi(a_1), \ldots, \chi(a_n))|.$$

But we have that $p(\chi(a_1), \ldots, \chi(a_n)) = \chi(p(a_1, \ldots, a_n))$, so $p(\chi(a_1), \ldots, \chi(a_n)) \in \mathrm{Sp}(p(a_1, \ldots, a_n))$ for every $\chi \in \mathfrak{M}(A)$. Then it follows that

$$|p(\chi(a_1), \ldots, \chi(a_n))| \leq \rho(p(a_1, \ldots, a_n)) \leq ||p(a_1, \ldots, a_n)||.$$

We see now that $\chi_0 : \mathbb{C}[a_1, \ldots, a_n] \to \mathbb{C}$ defined by $\chi_0(p(a_1, \ldots, a_n)) = p(x)$, for all n-variate polynomials p, is a multiplicative bounded linear functional such that $|\chi_0(y)| \leq ||y||$ for all $y \in \mathbb{C}[a_1, \ldots, a_n]$.

By the Hahn-Banach theorem, χ_0 can be extended to a linear functional χ on A. Moreover, χ is a character of A because $\mathbb{C}[a_1, \dots, a_n]$ is dense in A. Since $x = (\chi(p_1(a_1, \dots, a_n)), \dots, \chi(p_n(a_1, \dots, a_n)))$, where $p_i(x_1, \dots, x_n) := x_i$ $(i = 1, \dots, n)$ are n-variate polynomials, we have that $x \in K$. Hence K is polynomially convex. $\square$

3. Examples and further applications

3.1 The Wiener-Lévy Theorem

One of the most important applications of the above results is the proof of the Wiener-Lévy Theorem. We consider the Wiener algebra W, described in Example 1.1.9. In 1933, N. Wiener proved that if a function f has absolutely converging trigonometric series and f does not vanish at any point in $[0, 2\pi]$, then f is invertible in the Wiener algebra. This result was generalized by P. Lévy by composing such an f with a holomorphic function. A short proof of this result was given by I. M. Gelfand, using Banach algebras, as we will see in Corollary 3.1.2. We begin by describing the characters of the Wiener algebra. We have the following result.

3.1.1 Theorem ([1], Theorem 4.1.4)**.** *Let W be the Wiener algebra, then $\mathfrak{M}(W) = \{\chi_x : x \in [0, 2\pi]\}$, where $\chi_x(f) = f(x)$ for all $f \in W$ and $x \in [0, 2\pi]$.*

Proof. For $x \in [0, 2\pi]$, $\chi_x : W \to \mathbb{C},\ f \mapsto f(x)$ is obviously a character of W.

Conversely, let $f = \sum_{n=-\infty}^{\infty} a_n g^n \ \in W$ and $\chi \in \mathfrak{M}(W)$. Since χ is a continuous linear operator of norm 1, we have that

$$\begin{cases} |\chi(g)| \leq ||g|| = 1 \\ |\chi(g^{-1})| \leq ||g^{-1}|| = 1 \end{cases}$$

for any $t \in [0, 2\pi]$. However, by the multiplicative property of χ,

$$\chi(g)\chi(g^{-1}) = \chi(gg^{-1}) = \chi(\mathbf{1}) = 1.$$

So, $|\chi(g)\chi(g^{-1})| = |\chi(g)|\,|\chi(g^{-1})| = 1$. Hence $|\chi(g)| = 1$, and then $\chi(g) = e^{ix}$ for some[1] $x \in [0, 2\pi]$. Therefore $\chi(g^n) = e^{inx} = g^n(x)$ for any $n \in \mathbb{Z}$, and then,

$$\chi(f) = \sum_{n=-\infty}^{\infty} a_n g^n(x) = f(x) = \chi_x(f).$$

□

3.1.2 Corollary (Wiener-Lévy)**.** *Let $f \in W$, $g : V \to \mathbb{C}$ be holomorphic on V for some open neighborhood V of $f([0, 2\pi])$. Then $g(f) \in W$.*

Proof. By Corollary 2.2.2, and the preceding theorem, we have

$$\mathrm{Sp}(f) = \{\chi(f) : \chi \in \mathfrak{M}(W)\} = \{\chi_x(f) : x \in [0, 2\pi]\} = \{f(x) : x \in [0, 2\pi]\} = f([0, 2\pi]).$$

