Morphology and detection.

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23 May 2019

Submitted in partial fulfillment of a structured masters degree at AIMS South Africa

AIMS | African Institute for Mathematical Sciences SOUTH AFRICA

Abstract

This work studies the problem of object detection in an image. Many methods address such a problem in the literature including methods based on mathematical morphology. Shape is recognized as one of the main features that best describe an object contained in an image. Mathematical morphology is based on the principle of comparing an unknown structure, the image, to a set of shapes with known features. Thus, it provides morphological operators that can detect target shapes in an image. This essay introduces the basic morphological operators namely erosion and dilation as well as their combinations and some applications. Also, the dual property of dilation and erosion is emphasized by introducing Galois connections. Finally, the Hit-and-Miss Transform, which is based on erosion, is described as a powerful tool for shape detection. But using only shape as a feature for object detection may not provide an efficient outcome. We then look at ways to enhance detection based on morphology through the combination or sequential use of signals to form a multi-modal detection system.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Abigaïl Priscilla Mahugnon Djossou, 23 May 2019

Contents

List of Symbols

$\oplus$: Dilation

$\ominus$: Erosion

$\circ$: Opening

$\bullet$: Closing

$\circledast$: Hit-and-Miss Transform

1. Introduction

The world is essentially composed of 3D objects. The study of properties and relationships of 3D Objects gives birth to a wide range of applications which include computer vision and robotics. For instance, in a computer system, researchers investigate how to extract information from a high-level representation of digital images. Object manipulation is another instance of application in robotics. This project reviews properties of morphological operations in images from a computer vision point of view—artificial vision system.

Computer vision has many tasks including acquisition and processing. Another major computer vision task, which may require other tasks and seems to be in various related applications, is object detection—which consists of identifying some objects. To acquire information from the environment, various sensors could be used.

Recognizing objects in images is one of the most difficult problems in computer vision as it is a critical step in the implementation of many current applications. From its genesis, computer vision has focused on developing methods for automatically interpreting images, particularly methods for recognizing or identifying objects in images (Kurian and MV, 2011). Various methods have been proposed in the literature to address object identification and methods based on convolution neural networks seem to be the state of the art (Russakovsky et al., 2015). Object detection can also be viewed as a subtask of other computer vision tasks such as object tracking. The tracking of objects consists of dynamically locating moving objects (e.g., vehicles) in a scene (Bruijning et al., 2018). This work considers preliminary morphological operations that can be involved in object detection, which could be a first step to other tasks like tracking.

Different methods of image analysis have been developed. Linear image analysis uses transformations that remain invertible, but in mathematical morphology, transformations use set operators that commute with the greatest lower bound (GLB) and the least upper bound (LUB). The role of morphology is to transform a set into a generally simpler set of structures, without altering its main geometric features.

The rest of the essay is organised as follows. Chapter 2 introduces the binary mathematical morphology and gives the properties of the morphological operators, more particularly that concerning the duality, and then finishes with some examples. Chapter 3 tackles detection in a unimodal and multi-modal system. Finally, chapter 4 provides a concrete example of object detection in an image and presents a discussion about different issues raised during our research.

2. Mathematical morphology processes

In the physical world, we consider an object to be distinguishable from the background by its shape or its colour. Mathematical morphology is a field that studies the shape of an object based on set theory. It uses sets to describe the shapes of objects in an image. An image can be seen as an array of values and each cell in that array is called a picture element (pixel). If we have n rows and m columns in the image, the image size will then be $n * m$ pixels.

An image can be represented either in binary, in gray level or in RGB. A binary image is an image in which pixels are either black (0) or white (1). This image representation will be used in our work with the convention that the background is always set to black and the foreground to white.

Mathematical morphology operators are very useful for detecting objects in an image. For instance, it is used to draw a land map of an image in robot path planning in order to ensure a collision-free path for the robot. There are two basic mathematical morphology operators from which all the others can be derived, which are Erosion and Dilation. The way they operate is very similar to the convolution process on a binary image. As a probe, they use a kernel called a structuring element.

A structuring element is an important notion to understand while performing mathematical morphology operations. It is a binary array that has a geometric shape, an origin, and is much smaller than the image being processed. This structuring element may have any size or shape chosen according to the shape of the object we would like to detect in the image. Its origin is generally picked in the centre but this is not always the case. The structuring element slides over the image and proceeds to do a pixel transformation that depends on the morphological operation performed. Below are some examples of structuring elements with their origins at the centre. The 0's define the background while the 1's define the foreground.

1	1	1
1	1	1
1	1	1

0	1	0
1	1	1
0	1	0

0	0	1
0	1	0
1	0	0

1	1	1

0	0	1	0	0
0	1	1	1	0
1	1	1	1	1
0	1	1	1	0
0	0	1	0	0

The following notation will be used in the subsequent sections. Let E be a Euclidean space; A a binary image in E and B a structuring element. The complement of A is denoted A^c, where $A^c = \{x \in E \mid x \notin A\}$; the reflection of A across the origin is denoted $\hat{A}$, where $\hat{A} = \{x \in E \mid -x \in A\}$ and, the translation of A by z is denoted A_z, where $A_z = \{c \mid c = a + z, \forall\, a \in A\}$.

2.1 Mathematical operators

2.1.1 Dilation. The effect of dilation is to widen the object; the height and the width of the dilated object will be the sums respectively of the heights and widths of the original object and the structuring element. It is an operation that stretches the image by increasing its size. In practice, this means that for each pixel i present in the background of the image A, the origin of the structuring element B is superimposed. If at least one pixel of B overlaps A, then the pixel i is set to the foreground's state; otherwise it remains in the background.

