Pricing American Options using Least Squares Monte Carlo method

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23 May 2019

Submitted in partial fulfillment of a structured masters degree at AIMS South Africa

Abstract

In this study, we consider the Least squares Monte Carlo method introduced by Longstaff and Schwartz (2001) to value American put option. The particularity of this method is the approximation of the continuation value using Least squares method, unlike the Monte Carlo simulation introduced by Tilley (1993) which the continuation value is estimated for each group of simulated stock price ordered from the minimum to the maximum. Numerical experiments are performed to compare the Least squares Monte Carlo and the Monte Carlo method, and the efficiency of the method is considered.

Keywords: Least Squares Method, Monte Carlo Method, Option Pricing Theory, American options, Least Squares Monte Carlo Method.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Berthine Nyunga Mpinda, 23 May 2019

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1. Introduction

1.1 Background

Options pricing has played a crucial role and empowered activities of banks and financial market. Different style options have been proposed; among all options, American and European options are the most common. Options are financial instruments used on exchanges and financial institutions. An option is defined as a type of contract that gives to the owner the right, but not the obligation to buy or sell an underlying asset at a fixed date and fixed price in the contract (Hull, 2008).

Valuing options is a major area of interest within financial mathematics (Kazeem, 2014), and many corporate responsibility can be expressed in terms of options (Boyle, 1977). The difference between the American options and European options is that the first one can be exercised before the expiration date whereas the second one can only be exercised at the expiry date. American options have a pivotal role on exchanges (Hull, 2008), their valuation of optimal exercise remains one of the most challenging problems in derivatives, particularly when more than one factor affects the option value (Longstaff and Schwartz, 2001).

The Black-Scholes equation first suggested by Black and Scholes (1973) provides an analytical solution only for European-style options. Different studies have been done in order to find the American option value by using different techniques. In order to solve American options, Cox, Ross & Rubinstein (1979) introduced the Binomial model for pricing options. However numerical methods such as Finite Difference method were suggested to price American options by approximating Black-Scholes equation (Stentoft, 2004). But for options whose price depends on multiple stochastic factors such as interest rate, and volatility, and with complicated features, these techniques become complex to evaluate.

Therefore, complex models that are difficult to analyse can be approximated by using simulation techniques such as Monte Carlo methods. It was first shown by Boyle that the Monte Carlo simulation can be used for pricing American options (Boyle, 1977). The Monte Carlo approach is a helpful alternative to the binomial method and presents many advantages as a framework to value risk management and optimally exercise American options (Longstaff and Schwartz, 2001). Also the technique is flexible and assumes variables follow stochastic processes. However, Monte Carlo method is relatively slow.

Longstaff and Schwartz (2001) introduced a new approach known as the Least squares Monte Carlo method to evaluate American options, that combines Monte Carlo simulation and Least squares regression in American option value calculation. Several analysis have been done, to prove the convergence of the Least square Monte Carlo algorithm to the true value (Clément et al., 2002), (Glasserman, 2013). Huang and Huang (2009), used different polynomial families to approximate the continuation value and compare it to an existing options calculator. Svensson (2004), used simple polynomials as basis functions and compared the results to a finite difference method.

1.2 Objective

In this study, we are going to evaluate American put option by using three different methods, the Binomial method, the Monte Carlo simulation introduced and the Least Squares Monte Carlo method (LSMC).

The Least squares Monte Carlo method proceeds in two steps, the Least squares method to approximate the conditional expectation value and the Monte Carlo methods to simulate the multiple stock price paths and to get the mean option value for all paths. The next Chapter presents basic notions in finances and option theory that will be used all throughout our study. The third Chapter presents principles of Monte Carlo methods and the Least squares method. The fourth describes pricing options methods such as the Binomial method, the Monte Carlo simulation and Least Squares Monte Carlo method, which is the main subject of our study. Thus, we will compare the numerical results obtained by using the methods enumerated above.

2. Basic Notions in Finance and Option Theory

In this Chapter, we introduce important basic definitions, concepts, and notations used in Probability Theory, Stochastic Processes, and Options Theory. All these terms will be important to better understand the rest of our study.

2.1 Probability Theory

2.1.1 Definition: Probability measure. (Shreve, 2004) Let Ω be a nonempty set called a sample space and E an event. A probability measure P is defined as a function which satisfies the following assumptions

- A probability value of an event belongs in the interval $[0, 1]$,
- The probability of the nonempty set is equal to one $(P(\Omega) = 1)$,
- The probability of an impossible event is equal to zero $(P(\emptyset) = 0)$,
- If a sequence of events A_s are disjoint sets, then we have

$$P(\bigcup_{s=1}^{\infty} A_s) = \sum_{s=1}^{\infty} P(A_s).$$

2.1.2 Definition: $\sigma-$ algebra . (Shreve, 2004) Let Ω be a set with finitely many elements. A σ-algebra denoted by $\mathcal{F}$ is a collection of subsets of Ω which satisfy the following conditions

- The impossible event $\emptyset$, belongs in the collection of subset,
- If an event A is in the σ-algebra then its complement (A^c) is also in σ-algebra,
- If a sequence of events A_s are in a σ-algebra then their union, $\bigcup_{s=1}^{\infty} A_s$ are in the σ-algebra.

2.1.3 Definition: Filtration. (Shreve, 2004) Let $\mathcal{F}_n$ be a succession of sub $\sigma-$algebras such that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \mathcal{F}$. A filtration is define by

$$F := (\mathcal{F}_n)_{n \in \mathbb{N}}.$$

2.1.4 Definition: Probability space . (Shreve, 2004) The triple space $(\Omega, \mathcal{F}, P)$ is a probability space composed of the following elements:

(i) A non empty set Ω, which contains all possible results of a random experiment,

(ii) A σ-algebra $\mathcal{F}$ such that $\mathcal{F} \in \Omega$,

(iii) A probability measure P defined on the interval $[0, 1]$.

2.1.5 Definition: Expectation value. (Shreve, 2004) Let X be a random variable with the possible values x_s occurring with probabilities p_s.

$$E[X] = \sum_{s=1}^{n} x_s p_s$$

where $E[X]$ is an expectation value of X.

2.1.6 Definition: Conditional expectation. (Shreve, 2004) Let $\mathcal{F}$ be a σ-algebra having a sub σ-algebra $\mathcal{H}$. Let $X \in \mathcal{F}$ a random variable such that $E[|X|] < \infty$; $\exists$ a random variable $Y = E(X|\mathcal{H})$ such that

- Y is in $\mathcal{H}$
- For every set $A \in \mathcal{H}$, we have

$$\int_A Y dP = \int_A X dP.$$

This means considering the information in $\mathcal{H}$, Y is a good predictor of X. The option value will be expressed as a conditional expected value before being estimated. A few conditional expectation properties are demonstrated in (Shreve, 2004).

2.1.7 Definition: Martingale. (Bjork, 2009) Let $\mathcal{F}_n$ be the information available at time n and $M_0, M_1, \ldots, M_n$ be a process, which is stochastic. The process $\{M_n : n \geq 1\}$ is said to be a martingale if

$$E[|M_n|] < \infty$$
$$E(M_{n+1}|\mathcal{F}_n) = M_n, \text{for each } n \geq 1.$$

2.1.8 Definition: Equivalent Martingale measure. (Bjork, 2009) An equivalent martingale measure Q for a market model

- Is a measure of probability,
- The normalized asset price process, Z_t^i is martingale under Q.

$$Z_t^i = \frac{S_t^i}{S_t^1}$$

where S_t^i is an asset price, S_t^1 the asset fixed as the numeraire asset, for $i = 1, \ldots n$ and $t \in [0, T]$.