By holomorphic functional calculus, $g(f) = \frac{1}{2i\pi}\int_\gamma g(t)(t\mathbf{1} - f)^{-1}\mathrm{d}t$, where γ is any contour surrounding $\mathrm{Sp}(f)$, is an element of W. □

3.1.3 Corollary (Wiener)**.** *If $f : [0, 2\pi] \to \mathbb{C}$ is such that $f \in W$ and $f(x) \neq 0$ for any $x \in [0, 2\pi]$, then $\frac{1}{f} : [0, 2\pi] \to \mathbb{C},\ x \mapsto \frac{1}{f(x)}$ is an element of W.*

Proof. The function $h : \mathbb{C} \to \mathbb{C},\ x \mapsto \frac{1}{x}$ is holomorphic in any open set not containing 0. Hence, h is holomorphic in any open neighbourhood of $f([0, 2\pi])$ in $\mathbb{C} \setminus \{0\}$. By the preceding corollary, $h(f) = \frac{1}{f} \in W$. □

[1] $\chi(g)$ is in the unit circle of the complex plane

3.2 Automatic continuity

Given two normed spaces X and Y, if X is finite dimensional, then a linear map from X to Y is always continuous. Such automatic continuity holds for some infinite dimensional commutative Banach algebras as we show in Theorem 3.2.2. The following lemma is useful in the present and the next section.

3.2.1 Lemma. *Let A and B be commutative Banach algebras, and $T : A \to B$ a homomorphism. If $\chi \in \mathfrak{M}(B)$, then $\chi \circ T \in \mathfrak{M}(A)$.*

Proof. Let $\chi \in \mathfrak{M}(B)$ and $x,\ y\ \in A$. By definition of a homomorphism, $T(xy) = T(x)T(y)$. So $\chi(T(xy)) = \chi(T(x)T(y)) = \chi(T(x))\chi(T(y)) = (\chi \circ T(x))(\chi \circ T(y))$. Then $\chi \circ T$ is multiplicative. Since $T(\mathbf{1}) = \mathbf{1}$, we have in addition that $\chi \circ T(\mathbf{1}) = 1$. □

3.2.2 Theorem. *Let A and B be commutative Banach algebras such that B is semisimple. Then every homomorphism $T : A \to B$ is continuous.*

Proof. Let A, B and T as stated. We know that a linear operator is continuous if and only if it is continuous at a single point. So, it suffices to prove that T is continuous at 0. Let $(x_n)_{n\in\mathbb{N}}$ a sequence converging to 0 and suppose that the sequence $(T(x_n))_{n\in\mathbb{N}}$ converges to a limit l. Let $\chi \in \mathfrak{M}(B)$. Then $\chi \circ T \in \mathfrak{M}(A)$ (by Lemma 3.2.1) and we have that

$$\begin{aligned}\chi\,(l) &= \lim_{n\to\infty} \chi(T(x_n)) \\ &= \lim_{n\to\infty} \chi \circ T(x_n) \\ &= 0,\end{aligned}$$

since χ is continuous (Proposition 2.1.5), $\chi \circ T \in \mathfrak{M}(A)$ and $x_n \to 0$. Therefore, $l \in \ker \chi$ for all $\chi \in \mathfrak{M}(B)$, and then, by Corollary 2.2.3, l belongs to the radical of B. Since B is semisimple, $l = 0$. Thus, we conclude form the Closed Graph Theorem that T is continuous. □

The above result has the following corollary regarding equivalence of norms.

3.2.3 Corollary. *Let A be a commutative semisimple Banach algebra and $||\cdot||_1$ and $||\cdot||_2$ two (Banach algebra) norms on A. Then, $||\cdot||_1$ and $||\cdot||_2$ are equivalent, that is, there exist two positive constants $c_1,\ c_2$ such that, for all $x \in A$, $c_1||x||_1 \le ||x||_2 \le c_2||x||_1$.*

Proof. Let $I : (A, ||\cdot||_1) \to (A, ||\cdot||_2)$ be the identity function. Since A is semisimple, by Theorem 3.2.2, I is continuous. So, there exists a positive constant c_2 such that, for all $x \in A$, $||x||_2 = ||I(x)||_2 \le c_2||x||_1$.