Mathematically, we can define the dilation of A by the structuring element B by:

$$A \oplus B = \{a + b \mid a \in A, b \in B\}.$$

It can also be viewed as the union of shifted point sets :

$$A \oplus B = \bigcup_{b \in B} A_b \quad \text{where } A_b \text{ is the translation of } A \text{ by } b.$$

Dilation satisfies the following properties.

- Translation invariant.

  $A_z \oplus B = A \oplus B_z = (A \oplus B)_z.$

- Increasing in A. If the image A is a subset of C, the dilation of A by B is also subset of the dilation of C by B.

  If $A \subseteq C$ then $A \oplus B \subseteq C \oplus B$.

- Distributive over set union. Dilating A by the union of two structuring elements B and C is equivalent as dilating A by B, A by C and then perform their union.

  $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C).$

- Commutative.

  $A \oplus B = B \oplus A$

- Associative.

  $(A \oplus B) \oplus C = A \oplus (B \oplus C).$

- Dual with Erosion. The dilation of an image A is the complemented erosion of the complementary of the image A. In other terms, dilating the foreground is the same as eroding the background, but the structuring element reflects between the two.

  $(A \oplus B)^c = A^c \ominus \hat{B}.$

- Identity set. Dilation has an identity set I.

  $A \oplus I = A.$

- Dilation has an empty set.

  $A \oplus \emptyset = \emptyset.$

Dilation closes holes smaller than the structuring element, fills narrow channels and welds two shapes together. For example, in the case of an old manuscript whose letters would no longer be very visible, the dilation would make it possible to reconstitute each letter clearly by filling in the holes.

In a distributed detection system, the distributive property implies that if we take the dilation of two different sets and we apply the same structuring element by taking the union of the results after, then it is the same as taking the union of the sets and dilating it with the structuring element. If we have a large structuring element of size n, dilating with that structuring element is equivalent to dilate n times with the corresponding structuring element of size 1. Thus, the dilation by a structuring element can be broken down into a combination of dilations by simpler structuring elements. We next give an example.

Let us consider the following example where aiming to dilate the image A (left image) by the structuring element B (right image). The origin of B is its centre.

0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0
0	1	1	1	1	0	0	0	0	0
0	0	1	1	0	0	1	1	1	0
0	0	0	0	0	1	1	1	1	0
0	0	0	0	1	1	1	1	1	0
0	0	0	1	1	1	1	1	0	0
0	0	1	1	1	0	0	0	0	0
0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0

0	1	0
1	1	1
0	1	0

After dilation, we obtain:

0	0	1	1	0	0	0	0	0	0
0	1	1	1	1	0	0	0	0	0
1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	0	0
0	0	1	1	1	1	1	0	0	0
0	0	0	1	1	1	0	0	0	0

This image has been obtained by putting the origin of B on each white pixel in A. Any black pixel in A touched by a white pixel of B is set to the value 1.

Another way to view dilation is as a boolean algebra operation when the $\oplus$ represents the logical disjunction. The origin of the structuring element would be slided on each pixel of the image and the logical disjunction would be performed between each pixel of the structuring element and the pixel under it in the image.

2.1.2 Erosion. Erosion is the dual operation of dilation with respect to the passage to the complement. Unlike dilation, erosion shrinks the image by reducing its size. In practice, this means that for each pixel i in A, the origin of B is superimposed. If at least one pixel of B is contained in the background of A, then the pixel i is set to the background's state; otherwise it remains in A. In other words, for each position occupied by the centre of the structuring element B, we ask ourselves this question: Is B completely included in A? Positive answers will form the eroded set.

Let A and B be an image and a structuring element respectively. Let $A \ominus B$ denote the erosion of A by B. Erosion is defined as follows:

$$A \ominus B = \{z \in E \mid b + z \in A, \ \forall \ b \in B\}.$$

Equivalently, we may have

$$A \ominus B = \bigcap_{b \in B} A_{-b} \quad \text{where } A_{-b} \text{ is the translation of } A \text{ by } -b.$$

Erosion satisfies the following properties.

- Translation invariant.

  $$A_z \ominus B = A \ominus B_z = (A \ominus B)_z.$$

- Increasing in A. If the image A is contained in another image C, the erosion of A by B is also contained in the erosion of C by B.

  If $A \subseteq C$ then $A \ominus B \subseteq C \ominus B$.

- Decreasing in B. If we have two structuring elements B_1 and B_2 such that B_1 is contained in B_2, the erosion of A by B_1 will contain the erosion of A by B_2.

  If $B_1 \subset B_2$ then $A \ominus B_1 \supset A \ominus B_2$.

- Distributive over set intersection.

  $$A \ominus (B \cap C) = (A \ominus B) \cap (A \ominus C).$$

- Dual with Dilation. An image A eroded is the complemented dilation of the complementary of the image A. In other terms, eroding the foreground is the same as dilating the background, but the structuring element reflects between the two.

  $$(A \ominus B)^c = A^c \oplus \hat{B}.$$

- Identity set. Erosion has an identity set I.

  $$A \ominus I = A.$$

- Satisfies $(A \ominus B) \ominus C = A \ominus (B \oplus C)$.