2.1.9 Definition: Arbitrage. Zastawniak and Capinski (2003) Arbitrage is a practice that aim to take advantage of the price difference between two or more markets. It could be defined as a portfolio of values $\gamma_0, \ldots, \gamma_s$ satisfying the property

- $\gamma_0 = 0$ (no initial investment)
- $P(\gamma_s \geq 0) = 1$ (always win)
- $P(\gamma_s > 0) > 0$ (making a positive return on investment)

where P is a probability measure. The "no arbitrage" hypothesis is used to compute a unique risk neutral price for derivatives. It is under this assumption that an American option will be evaluated. However, we have an asset price's fundamental theorem below.

2.1.10 Theorem. *(Shreve, 2004) The existence of risk-neutral measure implies a no arbitrage.*

For the proof see (Shreve, 2004).

2.1.11 Definition: Discount factor process . [1] A discount factor process is the overall measure of uncertain future market expectations, that could be defined by

$$D(t) = e^{(-\int_0^t r(s)ds)}$$

where $r(t)$ is an interest rate, $t \in [0, T]$ an interval of time.

2.1.12 Definition: Risk neutral measure. [2] A risk neutral measure is a measure equivalent to a martingale measure for which the discounted stock price denoted by $D(t)S(t)$ is a martingale. The measure is useful for price derivative securities and its existence implies there is no arbitrage. Also, the uniqueness of the measure implies every derivative can be hedged.

2.2 Stochastic Process

2.2.1 Definition: Stochastic Process. (Shreve, 2004) Let $(\Omega, \mathcal{F}, P)$ be a probability space. A stochastic process denoted by $\{X_t : t \in T\}$ is a collection of random variables, where $T \subset \mathbb{R}_+ = [0, \infty)$.

2.2.2 Definition: Stochastic Differential Equation. (Bjork, 2009) The complete theory used to model asset prices at continuous times is a diffusion's process and a stochastic differential equation. X is said to be a diffusion if its local dynamics can be approximated by a stochastic differential equation (SDE)

$$X(t + \Delta t) - X(t) = \mu(t, X(t))\Delta t + \sigma(t, X(t))Z(t) \tag{2.2.1}$$

where $Z(t)$ is a vector of independent and identically distributed components, μ and σ are deterministic functions respectively called *drift* and *diffusion* terms over the interval $[t, t + \Delta t]$. The two terms of Equation (2.2.1) represent

- $\mu(t, X(t))$ the average rate at which the asset increases,
- $\sigma(t, X(t))$ the measure of dispersion of asses returns.

A stock price follows a random walk, which is a stochastic process. Over time the price changes with a particular probability by going up or down.

2.2.3 Definition: Markov process. (Shreve, 2004) A Markov process is a specific stochastic process where only the value in the moment can predicted the future, not the path followed in the past (Hull, 2008).

We assume a filtration $F = \{\mathcal{F}_t : t \in T\}$. We suppose $X = \{X_t : t \in T\}$ is a stochastic process on a space of probability. X represents a Markov process if there exists K, a state space such that $X_t \in K$ and respects the property below

$$E[f(X_{k+t})|\mathcal{F}_k] = E[f(X_{k+t})|X_k]$$

where $k, t \in T$ and $f : K \to R$ is a function.

Throughout this study, the stock price will satisfy the Markov property and we will use this property to compute the continuation value.

[1] Adopted from, http://www.math.cmu.edu/~gautam/sj/teaching/2016-17/944-scalc-finance1/pdfs/ch4-rnm.pdf, Accessed on 4 Mai 2019

[2] Adopted from, http://www.math.cmu.edu/~gautam/sj/teaching/2016-17/944-scalc-finance1/pdfs/ch4-rnm.pdf, Accessed on 4 Mai 2019

2.2.4 Brownian motion . (Glasserman, 2013) A Brownian motion is a stochastic continuous process. Let $\{W(t)\}$ be a stochastic process on the interval $0 \leq t \leq T$. $\{W(t)\}$ process is said to be a Brownian (Wiener) motion if the assumptions below are fulfilled

(i) $W(0) = 0$.

(ii) The increment $W(t)$ is continuous.

(iii) The process has independent increments, i.e, for $0 \leq s < t \leq u < v, W(t) - W(s)$ and $W(v) - W(u)$ are independents.

(iv) For $0 \leq u < t \leq T, W(t) - W(u) \sim \mathcal{N}(0, t-u)$.

If the assumptions (i) and (iv) are fulfilled then $W(t)$ is a normally distributed variable, which could be written

$$W(t) \sim \mathcal{N}(0, t), \; 0 < t \leq T.$$

The Brownian motion is defined on a continuous interval of time, it will be precisely modelled if the time is discretized in steps by constructing a random walk.

2.2.5 Random walk construction . (Glasserman, 2013) The Brownian motion simulation is used to generate $W(t_1), \ldots, W(t_n)$ values on the interval $0 < t_1 < \cdots < t_n$. Let $Z_1, \ldots, Z_n$ be a standard normal variable. Using the Brownian motion assumptions and setting $t_0 = 0$, the values could be generated by

$$W(t_{j+1}) = W(t_j) + \sqrt{t_{j+1} - t_j} Z_{j+1}, \; j = 0, 1, \ldots, n-1. \tag{2.2.2}$$

A random walk simulated in 1000 times step using Equation (2.2.2) is illustrated in Figure 2.1 below

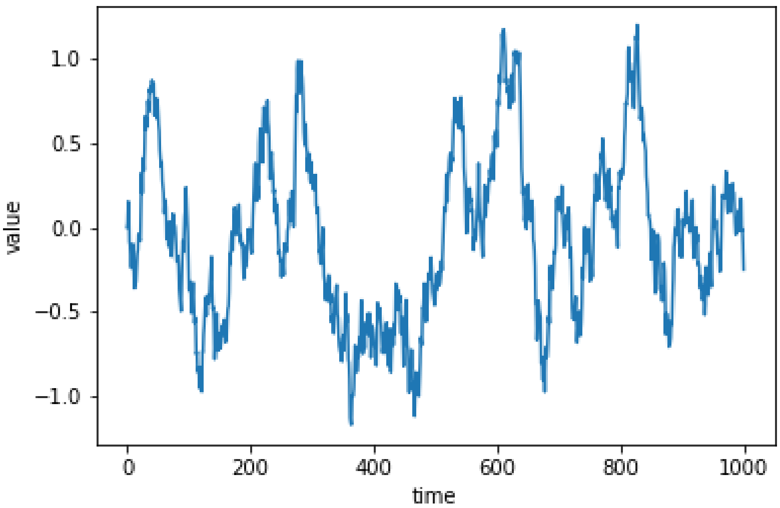


Figure 2.1: Random walk illustration.

In finance the assumptions presume that stock price variations in the market are the random events. They are independent and have the same probability distribution, therefore stock prices follow a random walk (Glasserman, 2013). Notice that the values of a Brownian motion can be negative, so it could not be considered as a model for an asset price, which will always be non-negative. For this reason, let us introduce geometric Brownian motion.

2.2.6 Geometric Brownian Motion. (Bjork, 2009) A geometric Brownian motion (GBM) is a model for the variation in a stochastic process, relative to value of X. Consider $\{X_t : t \in T\}$ to be a process, X_t follows a GBM if the stochastic differential equation (SDE) below is satisfied

$$\mathrm{d}X(t) = \mu X \mathrm{d}t + \sigma X \mathrm{d}W(t) \tag{2.2.3}$$

where μ, σ are drift and diffusion terms and $W(t)$ is a Brownian motion. To solve Equation (2.2.3) some calculations are needed. Particularly we will need Itô's lemma (Bjork, 2009).