Considering the identity function $J : (A, ||\cdot||_2) \to (A, ||\cdot||_1)$ and the same argument as above, there exists a positive constant c such that, for all $x \in A$, $||x||_1 = ||J(x)||_1 \le c||x||_2$.

We can take $c_1 = \frac{1}{c}$, and then we get

$$c_1||x||_1 \le ||x||_2 \le c_2||x||_1,$$

for all $x \in A$, which means that the two norms are equivalent. □

Corollary 3.2.3 also has a consequence (see Corollary 3.2.9) concerning the continuity of involutions in commutative Banach algebras.

3.2.4 Definition (Involution)**.** Let A be a Banach algebra. An *involution* on A is a mapping $*: A \to A$ satisfying

(i) $(\lambda x + y)^* = \overline{\lambda} x^* + y^*$

(ii) $(xy)^* = y^* x^*$

(iii) $(x^*)^* = x$

for all $x, \ y \in A$ and $\lambda \in \mathbb{C}$.

A trivial example of involution is the complex conjugation on $\mathbb{C}$. More generally, we can also consider the matrix conjugation, as shown in the following example.

3.2.5 Example. Let $A = M_n(\mathbb{C})$. The mapping $A \to A$ defined by $x^* = (\overline{x})^t$ where $(.)^t$ is the matrix transposition, is an involution.

3.2.6 Example. Let K be a compact set. The mapping $*: \mathcal{C}(K) \to \mathcal{C}(K)$ defined by $f^* = \bar{f}$ (where $\bar{f}(x) = \overline{f(x)}$ for all $x \in K$) is an involution.

Even if they are not linear, involutions still have properties in common with homomorphisms.

3.2.7 Lemma. *Let A be a Banach algebra and $*$ an involution on A. The mapping $|||\cdot||| : A \to A$ such that $|||x||| = ||x^*||$ for all $x \in A$ defines a Banach algebra norm on A.*

Proof. Obviously, $|||\cdot|||$ is a norm on A. Moreover, if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(A, |||\cdot|||)$, then $(x_n^*)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(A, ||\cdot||)$. So there exists some $x \in A$ such that $||x_n^* - x|| \underset{n\to\infty}{\longrightarrow} 0$. But $||x_n^* - x|| = ||(x_n - x^*)^*||$, and then $||(x_n - x^*)^*|| = |||x_n - x^*||| \underset{n\to\infty}{\longrightarrow} 0$, so that $(x_n)_{n\in\mathbb{N}}$ converges to x^* for the norm $|||\cdot|||$.

In addition, for $x, \ y \in A$, we have

$$|||xy||| = ||(xy)^*|| = ||y^* x^*|| \leq ||y^*|| \; ||x^*|| = |||x||| \; |||y|||,$$

and finally,

$$|||\mathbf{1}||| = ||\mathbf{1}^*|| = ||\mathbf{1}^*\mathbf{1}|| = ||(\mathbf{1}\mathbf{1}^*)^*|| = ||(\mathbf{1}^*)^*|| = ||1|| = 1.$$

□

3.2.8 Lemma. *An involution $*$ in a Banach algebra A is continuous if and only if there exists a positive constant c such that, for all $x \in A$, $||x^*|| \leq c||x||$.*

Proof. This lemma can be proved in the same way as the continuity of bounded linear operators. See [3], proof of Theorem 2.7-9-(a). □

3.2.9 Corollary. *Let A be a commutative semisimple Banach algebra. Then every involution on A is continuous.*

Proof. Let $(A, ||\cdot||)$ be a commutative semisimple Banach algebra, and $*$ an involution on A. Since A is semisimple, by Corollary 3.2.3, the norm $||\cdot||$ and the norm $|||\cdot|||$ defined in Lemma 3.2.7 are equivalent. So, there exists a positive constant c such that, for all $x \in A$,

$$|||x||| = ||x^*|| \leq c||x||.$$

By Lemma 3.2.8, the involution $*$ is continuous. □

3.3 Existence of Banach algebra norms

We refer to ([5], E 2.1.9, p.51) for the following result. For every positive integer n, the set $\mathcal{C}^n([a,b])$ of n times continuously differentiable functions on the interval $[a,b]$ is a Banach algebra with pointwise addition and multiplication, and the norm defined by

$$||f|| = \sum_{k=0}^{n} ||f^{(k)}||_\infty,$$

where $f^{(0)} = f$ and $f^{(k)}$ the k-th derivative of f for $k \geq 1$. We will see in Theorem 3.3.2 that such a construction cannot be extended to the case where $n \to \infty$. Since the algebra $\mathcal{C}^\infty([a,b])$ of infinitely differentiable functions on $[a,b]$, is a subalgebra of $\mathcal{C}([a,b])$, we will need some properties of $\mathcal{C}([a,b])$.

