Approximation for Total-Value Adjustment of European Put Option using Central Finite Difference Method

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Abstract

After the global financial crisis in 2007-2009, traditional option pricing models were regarded as inappropriate. New option pricing models that include total-valuation adjustments are being developed. In this work we derive a Bilateral risky PDE model, which extends the Black-Scholes PDE model to include total-valuation adjustments such as bilateral counterparty risk and funding costs. We apply a central-finite difference method to approximate the solution of the total-valuation adjustments for European put options.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

N. Yuta

Noluvuyo Yumata, 26 October 2017

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1. Introduction

Derivatives have become very important in finance. A derivative can be defined as a financial contract that derives its value from an underlying asset such as stocks, bonds, commodities, currencies, interest rates and market indexes. Derivatives are used to transfer risk in mortgages from original lenders to investors or act as a protection to insure the risk of adverse changes in asset prices . Also, they can serve as a collateral for another derivative (Kariya and Liu, 2003). Only few derivatives are traded on exchanges, the bulk of the market is simply private contracts between banks and sophisticated investors. The private contracts are referred as over-the-counter (OTC) market.

Exchange derivatives are traded through a central-exchange which act as an intermediary between the counterparties that are trading a derivative. An exchange market has listing requirements and publicly visible prices unlike the OTC market. The OTC market is very flexible as it allows even small firms who cannot meet exchange listing requirements to trade. But it carries high risk for the counterparty compared to the exchange, because of less prices transparency (Canabarro and Duffie, 2003).

Many institutions and financial analysts claim that the 2007 global financial crisis was the consequence of ignoring counterparty risk in the OTC derivative market. Counterparty risk is the risk that each party involved in a derivative trade may not meet the full obligations of the contract. Inappropriate pricing of OTC derivatives and consideration of low probability of default were considered as the main causes of the crisis (Arregui et al., 2017).

The traditional derivative pricing model that relies on the assumption that one can borrow and lend at a risk-free interest rate, did not take into account an effect of counterparty risk. After the crisis, new pricing models which include total-valuation adjustments (XVA) were developed. Total-valuation adjustments aim to capture "counterparty risk" in derivative pricing and also serve as a protection in order to avoid another global financial crisis.

The main purpose of this work is to approximate the value of XVA for European put options using central finite-difference method and Crank Nicolson method. Firstly, we derive the bilateral risky PDE model which include counterparty risk and funding costs. In which the counterparty risk and funding costs represent XVA. We then decompose the risky PDE model into two parts: a risk-free derivative and XVA models. Numerical methods are then applied to approximate the solution of the XVA PDE model. The central-finite difference method is applied to discretize the space (asset prices) and Crank Nicolson method to discretize time.

This paper is structured as follows. In Chapter 2, we study the standard Black-Scholes pricing model and total-valuation adjustments. In Chapter 3, the bilateral risky PDE model and XVA model are derived. We present the space discretization, time discretization and discuss numerical results in Chapter 4. We then conclude in Chapter 5.

2. Option pricing

In this chapter we discuss basic notions of option pricing including the Black-Scholes standard pricing model. We first introduce mathematical tools such as probability theory and stochastic processes which are fundamental in financial models.

2.1 Basic notions on probability theory

In this section we introduce basic probability theory which is essential in building financial models.

2.1.1 Definition. [**Probability space**] [σ - *field, measurable space*]

A class $\mathcal{F}$ of subsets of a given set Ω is a $\sigma-$field of subsets of Ω if:

1. $\Omega \in \mathcal{F}$.
2. If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$, where $A^c = \Omega \backslash A$ is the complement of A in Ω.
3. If $A_n \in \mathcal{F}$ then $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$.

The couple $(\Omega, \mathcal{F})$ is called a measurable space.

Moreover, the σ-field, $\sigma(\mathcal{F})$, generated by the class $\mathcal{F}$ is the smallest σ-field containing the class $\mathcal{F}$.

$$\sigma(\mathcal{F}) = \cap\{\xi : \xi \;\; \sigma - field, \mathcal{F} \subset \xi\} \tag{2.1.1}$$

The σ-field generated by the class of intervals in $\mathbb{R}$ is called the Borel σ-field denoted by $\mathcal{B}$. see (Janssen and Manca, 2007)

2.1.2 Definition. [Probability measure]

A probability measure $\mathbb{P}$ on a probability space $(\Omega, \mathbb{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ such that

1. $\mathbb{P}(\emptyset) = 0, \quad \mathbb{P}(\Omega) = 1.$
2. If $A_1, A_2, ..., A_n \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint then

$$\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i) \tag{2.1.2}$$

3. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

see Janssen and Manca (2007).

2.1.3 Definition. [Random variable]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable X is an application $X : \Omega \to \mathbb{R}^n$ such that

$$\forall B \in \mathcal{B}, X^{-1}(B) = \{\omega : X(\omega) \in B \in \mathcal{F}\} \tag{2.1.3}$$

where $\mathcal{B}$ is the Borel $\sigma-$field (Janssen and Manca, 2007).

2.1.4 Definition. [Independence of random variables]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X_1 and X_2 be two random variables. The random variables X_1 and X_2 are said to be independent if for every choice of Borel sets B_1, B_2 we have

$$\mathbb{P}(X_1 \in B_1, X_2 \in B_2) = \mathbb{P}(X_1 \in B_1) \times \mathbb{P}(X_2 \in B_2) \quad (2.1.4)$$

see (Janssen and Manca, 2007)

2.2 Basic notions on stochastic process

The models in financial mathematics are based on stochastic process which include stochastic calculus. Let us define some useful notions by introducing the very important Ito's formula for jump diffusion process.

2.2.1 Definition. [*Stochastic process*]

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(S, \sum)$ and a measurable space. A stochastic process with values on the measure space S is a family of random variables

$$\{X_t, t \in T\} \quad (2.2.1)$$

where for all $t, X_t : \Omega \rightarrow E$ is $\mathcal{F}$-measurable, i.e

$$\forall \in S, X_t^{-1}(B) = \{\omega : X_t(\omega) \in B\} \in \mathcal{F} \quad (2.2.2)$$

The set T is called a parameter of a stochastic process.

For every $\omega \in \Omega$, the mapping

$$t \rightarrow X_t(\omega) \quad (2.2.3)$$

defined on the parameter T is called the trajectory or the sample path of the process. Fore more details see (Øksendal, 2003)

2.2.2 Definition. [*Increments of stochastic process*]

An increment of a stochastic process $\{X_t, t \in T\}$ is the random variable $X_t - X_s$ with $t \neq s$ where $t, s \in T$. A stochastic process $\{X_t, t \in T\}$ has independent increment if for all $t_1 < t_2 < \cdots < t_k$, the random variables $X_{t_1}, X_{t_2} X_{t_1}, \cdots, X_{t_k} - X_{t_{k-1}}$ are independent (Canabarro and Duffie, 2003).

2.2.3 Definiton. [*Weiner process*]

A standard (one-dimensional) Wiener process (also called Brownian motion) is a stochastic process $\{W_t, t \geq 0\}$ indexed by non-negative real number t with the following properties

1. W_0=0
2. With probability 1, the function $t \rightarrow W_t$ is continuous in t.
3. The process $\{W_t, t \geq 0\}$ has a stationary independent increments.
4. The increment $W_t - W_s$ are Normally distributed $N(0, t)$.

A standard d-dimensional Wiener process is a vector-valued stochastic process

$$W_t = \left(W_t^1, W_t^2 \cdots W_t^d\right) \tag{2.2.4}$$

whose components are independent, standard 1 dimensional Weiner processes.

Remark: Brownian motion is used in financial modelling to represent the movement of some financial instruments/assets. see (Canabarro and Duffie, 2003)

2.2.4 Definition. [*A Poisson process*]

A Possion process is a stochastic process which counts the number of events and the time these events occur in a given time interval. The Poisson process $\{N_t, t \geq 0\}$ holds the following properties

1. $N_0 = 0$.
2. N_0 has independent increment.
3. No counted occurrences are simultaneous and N_t follows a poisson distribution given by

$$\mathbb{P}[N(t+\epsilon) - N(t)] = \frac{(a\lambda)^k \times \exp^{-a\lambda}}{k!}, \qquad k \in \mathbb{N} \tag{2.2.5}$$

where $N(t+a) - N(t)$ is the number of events in time interval $[t, t+a]$ (Canabarro and Duffie, 2003).