2.2.7 Itô's Lemma. (Bjork, 2009) Suppose $\{X_t\}$ is a process satisfying the Equation (2.2.3). Consider $h(X,t)$ to be a twice differentiable function. Then h's SDE is given by

$$dh(X(t),t) = \left(\frac{\partial h}{\partial t} + \mu\frac{\partial h}{\partial X} + \frac{\sigma^2}{2}\frac{\partial^2 h}{\partial X^2}\right)dt + \sigma\frac{\partial h}{\partial X}dW(t). \tag{2.2.4}$$

Using Itô's Lemma from Equation (2.2.3) we get

$$dh(X,t) = \left(\frac{\partial h}{\partial t} + \mu X\frac{\partial h}{\partial X} + \frac{1}{2}\sigma^2 X^2\frac{\partial^2 h}{\partial X^2}\right)dt + \sigma X\frac{\partial h}{\partial X}dW(t). \tag{2.2.5}$$

Assuming the initial value equal $h(X,t) = \ln(X,t)$ and $\ln(X(0))$ we obtain (Glasserman, 2013).

$$\frac{\partial h}{\partial t} = 0, \quad \frac{\partial h}{\partial X} = \frac{1}{X}, \quad \frac{\partial^2 h}{\partial X^2} = -\frac{1}{X^2}. \tag{2.2.6}$$

By substituting the Equation (2.2.6) into (2.2.5) we get

$$\begin{aligned}
d\ln(X) &= (\mu - \frac{1}{2}\sigma^2) + \sigma dW(t)\\
\int_{t_0}^{t_1} d\ln(X) &= \int_{t_0}^{t_1}(\mu - \frac{1}{2}\sigma^2) + \int_{t_0}^{t_1}\sigma dW(t)\\
\ln(X(t_1)) - \ln(X(t_0)) &= (\mu - \frac{1}{2}\sigma^2)(t_1 - t_0) + \sigma(W(t_1) - W(t_0))\\
\ln(X(t_1)) &= (\mu - \frac{1}{2}\sigma^2)(t_1 - t_0) + \sigma(W(t_1) - W(t_0)) + \ln(X(t_0)).
\end{aligned}$$

We can exponentiated each side to get

$$X(t_1) = X(t_0)e^{((\mu - \frac{1}{2}\sigma^2)(t_1 - t_0) + \sigma(W(t_1) - W(t_0)))}$$

Assigning $t_0 = 0$ and $t_1 = t$ we get

$$X(t) = X(0)e^{((\mu - \frac{1}{2}\sigma^2)t + \sigma W(t))}. \tag{2.2.7}$$

By Brownian motion properties and using random walk construction, we can simulate $X(t)$ values by the recursive formula

$$\begin{cases} X(t_i) &= X(t_{i-1})e^{((\mu - \frac{1}{2}\sigma^2)(t_i - t_{i-1}) + \sigma\sqrt{t_i - t_{i-1}}Z_i)}\\ X(t_0) &= X_0 \end{cases} \tag{2.2.8}$$

for $0 = t_0 < t_1 < \cdots < t_n$, $i = 0, 1, \ldots, n-1$ and independent standard normals $Z_1, \ldots Z_n \sim \mathcal{N}(0, 1)$. GBM is a fundamental model of the value of a financial asset (Glasserman, 2013). Figure 2.2 shows ten simulated GBM paths, given $\mu = 0.06$, $\sigma = 0.2$ at initial time $X_0 = 90$ in 300 time steps.

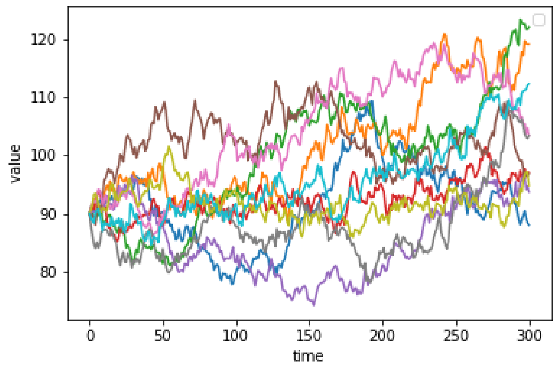


Figure 2.2: Simulation of GBM.

2.3 Option Theory

An option is an agreement between two parties, within specified conditions, that gives the owner the right, but not an obligation to buy or to sell an underlying asset at a specified price (Hull, 2008). The underlying asset could be a stock, property, change or other financial instrument. In this study, the underlying will be referred to as a stock. An option is not an obligation, the owner may choose either to exercise his right, or let it expire (Hull, 2008).

Options are particular derivatives, that is to say, financial instruments which promise some payment in the future and their derived values are from the underlying stocks (Kazeem, 2014). Options are a good instrument for investment risk management, also called hedging, and whenever they are applied to buy or sell shares, we will say that they are exercised.

The price in the agreement is called the *strike price* or *exercise price*, it will be denoted by K and the time when the option will be exercised is called *expiration date*, *exercise time* or *maturity*, it will be denoted by T. In return for granting the option, the buyer should be paid an amount called a premium, and if the option will not be exercised then the seller's profit will be this premium. This is the main difference between an option and a forward contract, which the buyer has the obligation to exercise (Zastawniak and Capinski, 2003).

There exist two principal types of options (Zastawniak and Capinski, 2003):

- *Call option* where the holder has the right to buy the underlying asset for a specified price K.
- *Put option* where the holder has the right to sell the underlying asset for a specified price K.

An option is determined by its pay-off which is defined by $max(S(T) - K, 0)$ for a call and $max(K - S(T), 0)$ for a put, where $S(T)$ represents the stock price at time T. This explains why an investor purchasing an option could not lose money.

Options can have different types of exercise time. An European option can only be exercised at the expiry time. An American option could be exercised at any time before or at expiry date. Therefore we can have a European call or put options, American call or put options. There is also Bermudan options, exercised at a predetermined date before the expiration date, and Asian options, exercised when the pay-off reaches a value that belongs on the mean stock price during a defined time period.

American and European are the most used options. Also, American style-options are the most traded on exchanges. Our study will focus on American style-options, because they can exercised at any time.

The advantage of American style-options is that the holder has a choice of when to exercise, unlike the European options for which the holder should wait until the end to the exercise. Many factors affect the price of options apart from the exercise price and expiration date such as

- The stock price at present time S_0, called the underlying stock price;
- The risk-free interest rate;
- The volatility;
- The dividend expected during the life of the options. But in our study, we are considered the stock that does not pay dividends.

But it is the market value and the exercise price that have the most influence in the computing of an option's value.

The volatility of the stock price is a measure of our uncertainty about the returns provided by the stock (Hull, 2008). Usually, for a stock, it varies between 15% and 60%. When the return is expressed as a continuous variable, the volatility can be seen as the standard deviation of the return supplied in one year by the stock. Thus, increase of the volatility affect positively the probability that the stock will be well. (Hull, 2008).

2.3.1 Definition: Risk-free interest rate. (Hull, 2008) The risk-free interest rate is the theoretical return rate expected by an investor from their investment without risk. If the interest rate increases in the economy then the expected return from the stock by investors will increase.

Investors who decide to exercise an option should know what they are paying for, risk and pay-off. Thus, we define the two concepts below.