  Vermani and Vermani (2011) show this property:

$$\begin{aligned}
(A \ominus B) \ominus C &= \left[(A \ominus B)^c \oplus \hat{C}\right]^c && \textit{(Duality principle between } (A \ominus B) \textit{ and } C.) \\
&= \left[(A^c \oplus \hat{B}) \oplus \hat{C}\right]^c && \textit{(Duality principle between } A \textit{ and } B.) \\
&= \left[A^c \oplus (\hat{B} \oplus \hat{C})\right]^c && \textit{(Associativity of Dilation.)} \\
&= \left[A \ominus \widehat{(\hat{B} \oplus \hat{C})}\right] && \textit{(Complementation rule.)} \\
(A \ominus B) \ominus C &= A \ominus (B \oplus C).
\end{aligned}$$

Furthermore, erosion eliminates related components smaller than the structuring element, widens channels and holes. Erosion is very effective when it comes to eliminate noise in an image. It would be enough to use a structuring element of size slightly larger than the elements to be removed.

In a distributed detection system, the distributive property of erosion implies that if we take the erosion of two different sets and we apply the same structuring element by taking the intersection of results after, then it is the same as taking the intersection of the sets and eroding it with the structuring element.

We could also decompose an erosion with a large structuring element into a sequence of operations with smaller structuring elements.

As an example, let us erode A (left image) by B (right image):

0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0
0	1	1	1	1	0	0	0	0	0
0	0	1	1	0	0	1	1	1	0
0	0	0	0	0	1	1	1	1	0
0	0	0	0	1	1	1	1	1	0
0	0	0	1	1	1	1	1	0	0
0	0	1	1	1	0	0	0	0	0
0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0

0	1	0
1	1	1
0	1	0

After erosion, we obtain:

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	1	0	0
0	0	0	0	0	1	1	1	0	0
0	0	0	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

This image has been obtained by putting the origin of B on each white pixel i in A. If any white pixel in B touches any black pixel in A, then i is set to the value 0.

Erosion and Dilation are the basic operators in Mathematical morphology, but their composition or combination can create other types of operators like Opening and Closing.

2.1.3 Opening and Closing. Once an image is eroded, it is not generally possible to get the original image back with an inverse transformation. Applying dilation on an eroded image can help us to recover some parts of the original image. This process of erosion followed by dilation is called Opening. Opening removes small and isolated objects from the foreground, it breaks narrow sections and sharp peaks. As for closing, it fills small holes in the foreground and combine narrow breaks. Closing is dilation followed by erosion. It has the property of filling all the holes that are smaller than the structuring element while opening aims to smooth out shapes and remove all related components that are not bigger than the structuring element.

Dilation and erosion composed can give rise to opening and closing but in different procedures. To get opening, an image is first eroded then dilated. While closing dilates the image first before eroding it.

Mathematically, opening and closing can be defined as follows. Let $A \circ B$ be the opening of A by the structuring element B and $A \bullet B$ be the closing of A by the structuring element B.

$$A \circ B = (A \ominus B) \oplus B.$$
$$A \bullet B = (A \oplus B) \ominus B.$$

Opening has the following properties:

- Translation invariant.

  $A_z \circ B = A_z \circ B = (A \circ B)_z.$

- Increasing in A. If the image A is contained in another image C, the opening of A by B is also contained in the opening of C by B.

  If $A \subseteq C$ then $A \circ B \subseteq C \circ B$.

- Anti-extensive. The opening of A by B is contained in A.

  For every A, $(A \circ B) \subseteq A$.

- Idempotent. Applying the opening many times on the same image has no additional effect.

  $(A \circ B) \circ B = A \circ B.$

- Dual with Closing.

  $(A \circ B)^c = A^c \bullet \hat{B}.$

  Vermani and Vermani (2011) show this property.

  We know that $A \circ B = (A \ominus B) \oplus B$.

$$\begin{aligned}
(A \circ B)^c &= [(A \ominus B) \oplus B]^c \\
&= (A \ominus B)^c \ominus \hat{B} \quad \textit{(Complementation rule between } (A \ominus B) \textit{ and } B.) \\
&= (A^c \oplus \hat{B}) \ominus \hat{B} \quad \textit{(Complementation rule between } A \textit{ and } B.) \\
&= A^c \bullet \hat{B} \\
(A \circ B)^c &= A^c \bullet \hat{B}.
\end{aligned}$$

Closing has the following properties:

- Translation invariant.

  $A_z \bullet B = A_z \bullet B = (A \bullet B)_z.$

- Increasing in A. If the image A is contained in another image C, the closing of A by B is also contained in the closing of C by B.

  If $A \subseteq C$ then $A \bullet B \subseteq C \bullet B$.

- Extensive. The closing of A by B contains A.

  For every A, $(A \bullet B) \supseteq A$.

- Idempotent. Applying the closing many times on the same image has no additional effect.

  $(A \bullet B) \bullet B = A \bullet B.$

- Dual with Opening.

$$(A \bullet B)^c = A^c \circ \hat{B}.$$

Vermani and Vermani (2011) show this property.

We know that $A \bullet B = (A \oplus B) \ominus B$.

$$\begin{aligned}(A \bullet B)^c &= [(A \oplus B) \ominus B]^c \\ &= (A \oplus B)^c \oplus \hat{B} \quad \textit{(Complementation rule between } (A \oplus B) \textit{ and } B.) \\ &= (A^c \ominus \hat{B}) \oplus \hat{B} \quad \textit{(Complementation rule between } A \textit{ and } B.) \\ &= A^c \circ \hat{B} \\ (A \bullet B)^c &= A^c \circ \hat{B}.\end{aligned}$$

Using the preceding example, opening and closing of A by B give respectively:

0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0
0	1	1	1	1	0	0	0	0	0
0	0	1	1	0	0	1	1	0	0
0	0	0	0	0	1	1	1	1	0
0	0	0	0	1	1	1	1	1	0
0	0	0	1	1	1	1	1	0	0
0	0	1	1	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0
0	1	1	1	1	0	0	0	0	0
0	0	1	1	1	1	1	1	1	0
0	0	0	1	1	1	1	1	1	0
0	0	0	1	1	1	1	1	1	0
0	0	0	1	1	1	1	1	0	0
0	0	1	1	1	1	1	0	0	0
0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0

In general, when applying opening, we do not get back the original image because part of the shape eliminated by erosion can not be recreated by dilation. These transformations do not preserve connectivity. When applying opening and closing to an image repeatedly, there is no further modification. They can be seen as fixed points. This idempotence property is very important for a filter because it ensures that the image will not be modified further even if we repeat the transformation process.