2.2.5 Definition. A compound Poisson process J_t with intensity $\lambda \geq 0$ and jump size distribution f and $N(t)$ is a stochastic process defined as

$$J_t = \sum_{n=1}^{N_t} X_n \tag{2.2.6}$$

where jumps size X_n are identically independent distribution f and N_t is a poisson process with intensity λ, independent from $(X_n)_{n\geq 1}$ (Canabarro and Duffie, 2003).

2.2.6 Definition. [*Ito's Process*]

An Ito process is a stochastic process X that solves

$$X_t = X_0 + \int_0^t a(X_s, s)ds + \int_0^t b(X_s, s)dW_s, \quad t \geq 0 \tag{2.2.7}$$

- X_0 is a scalar point
- $\{a(X_t, t) : t \geq 0\}$ and $\{b(X_t, t) : t \geq 0\}$ are stochastic processes satisfying certain regularity conditions.

In differential equations, it can be written as follows

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t \tag{2.2.8}$$

where W_t is the Weiner process with instantaneous drift $a(X_t, t)$ and an instantaneous volatility $b(X_t, t)$. see Canabarro and Duffie (2003)

2.2.7 Theorem:. [*Ito's formula*]

Let X_t be an Ito's process given by

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{2.2.9}$$

Let $g(t,x) \in \mathbb{C}^2\Big([0,\infty] \times \mathbb{R}\Big)$. That is g is twice continuously differentiable on $[0,\infty) \times \mathbb{R}$. Then $Y_t = g(t, X_t)$ is again an Ito's process and

$$dY_t = \frac{\partial g(t,X_t)}{\partial t}dt + \frac{\partial g(t,X_t)}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2 g(t,X_t)}{\partial X_t^2}(dX_t)^2 \tag{2.2.10}$$

where $(dX_t)^2$ is computed by the rules

$$dt \times dt = dt \times dW_t = dW_t \times dt = 0,$$
$$dW_t \times dW_t = dt$$

For proof details see (Øksendal, 2003).

2.2.8 Theorem. [*Ito's formula for jump-diffusion process*]

Let X_t be a jump-diffusion process defined as

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \sum_{n=1}^{N_t} \Delta X_n \tag{2.2.11}$$

where b_t and σ_t are continuous non-anticipating process with

$$\mathbb{E}\Big[\int_{t_0}^t \sigma_t^2 dt\Big] < \infty$$

Then for any $C^{1,2}$ (*twice continuous differentiable*) function $f : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$, the process $Y_t = f(t, X_t)$ can be denoted as

$$dY_t = \frac{\partial f(X_t,t)}{\partial t}dt + b_t\frac{\partial f(X_t,t)}{\partial x}dt + \sigma_t\frac{\partial f(X_t,t)}{\partial x}dW_t + \frac{1}{2}\sigma_t^2\frac{\partial^2 f(X_t,t)}{\partial x^2}dt + \Big[f(X_{t-} - \Delta X_t) + f(X_{t-})\Big] \tag{2.2.12}$$

see (Tankov, 2003)

2.3 Basic finance definitions

In this section we provide basic definitions for the reader to understand the terms used in this paper .

2.3.1 Definition. [Asset]

In finance, an asset is defined as a resource with economic value that an individual, corporation or country owns or controls, with the expectation that it will provide future benefit (Janssen and Manca, 2007).

2.3.2 Definition. [Portfolio]

A portfolio is a set of financial assets such as stocks, bonds and cash equivalents, as well as their mutual, exchange-traded and closed-fund counterparts. Portfolios are held directly by investors and/or managed by financial professionals. They are used for hedging (an investment strategy to reduce the risk of adverse price movements in an asset). see Janssen and Manca (2007)

2.3.3 Definition. [Self-financing portfolio]

A self-financing portfolio is characterised by the assumption that all trades are financed by selling or buying assets in the portfolio. No amount withdrawn or added after the initial forming of the portfolio. If we let Π_t be the value of the portfolio at time t, we have

$$\Pi(t) = \sum_i \lambda_i S_i(t)$$

So for the self-financing portfolio we have

$$d\Pi(t) = \sum_i \lambda_i dS_i(t)$$

where S_i present the financial assets (Siadat, 2016).

2.3.4 Definition. [Arbitrage portfolio]

The portfolio is called an arbitrage portfolio if the value of a portfolio $\Pi(t)$ has the following properties

$$\begin{aligned} &\Pi(0) = 0 \\ &\Pi(1) = 1 \quad \text{with probability 1} \end{aligned}$$

Basically, having an arbitrage in a portfolio means a portfolio makes a positive amount out of nothing, (Siadat, 2016).

Remark: An existence of arbitrage is considered as serious case of mispricing.

2.3.5 Definition. [Hedging Portfolio]

Let V be the financial derivative (claim) defined as $V = \phi(Z)$, where Z is the stochastic variable driving the stock price process. Then a financial derivative V is a said to be reachable if there exist a portfolio such that

$$\Pi(t) = V$$

with probability 1. In this case, the portfolio is said to be hedging portfolio or replication portfolio. If all claims can be replicated then the market is complete (Siadat, 2016).

2.3.6 Difinition. [Dividend]

A dividend is a payment made by a corporation to its shareholders, usually as a distribution of profits.

2.3.7 Definition. [Collateral]

A collateral is the asset used to pledge for repayment of a derivative, to be forfeited in the event of a default.

2.3.8 Definition. [Out-of-money]

"Out-of-money" is a phrased used to describe the call option with strike prices higher than the market price or the put option with strike prices less than the market price.

2.3.9 Definition. [Risk-free rate]

A risk-free rate represent the interest the investor would expect from an investment with zero risk over a specified period of time.

2.3.10 Definition. [Mark-to-Market]

Mark-to-market refers to the daily settling of gains/ losses due the changes in the market price. If the market value of the derivative goes up on a given trading day, the party who bought the derivative (**long position**) collects the money-equal to the derivative change in value from the party who sold the derivative (**short position**). If the value of the derivative goes down on a given trading day, the party who sold the derivative collects money-equal to the security change in value-from the party who bought the derivative. The value of the derivative does not change at maturity as a result of daily fluctuations. However the parties pay losses or gains to each other at the end of each trading day (Green, 2015)

2.3.11 Definition. [close-outs netting agreement]

Close-out netting agreement occurs between a defaulting and non-defaulting party. It is a technique used to mitigate counterparty risk. A **close out** refers to a process of terminating a contract with a defaulting party. The process involves evaluating the negative (those owed by the defaulting party) and positive (those owed by the non defaulting party) replacement costs of each transaction. Then net them against each other into a single net payable/receivable to determine the value of the derivative at default. see (Canabarro and Duffie, 2003)

2.3.12 Definition. [A repurchase agreement]

A repurchase agreement is a form of transaction between two parties where one party sells an asset at a specified price and commits to repurchase the asset from them other party at different higher price at a future date. The difference between the price paid in the future for buying back an asset and a money received now for an asset is an interest. The party selling an asset and agreeing to repurchase it the near future is a **repo**. The other party that is buying an asset and agreeing to sell it in the near future, is called a reversed-repo. If the party that that sold the asset, does not buy back it back in the future as per agreement, the other party has has a right to sell the asset to a 3rd party (Siadat, 2016). This process is called **collaterised agreement**.

2.3.13 Definition. [Zero-Coupon Bond]

A zero coupon bond with maturity date T, is a contract which guarantees the holder 1 dollar (face value) to be paid on the date T. The convention that face value, equals 1 is made for computational convenience while in reality different face values are used , e.g $1000 is a common face value for zero-coupon bond, (Siadat, 2016).

2.4 Basic notions on option pricing

In this section we briefly introduce basic notions of option pricing. We are following the book (Kariya and Liu, 2003).