2.3.2 Definition: Intrinsic value . (Zastawniak and Capinski, 2003) The intrinsic value is defined as the current worth of the option. It is given by $(S(t) - K)^+$ if the option is a call and $(K - S(t))^+$ if the option is a put at time $t < T$, where $(Z)^+ = max(Z, 0)$.

2.3.3 Definition: Time value option. (Zastawniak and Capinski, 2003) The time value describes the possibility that the option could increase in value prior to its expiry date.

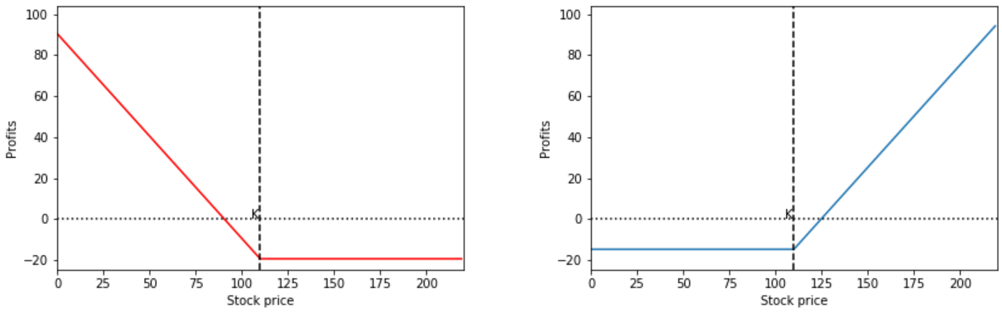
A put option is said to be in the money (ITM), which means profitable when the current price is less than the exercise price, at the money (ATM) when the current price and the exercise are equal, out of the money (OTM) when the current price is greater than the exercise price. Unlike for a call option, which is ITM when the current price is greater than exercise price, and it is OTM when the current price is less than the exercise price.

Let K be the exercise price and $S(t)$ be the current stock price at time t. The time value can be summarised in the table below

	Put option	Call option
In the money	$S(t) < K$	$S(t) > K$
At the money	$S(t) = K$	$S(t) = K$
Out the money	$S(t) > K$	$S(t) < K$

Table 2.1: Time value option

Thus, the intrinsic value will be zero when the options are out or at the money. For American options, it will be better to exercise when they are in the money than when they are out of the money. We can illustrated by Figure 2.3 below



(a) Put option (b) Call option

Figure 2.3: Intrinsic option value graph with exercise price $K = 110$

As we can see Figure 2.3 above displays the way in which the pay-off of an American call and put varies with $S(t)$. For a put option, with interest rate $r > 0$, it is optimal to exercise immediately when the stock price is less than the option's exercise price K. In the case of a call, it is better to exercise when the stock price is greater than option's exercise price K.

2.3.4 Definition: Hedging. (Zastawniak and Capinski, 2003) Hedging is a strategy used by an investor to minimize the risk of their investment which leads to lower profit. Thus the hedgers use options to lower the risk caused by potential future movements in an uncertain market. Next, we give the common model used to price options.

2.3.5 Option Pricing Model. In the stock market, the common model used to get the fair value of an option in continuous price process is the Black-Scholes model, which makes the following assumptions about the market (Black and Scholes, 1973)

- A stock price is modelled as a GBM[3] given by

$$dS(t) = rS(t)dt + \sigma S(t)dW(t) \tag{2.3.1}$$

  where r is the risk free interest rate and σ the volatility, both constant (Glasserman, 2013).

- A stock price has no dividend during the life of an option.

- Market movements could not be predicted.

- No transaction costs on the specific stock.

[3] GBM: Geometric Brownian motion

- The returns are normally distributed.

Thus, The Black-Scholes equation for European option price (Black and Scholes, 1973) is defined as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.3.2}$$

where

- $V = V(S, t)$ is the option value at time t,
- $S = S(t)$ is the current stock price,
- r is the risk-free interest rate,
- σ is the volatility,
- t is the time in the interval $[0, T]$.

The boundary condition of Equation 2.3.2 is that at the maturity date, the option value is equal to pay-off. This means

$$V(S, T) = max(K - S_T, 0). \tag{2.3.3}$$

Equation (2.3.3) is the option at maturity, and corresponds to the value of the European put option. However, analytical solution of Equation 2.3.2 is given by

$$V(S, t) = Ke^{-r(T-t)}\mathcal{N}(-d_2) - S\mathcal{N}(-d1) \tag{2.3.4}$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$
$$d_2 = d_1 - \sigma\sqrt{T - t}$$

and $\mathcal{N}(.)$ represents the standard normal cumulative distribution function, σ is the standard deviation on the stock returns, t is the current time and T is the maturity time (Black and Scholes, 1973). Equation 2.3.4 is the value of a European option at time t.

2.3.6 American Options . The American options give the owner the right to exercise at any time up until the expiration date. Given T the maturity and $t = 0$ the present time, h will represent the pay-off function. To be able to understand, a stock price at maturity will be denoted $S(T)$, a stock price at current time $t < T$ by $S(t)$ and K the exercise price.

We will consider a put option, which gives the holder the right to sell the stock at a fixed exercise price K at a future time. In this case the stock price $S(T) < K$. At the end of the contract the holder decides to exercise if it is in the money, otherwise he lets it expire.

Unlike the European option, the American holder has the choice to exercise before the final time T or wait until maturity. The pay-off of the option at time T is the intrinsic value given by

$$(K - S(T))^+ = max(K - S(T), 0). \tag{2.3.5}$$

Knowing the pay-off at maturity, the value of the option at the present time, denoted by $V(S, 0)$ is defined as the expectation discounted pay-off, given by

$$V(S, 0) = E^Q[e^{-rT}(K - S(T))^+] \tag{2.3.6}$$

where $E^Q[\,.\,]$ is the conditional expectation taken under the risk-neutral pricing, Q a martingale measure, e^{-rT} is the discount factor and r the risk-free interest rate. We assume r is discrete and constant (Longstaff and Schwartz, 2001). The expectation value defined above will be significant if we know the stock price distribution at maturity.

Recall that the Black-Scholes model assumes that the stock follows a GBM with mean μ and standard deviation σ^2 in continuous time (Barola, 2014). This means, the stock price has a log normal distribution (Glasserman, 2013). The risk neutral measure Q is used to express the derivative price. Equation 2.3.6 is an integral of the lognormal density of $S(T)$ with respect to time which can be evaluated in terms of the standard normal cumulative distribution function (Glasserman, 2013). Besides, an American option value at time $t_0 = 0$ is defined by

$$V(S,0) = \operatorname*{Sup}_{t\in[0,T]} E^Q[e^{-rt}h(S(t))] \tag{2.3.7}$$

where $h(S(t)) = (K - S(t))^+$ is the non-negative intrinsic value at time t.

As previously reported, an American option is not straightforward to analyse as a European option because the holder should decide on an optimal exercise strategy, when the discount pay-off gives the best value. The reason is the no-arbitrage assumption. This means, we should find the optimal stopping time of exercise and, estimate the present value at that time. For an unknown exercise boundary b^*, the stopping time is defined by

$$\tau = \inf\,\{t \geq 0 | S(t) \leq b^*(t)\}. \tag{2.3.8}$$

The optimal time is the first in time, at which the option price becomes smaller than the pay-off. The situation with American options is rather more complex than European options because at each time step before expiration date, the holder of the option has the choice between exercising and holding on to the contract. He should exercise immediately if the option is in the money. Thus, the present value of the American option on optimal stopping time is given by

$$V(S,0) = \operatorname*{Sup}_{\tau\in[0,T]} E^Q[e^{-r\tau}h_\tau(S_\tau)] \tag{2.3.9}$$

where τ is the stopping time, S_τ is the price process stopped at $t = \tau$, $e^{-r\tau}$ is the discount factor favourable for stopping time, and h_τ the non-negative intrinsic value on the stopping time.