3.3.1 Lemma. *Let $a, b \in \mathbb{R}$ such that $a < b$. Then the Banach algebra $\mathcal{C}([a,b])$ is semisimple.*

Proof. By Theorem 2.2.7, every character $\chi \in \mathfrak{M}(\mathcal{C}([a,b]))$ is such that $\chi = \chi_x$, for some $x \in [a,b]$. By 2.2.3, we have that

$$\mathrm{Rad}(\mathcal{C}([a,b])) = \bigcap_{\chi \in \mathfrak{M}(\mathcal{C}([a,b]))} \ker \chi = \{f \in \mathcal{C}([a,b]) : f(x) = 0 \text{ for all } x \in [a,b]\} = \{0\}.$$

□

3.3.2 Theorem ([1], Corollary 4.1.12). *Let $a,\ b \in \mathbb{R}$ such that $a < b$. There is no Banach algebra norm in the algebra $\mathcal{C}^\infty([a,b])$.*

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$, and $||\cdot||$ a Banach algebra norm in $\mathcal{C}^\infty([a,b])$. We denote the supremum norm in $\mathcal{C}([a,b])$ by $||\cdot||_\infty$. Since $\mathcal{C}([a,b])$ is semisimple (Lemma 3.3.1), by Theorem 3.2.2, the identity homomorphism $I : \mathcal{C}^\infty([a,b]) \to \mathcal{C}([a,b])$ is continuous. Then there exists a positive constant c such that $||f||_\infty \leq c||f||$ for all $f \in \mathcal{C}^\infty([a,b])$.

Now let $D : \mathcal{C}^\infty([a,b]) \to \mathcal{C}^\infty([a,b])$ be the homomorphism defined by $D(f) = f'$ for all $f \in \mathcal{C}^\infty([a,b])$. D is a continuous homomorphism. Indeed, given a sequence $(f_n)_{n\in\mathbb{N}}$ converging to 0, if $||f_n' - g|| \to 0$ for some $g \in \mathcal{C}^\infty([a,b])$, then $||f_n' - g||_\infty \to 0$. Since $||f_n||_\infty \to 0$, the function $G : [a,b] \to \mathbb{C}$ defined by $G(t) = \int_a^t g(u)\mathrm{d}u$ is identically 0, and then $g = 0$. Indeed, G coincides with the uniform limit of the function sequence $(f_n)_{n\in\mathbb{N}}$, and $g = G'$. Thus, by the Closed Graph Theorem, D is continuous. Then there exists $\beta > 0$ such that, for every $f \in \mathcal{C}^\infty([a,b])$, $||f'|| \leq \beta||f||$.

For $\lambda \in \mathbb{C}$, let

$$T_\lambda(f) := \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f^{(n)}.$$

Since $||f'|| \leq \beta||f||$, it follows (by induction) that $||f^{(n)}|| \leq \beta^n||f||$ for all $f \in \mathcal{C}^\infty([a,b])$ and $n \geq 1$. So $\frac{|\lambda|^n}{n!}||f^{(n)}|| \leq \frac{|\lambda|^n\beta^n}{n!}||f||$ for all $n \in \mathbb{N}$. Since $\sum_{n=0}^{\infty} \frac{|\lambda|^n\beta^n}{n!}$ is convergent, the series defined by T_λ is convergent for every $\lambda \in \mathbb{C}$. Moreover T_λ is a homomorphism from $\mathcal{C}^\infty([a,b])$ into itself. Indeed, the linearity is obvious, and given $f, g \in \mathcal{C}^\infty([a,b])$, we have that

$$T_\lambda(fg) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}(fg)^{(n)}.$$

By Leibniz formula

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)},$$

so

$$\begin{aligned} T_\lambda(fg) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)} \\ &= \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f^{(n)} \right) \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} g^{(n)} \right) \\ &= T_\lambda(f) T_\lambda(g). \end{aligned}$$