An option represents a contract sold by one party (**option writer**) to another party (**option holder/buyer**). This contract gives the option holder a right but not an obligation, to buy or sell the underlying asset at a fixed agreed-upon price (**strike price**) during a certain period of time or fixed date in the future. If the option holder decides to buy or sell the underlying asset, he is **exercising** the option.

There exist several types of options but the most common are American options and European options. **European options** refer to option contracts that can only be exercised at a fixed date in the future, known as maturity date. **American options** refer to options that can be exercised any time during a certian period of time, that is from the opening of the contract until maturity. In this work we will focus on European options.

Options are under two forms namely: call and put. An **European call** option gives the option holder a right to buy an underlying asset at a strike price at maturity date. Whereas, an **European put** option gives a right to sell an underlying asset at a strike price at maturity date.

Since the contract confers the option holder a right with no obligation, it has a value/price. At the opening of the contract, the option holder must pay a **premium** to the writer as an initial amount to buy this right. The premium paid by the option holder is a non-refundable payment. The writer of an option contract keeps the premium received and is obligated to fulfil the contract's obligations should the option holder decides to exercise the contract. The main goal of option pricing is to determine the "fair" value of the premium.

The **payoff** of an option contract is one of the important concepts in option pricing. The payoff represent the value of the option contract at maturity. In order to comprehend this, let us consider the case of a European call and European put.

Payoff for European call option: Let S be the market price of an underlying stock, K be the strike price of an option contract and T be the maturity date to exercise an option.

- If $S < K$, the option is unlikely to be exercised. It is sensible for the holder of the option not to pay a high strike price K, when the the same underlying asset can be bought at market price S cheaper somewhere else. So in general the option contract expires without being exercised. Thus, the payoff or the value of an contract at maturity T is zero.
- On the other hand, if $S > K$, the option will be exercised. So, the payoff of an option at time T is given by $S - K$.

Therefore, the payoff of the European call option at maturity T is given by

$$V(T, S) = max[S - K, 0]$$

Payoff for European put option: Similar argument to that given above for a call-option leads to the payoff for a put option. At maturity T

- If $S > K$, the put option will not be exercised and the option will expire worthless, so the payoff is zero.

- On the other hand, if $S < K$, the put option will be exercised, so the payoff of the put will be $K - S$.

Thus, the payoff of the European put option at maturity T is given by

$$V(T, S) = max[K - S, 0] \tag{2.4.1}$$

We illustrate the payoff of both call and put in the following diagrams

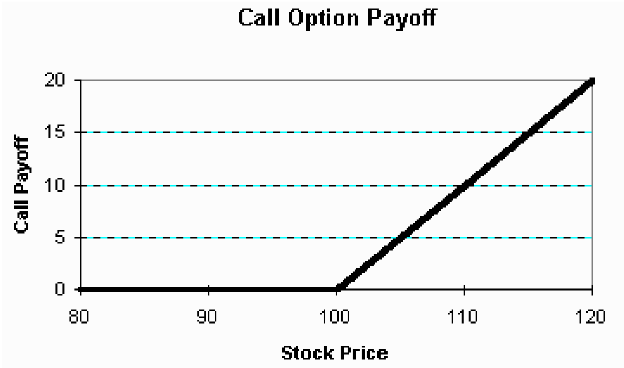


Figure 2.1: The payoff of an European call option where $K = 100$.

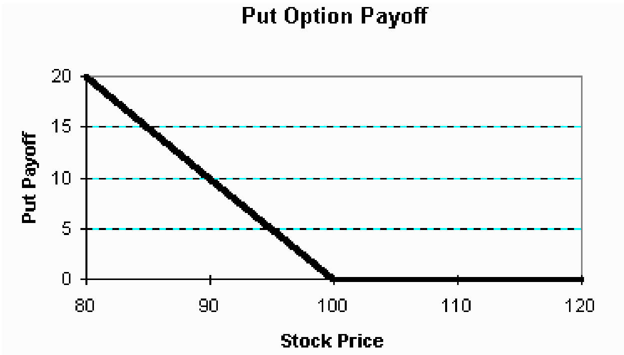


Figure 2.2: The payoff of an European put option where $K = 100$.

2.5 Black-Scholes framework of pricing

In 1973, Black, Scholes and Merton came up with an original approach of modelling the value of an European option. In their paper they developed a PDE model that has a closed form solution, see (Black and Scholes, 1973). The solution gives a price of European option. In the Black-Scholes framework of valuation/pricing, the value of an option is modelled using a hedging portfolio. Basically, the Black-Scholes model states that by continuously adjusting the proportions of underlying assets and options in a portfolio, an investor can create a riskless hedging portfolio. The ability to create such portfolio lies on the assumption of continuous trading and continuous sample paths of asset prices. It is assumed in an efficient market with no arbitrage opportunities, any riskless hedging portfolio must have an expected rate of return equal the risk-free rate. This approach led to a partial differential equation(PDE) model.

Model Assumptions: The following conditions in the market for an underlying asset prices and the option are assumed:

- The market is efficient. This assumption suggests that people cannot consistently predict the direction of the market or an individual asset.
- The asset prices follows a geometric brownian motion, with constant volatility:

$$dS = \mu S dt + \sigma S dW_t \tag{2.5.1}$$

  where μ is the return on the underlying asset, σ is the volatility and W_t is the Weiner process.
- An option written on an underlying asset prices is European option.
- There are no arbitrage opportunities.
- The underlying asset pays no dividend during the life of the option.
- The continuous compounding risk-free rate r is constant.
- The institutions involved in option trading can only borrow or lend at unique risk-free rate.
- Short selling is permitted and assets are perfectly divisible. This means assets that are not owned can be sold and any amount(not necessarily an integer) of the underlying asset can be bought or sold.
- There are no transactional costs associated with buying or selling the financial instruments in a hedging portfolio.

Derivation of the PDE model:

Let S be the underlying asset prices and V be the an option written on the underlying asset prices. Set up a self-financing portfolio Π, that is comprised of an amount Δ of underlying asset S and one option V written on the underlying asset. In this portfolio, we buy the option and short the Δ amount of underlying assets. Thus, the value of the portfolio at time t is given by

$$\Pi = V - \Delta S \tag{2.5.2}$$

Since the portfolio is self-financing, this assumption implies

$$d\Pi = dV - \Delta dS \tag{2.5.3}$$

V is a random variable, we apply Ito's theorem,

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 \tag{2.5.4}$$

where

$$(dS)^2 = (\mu S dt + \sigma S)^2 \tag{2.5.5}$$

$$= \mu^2 S^2 dt^2 + 2\sigma\mu S^2 dt dW_t + \sigma^2 S^2 dW_t^2 \tag{2.5.6}$$

$$= \sigma^2 S^2 dt \tag{2.5.7}$$

note $dt \times dt = dt \times dW_t = 0$ and $dW_t \times dW_t = dt$.

Substitute the equations (2.5.1) and (2.5.7) into (2.5.4). We obtain that an option V evolves in accordance with Ito's process

$$dV = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \left(\sigma S\frac{\partial V}{\partial S}\right)dW_t \tag{2.5.8}$$

where W_t is the Weiner process. Thus, a self-financing portfolio can be simplified to

$$d\Pi = dV - \Delta dS \tag{2.5.9}$$
$$= \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - \Delta\mu S\right)dt + \left(\sigma S\frac{\partial V}{\partial S} - \Delta\sigma S\right)dW_t \tag{2.5.10}$$

The portfolio Π must hold two properties. Firstly, it must be riskless. This assumption implies the term involving the Weiner process dW_t must be zero. So we choose $\Delta = \frac{\partial V}{\partial S}$ to eliminate the risk in the portfolio.