Opening and closing are not the only operators that use a combination of dilation and erosion. There are other types of transformations like Hit-and-Miss Transform that can also describe objects shapes, using a pair of structuring elements rather than a single structuring element.

2.1.4 Hit-and-Miss Transform. Hit-and-Miss Transform is a classical operator of mathematical morphology used to detect particular shapes in an image. It involves the application of a double erosion on the image A and its complement A^c with two disjoint structuring elements B_1 and B_2, having the same origin.

The Hit-and-Miss Transform (HMT) of A by (B_1, B_2) is the set of points i such that the translation of B_1 by i fits into A and the translation of B_2 by i fits into the background of A. When sliding the composite structuring element to every position in the image, we should consider the following question: "Does the first structuring element fit the foreground, while at the same time the second structuring element fits the background?". The positive answers form the HMT set.

HMT is usually used to identify particular features and patterns in images like isolated foreground pixels, foreground endpoints or foreground contour points. For instance, for a corner detection using HMT, we could use the following structuring elements where x represents the "don't care" cells. It means the x cells could be either 1 (foreground) or 0 (background). Below are four structuring elements that can detect right-angle corner points in a binary image (Bottom left corner, Bottom right corner, Top right corner, Top left corner).

x	1	x
0	1	1
0	0	x

x	1	x
1	1	0
x	0	0

x	0	0
1	1	0
x	1	x

0	0	x
0	1	1
x	1	x

Let $A \circledast B$ be the HMT of A by the structuring element B such that $B = (B_1, B_2)$ with $B_1 \cap B_2 = \emptyset$.

$$A \circledast B = (A \ominus B_1) \cap (A^c \ominus B_2).$$

It is also equivalent to:

$$A \circledast B = \{z \mid (B_1)_z \subseteq A, (B_2)_z \subseteq A^c\}$$

The HMT can also be viewed as the difference of the erosion of A by B_1 and the dilation of A by B_2 reflected.

$$A \circledast B = (A \ominus B_1) - (A \oplus \hat{B}_2).$$

Let us illustrate how it works by applying the Hit-and-Miss Transform on A (left image) with the structuring element B (right image) composed of B_1 (white cells) and B_2 (black cells).

0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0
0	1	1	1	1	0	0	0	0	0
0	0	1	1	0	0	1	1	1	0
0	0	0	0	0	1	1	1	1	0
0	0	0	0	1	1	1	1	1	0
0	0	0	1	1	1	1	1	0	0
0	0	1	1	1	0	0	0	0	0
0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0

0	0	x
0	1	1
x	1	x

The first step consists in eroding the foreground with B_1.

0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0
0	0	0	0	0	0	1	1	0	0
0	0	0	0	0	1	1	1	0	0
0	0	0	0	1	1	1	1	0	0
0	0	0	1	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

The second step consists in eroding the background with B_2.

0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

In order to find the final image of the transformation by HMT, we should do the intersection of the two images obtained previously.

0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

We notice that HMT retains only the pixels that fit exactly with the foreground and background of the structuring elements. Erosion is a special case of HMT where B_2 is an empty set.

Almost all other binary morphological operators such as thinning and skeletonisation can be derived from HMT (Shih, 2017).

2.2 Application

HMT has many applications in target shapes detection which include thinning and skeletonisation. Thinning is the residue between the original image and its Hit-and-Miss Transform. In other words, it is the arithmetic difference between the original image and its HMT. It provides a simple representation of the shape of an object. Thinning can be used in biomedical diagnosis, fingerprint recognition and industrial tools inspection. The thinning of A by B denoted by $A\phi B$ is defined as

$$\begin{aligned} A\phi B &= A - (A \circledast B) \\ &= A \cap (A \circledast B)^c. \end{aligned}$$

The thinning operation is processed by translating the origin of the structuring element to each possible pixel position in the image, and at each such position comparing it with the underlying image pixels. The pixel under the origin of the structuring element is set to 0 if there is a perfect match between the foreground and background pixels in the image, and the foreground and background pixels in the structuring element. Otherwise, there is no change.

The skeleton of A is the medial axis of A. It helps to find a minimal representation of an image by reducing foreground regions and also find intersection points. It is used to remove selected pixels in images foreground in order to obtain a single pixel wide. We can find the skeleton of an image by applying successively thinning until we reach stability. It can also be expressed in terms of erosions and openings where B is the structuring element and $(A \ominus_k B)$ represents k successive erosions of A. The skeleton of A by B denoted by $S_k(A)$ is defined as:

$$S_k(A) = (A \ominus_k B) - (A \ominus_k B) \circ B.$$

We have seen above that dilation and erosion are dual with respect to complementation. Galois connections are often used in mathematical morphology to establish a dual isomorphism between two types of structures (Bloch et al., 2007). We will see in the following section how dilation and erosion form a Galois connection.

2.3 Galois connection

Galois connection is a notion that is typically used to define a relatively complex function in terms of another relatively simple function (Backhouse, 2002).