Therefore, the general practice is the estimation of the present option price. This means to hedge the present value of the underlying asset in the future. This is the aim of our study, we will use the Least Squares Monte Carlo method to estimate this value. Besides the Black-Scholes equation for American options is an inequality given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S^2} + rS\frac{\partial V}{\partial S} - rV \leq 0 \tag{2.3.10}$$

under the boundary conditions

$$V(S,T) = max(K - S, 0) \text{ and } V(S,t) \geq max(K - S, 0)$$

Equation 2.3.10 under these conditions doesn't have an analytic solution which will help the holder to value the option and to make good decision. Also, to find the optimal time to exercise and get a good pay-off, numerical methods like finite difference are used to solve 2.3.10 (Brandimarte, 2013).

3. Monte Carlo Method

In this Chapter, we are going to give in general the basic concepts behind Monte Carlo method and Least Squares method.

3.1 Monte Carlo Method

The common idea behind the Monte Carlo method is to simulate a sample of something you are interested in, and compute the mean to find the true value (Glasserman et al., 2004). Generally, the price of derivatives is expressed as an expected value. Thereby, pricing derivatives lead to compute expectation which can be written as multidimensional integrals. Besides, Monte Carlo methods are very efficient for approximating high dimensional integrals, hence their importance in finance (Glasserman, 2013).

3.1.1 Principles. Consider u given by the integral

$$u = \int_0^1 f(x)dx. \tag{3.1.1}$$

To estimate the integral u over the interval $[0,1]$, $f(x)$ can be written as a product of an arbitrary function $g(x)$ and some probability density function $q(x)$. Thus, u could be expressed as the expected value $u = E[g(x)]$ as

$$\mathbb{E}[g(x)] = \int_0^1 g(x)q(x)$$

Suppose the probability density is generated by the sample $x_i, i = 1, \ldots, n$ which are independent and identically distributed (Barola, 2014), (Glasserman, 2013). Therefore, the expected value could be estimated at n random points as

$$\hat{u_n} = \frac{1}{n}\sum_{i=1}^{n} g(x_i). \tag{3.1.2}$$

3.1.2 Convergence. If $g(x)$ is integrable over an interval $[0,1]$ then the strong law of large numbers ensures that the estimated value converges to the true value when the number of x_i increases (Barola, 2014), (Glasserman, 2013). This means

$$\hat{u_n} \to u \text{ with probabily 1 as } n \to \infty.$$

Also, if f is square integrable over the interval $[0,1]$ and the variance is define by

$$\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(g(x_i) - \hat{u_n})^2 \tag{3.1.3}$$

then the error $\hat{u} - u$ could be approximated by a normal distribution with mean 0 and standard deviation $\frac{\sigma}{\sqrt{n}}$. The convergence rate $\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ is implied by the $\sqrt{n}$ of standard deviation (Glasserman, 2013).

3.2 Least Squares Method (LS)

The Least squares method (LS) consists of determining the best fit line of a data set by minimizing the sum of squared errors. Let $\{(x_1, y_1), \dots, (x_n, y_n)\}$ be a data set, where y_i are the values of dependent variables, and x_i the independent variables, for $i = 1, \dots, n$. We expect to find linear relationships between x_i and y_i. Let us group all the observations (x_i, y_i) into vector X, Y respectively and consider the model function on the variables define as

$$Y = \beta_0 + \beta_1 X + e_i \tag{3.2.1}$$

where e_i represent the error of the relationship between X and Y and β_0, β_1 the parameters (Rao et al., 2007). We can illustrate by Figure 3.1 below

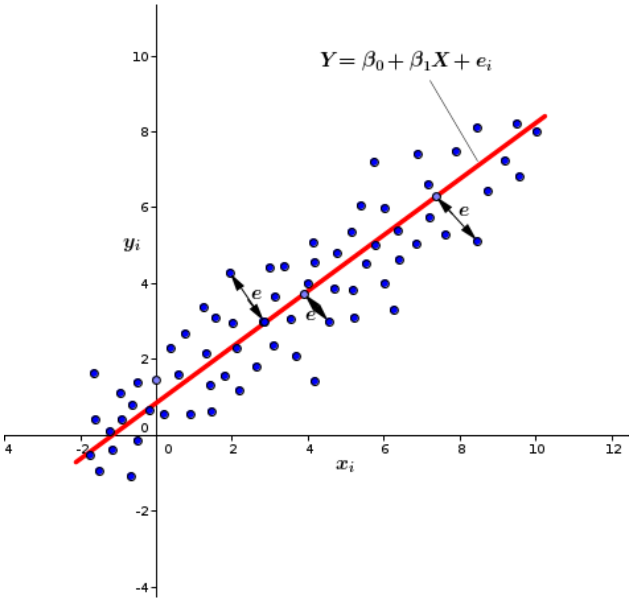


Figure 3.1: Example Least squares method.

The main problem is to find the parameter values that best fits the data, that is, to find parameter values that minimize a sum of squared errors (SSE).

$$SSE(\beta) = \sum_{i=1}^{n} e_i^2 = \text{min} \ \sum_{i=1}^{n} (Y - (\beta_0 + \beta_1 X))^2 \tag{3.2.2}$$

with β a vector of β_0, β_1. Equation 3.2.2 can be written in this form [1]

$$SSE(\beta) = \sum_{i=1}^{n} e_i^2(\beta) = \mathbf{e}^T \mathbf{e}. \tag{3.2.3}$$

Expanding Equation 3.2.3 we get

$$\begin{aligned}
\mathbf{e}^T \mathbf{e} &= (Y - X\beta)^T (Y - X\beta) \\
&= (Y^T Y - Y^T X\beta - \beta^T X^T Y + \beta^T X^T X\beta) \\
&= (Y^T Y - 2\beta^T X^T Y + \beta^T X^T X\beta)
\end{aligned}$$

[1] Adopted from, https://www.stat.cmu.edu/~cshalizi/mreg/15/lectures/13/lecture-13.pdf, Accessed on 4 Mai 2019

where $(Y^T X\beta)^T = \beta^T X^T Y$ is a vector. The minimum is determined by finding the gradient of SSE with respect to β.

$$\begin{aligned}\nabla SSE(\beta) &= (\nabla Y^T Y - 2\nabla \beta^T X^T Y + \nabla \beta^T X^T X\beta) \\ &= 2(-X^T Y + X^T X\beta)\end{aligned}$$

Setting $\nabla SSE(\beta) = 0$ we obtain

$$X^T X\hat{\beta} - X^T Y = 0 \ . \tag{3.2.4}$$

where $\hat{\beta}$ is the estimated coefficient. Equation 3.2.4 will be used to compute the estimated coefficients $\hat{\beta}_0, \hat{\beta}_1$ by

$$\hat{\beta} = (X^T X)^{-1} X^T Y. \tag{3.2.5}$$

The LS is generalisable, instead of finding the best fit line, we can find the best fit given by any finite linear combinations of specified functions. This means, given functions $f_1, \ldots, f_k$, find values $\beta_1, \ldots, \beta_k$ such that the linear combination [2] is the best approximation to the data.

$$Y = \beta_0 f_0(x) + \cdots + \beta_k f_k(x) = \sum_{j=0}^{k} \beta_j f_j(x_i), \quad i = 1, \ldots, n. \tag{3.2.6}$$

Suppose $F_{i,j} = f_j(x_i), i = 1, \ldots, n$ and $j = 0, \ldots, k$. The coefficients $\hat{\beta}_j$ are estimated using the Equation 3.2.5 as

$$\hat{\beta}_j = (F^T F)^{-1} F^T Y. \tag{3.2.7}$$

Now the Monte Carlo method and Least squares method have been presented, we need to describe the methods of pricing American options. That is what we will do in the next Chapter.