We substitute $\Delta = \frac{\partial V}{\partial S}$ into equation (2.5.10) and obtain that

$$d\Pi = \left(\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt \tag{2.5.11}$$

Secondly, any riskless portfolio must earn a risk-free rate. This means a return on the portfolio Π would face a growth rate $r\Pi dt$ over time. Thus

$$d\Pi = r\Pi dt \tag{2.5.12}$$
$$\left(\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt = r\left(V - \frac{\partial V}{\partial S}\right)dt \tag{2.5.13}$$

We rearrange the equation (2.5.13) and obtain the following Black-Scholes PDE model

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.5.14}$$

Remarks: An important observation is that the PDE model depends on the volatility σ of asset prices. This observation makes the volatility an important parameter to consider when pricing an option.

2.5.1 Boundary Conditions. In order to find a unique solution of PDE model which is the value of an option, boundary conditions were introduced.

For the case of a European put option, with value now denoted by $P(S,t)$, with strike price K and maturity T. At $t = T$, the value of a put option is given by the payoff which is

$$P(S,T) = max(K - S, 0)$$

This is the final condition at time T.

- If $S = 0$, the value of an option at maturity is K. To determine $P(0,t)$, we calculate the present value of an option received at maturity T. The present value is calculated by discounting the future value K, using continuously compounded interest. Assuming a constant interest rate r, money in a risk-free investment $M(t)$ grows exponentially according to

$$\frac{dM}{dt} = rM \tag{2.5.15}$$

The solution to (2.5.15) is given by

$$M = Ke^{-r(T-t)}$$

since $M = K$ at t=T. Therefore the boundary condition at $S = 0$ is given by

$$P(0,t) = Ke^{-r(T-t)}$$

- If $S \to \infty$ we saw earlier that the put option is won't be exercised. Thus

$$P(S,t) \to 0 \quad \text{as} \quad S \to \infty$$

Similar arguments described above are applied for the European call option. Therefore, the premium value/ price of a call and put are given by the following equations

- European Call: $C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$
- European Put $P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)$

  where

$$d_1 = \frac{log(\frac{S}{K}) + (r + \frac{1}{2}\sigma)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_1 = \frac{log(\frac{S}{K}) + (r - \frac{1}{2}\sigma)(T-t)}{\sigma\sqrt{T-t}}$$

  $N(x)$ is the standard normal cumulative distribution.

The Black-Scholes model has limitations as some of the assumptions are not valid in real life. For example, the underlying assets do pay dividends to the shareholders. Also when hedging a portfolio, there are transactional costs involved for buying and selling assets in the portfolio. The volatility of asset prices is not constant over time. However, the original model had been adjusted to overcome these limitations.

The Black-Scholes model played an important role in option pricing. It is considered as a traditional way for pricing European options. The Black-Scholes option price is referred as a no-default value. It assumes that both parties involved in an option contract/ trade will live up to their obligations. This assumption of parties not defaulting was valid, until parties involved in option trading started defaulting during the global financial crisis in 2007-2008. So after crisis, the Black-Scholes model can no longer be used for pricing options.

2.6 Total-valuation adjustments

Since the crisis, when financial entities went bankcrupt, the counterparty risk has become an important concern in pricing derivatives. Counterparty risk can be described as a risk that each party involved in a financial contract may not meet the full obligations of the contract.

Many techniques in derivate trading have been introduced to mitigate counterparty risk. Include techniques such as close-out netting agreements to end a trade in the event of one party defaults. Also, collaterized agreements to make sure a collateral is posted when trading any derivative (Bliss and Kaufman, 2006). However, these risk mitigants do not fully eliminate the counterparty risk. After the crisis, total valuation adjustments were then introduced in derivative pricing to avoid another global financial crises.

The total-valuation adjustment(XVA) include credit-valuation adjustment (CVA), debit-valuation adjustment (DVA), funding value adjustment (FVA). These adjustment are added to non-default value of a derivative (price before the crisis) to capture the effect of counterparty risk and funding (transaction) costs of trading a derivative in today's financial market conditions.

- **Credit-valuation adjustment (CVA)**

Suppose a lender(B) enters into a derivative trade with a counterparty (C). The CVA is designed to take into account the risk that the counterparty C defaults and causes losses to the lender. Derivatives are marked-to-market. The value of the derivative upon defaults, depends on the sign of the mark-to-market value.

So when pricing a derivative one needs to take this risk into account. Basically, the value of the derivative is calculated without considering the possibility of default (risk-free derivative) and then it is subtracted by CVA to reflect the possibility of one party default.

- **Debit-valuation adjustment (DVA)**

Counterparty risk is considered as bilateral, since both parties involved in a derivative trade have a possibility of defaulting. DVA is simply CVA from the counterparty's perspective. It reflects a risk faced by a counterparty (C) when a lender (B) defaults. So if B enters into a derivative trade with C then

$$(CVA)_B = (DVA)_C \qquad (CVA)_C = (DVA)_B$$

However, in practice this symmetry might not be seen since different traders use different CVA models. CVA and DVA is what is known as bilateral counterparty risk. To incorporate the bilateral counterparty risk, from the non-default value of a derivative the CVA is subtracted and DVA is added.

DVA is known to record a "gain" at a lender's own default. For example, imagine a case where a lender trades an **uncollaterised** OTC derivative that is "out-of money". When the lender defaults, the counterparty will recover a proportion of the mark-to-market value of the derivative, the rest of the money is with the lender's bondholder (Green, 2015).

- **Funding-valuation adjustment (FVA)**

FVA is an adjustment to the value of a derivative or a portfolio derivative that is designed to make sure that a lender recovers his average funding cost when he trade or hedges a derivative. For example, when trading a derivative there are transaction costs involved, the lender must pay a broker. Also, there are trading spreads associated with hedging a derivative.

There exist many other adjustments related to XVA. However in this work we restrict to CVA,DVA and FVA. Therefore, mathematically the XVA can be defined as

$$XVA = DVA - CVA + FVA$$

2.7 Problem statement

Traditional derivative pricing models such as Black-Scholes which were used before the crisis, did not take into account the counterparty risk. Ignorance of counterparty risk in derivative pricing models is regarded as a main ingredient of the crisis. The consequences of this risk resulted to a development of total-valuation adjustments (XVA). Therefore, after the crisis new pricing models which include total-valuation adjustments are being developed. In this work we consider the bilateral risky PDE model for pricing European options proposed by Burgard and Kjaer (2010). This bilateral risky PDE extends the Black-Scholes model to include XVA. The value of the risky derivative denoted by $\hat{V}$ is given as

$$\hat{V} = V + XVA \tag{2.7.1}$$

where V is the value of the risk-free European option from a Black-Scholes pricing model with dividends.

The main goal of this work is to approximate the value of the XVA for European options, using central finite-difference methods.

3. PDE representation for bilateral counterparty risk and funding cost

In this chapter we build the models setup. We first derive the Bilateral risky PDE model for pricing European options. Then decompose the risky PDE to get the model for the XVA. We are following the papers of (Burgard and Kjaer, 2010) and Arregui et al. (2017).

3.1 Derivation of bilateral risky PDE model

Let us consider the following traded financial assets:

- P_B : default risky, zero-recovery, zero-coupon bond of counterparty B.
- P_C : default risky, zero-recovery, zero coupon bond of counterparty C.
- S: underlying asset with no default risk

Take note the P_B and P_C are simple bonds that are very useful for simplifying complex models. The counterparty B (seller) pays 1 at maturity if it does not default and zero otherwise. The same situation is defined for counterparty C.

Due to the risk involved, stock and bond prices are modelled as stochastic processes which satisfy the following stochastic differential equations.

$$dS = r_R(t)Sdt + \sigma(t)SdW_t \tag{3.1.1}$$

$$dP_B = r_B(t)P_Bdt - P_BdJ_B \tag{3.1.2}$$

$$dP_C = r_C(t)P_Cdt - P_CdJ_C \tag{3.1.3}$$

Where W_t is a Weiner process, r_B and r_C are the yields of the risky zero-coupon bonds of counterparty B and C respectively. Assume $r_B(t) > 0$, $r_C(t) > 0$ and $\sigma(t)$ are deterministic functions of t. J_B and J_C are two independent jump 'Poisson' processes that change from 0 to 1 on default of B and C respectively.