On two partially ordered sets $(L, \leq)$ and $(L', \leq')$, let α and β be such that $\alpha : L \mapsto L'$ and $\beta : L' \mapsto L$. We will say that α and β form a Galois connection (Atif et al., 2013) if:

$$\forall\, x \in L,\ \forall\, y \in L',\ y \leq' \alpha(x) \Longleftrightarrow x \leq \beta(y).$$

Let us redefine Erosion and Dilation in a complete lattice theory. A lattice is a partially ordered set in which any two elements have an infimum (greatest lower bound) and a supremum (least upper bound) (Soille et al., 2011). We say that a lattice is complete if each subset has an infimum and a supremum.

In a complete lattice $(\mathcal{P}, \leq, \vee, \wedge)$, if an operator $\delta : L \mapsto L'$ conserves the smallest element and commutes with the supremum, then it is called a dilation (Atif et al., 2013):

$$\forall\, (x_i) \in L,\ \delta(\vee_i x_i) = \vee'_i \delta(x_i),$$

where $\vee$ is the supremum associated with $\leq$ and $\vee'$ the one associated with $\leq'$.

An operator $\epsilon : L' \mapsto L$ is an erosion if it conserves the greatest element and commutes with the infimum (Atif et al., 2013):

$$\forall\, (x_i) \in L',\ \epsilon(\wedge'_i x_i) = \wedge_i \epsilon(x_i),$$

with $\wedge$ the infimum associated with $\leq$ and $\wedge'$ the one associated with $\leq'$.

The operators ϵ and δ form an adjunction if:

$$\forall\, x \in L,\ \forall\, y \in L',\ \delta(x) \leq' y \Longleftrightarrow x \leq \epsilon(y).$$

We call (ϵ, δ) a *monotone Galois connection* and (α, β) is an *antitone Galois connection*. Obviously they are equivalent if we take an opposite order in one of the lattices (Atif et al., 2013).

It means that in a complete lattice, there is only a single erosion (similarly for the dilation) that satisfies the following property for each dilation (similarly for the erosion):

$$\forall\ X, Y \in \mathcal{P}, \quad \delta(X) \leq Y \Longleftrightarrow X \leq \epsilon(Y).$$

They satisfy also the following property:

$$\forall\ X \in \mathcal{P}, \quad \delta\epsilon\delta(X) = \delta(X) \text{ and } \epsilon\delta\epsilon(X) = \epsilon(X).$$

A simple example of Galois connection is given by the floor function (Backhouse, 2002). The floor function gives the greatest integer less than or equal to a real number x. It is denoted $\lfloor x \rfloor$.

$$n \leq \lfloor x \rfloor \Longleftrightarrow n \leq x$$
$$n \leq \text{ floor}(x) \Longleftrightarrow \text{real}(n) \leq x.$$

A Galois connection is defined by two partially ordered sets and two functions between the ordered sets. From the definition stated above, we can observe that two weak inverse functions are Galois connected (Backhouse, 2002). In our case, dilation and erosion are weak inverses. That means that we do not always recover the original image back by applying opening (erosion followed by dilation) or closing (dilation followed by erosion) on an image. However there are some interesting cases where dilation and erosion are true inverses. It is interesting for us to find a classification of the types or categories of images that would give true inverses and weak inverses. Knowing which types of images could never be recovered with dilation and erosion would allow us to hide some information in images.

In our research, we found that applying closing to an image A will give the original image back if:

- A is a single component that does not have an interior cavity smaller than the size of the structuring element B, or
- A is the union of components with each one of them without an interior cavity smaller than the size of the structuring element B, and the distance between each two of them should be greater than the size of the structuring element.

In other words, dilation and erosion are weak inverses only if there is a void or hole smaller than the structuring element in the original image or between separated components of the original image. The obvious reason is that once the dilation succeeds in filling holes in the image or between the components of the image, the erosion does not succeed in completely reconstituting these holes to get the original image back. In the case where we would like to hide some information in our image, creating some cavities in the original image could prevent someone from reconstructing the original image back. In the following examples, we would like to show some patterns of behaviour resulting from the composition of dilation and erosion.

• A is a single component that does not have an interior cavity smaller than the size of the structuring element B.

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0
0	1	1	1	1	1	0	0
0	1	1	1	1	1	0	0
0	0	1	1	1	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

0	1	0
1	1	1
0	1	0

The dilation of A by B (left image) followed by the erosion of $(A \oplus B)$ by B (right image) gives the original image A back. In this case, dilation and erosion are true inverses. This is more likely to work with images that have the same structure as A.

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0
0	1	1	1	1	1	0	0
1	1	1	1	1	1	1	0
1	1	1	1	1	1	1	0
0	1	1	1	1	1	0	0
0	0	1	1	1	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0
0	1	1	1	1	1	0	0
0	1	1	1	1	1	0	0
0	0	1	1	1	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

• Let us now take the case where A is a single component but has an interior cavity smaller than the size of the structuring element B.

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

0	1	0
1	1	1
0	1	0

The dilation of A by B (left image) followed by the erosion of $(A \oplus B)$ by B (right image) gives an image that contains the original image A. In this case, dilation and erosion are weak inverses and we get artifacts in the output image.

0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0
0	1	1	1	1	1	1	0
0	1	1	1	1	1	1	0
0	1	1	1	1	1	0	0
0	1	1	1	1	1	0	0
0	1	1	1	1	1	1	0
0	1	1	1	1	1	1	0
0	0	1	1	1	1	0	0
0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	0	0	0
0	0	1	1	1	0	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

As for opening, it does not behave in the same way as closing. Dilation and erosion are always true inverses in the case where the image and the structuring element are both convex components. The original image could also be composed of separated convex components. In other words, the fact that the original image is not convex would in most of the cases result in omissions in the output image, causing dilation and erosion to be weak inverses. Let us see some relevant examples.