[2] Adopted from, https://www.stat.cmu.edu/~cshalizi/mreg/15/lectures/13/lecture-13.pdf, Accessed on 4 Mai 2019

4. Options Pricing Methods

4.1 Binomial Method

The Binomial model is a model to value options using a discrete time. The general formulation is based upon the stock price process, in which, at any time, the stock price has a possibility to move up or down (Brandimarte, 2013). The value of options is determined by its current price on discrete time.

Let S to be a current price at time $t = 0$ and we assume in the future the price could take only two values Su or Sd with probability p and $1 - p$, where u, d represent "up" and "down" respectively. Thus, the model can be generalized by a binomial tree as in Figure 4.1, where each node in the tree represents a possible price of the asset at a given point in time.

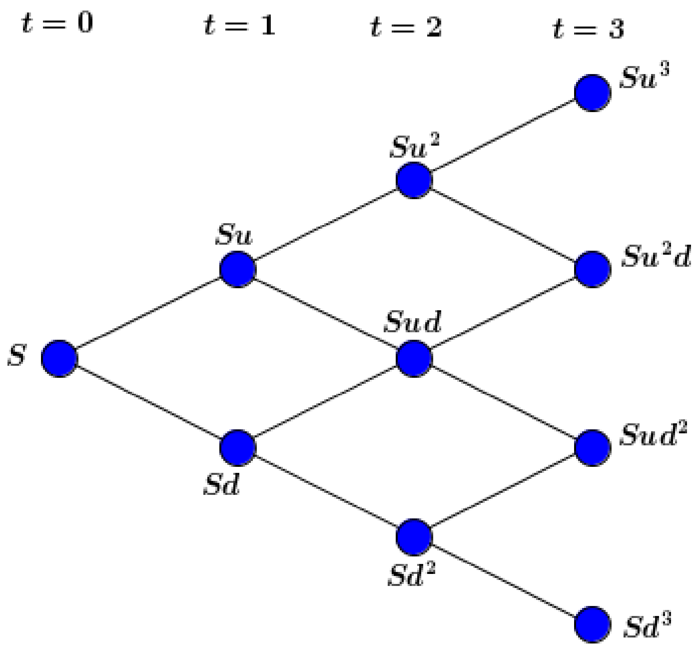


Figure 4.1: A Binomial tree illustration.

The value of an option is computed at each node in time by beginning at each of the final nodes, and moving backwards through the tree to the first node. The factors up, u, and down, d such that $u \geq 1, 0 < d \leq 1$ are given by

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}$$

where σ is the volatility and Δt is the time variation. The probability represents the risk-neutral defined by

$$p = \frac{e^{r\Delta t} - d}{u - d}.$$

Let us consider an American-style put option and let (i, t) be the point on the last layer. At the expiry date the intrinsic value at each final node is given by

$$f_{it} = \max\{K - S_{it}, 0\}$$

where $S_{it} = Su^i d^{t-i}$ is the underlying asset price on node i at time t and K the exercise price. The option value at each node is found by

$$C_{i,t-1} = e^{-r\Delta t}(pf_{i,t} + (1-p)f_{i+1,t}) \tag{4.1.1}$$

where $e^{-r\Delta t}$ is the discounted factor, p and $1-p$ are the probability to move up and down respectively, and $f_{i,t}$ is the value of the option on node i at time t. The $C_{i,t}$ which is the option's value for the i^{th} node at time t is called the continuation value, i.e, the value to hold the option.

However, the holder has to solve an optimal stopping problem, wherewith at each time step, he must observe the state and decide to exercise option if it is in the money.

Thus to do this, the holder will compared the immediate pay-off with the continuation value given in Equation 4.1.1. If the pay-off is better than the continuation value then he could exercise, else he can continue to wait for a better future value. The same argument will be repeated recursively for each node from time $t-1$ to $t=0$ (Brandimarte, 2013). Therefore, the American option value is

$$f_{i,t} = \max\left\{K - S_{i,t}, e^{-r\Delta t}(pf_{i+1,t+1} + (1-p)f_{i,t+1})\right\}. \tag{4.1.2}$$

4.2 Monte Carlo Simulation

The option price is expressed by the discounted expected value under risk neutral, Equation 2.3.9 and the value can not be calculated explicitly. Thus the Monte Carlo methods are introduced to evaluate the expected value of the pay-off function, which is a function of random variables (Glasserman et al., 2004).

Indeed to evaluate the price of an American option with the Monte Carlo simulation introduced by Tilley (1993), we firstly simulate the stock price path by using the solution of the geometric Brownian motion equation define by

$$S(t_i) = S(t_{i-1})e^{((r-\frac{1}{2}\sigma^2)(t_i-t_{i-1})+\sigma\sqrt{t_i-t_{i-1}}Z_i)} \tag{4.2.1}$$

Figure 4.2 below shows the 1000 simulated stock price using Equation 4.2.1, with an initial stock price $S_0 = 90, r = 0.06$ and $\sigma = 0.2$.

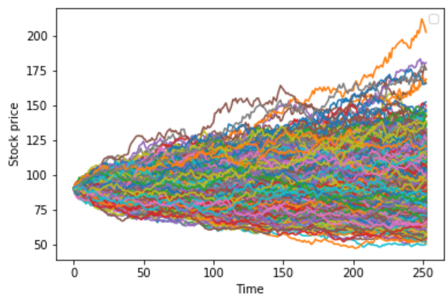


Figure 4.2: Stock price simulated

After, we evaluate the pay-off of each node of the simulated paths on the maturity. Secondly, reorder the stock price paths from the minimum price to the maximum price and classify into group with the equal number of stock, and compute the intrinsic value. Then for each group, estimate the continuation value which is compared to all the intrinsic value of that group. Thirdly, at each node, we compute V_i, for i from 1 to N, the maximum between the intrinsic value and the continuation value. Finally, the option value will be the average $\hat{V_0}$ of the V_i. Following the process described above, the Monte Carlo algorithm is

Algorithm 1 Monte Carlo Algorithm

Create random paths $j = 1, \ldots, N$.
Divide the continuous time interval time $[0, T]$ in time steps $t_0, \ldots, t_M$.
Simulate $S_j(t_i), i = 0, \ldots, M$.
Reorder $S_j(t_i)$ form the minimum to the maximum and classify all paths into equal group.
Compute the pay-off $g(S_j(t_i)) = max(K - S_j, 0)$.
Compute the continuation value $C_j(t_i)$.
for j from 1 to N
    $\hat{V_j} = max(g(S_j), C_j)$
end
The present value is the average of $\hat{V_j}$.

In the next section, we will develop the Least Squares Monte Carlo Method, to approximate the option value. The method will be applied to a put option, where the current stock price is less than the exercise price.

4.3 Least Squares Monte Carlo Method (LSMC)

4.3.1 Valuation framework of the LSMC methods. (Longstaff and Schwartz, 2001), (Clément et al., 2002) Consider a filtration F defined such that $F = \{\mathcal{F}_t; t \in [0, T]\}$ on the probability space where F is a family of σ-algebras, $\mathcal{F}_s \subset \mathcal{F}_t, s \leq t$ that could be generated by the stock price processes.