Now, consider a seller B and the counterparty C entering a derivative trade on the underlying asset S with the payoff $H(S)$ at maturity T. When $H(S) > 0$ to the seller, it means the seller$B$ receives cash or an asset from C. We denote the value of this derivative at time t by $\hat{V}(t, S, J_B, J_C)$. It can be seen that this derivative does not only depend on the underlying asset but also on the defaults states J_B and J_C of the seller B and counterparty C, so clearly it is risky. Let $V(t, S)$ denote a risk-free value of the derivative, where both parties are assumed to be non-defaulting.

Since the trade takes place between counterparties that may default, we need to incorporate the close-out issues (end of the trading session in the financial market) or claim position of the derivative contract as enforced by the International Swaps and Derivative Association (ISDA). Let $M(t, S)$ be the mark-to-market value of the derivative. In the event that one party defaults, the close-out is determined by $M(t, S)$. Generally $M(t, S) = V(t, S)$ as suggested by the ISDA agreement. Even though (Burgard and Kjaer, 2010) have considered also the scenario where $M(t, S) = V(t, S, J_B, J_C)$, for our work we restrict to the case where $M(t, S) = V(t, S)$. Please note that the positive values of the mark-to-market derivative value corresponds to the seller's assets and counterparty's liabilities.

Let $R_B \in [0,1]$ and $R_C \in [0,1]$ be the recovery rates (the value of a derivative when one party defaults) of derivative positions of parties B and C respectively. So we have the following boundary conditions:

- If the seller B defaults first

$$\hat{V}(t, S_t, 1, 0) = V^+(t, S) + R_B V^-(t, S) \tag{3.1.4}$$

- If the counterparty C defaults first

$$\hat{V}(t, S_t, 0, 1) = R_C V^+(t, S) + V^-(t, S) \tag{3.1.5}$$

Equation (3.1.4) explains a situation where the seller B defaults first, so if the mark-to-market is positive to the seller then $V^-(t,S) = 0$ and we have $V(t, S_t, 0, 1) = V^+(t, S)$ which means the counterparty C should pay the full mark-to-market $V^+(t, S)$ to the seller. If the mark-to-market is negative to the seller then $V^+(t, S) = 0$, hence $V(t, S_t, 1, 0) = R_B V^-(t, S)$ which means the recovery value of $(R_B V^-(t, S))$ should be paid to the counterparty, (Siadat, 2016).

Equation (3.1.5) can be analysed in a similar manner. In the event that the counterparty C defaults first, if the mark-to-market is positive to the seller then $V^-(t, S) = 0$, we have $V(t, S_t, 0, 1) = R_C V^+(t, S)$ which means the recovery value of $(R_C V^-(t, S))$ should be paid by the counterparty to the seller. If the mark-to-market is negative to the seller then $V^+(t, S) = 0$, hence $V(t, S_t, 0, 1) = V^-(t, S)$ which means the seller B should pay the full mark-to-market $V^+(t, S)$ to the counterparty.

Similar to the Black-Scholes framework, (Burgard and Kjaer, 2010) set up a self-financing portfolio to hedge the value of the risky derivative to the seller at time t, such that $\Pi_t + \hat{V}_t = 0$. The portfolio Π_t consists of all the risky assets such as $\delta(t)$ units of S, $\alpha_B(t)$ units of P_B, $\alpha_B(t)$ units of P_C and an amount of cash $\beta(t)$. So we have

$$-\hat{V}_t = \mathit{\Pi}_t = \delta(t)S_t + \alpha_B(t)P_B + \alpha_C(t)P_C + \beta(t) \tag{3.1.6}$$

The cash account β is for financial costs involved for buying or selling the trading assets in this derivative trade. The bonds P_B and P_C are used for hedging the counterpaty default risk. We are interested in the seller's point of view. In the event of default of the counterparties, as discussed in the arguments above, we saw that the seller might gain and lose depending on the sign of the mark-to-market V.

Suppose $V > 0$, the seller will incur a loss if the counterparty defaults, because the seller will only recover a proportion of the mark-to-market derivative. To hedge this loss, the bond P_C is borrowed by the seller through a repurchase agreement close to a risk-free rate r, known as the repo rate. Then, the seller short the bond P_C, hence $\alpha_C < 0$. The spread λ_C, which is the difference between the rate r_C on the bond P_C and the cost of financing the hedge is given by $\lambda_C = r_C - r$. This spread corresponds to the default intensity of C, recall we assumed P_B and P_C are zero-recovery bonds.

Furthermore, suppose that $V < 0$, the seller will gain at his own default. According to (Brigo and Morini, 2010), the seller can hedge his own default risk by buying back $\alpha_B P_B$ bonds, hence $\alpha_B > 0$. But in order for this hedging strategy to work, sufficient cash needs to be generated and any remaining cash (after purchasing the bonds of P_B) must be invested in a way that doesn't generate extra credit risk.

The portfolio Π_t is assumed to be self-financing, this assumption implies

$$-d\hat{V}_t = d\mathit{\Pi}_t = \delta(t)dS_t + \alpha_B(t)dP_B + \alpha_C(t)dP_C + d\beta(t) \tag{3.1.7}$$

where the growth in cash can be decomposed into:

$$d\beta(t) = d\beta_S(t) + d\beta_F(t) + d\beta_C(t) \tag{3.1.8}$$

1. $d\beta_S$ is the funding for the underlying asset which provides a dividend income $\delta(t)D_0(t)S(t)dt$ and financing costs for the underlying assets S given by $\delta(t)r_R(t)S(t)dt$. Thus

$$\beta_S(t) = \delta(t)(D_0 - r_R)(t)S(t)dt \tag{3.1.9}$$

   where r_R is the rate (repo-rate) paid on the underlying asset in a repurchase agreement. Assuming that it also depends on the risk-free rate.

2. Financing costs incurred for shorting the bond P_C :

$$d\beta_C(t) = -\alpha_C(t)r(t)P_C(t)dt \tag{3.1.10}$$

   The bonds of P_C are borrowed through a repo rate $r(t)$.

3. Remaining cash on the seller's bank account after buying own bond P_B:

   - If the **cash account is positive**, the remaining cash held by the seller after purchasing his own bond P_B must be invested in risk-free assets to avoid extra credit risk.
   - If the **cash account is negative**, the seller needs fund via external provider. The seller will be borrowing money and paying the funding rate r_F. For this case there are two possibilities. Firstly, if the derivative itself can be used as a collateral, then $r_F = r$. If the derivative cannot be used as a collateral, then $r_F = r + s_F$. Where $s_F > 0$ is the funding spread which corresponds to the yield of the unsecured bond P_B with recovery bond R_B given by $s_F = (1 - R_B)\lambda_B$. Thus the seller's cash account position is given by

$$\begin{aligned} d\beta_F(t) &= r(t)(-\hat{V} - \alpha_B P_B)^+ dt + r_F(-\hat{V} - \alpha P_B)^- dt &(3.1.11)\\ &= r(t)(-\hat{V} - \alpha_B P_B)^+ dt + s_F(-\hat{V} - \alpha P_B)^- dt &(3.1.12)\end{aligned}$$

   where $s_F \equiv r_F - r$. When the derivative is used as a collateral $s_F = 0$ and $s_F = (1 - R_B)\lambda_B$ when it is not used as a collateral.