- A is a single convex component as well as the structuring element B.

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0
0	0	1	1	1	0	0	0
0	0	1	1	1	0	0	0
0	0	1	1	1	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

0	0	0
1	1	1
0	0	0

The erosion of A by B (left image) followed by the dilation of $(A \ominus B)$ by B (right image) gives back the original image A. In this case, dilation and erosion are true inverses.

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0
0	0	0	1	0	0	0	0
0	0	0	1	0	0	0	0
0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0
0	0	1	1	1	0	0	0
0	0	1	1	1	0	0	0
0	0	1	1	1	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

- Let us see a case where A is not a convex component while the structuring element B is convex.

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	1	1	0	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	0	1	1	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

1	1	1
1	1	1
1	1	1

The erosion of A by B (left image) followed by the dilation of $(A \ominus B)$ by B (right image) gives an image contained in the original image A. In this case, dilation and erosion are weak inverses and we have some omissions in the output image.

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	1	1	0	0	0
0	0	0	1	1	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

However there are some special cases where both closing and opening give back the original image. Let us illustrate one of these cases. Here, the original image A is not convex, neither is the structuring element B.

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0
0	1	1	1	1	1	0	0
0	0	1	1	1	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

0	1	0
1	1	1
0	1	0

The erosion of A by B (left image) followed by the dilation of $(A \ominus B)$ by B (right image) gives back the original image A (opening).

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0
0	1	1	1	1	1	0	0
0	0	1	1	1	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

The dilation of A by B (left image) followed by the erosion of $(A \oplus B)$ by B (right image) gives back the original image A (closing).

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0
0	1	1	1	1	1	0	0
1	1	1	1	1	1	1	0
0	1	1	1	1	1	0	0
0	0	1	1	1	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0
0	1	1	1	1	1	0	0
0	0	1	1	1	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

Another interesting question that deserves further investigation is that of the fixed points. Would there be a set of fixed points that would always remain unchanged regardless of the sequence of operations that would be applied to an image A? The answer to that question may give a good research framework to classify shapes according to their fixed points.

This chapter introduced the notion of mathematical morphology in relation to the shape of objects in an image. It helped us to understand the processes of the two basic operators (dilation and erosion) and their derivates. We discovered that their different compositions (opening and closing) may have different effects on images. We were also interested in the notion of duality between dilation and erosion. This allowed us to express them as a Galois connection and to navigate through several examples to discover certain patterns of behaviour. We saw that dilation and erosion are weak inverses in closing only if the original image has a cavity smaller than the structuring element; while opening gives true inverses if the original image and the structuring element are both convex components. In the following chapter, we will use the notions acquired previously to design a detection system based on morphology.

3. Shape detection

Object detection is a challenging problem in computer vision because it implies identifying a particular shape of an object in an environment filled with many other objects. As an example, it might be easy for a robot to detect a particular tool on a cleared table, but it might be a more difficult detection task if the table is cluttered by other tools that could hide certain parts of the desired tool. Detection problems can be addressed by mathematical morphology operations but there are also other methods used in image processing and computer vision. The focus of this essay is on morphology operations. In this section, we will tackle the problem of detection by unimodal and multi-modal methods.

3.1 Unimodal detection method

We have previously seen that dilation and erosion can be distributed respectively with respect to the union and the intersection. This is a very important property that we can use in many real-world applications. In the context of the analysis of a geographical map (for example a satellite image), at first, we could divide the map into several sub-regions. Then we could apply the morphological operators in each subregion before making a union or an intersection depending on the operator used. This can enable parallelization of tasks, yet reduce time complexity.

In our study, we are concerned with the detection of shapes in binary images. Hit-and-Miss Transform (HMT) is a suitable operator for this detection. The reason of using HMT is because of its double structuring element: one to erode the object and the second one to erode the background of the object. Those two structuring elements are disjoint and it allows detecting individual objects. In the case where we are not interested in detecting individual objects, the background will not be required and the HMT could be reduced to a simple erosion.

The word detection often makes us think of discovery or localization. We would be interested in objects that we know and would like to be able to detect in an image. We have seen how the Hit-and-Miss transform could help us to achieve this goal. Since we know the exact shape of the object, we could build a pair of disjoint structuring elements in order to detect the object we are looking for. We could also reduce this problem to an enumeration problem of all the objects contained in a particular image. In case there are several instance of objects, it is necessary to build different pairs of structuring elements adapted to each type of object.

Now, let us imagine that we are not interested in objects whose shapes we know, but rather in those we do not know. For instance, let us say that we have two images of a room in a museum taken at different times. We would like to know if there has been any change in the room, precisely if no object has disappeared in the meantime. A simple difference between the first image and the second image would allow us to notice any changes that may have occurred and to detect any missing object.

The problem of shape detection that leads us to look for a perfectly known object in an image implies the choice of a suitable structuring element. Unfortunately, in most cases, detecting the shape of an object does not give us enough information about the nature of that object, especially if we would like to act on it. In this case, adding another type of signal can be very beneficial.

3.2 Multi-modal detection method

A multimodal detection system contains information that can be complementary in a way that could improve the detection performance of a unimodal system. In order to improve the task of detecting objects in an image, we could add a second signal that will strengthen the detection model. For better exposition of multi-modal detection, we consider two examples next. The first example uses a single data type but in two different ways. The second one shows sequence processes of two signals.