Let $\chi \in \mathfrak{M}(\mathcal{C}^\infty([a,b]))$. By Lemma 3.2.1, $\chi \circ T_\lambda$ is also a character of $\mathcal{C}^\infty([a,b])$. So it is continuous of norm 1 (Proposition 2.1.5), and then

$$||\chi \circ T_\lambda(f)|| \leq ||f|| \text{ for all } f \in \mathcal{C}^\infty([a,b]).$$

Now let $f \in \mathcal{C}^\infty([a,b])$ and $\phi : \mathbb{C} \to \mathbb{C}$ be the function defined by $\phi(\lambda) = \chi \circ T_\lambda(f)$. By Proposition 1.4.2, ϕ is an entire function. Moreover ϕ is bounded, so by Liouville's Theorem, ϕ is a constant function. Hence, $\phi(\lambda) = \phi(0)$ for all $\lambda \in \mathbb{C}$. We have that

$$\phi(\lambda) = \chi\left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f^{(n)} \right) = \chi(f) + \chi\left(\sum_{n=1}^{\infty} \frac{\lambda^n}{n!} f^{(n)} \right),$$

and $\phi(0) = \chi(f)$. So for all $\lambda \in \mathbb{C}$,

$$\chi\left(\sum_{n=1}^{\infty} \frac{\lambda^n}{n!} f^{(n)} \right) = 0.$$

Since it is true for any $\chi \in \mathfrak{M}(\mathcal{C}^\infty([a,b]))$, it follows from Corollary 2.2.3-(ii) that $\sum_{n=1}^{\infty} \frac{\lambda^n}{n!} f^{(n)} \in \mathrm{Rad}(\mathcal{C}^\infty([a,b]))$.

But $\mathcal{C}^\infty([a,b])$ is semisimple (for the same reason as in Lemma 3.3.1), so for all $\lambda \in \mathbb{C}$,

$$\sum_{n=1}^{\infty} \frac{\lambda^n}{n!} f^{(n)} = 0.$$

This implies that $f' = 0$, hence f is a constant function, which is absurd since f was chosen arbitrarily. □

3.4 Uniform continuity of the spectrum

We consider the set $\mathfrak{C}$ of all compact subsets of $\mathbb{C}$ together with the Hausdorff distance Δ defined by

$$\Delta(K_1, K_2) := \max\left\{ \sup_{z \in K_2} \mathrm{d}(z, K_1), \sup_{z \in K_1} \mathrm{d}(z, K_2) \right\},$$

where, for a compact set $K \subseteq \mathbb{C}$ and $z \in \mathbb{C}$, $\mathrm{d}(z, K) := \inf_{x \in K} |z - x|$. We also introduce the following notation. Let $\alpha > 0$ and $K \in \mathfrak{C}$. We denote by $K + \alpha$ the set

$$K + \alpha := \{z \in \mathbb{C} : \mathrm{d}(z, K) \leq \alpha\}.$$

Our objective is to show that if A is a *commutative* Banach algebra, the function $\sigma : A \to \mathfrak{C}$ defined by $\sigma(x) = \mathrm{Sp}(x)$ for all $x \in A$, is uniformly continuous (see Corollary 3.4.3).

We remark that σ is:

- *Continuous at a point* $a \in A$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ with $||x - a|| \leq \delta$, then $\Delta(\sigma(x), \sigma(a)) \leq \epsilon$.
- *Continuous in a subset* $U \subseteq A$ if and only if it is continuous at each point of U.
- *Uniformly continuous* in a subset $U \subseteq A$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in U$, $||x - y|| \leq \delta$ implies $\Delta(\sigma(x), \sigma(y)) \leq \epsilon$.