We substitute equations (3.1.9), (3.1.10), (3.1.12) into equation (3.1.8) and dP_B and dP_C from equations (3.1.1) and (3.1.3). For notational convenience we drop the (t)'s.

$$\begin{aligned} d\hat{V} &= \delta dS + \alpha_B dP_B + \alpha_C dP_C + d\beta_S + d\beta_C + d\beta_F \\ &= \delta dS + \alpha_B P_B(r_B dt - dJ_B) + \alpha_C P_C(r_C dt - dJ_C) - \delta r_R S dt - \alpha_C P_C dt \\ &+ r(-\hat{V} - \alpha_B P_B)^+ dt + s_F(-\hat{V} - \alpha P_B)^- dt \\ &= \Big\{\alpha_B r_B P_B + \alpha_C r_C P_C - \alpha_B r P_B - \alpha_C r P_C + (D_0 - r_R)\delta S - r\hat{V} + s_F(-\hat{V} - \alpha_B P_B)^-\Big\}dt \\ &+ \delta dS - \alpha_B P_B dJ_B - \alpha_C P_C dJ_C \\ &= \Big\{ - r\hat{V} + \alpha_B P_B(r_B - r) + \alpha_C P_C(r_C - r) + (D_0 - r_R)\delta S + s_F(-\hat{V} - \alpha_B P_B)^-\Big\}dt \\ &+ \delta dS - \alpha_B P_B dJ_B - \alpha_C P_C dJ_C \end{aligned} \tag{3.1.13}$$

We apply the Ito's lemma for jump diffusions, assuming no simultaneous jump between the counterparties. We then have

$$\begin{aligned} d\hat{V} &= \partial_t \hat{V} dt + \partial_S \hat{V} dS + \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V} dt + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C \\ &= \Big(\partial_t \hat{V} + \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V}\Big) dt + \partial_S \hat{V} dS + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C \end{aligned} \tag{3.1.14}$$

where $\Delta \hat{V}_B$ and $\Delta \hat{V}_C$ (computed from the boundary conditions in (3.1.5) and (3.1.4)) are given by

$$\begin{aligned} \Delta \hat{V}_B &= \hat{V}(t,S,1,0) - \hat{V}(t,S,0,0) \\ &= -\Big[\hat{V}(t,S,0,0) - \hat{V}(t,S,1,0)\Big] \\ &= -\Big[\hat{V} - \Big(V^+ + R_B V^-\Big)\Big] \end{aligned} \tag{3.1.15}$$

$$\begin{aligned} \Delta \hat{V}_C &= \hat{V}(t,S,0,1) - \hat{V}(t,S,0,0) \\ &= -\Big[\hat{V}(t,S,0,0) - \hat{V}(t,S,0,1)\Big] \\ &= -\Big[\hat{V} - \Big(R_C V^+ + V^-\Big)\Big] \end{aligned} \tag{3.1.16}$$

We replace $d\hat{V}$ from (3.1.13) by (3.1.14) to find out the coefficients of the portfolio. Hence we eliminate all the risk in a portfolio by choosing the coefficients as follows:

For δ

$$\begin{aligned} \delta dS &= -\partial_S \hat{V} dS \\ \delta &= -\partial_S \hat{V} \end{aligned} \tag{3.1.17}$$

For α_B

$$\begin{aligned} \alpha_B P_B dJ_t^B &= -\Delta \hat{V}_B dJ_t^B \\ \alpha_B &= \frac{-\Delta \hat{V}_B}{P_B} \end{aligned} \tag{3.1.18}$$

For α_C

$$\begin{aligned} \alpha_C P_C dJ_t^C &= -\Delta \hat{V}_C dJ_t^C \\ \alpha_C &= \frac{-\Delta \hat{V}_C}{P_C} \end{aligned} \tag{3.1.19}$$

If we introduce the parabolic differential operator A_t as

$$A_t V \equiv \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + (D_0 - r_R) S \partial_S V \tag{3.1.20}$$

it follows that $\hat{V}$ is the solution of the following PDE

$$\begin{cases} \partial_t \hat{V} + A_t \hat{V} - r\hat{V} &= s_F\Big(\hat{V} + \Delta \hat{V}_B\Big)^+ - \lambda_B \Delta \hat{V}_B - \lambda_C \Delta \hat{V}_C \\ \hat{V}(T,S) = H(S) \end{cases} \tag{3.1.21}$$

where $\lambda_B \equiv r_B - r$ and $\lambda_C \equiv r_C - r$. We substitute (3.1.15) and (3.1.16) into (3.1.21), the PDE model simplifies to

$$\begin{cases} \partial_t \hat{V} + A_t \hat{V} - r\hat{V} & = s_F V^+ + (\lambda_B + \lambda_C)\hat{V} - \lambda_B\Big(R_B V^- + V^+\Big) - \lambda_C\Big(V^- + R_C V^+\Big) \\ \hat{V}(T,S) = H(S) \end{cases} \tag{3.1.22}$$

where we used the fact that $\Big(\hat{V} + \Delta\hat{V}_B\Big)^+ = (V^+ + R_B V^-)^+ = V^+$.

The above Bilateral risky PDE given in equation (3.1.22) is linear. If we compare the bilateral risky PDE model to the Black-Scholes PDE model given by

$$\begin{cases} \partial_t V + A_t V - rV & = 0 \\ V(T,S) = H(S) \end{cases} \tag{3.1.23}$$

we see that the bilateral risky PDE has few adjustments. The first term on the right shows the funding cost, the 2nd, 3rd and 4th are related to the bilateral counterparty risk.

3.2 XVA PDE model

According to (Brigo and Morini, 2010), the value of the derivative with bilateral counterparty risk can be written as follows

$$\hat{V} = V + U \tag{3.2.1}$$

where U is the total value adjustment and V is the risk-free value of the derivative which satisfy the Black-Scholes PDE model in equation (3.1.23).

If we substitute $\hat{V}$ into the bilateral risky PDE in (3.1.22), we obtain

$$\begin{aligned} \partial_t(V+U) + A_t(V+U) - r(V+U) = s_F V^+ + (\lambda_B + \lambda_C)(V+U) - \lambda_B\Big(R_B V^- + V^+\Big) \\ - \lambda_C\Big(V^- + R_C V^+\Big) \end{aligned} \tag{3.2.2}$$

we split V and U to find the PDE model for each. It is observed that V satisfy the Black-Scholes PDE model given in (3.1.23). We then simplify the remaining terms in (3.2.2) and obtain the following:

$$\begin{aligned} \partial_t U + A_t U - (r + \lambda_B + \lambda_C)U &= s_F V^+ + (\lambda_B + \lambda_C)V - \lambda_B\Big(R_B V^- + V^+\Big) - \lambda_C\Big(V^- + R_C V^+\Big) \\ \partial_t U + A_t U - (r + \lambda_B + \lambda_C)U &= s_F V^+ + (\lambda_B + \lambda_C)(V^+ + V^-) - \lambda_B\Big(R_B V^- + V^+\Big) - \lambda_C\Big(V^- + R_C V^+\Big) \\ &= s_F V^+ - \lambda_B R_B V^- - \lambda_C R_C V^+ + \lambda_B V^- + \lambda_C V^+ \\ &= s_F V^+ + (1 - R_B)\lambda_B V^- + (1 - R_c)\lambda_c V^+ \end{aligned}$$

Note that $V = V^- + V^+$ where $V^- = min(V,0)$ and $V^+ = max(V,0)$.

Thus, we obtain the following linear PDE model for U given by

$$\begin{cases} \partial_t U + \mathcal{A}_t U - (r + \lambda_B + \lambda_c)U = s_F V^+ + (1 - R_B)\lambda_B V^- + (1 - R_c)\lambda_c V^+ \\ U(T,S) = 0 \end{cases} \tag{3.2.3}$$

where the variable S lies in the unbounded domain $[0, \infty)$ and $t \in [0, T]$.

We consider the change of time variable $\tau = T - t$ in order to write (3.2.3) forward in time. Note $d\tau = -dt$. Let $\phi = (r + \lambda_B + \lambda_C)$ and

$$f(V) = (1 - R_B)\lambda_B V^- + (1 - R_c)\lambda_c V^+ + s_F V^+ \tag{3.2.4}$$

be the right-hand side of the equation (3.2.3). Then we can write the initial value problem in (3.2.3) as follows

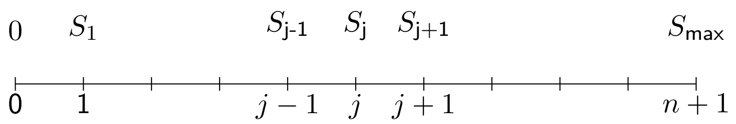
$$\begin{cases} -\partial_\tau U + \mathcal{A}_\tau U - \phi U = f(V) \\ U(0, S) = 0 \end{cases} \tag{3.2.5}$$

4. Numerical methods and results

In this chapter we present numerical methods to approximate the solution of the XVA model derived in the previous section. Firstly, we discretize the space using central finite difference method. Then apply Crank-Nicolson for time-discretization.