Let us take the example of an office equipment factory. We would like a robot to put pens in packages according to their different colours. Two features need to be detected here: shape and colour. Assuming that the robot can recognize the shape of a pen, it will be able to detect pens. On top of shapes, the robot must distinguish the pens by colour. Thus instead of using a binary signal alone, one could do detection with the binary image, and then augment that detection with reference to a colour image (RGB). When distinction of colours is not required, this example is then not a multi-modal detection.

For the second example, we consider a realistic case which uses two different signals in sequence namely infrared and vision. Owyhee Air Research (OAR) is a company that works with state agencies and private consultants to track and analyse animal movements in a wild environment (Owyhee Air Research). In order to find certain animal species, OAR uses infrared as the first signal. Infrared aerial is a very interesting method because it allows to easily detect the infrared emissions of target animals even in the middle of the forest. After locating these animals, it then uses the camera's vision (colour) to confirm their species. We could also design a detection system using two or more signals in parallel.

4. Results and discussion

One way to characterize a spatial structure is to describe the size of this set using a criterion that is sensitive to the surface of its connected components. In this case, the size will be described not by an average, but by a distribution. To define a size criterion, the theorists of mathematical morphology (Matheron, 1974) were inspired by the analysis of powdery materials. One of the conventional techniques used in this case is to sieve the image, separating the structures of the image according to their size. The idea of granulometry by mathematical morphology is to apply to the image, some filters by a structuring element of a given form, and whose size varies. In this chapter, we will briefly introduce the field of granulometry and see some practical examples of objects detection in a binary image.

4.1 Experiment setup

For experimental investigation, we implemented our methods in Python and OpenCV. The source code of this experiment can be found here. OpenCV (Open Source Computer Vision) is a library that offers a set of more than 2500 computer vision algorithms, accessible through APIs for C, C ++, and Python languages. It is distributed under a BSD (free) license for Windows, GNU/Linux, Android and MacOS platforms. (Bradski and Kaehler, 2008).

4.2 Experiment and results

Opening makes objects disappear from a binary image when they do not contain the structuring element. With this concept, it is therefore possible to sieve a set of particles by simply considering a family of openings associated with structuring elements of increasing size (Vachier, 1995). The notion of granulometry and particle size transformation was introduced by Georges Matheron in 1967 (Matheron, 1967). Granulometric analysis is the study of the size of objects based on the principle of sieving. A particle size is calculated using a filter pyramid in which each opening acts according to a given size. For example, granulometry can be used to characterize textures or to obtain information about an image without segmentation (Serra, 1969). One can also use granulometry to separate blood cells in an image in order to count them.

In the following example, we detect the shapes of each of the objects in the image using Hit-and-Miss Transform. Let us consider the following image.

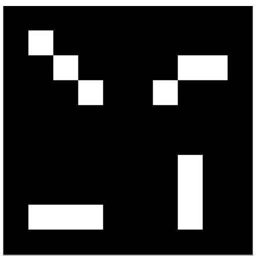
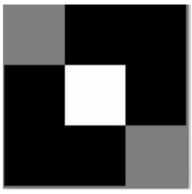
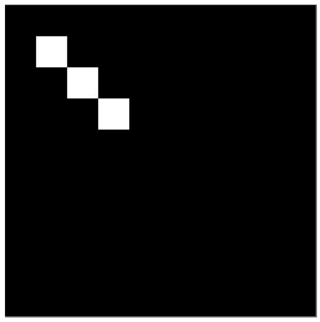


Figure 4.1: Original image.

The structuring element is illustrated as follows. It was said earlier that a structuring element has three types of cells: foreground, background and "don't care". The white pixels represent the foreground that should match the object, the black pixels represent the background and the grey pixels are the cells we do not care about. To detect the first object (upper left in the original image) with the Hit-and-Miss Transform, we could use the structuring element (Figure 4.2a) and get the object in Figure 4.2b. We can see that the structuring element matches only all the pixels of the first target object and all the other objects are left out.



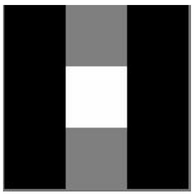
(a) Structuring element.



(b) Object detected.

Figure 4.2: Detection of the first object.

To detect the second object (bottom right in the original image) with the Hit-and-Miss Transform, we could use the structuring element 4.3a and get the object 4.3b.

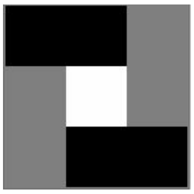


(a) Structuring element.

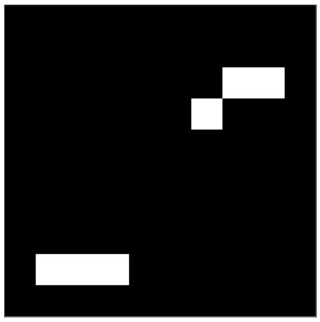
(b) Object detected.

Figure 4.3: Detection of the second object.

Unfortunately, to detect the remaining objects in the image, we could not find different structuring elements to detect the two remaining objects separately (4.4b). To be able to detect the upper right object alone in Figure 4.2b, we were obliged to not take care of some cells in our structuring element. Unfortunately, those "don't care" cells have allowed another object to come out.



(a) Structuring element.



(b) Object detected.

Figure 4.4: Detection of the remaining objects.