3.4.1 Theorem ([1], Theorem 3.4.1)**.** *Let A be a Banach algebra and $x,\ y \in A$ such that $xy = yx$. Then*

$$\mathrm{Sp}(x) \subseteq \mathrm{Sp}(y) + \rho(x - y).$$

Proof. Let x and y be as stated. Suppose that there exists an element $\lambda \in \mathrm{Sp}(x)$ such that $\lambda \notin \mathrm{Sp}(y) + \rho(x - y)$. Then $\mathrm{d}(\lambda, \mathrm{Sp}(y)) > \rho(x - y)$, so $\lambda \notin \mathrm{Sp}(y)$ as well. By Proposition 1.3.8, we have

$$\mathrm{d}(\lambda, \mathrm{Sp}(y)) = \frac{1}{\rho((\lambda\mathbf{1} - y)^{-1})}.$$

So $\frac{1}{\rho((\lambda\mathbf{1} - y)^{-1})} > \rho(x - y)$, that is $\rho(x - y)\rho((\lambda\mathbf{1} - y)^{-1}) < 1$. Therefore, by Corollary 2.4.7,

$$\rho((x - y)(\lambda\mathbf{1} - y)^{-1}) < 1.$$

Consequently, $1 \notin \mathrm{Sp}((x - y)(\lambda\mathbf{1} - y)^{-1})$, that is $\mathbf{1} - (x - y)(\lambda\mathbf{1} - y)^{-1} \in A^{-1}$. Multiplying by $\lambda\mathbf{1} - y$, we have that

$$(\lambda\mathbf{1} - y)\left(\mathbf{1} - (x - y)(\lambda\mathbf{1} - y)^{-1}\right) = \lambda\mathbf{1} - y - x + y = \lambda\mathbf{1} - x,$$

which is a contradiction since $\lambda\mathbf{1} - x$ is not invertible, but it is written as product of two invertible elements. □

3.4.2 Corollary. *Let A be a Banach algebra and $x,\ y \in A$ such that $xy = yx$. Then*

$$\Delta(\mathrm{Sp}(x), \mathrm{Sp}(y)) \leq \rho(x - y) \leq ||x - y||.$$

Proof. Let x and y be as stated. Since $\mathrm{Sp}(x) \subseteq \mathrm{Sp}(y)) + \rho(x - y)$ (Theorem 3.4.1), for all $z \in \mathrm{Sp}(x)$, we have that $\mathrm{d}(z, \mathrm{Sp}(y)) \leq \rho(x - y)$. Swapping x and y in Theorem 3.4.1, we have also that $\mathrm{d}(z, \mathrm{Sp}(x)) \leq \rho(x - y)$ for all $z \in \mathrm{Sp}(y)$. Then

$$\Delta(\mathrm{Sp}(x), \mathrm{Sp}(y)) = \max\left\{\sup_{z \in \mathrm{Sp}(x)} \mathrm{d}(z, \mathrm{Sp}(y)),\ \sup_{z \in \mathrm{Sp}(y)} \mathrm{d}(z, \mathrm{Sp}(x))\right\} \leq \rho(x - y).$$

By Corollary 1.3.7, we have also $\rho(x - y) \leq ||x - y||$. □

3.4.3 Corollary. *Let A be a commutative Banach algebra. Then the function σ defined above is uniformly continuous.*

Proof. Let $x,\ y \in A$. Since A is commutative, by Corollary 3.4.2, we have

$$\Delta(\mathrm{Sp}(x), \mathrm{Sp}(y)) \le ||x - y||.$$

Then, for any $\epsilon > 0$, there exists $\delta\ (= \epsilon)$ such that $||x-y|| \le \delta$ implies $\Delta(\sigma(x), \sigma(y)) = \Delta(\mathrm{Sp}(x), \mathrm{Sp}(y)) \le \epsilon$. Hence, σ is uniformly continuous. □

Conclusion

We have seen in Theorem 2.2.7 that, given a compact set K, the characters of $\mathcal{C}(K)$ $(\mathfrak{M}(\mathcal{C}(K)))$ can be identified with K. For a commutative Banach algebra A, the set of characters $\mathfrak{M}(A)$ is a compact set in the Gelfand topology. We can thus consider the Banach algebra $\mathcal{C}(K)$, where $K = \mathfrak{M}(A)$. If, in addition, A is semisimple, then A is isomorphic to some subalgebra of $\mathcal{C}(K)$ (Corollary 2.4.6). A stronger (in terms of identification) result, given in Theorem 2.5.4, says that this isomorphism is isometric if every element $x \in A$ satisfies $||x^2|| = ||x||^2$. This is the Gelfand representation theory for commutative Banach algebras. This theory has several applications, some which are presented in Chapter 3. One of the most interesting applications is the short proof of the Wiener Theorem as shown in Corollary 3.1.3.

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