4.1 Space discretization using central-finite difference method

In this section we apply a central-finite difference method to discretize the XVA PDE model in (3.2.5). Discretization is any method of reducing a continuous partial differential equation to a discrete set of differences that can be solved on a computer. Since the underlying asset S is defined over the unbounded domain $[0, \infty)$, we truncate the unbounded domain to the bounded domain $[0, S_{max}]$. We consider the subdivision of $[0, S_{max}]$ in $n+1$ intervals as given below



where

$$S_j = j \cdot h, \quad j = 0, \dots, n+1 \quad \text{with} \quad h = \frac{S_{max}}{n+1} \tag{4.1.1}$$

and h is the step size, that is a distance between S_j and S_{j-1} or S_j and S_{j-1}.

4.1.1 Central-finite difference method. Finite-difference methods are numerical methods used to solve differential equations. They approximate partial derivatives. The central-finite difference method states that if the function $U(S_j)$ can be evaluated at values that lie to the left and right of S_j, then the best two-point formula will involve abscissas that are chosen symmetrically on both sides of S_j (Wilmott et al., 1993). Assume that $U \in C^3[0, S_{max}]$ and $S_j - h, S_j, S_j + h \in [0, S_{max}]$, for $0 \leq j \leq n+1$.

- **The first-order approximation**:

$$\left.\frac{\partial U}{\partial S}\right|_{S=S_j} \approx \frac{U(S_j - h) - U(S_j + h)}{2h} \tag{4.1.2}$$

  For simplicity let $U(S_j - h) = U_{j-1}$ and $U(S_j + h) = U_{j+1}$. Note there exist $c = c(S_j) \in [0, S_{max}]$ such that

$$\left.\frac{\partial U}{\partial S}\right|_{S=S_j} = \frac{U_{j+1} - U_{j-1}}{2h} + O(h^2) \tag{4.1.3}$$

  where $O(h^2) = \frac{-h^2 U^{(3)}(c)}{3!}$ is the truncation error

- **The second-order approximation**:

$$\left.\frac{\partial^2 U}{\partial S^2}\right|_{S=S_j} \approx \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} \tag{4.1.4}$$

Note also, the second-order approximation has a truncation error (computational) of $O(h^2)$. The first and second-order errors are balanced such that the total truncation error is very small, see (Wilmott et al., 1993).

We apply the first and second order partial derivative approximations given by equations (4.1.2) and (4.1.4) respectively, into (3.2.5) and we obtain the following

$$\partial_\tau U_j = \mathcal{A}_\tau U_j - \phi U_j - f(V_j) \tag{4.1.5}$$

$$\partial_\tau U_j = \frac{1}{2}\sigma^2 S^2\left(\frac{\partial^2 U_j}{\partial S^2}\right) + r_R S_j\left(\frac{\partial U_j}{\partial S}\right) - \phi U_j - f(V_j) \tag{4.1.6}$$

$$\partial_\tau U_j = \frac{1}{2}\sigma^2 S_j^2\left(\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}\right) + r_R S_j\left(\frac{U_{j+1} - U_{j-1}}{2h}\right) - \phi U_j - f(V_j) \tag{4.1.7}$$

$$\partial_\tau U_j = \left(\frac{\sigma^2 S_j^2}{2h^2} + \frac{r_R S_j}{2h}\right)U_{j+1} + \left(-\frac{\sigma^2 S_j^2}{h^2} - \phi\right)U_j + \left(\frac{\sigma^2 S_j^2}{2h^2} - \frac{r_R S_j}{2h}\right)U_{j-1} - f(V_j) \tag{4.1.8}$$

Let $A_j = \left(\frac{\sigma^2 S_j^2}{2h^2} + \frac{r_R S_j}{2h}\right)$, $\quad B_j = \left(-\frac{\sigma^2 S_j^2}{h^2} - \phi\right)$ and $C_j = \left(\frac{\sigma^2 S_j^2}{2h^2} - \frac{r_R S_j}{2h}\right)$

Then we have

$$\partial_\tau U_j = A_j U_{j+1} + B_j U_j + C_j U_{j-1} - f(V_j) \tag{4.1.9}$$

The above equation in (4.1.9) can be written as the following system of equations

$$\begin{cases} \partial_\tau U_1 & = A_1 U_2 + B_1 U_1 + C_1 U_0 - f(V_1) \\ \partial_\tau U_2 & = A_2 U_3 + B_2 U_2 + C_2 U_1 - f(V_2) \\ & \quad . \\ & \quad . \\ & \quad . \\ \partial_\tau U_{n-1} & = A_{n-1} U_n + B_{n-1} U_{n-1} + C_{n-1} U_{n-2} - f(V_{n-1}) \\ \partial_\tau U_n & = A_n U_{n+1} + B_2 U_n + C_n U_{n-1} - f(V_n) \end{cases} \tag{4.1.10}$$

but before we continue let us first introduce boundary conditions.

4.1.2 Boundary conditions. We apply the boundary conditions similar to (Arregui et al., 2017). For the boundary condition at $S = 0$, we replace $S = 0$ in equation (3.2.5). The term $\mathcal{A}_\tau U = 0$, we obtain the ordinary differential equation (ODE) given below

$$\partial_\tau U_0 = -\phi U_0 - f(V_0) \tag{4.1.11}$$

For the boundary condition $S = S_{max} = S_{n+1}$, we use the following property

$$\lim_{S\to\infty} \frac{\partial^2 U}{\partial S^2} = 0 \tag{4.1.12}$$

which implies

$$\frac{\partial^2}{\partial S^2} U(\tau, S_{n+1}) = 0 \tag{4.1.13}$$

$$\left.\frac{\partial^2 U}{\partial S^2}\right|_{S=S_{n+1}} \frac{U_{n+1} - 2U_n + U_{n-1}}{h^2} = 0$$

Therefore

$$U_{n+1} = 2U_n - U_{n-1} \tag{4.1.14}$$

After considering the boundary conditions (4.1.11) and (4.1.14), we insert the lower bound $\partial_\tau U_0$ in the systems of equations given above and replace U_{n+1} by its expression. A new system of equations is given by

$$\begin{cases} \partial_\tau U_0 & = -\phi U_0 - f(V_0) \\ \partial_\tau U_1 & = A_1 U_2 + B_1 U_1 + C_1 U_0 - f(V_1) \\ \partial_\tau U_2 & = A_2 U_3 + B_2 U_2 + C_2 U_1 - f(V_2) \\ & \quad . \\ & \quad . \\ & \quad . \\ \partial_\tau U_{n-1} & = A_{n-1} U_n + B_{n-1} U_{n-1} + C_{n-1} U_{n-2} - f(V_{n-1}) \\ \partial_\tau U_n & = (2A_n + B_n) U_n + (C_n - A_n) U_{n-1} - f(V_n) \end{cases} \tag{4.1.15}$$

Thus, we obtain the following ordinary differential equation depending on time

$$\partial_\tau U = MU + F \tag{4.1.16}$$

where

$$M = \begin{bmatrix} -\phi & 0 & & \cdots & \cdots & \cdots & 0 \\ C_1 & B_1 & A_1 & & & & \vdots \\ 0 & C_2 & B_2 & A_2 & & & \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ & & & & C_{n-1} & B_{n-1} & A_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & (C_n^*) & (B_n^*) \end{bmatrix}, \quad U = \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ \vdots \\ \vdots \\ U_{n-1} \\ U_n \end{bmatrix}, \quad F = \begin{bmatrix} -f(V_0) \\ -f(V_1) \\ \vdots \\ \vdots \\ \vdots \\ -f(V_{n-1}) \\ -f(V_n) \end{bmatrix}$$

Note that in the matrix M, the elements $B_n^* = 2A_n + B_n$ and $C_n^* = C_n - A_n$

4.2 Time discretization using Crank Nicolson method

For discretizing the time $[0, T]$ in $K+1$ intervals, we have

$$\tau_m = m \cdot d\tau \quad m = 0, \ldots, K+1 \quad \text{with} \quad d\tau = \frac{T}{K+1} \tag{4.2.1}$$

We use the θ-method where $\theta = 0.5$. This method is known as Crank- Nicolson method.