Mathematical morphology offers powerful tools for analysing the shapes of objects. In the previous chapter, we saw the two basic morphological operators namely dilation and erosion. From these two operators flow all the others. The duality that exists between erosion and dilation is a very important property that allows us to express one according to the other. Indeed, dilating the foreground is equivalent to eroding the background. As an image can be seen as a set, this duality is even more noticeable when redefining erosion and dilation in a complete lattice. It is observed that they form a Galois connection. Galois connections are characterized by a relationship linking two weak inverses (Backhouse, 2002). Nevertheless, there are some particular cases where we encounter true inverses. This says a lot about erosion and dilation as two functions. Indeed, we already knew that they are not inverses of each other because applying erosion (similarly for dilation) on a dilated image (similarly on an eroded image) does not necessarily give back the original image. Yet knowing that dilation and erosion are weak inverses and form a Galois connection, allows us to state with certainty that there are some cases where they behave as true inverses. In these cases, we certainly find the original image. We have studied some of these cases in the previous chapter.

4.3 Discussion

We studied some patterns that guaranteed a return to the original image after applying closing or opening to an image. Closing being a dilation followed by an erosion, it was easy to see that when dilation completely closed a cavity present in the original image, the erosion did not manage to reopen it completely, and we finally obtain an image larger than the original image. But it is often what we seek by applying closing to an image, we would like to fill in small holes that appeared and connect narrow sections that had been separated. This is a way of cleaning the image with a filter. Knowing that closing could sometimes give us the original image back allows us to conclude that certain types of images can never be filtered with closing.

As for the opening, it proceeds first erosion before applying the dilation. As the closing closes the small holes in the foreground, opening eliminates small islands and breaks thin sections. It is also a way of cleaning the image. As for the final object obtained, it is often smaller than the original image. Nevertheless, as we have seen in examples, the fact that the original image and the structuring element are both connected components, would ensure us to get the original image back.

During our investigations, we also encountered a case in which neither closing nor opening had managed to clean the image by transforming it. This is a very interesting case because it would imply that there would be non-filterable images. The idempotence property of opening and closing ensures that nothing would change even if they were applied multiple times to this type of images. These images are therefore invariant with respect to closing and opening.

Hit-and-Miss Transform is an operator that can detect the shape of an object in an image. We have seen that it is an operator that uses a combination of two disjoint structuring elements in order to perform a double erosion, one in the foreground and the other in the background. What differentiates it mainly from the other operators is the presence of the “don’t care” cells (the grey cells in Figures 4.2a, 4.3a and 4.4a). These cells allow any binary value at their location. This kind of weakening for some of the structuring element’s values guarantees a flexibility around the object and gives a margin to not miss it. A practical example has shown us that, when detecting a particular object, it may happen that other objects or small artifacts also creep into the output image (Figure 4.4b).

In summary, when we consider detecting an object whose shape is already known with the mathematical morphology operators we have studied, we could end up in one of the following four cases:

- The output image contains exactly the object we wanted to detect. In this case we will say that our detection has succeeded;
- the object obtained is not at all the one we expected. This case may occur with HMT when there are objects in the image that have a shape similar to our object and may have passed through the mesh of our filter because of the “don’t care” in the structuring element. At the same time, some occlusions could have hidden parts of the object that we would have liked to detect;
- we get exactly our object but with some artifacts in the image. This case is similar to the previous case without occlusions of the object. We could use opening as a filter to clean these artifacts;
- we get our object but with omissions. This means that the shape of the object is not complete, some parts are missing.

HMT is a very powerful tool that allows us to detect target objects in an image because of the structure of its pair of structuring elements that allows us to give a certain margin to not miss the target object.

5. Conclusions and further work

The aim of this essay was to be an introduction to the field of mathematical morphology and an application to object detection in images. After exposing the basic operators of mathematical morphology, namely dilation and erosion, we presented the elementary combinations of these two basic operators: opening and closing; then we introduced the Hit-and-Miss Transform operator (which allowed us to detect an individual object in an image) and some of its applications. Subsequently, we discovered that the basis of the definition of morphological operators on complete lattices comes from the notion of adjunction (Galois connection). This allows us to establish an isomorphism between the set of erosions and that of dilations.

Detection of an object is a challenging problem in computer vision especially in an environment filled with objects that could hide the target object or have a shape similar to this object. This allowed us to conclude that a detection system based on shape alone is not enough. A detection system would be more robust by adding colour as a feature for example.

Many interesting investigations were conducted during this research. Many questions remain unanswered which can be addressed by the use of some mathematical concepts. For instance, we were not able to fully characterise types of images in which erosion and dilation are true inverses. Future work could then be to investigate mathematical ways of classifying images under same properties.

Acknowledgements

"Not that we are sufficient in ourselves to claim anything as coming from us, but our sufficiency is from God" (2 Corinthians 3:5). This verse sums up pretty well how I feel at the end of writing this essay. God has been faithful throughout this journey and I feel very grateful for His blessings. I thank AIMS for the privilege to be part of this great scientific family.

First of all, I would like to express my gratitude to my supervisor, Dr. Simukai Utete, for her patience, her availability and above all her wise advice, which contributed to my reflections. I would also like to thank Prof. Jeff Sanders, who was the first to introduce me to the notion of Galois connections and its link to my research topic. I would also like to thank the efforts of Dr. Yaé Ulrich Gaba, who was kind enough to answer my questions, provide necessary explanations and open my mind to innovative ideas.

A big thanks to my dear parents Jacob and Esther for their love and their unconditional support. I thank my sisters Shélémia, Anne and Lydie for their encouragements.

I would like to express my gratitude to the friends and tutors who gave me their moral and intellectual support throughout my journey. I would like to thank Juliana, Audace, Rock, Kenneth and Taboka. Many thanks to Jordan for his advice on writing my essay.

Finally, I wish to express my gratitude to D'Jeff, Jean, Hewan, Salomon, Berthine, Joram, Banan and Samah for their invaluable support. You have played an important role in my life and I will never forget it. You are the reason why my stay in South Africa has been great. I will miss you all.

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