We suppose $[0, T]$ is a finite time interval and the existence of an equivalence martingale measure denoted by Q. The pay-off derivatives belong on $L^2(\Omega, \mathcal{F}, Q)$, which is the space of square-integrable functions (Longstaff and Schwartz, 2001). We assume an adapted pay-off process $h = (h_i)_{i=0,\ldots,T}$, which is a sequence of square integrals with real values defined by

$$E[h_i^2] = \int_\Omega h_i(\omega)^2 dP(\omega) < \infty \quad , \omega \in \Omega. \tag{4.3.1}$$

We assume a price process $S = \{S_t : t \in T\}$ modelled as a GBM, see Equation 4.2.1. At maturity the pay-off is equivalent to the option value. What is the price at time $t, t < T$? This can help the holder to make a decision before exercising.

4.3.2 Description of Algorithm. In addition to the framework defined previously, we assume that for all $t < s \leq T$, the holder pursue an optimal strategy. However, it is better to exercise at time τ, Equation 2.3.8, and realize maximum profit from the options.

We need to approximate the option value by taking the interval of time to be sufficiently large, knowing that the option could be exercised only at a discrete time (Longstaff and Schwartz, 2001). However, to get the optimal stopping time, the dynamic programming approach is used, which consist of dividing the continuous interval into a set of finite time $\{t_0, t_1, \dots t_M\}$, i.e, subintervals $[t_i, t_{i+1}], i = 1, \dots, M$, with length $dt = \frac{T}{M}$. Note that the option cannot be exercised at time t_0 and $t_M = T$. For each time t_i decide if it is better to exercise than to hold on it.

Consider an ensemble of random paths $j = 1, \dots, N$, and time step t_i, wich is discrete. The price process $S_j(t_i)$ will be generated using Equation 4.2.1. Under the hypothesis that prior to the expiration date, the option is not exercised. At the maturity the option value is equal to the pay-off.

$$V_j(T) = h(S_j(T)). \tag{4.3.2}$$

At each exercise date, except the first and the last, make a backward iteration in time by discounting the price and estimating the expectation value, which is the worth of holding on to the option at time t_i. This expectation value at time t_i is a conditional expectation value which uses the Markov property on S_{t_i}. It is defined by

$$C_{t_i} = E^Q[e^{-r\Delta t} V(S(t_{t+1}))|S(t_i) = S] \tag{4.3.3}$$

where $\Delta t = t_{i+1} - t_i$ and $e^{-r\Delta t}$ represents the discount factor. Equation 4.3.3 is the value of keeping alive an option instead of exercising. At the maturity date, the continuation value is null because the option is no longer available.

To compute the continuation value is more complex. So, we can approximate it by using the least squares method from time t_{M-1} to t_1, (Longstaff and Schwartz, 2001). We know that the least squares method is used to estimate the best possible coefficients for the approximation, by minimizing the mean squared errors, Equation 3.2.5.

The method assumes that, with a set of simulated Markov chain sample paths, the estimated continuation value can be expressed as a linear combination of basis functions from a countable measurable set. This assumption is justified because of the conditional expectation to belong to L^2, a square-integrable functions space (Longstaff and Schwartz, 2001). Consider a set of realised paths $S_j(t_i), j = 1, \dots, N$ that are in the money at time t_i, i.e, $h(S_j(t_i)) > 0$. The conditional expectation can be estimated as

$$\hat{C}_{t_i} = \sum_{m=0}^{k} \hat{\beta}_m \phi_m(S_j(t_i)) \tag{4.3.4}$$

where $\phi_m(S_j) = (\phi_1(S_j), \phi_2(S_j), \dots, \phi_m(S_j))$ is the orthogonal basis function which we will choose and $\hat{\beta}_0, \dots, \hat{\beta}_k$ are the estimated regression coefficients using Equation 3.2.7. Longstaff-Schwartz used weighted Laguerre polynomials as basis functions (Longstaff and Schwartz, 2001). Throughout our study, we choose to use Laguerre polynomials.

4.3.3 Basis function. Since $C_{t_i} \in L^2$, this means L^2 has a finite countable orthogonal basis in which all elements can be written as a linear combination of a set of basis functions (Clément et al., 2002). There are many orthogonal polynomials that can be used as basis functions such as simple polynomials, Hermite polynomials, Chebychev polynomials, Legendre polynomials, Jacobi polynomials, weighted Laguerre and Laguerre polynomials, which we choose as basis function.

4.3.4 Laguerre polynomials. [1] Let a Laguerre differential equation be given by

$$uv'' + (1-u)v' + \lambda v = 0 \tag{4.3.5}$$

[1] Adopted from, http://mathworld.wolfram.com/LaguerrePolynomial.html, 4 Mai 2019

where $u \in [0,1], \lambda > 0$. Laguerre polynomials, $L_n(u)$ are solutions of Equation 4.3.5 obtained by

$$L_n(u) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} u^k. \tag{4.3.6}$$

The basis functions L_m with degree m take the form

$$\begin{aligned}
L_0(u) &= 1 \\
L_1(u) &= 1 - u \\
L_2(u) &= \frac{1}{2}(u^2 - 4u + 2) \\
L_3(u) &= \frac{1}{6}(-u^3 + 9u^2 - 18u + 6) \\
L_4(u) &= \frac{1}{24}(u^4 - 16u^3 + 72u^2 - 96u + 24).
\end{aligned}$$

Therefore, the basis functions ϕ_m which are the Laguerre polynomials, are complete, linearly independent and have the property of being mutually orthogonal on the interval $[0, \infty)$.

The coefficients $\hat{\beta}_m$ are obtained by a least squares regression, Equation 3.2.5 between discounted pay-off $y_j = e^{-r\Delta t}\, h(S_j(t_i))$ and underlying assets $x_j = S_j(t_i)$.

Consider the Laguerre polynomials as basis function, i.e, $\phi_m = L_m$, where m is the degree of the polynomials L_m. Then, $\hat{\beta_m}$ are solution of Equation 3.2.7

$$(\hat{\beta}_0, \ldots, \hat{\beta}_m) = (L^T L)^{-1} L^T (y_1, \ldots, y_N) \tag{4.3.7}$$

where $L = L_m(x_j),\ j = 1, \ldots, N$ and $m = 0, \ldots, k$.

Then, for each path j compare the values of immediate exercise $h(S_j(t_i))$ with the estimated continuation value. The greater of the two gives the value of the option. The early exercise decision is taken by comparing the continuation value estimated with the pay-off as given below

$$\hat{V}_{j,t_i} = \begin{cases} h(S_j(t_i)) & \text{if } h(S_j(t_i)) > \hat{C}_{t_i}(S_{j,t_j}) \\ e^{-r\Delta t} h(S_j(t_{i+1})) & \text{if } h(S_j(t_i)) \leq \hat{C}_{t_i}(S_{j,t_j}) \end{cases} \tag{4.3.8}$$

where $h(S_j(t_i))$ is the pay-off option of the asset price at time t_i. The term $e^{-r\Delta t} h(S_j(t_{i+1}))$ in the system 4.3.8 is the non-exercised value of holding on to the option.