4.2.1 Euler-θ-Method. From the ordinary differential equation given by (4.1.16) we have,

$$\partial_\tau U = MU + F$$

using approximation of $\partial_\tau U$ for Euler, we have

$$\begin{aligned}
\frac{U^{m+1} - U^m}{d\tau} &= \theta[MU^{m+1} + F^{m+1}] - (1-\theta)[MU^m + F^m] \\
\Big[1 - d\tau\theta M\Big]U^{m+1} &= \Big[I + d\tau(1-\theta)M\Big]U^m + (1-\theta)d\tau F^m + d\tau\vartheta F^m \\
U^{m+1} &= \Big[1 - d\tau\theta M\Big]^{-1}\Bigg[\Big[I + d\tau(1-\theta)M\Big]U^m + (1-\theta)d\tau F^m + d\tau\vartheta F^m\Bigg]
\end{aligned}$$

where $U(S, \tau_m) = U^m$.

Before we proceed to the next section of numerical results, let us note that the value of risk-free derivative V is known. It is a solution for the Black-Scholes model of options with dividends

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_t}{\partial S_t^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV &= 0 \\ V(T, S) = H(S) \end{cases} \tag{4.2.2}$$

where $D_0 \equiv r - r_R$. In this project we are focusing on the European put, so the value for the put-option is given by

$$V = Ke^{-r(T-t)}N(-d_2) - Se^{-D_0(T-t)}N(-d_1) \tag{4.2.3}$$

where

$$d_1 = \frac{log(\frac{S}{K}) + (r - D_0 + \frac{1}{2}\sigma)(T-t)}{\sigma\sqrt{T-t}} \tag{4.2.4}$$

$$d_1 = \frac{log(\frac{S}{K}) + (r - D_0 - \frac{1}{2}\sigma)(T-t)}{\sigma\sqrt{T-t}} \tag{4.2.5}$$

For more details on the derivation, see (Wilmott et al., 1993).

4.3 Numerical results

In this section we present numerical results for the total-value adjustment in the case of the European put option. Numerical analysis was performed using MATLAB software.

We assumed that a bank (counterparty B) buys a put option from counterparty C. The strike price depends on the repo-rate and a maturity period of 0.5 years. The bank must pay a premium to C. So the bank borrows this premium from a derivative trading desk/ central bank. Assuming that this derivative is not secured by any collateral, the derivative trading desk will charge an extra interest rate (spread). This represents an effect of funding costs. For the premium, the bank will subtract the value of the funding costs and also subtract the effect of the default risk of counterparty C from a risk-free derivative value. The risk-free value denotes the value of the put-option before the crisis, in which the counterparties were considered to be default free. Figure 5.1 shows the plot of the total-value adjustment. From figure 5.1 we observe that the XVA is negative. This represent the decrease in the value of the risk-free put-option due to the probability of that counterparty C may default and also funding costs.

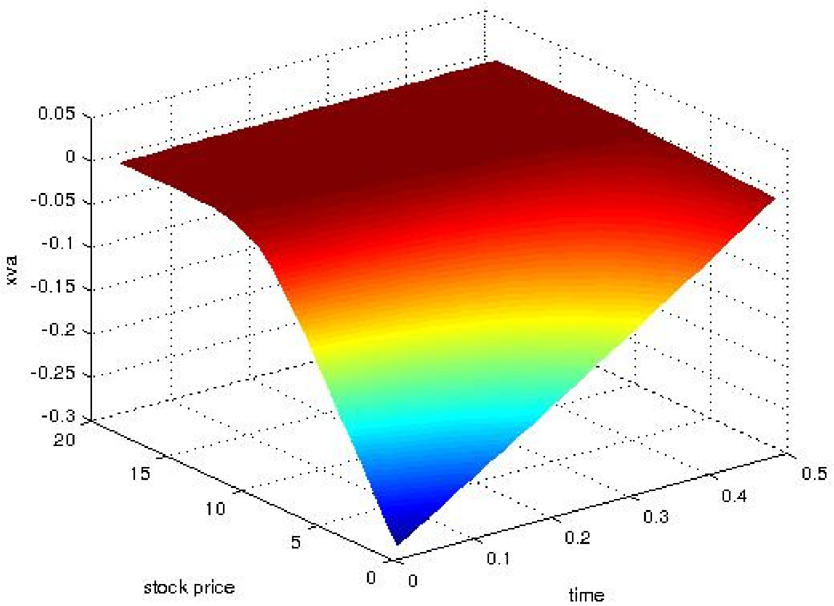


Figure 4.1: XVA surface for European put with input arguments: $S \in [0, 20]$, $K = 10e^{r_R T}$, $r = 0.04$, $T = 0.5$, $r_R = 0.06$, $r_{P_B} = 0.08$, $r_{P_C} = 0.08$, $r_B = 0.3$, $r_C = 0.3$.

We continue and plot the risk-free value and the risky value of the European put option. The risky value denotes the value an option after the crisis. In this case, the counterparty risk is considered, also the funding costs is included. From figure 5.2 we can observe that when the XVA is added, the value of the risky European put after the crisis becomes lesser than before the crisis.

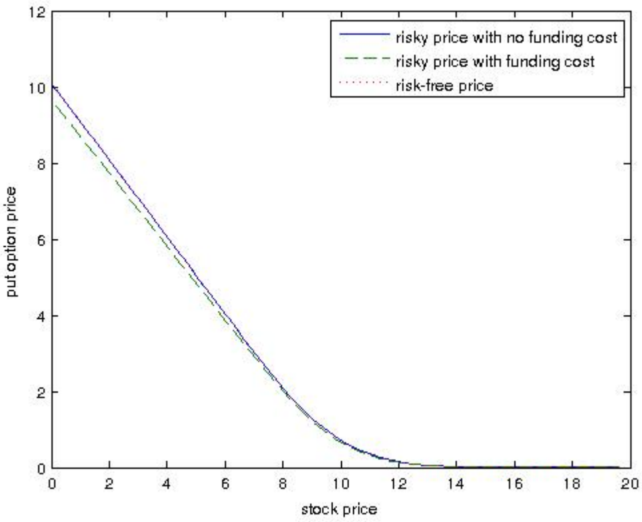


Figure 4.2: The plot for the European option with and without XVA .

5. Conclusion

In this work we derived a bilateral risky PDE model for pricing European options. This risky PDE model extends the Black-Scholes to include XVA. For model derivation, it is assumed that two counterparties B and C enter into a derivative trade. Where each party has a probability of defaulting during this trade. The following traded assets: an underlying asset in which this risky derivative (European option) is written on, one bond from each party and also a seller's (B) cash account were considered. The bonds were used for hedging a default risk of the counterparties and the cash account was meant for all the financing costs involved for hedging and trading this derivative. Hedging arguments were then applied to derive a bilateral risky PDE model for pricing European options. The risky PDE model was found to be linear, due to the mark-to-market derivative used, which was assumed to be a risk-free value of derivative as suggested by the ISDA. The bilateral risky PDE model was then decomposed to determined a PDE model for the XVA.

A central-finite difference method was applied to discretize the space of the XVA PDE model, which then led to an Ordinary differential equation (ODE) and was able to be solved using Crank Nicolson method. Finally we performed numerical simulations using Matlab Software. The approximated value of the total-value adjustment was found to be negative. This is due to the effect of counterparty risk and funding costs, which decreases the value of the risk-free European option. Risk-free value refers to the value of the European put option before the crisis (counterpaty risk not considered).

Following the paper of (Arregui et al., 2017), the total-valuation adjustment for European options has an analytical solution. For further research, we can analyse the error analysis of the central-finite difference method that we have applied. Also consider other numerical methods such as finite-volume method and compare it with results obtained for central-finite difference.

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