The present price below is calculate using Monte Carlo methods 3.1.2, i.e, by taking the average over all paths.

$$\hat{V}_0 = \frac{1}{N} \sum_{j=1}^{N} e^{-r\Delta t} \hat{V}_{j,t_i} \tag{4.3.9}$$

this is the expected value at present time. The Least Squares Monte Carlo (LSMC) algorithm is described in Algorithm 2 below

Algorithm 2 LSMC Algorithm

Create random paths $j = 1, \ldots, N$.
Divide the continuous tilme interval time $[0, T]$ in time steps $t_0, \ldots, t_M$.
Simulate $S_j(t_i), i = 0, \ldots, M$.
Suppose $H_j \longleftarrow h(S_j(t_M))$ for all j,
for t from t_{M-1} to t_1 **do**
    Find paths $\{j_1, \ldots, j_s\}$ that are in th money, i.e, $h(S_j(t_M)) > 0$
    Let the paths $\gamma \longleftarrow \{j_1, \ldots, j_m\}$
    Let $x_j = S_j(t)$ and $y_i = e^{-\Delta t} H_j$ for $j \in \gamma$- paths
    Use the least squares method on x, y to get the estimated coefficients $\hat{\beta}_0, \ldots, \hat{\beta_m}$.
    Approximate the continuation value $\hat{C}(S_j(t)) = \sum_{m=0}^{k} \hat{\beta}_m \phi_m(S_j(t_i))$.
    for j from 1 to N **do**
        if $j \in \gamma$ and $h(S_j(t_M)) > \hat{C}(S_j(t))$
            $H_j \longleftarrow h(S_j(t_i))$
        else
            $H_j \longleftarrow e^{-r\Delta t} h(S_j(t_{i+1}))$
        end
    end
end
Use the average to approximate the option value
option price $\longleftarrow \frac{1}{N} \sum_{j=1}^{M} e^{-r\Delta t} H_j$.

4.3.5 Convergence. There is an estimated error when we approximate the expected continuation value, Equation 4.3.3 by the Equation 4.3.4. In Clément et al. (2002), it is shown that at certain rates

$$\lim_{k \to \infty} \hat{C}(S(t)) = C(S(t))$$

In Huang and Huang (2009), it is shown that the price estimated will converge to the true option value if the approximated continuation value converges to the true expectation function. Also, Longstaff and Schwartz (2001), showed that the Least Squares Monte Carlo algorithm is convergent with a sufficient number of basis functions and a large number of simulation.

In the next Chapter, we will present the numerical results for all of the previous methods.

5. Numerical Results

5.1 Implementation

The implementation of the three options methods is done in Python. For the Least Squares Monte Carlo method (LSMC), we use the three first Laguerre polynomials as basis functions in Equation 4.3.6 to approximate the continuation value and the stock price simulated using a geometric Brownian motion in Equation 4.2.1.

5.2 Results

The table below shows the values of American put option using the Binomial method, Monte Carlo method (MC) and LSMC. The prices are obtain for different current stocks with exercise price $K = 100$, the stock volatility at 20% and the risk free interest rate at 6% in one year, which is the expiration date T. The LSMC and MC solutions are obtained for 1000 simulations and 252 exercise dates. While Binomial solutions are obtained for a tree with 252 steps.

Stock price	Binomial	MC	LSMC
60	40.00000000	39.88360011	39.88727041
70	30.00000000	29.85799861	29.86700293
80	20.00000000	19.85839718	19.87487220
90	11.21709288	9.89989889	11.10481464
100	5.79819565	1.04688985	5.69180536
110	2.78359189	0.00031550	2.60726455
120	1.24944663	0.00000000	1.06033163
130	0.52981984	0.00000000	0.39897257

Table 5.1: American put option prices.

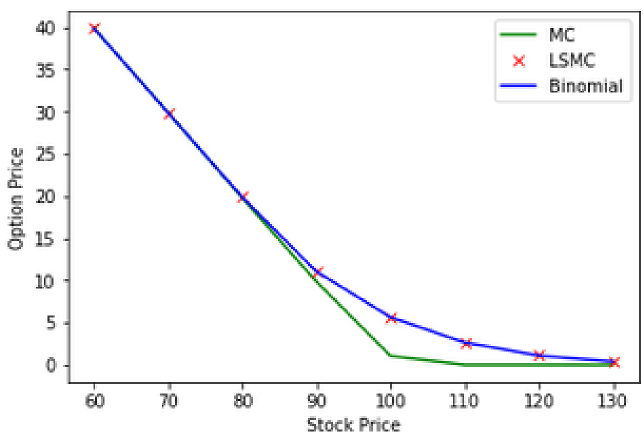


Figure 5.1: Plot American put option prices.

Now let's compare the running time (in second) for Monte Carlo and Least Squares Monte Carlo method.

Number of simulation	MC	LSMC
10	0.004	0.021
100	0.017	0.023
1000	0.043	0.036
10000	0.414	0.182
100000	3.933	1.774

Table 5.2: Table of Execution Time.

5.2.1 Approximation Error. Here, we use the relative error with respect to Binomial method solution to compare the Monte Carlo method and the Least Squares Monte Carlo method. The errors are given in the table below:

Stock Price	MC	LSMC
60	0.29099	0.28182
70	0.47334	0.44332
80	0.62563	0.70801
90	11.74273	1.00095
100	81.94455	1.83489
110	99.98866	6.33452
120	100.0	15.13590
130	100.0	24.69655

Table 5.3: Table of Errors.

5.2.2 Discussion. From Figure 5.1, we can see that the LSMC values are close to Binomial method. From Table 5.2, it is clear that LSMC method is faster than the MC method. Moreover, from Table 5.3 we can see that LSMC method is more accurate than MC.

6. Conclusion

In this study, we presented the Least Square Monte Carlo method which is a combination of the least squares and the Monte Carlo method for pricing American put options. We used the Least Squares method to approximate the continuation value.

Moreover, the Binomial method, the Monte Carlo simulation and Least Squares Monte Carlo method (LSMC) were used to find the price of an American option. Numerical experiments have shown that the LSMC method is faster and more accurate than the Monte Carlo method.

For future work, we plan to use the Artificial Neural Network method (ANNs), which is machine learning technique, to price American options and compare it to the Least Squares Monte Carlo method.

Acknowledgements

"I will sing of your love and justice; to you, LORD, I will sing praise ". Foremost, I would like to thank GOD Almighty for giving me the strength, knowledge to undertake this research study. Without his blessings, this achievement would not have been possible.

Firstly, I would like to express my sincere gratitude to Professor Philip Mashele my supervisor for the guidance, constructive suggestions during this research. My grateful to Rock Koffi, for assistance and objective explanations on various aspects of this research. I would also like to thank my tutor Lila Chergui, and all of the AIMS tutors specially Jordan Masakuna, Kenneth, Dinna for their guidance and valuable support. Many thanks to Noluvuyo for her assistance, she has been a great help.

Secondly, My gratitude goes to Prof. Turok Neil, Prof. Barry Green, Prof. Jeff Sanders, Jan Groenewald and all AIMS family. It is a pleasure for me to be part of such team.

Thirdly, I would like to thank my family Michel Bakajika for the assistance, prayer and affection throughout my study. Particularly to my mum Germaine Odia and my grandmother Berthine Nyunga. My special thanks to P. Gerard Nkongolo, Richard Ngandu, Michel Bakajika, Jeancy Tshitadi, Prof. Gaylord Kabasele, Jacky M., Pa Ambroise N., Agnes M., Martin Kanku, Alpho K., Marie M., Beatrice M. and Jully M. for their endless support.

Finally, I would like to thank my friends who played a important role throughout my stay at AIMS Salomon, Abigail, D'jeff, Joram, Jean, Hewan, Ghyslain, Veronica, Irene and my friend Yves Mazol. I would like to acknowledge several people who have contributed directly and indirectly to this research.

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