Renormalization Group Methods in Asymptotics

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Abstract

The renormalization group (RG) method is one of the perturbation methods used to obtain approximate solutions of a differential equation (DE). At a purely technical level, the starting point of this method is the removal of divergences in order to extract global information from the perturbation expansion. The main importance of the proposed method is that the secular terms in naive perturbation expansion are eliminated in a logical manner. As an application of the method, we illustrate the idea of 3-Steps RG method using many non-trivial examples and obtain uniformly valid asymptotic solutions of the boundary value problems.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Tasiu Abdullahi Yusuf, 26 October 2017

Contents

1. General Introduction

Most differential equations can't be solved exactly and can only be handled by various perturbation or asymptotic analysis. This is why perturbation theory and asymptotic analysis constitute such an important topic in mathematical physics and have applications to various natural sciences (O'Malley, 2012). Perturbation theory usually refers to collection of iterative methods for the systematic analysis of global behaviour of differential equations. It usually proceeds by an identification of a small parameter, say ϵ, in the problem such that when $\epsilon = 0$, the problem is exactly solvable. The global solution to the problem then can be studied via local analysis about ϵ and solution can be expressed by a regular perturbation expansion:

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \epsilon^3 x_3(t) + \cdots . \tag{1.0.1}$$

Such a series is called a perturbation series where $x_n(t)$ can always be computed in terms of $x_0, x_1, \cdots, x_{n-1}$ as long as the $\epsilon = 0$ problem is exactly solvable. Usually when ϵ is small, it's expected that only a few terms of the perturbation series are enough for a well approximated solution.

When the highest order derivative of a given differential equation is multiplied by a small parameter, ϵ, then the equation lead to narrow regions of rapid variation called boundary layers. Such cases constitute yet another class of problems where regular perturbation theory fails. In cases where the small parameter, $\epsilon \longrightarrow 0$, boundary-layer techniques can be employed. In this essay however, we will be employing a unified approach based on ideas from renormalization group theory to deal with a wide range of differential equations which are not easy to solve analytically.

The recently developed of renormalization group (RG) method introduced by (Chen et al., 1996), opened a new direction of research in non-linear dynamics. They showed that RG can be used as that global and asymptotic analysis tool for ODEs and PDEs. What makes the method so powerful is it starts with a regular perturbation expansion and substitutes in the equation, then uses the renormalization transform that will deals with the secular terms and applies RG condition to obtain a valid solution.

It should be noted here that, (Bricmont and Kupiainen, 1995) in an independent work applied a scaling transformation to obtain the asymptotic behaviour of nonlinear diffusion equations which is in some sense equivalent to what (Goldenfeld et al., 1989) have done. Subsequently, (Kunihiro, 1997) showed that RG for dynamical systems may be understood in terms of classical theory of envelopes. He suggested that the RG equation may be interpreted as the basic equation for construction of envelopes for a family of curves (or surfaces in case of partial differential equations). Chen, Goldenfeld, Oono's RG (CGO-RG) has been successfully applied to a wide range of problems. To mention a few, (Maruo et al., 1999) derived Kuramoto-Sivashinsky (KS) equation and (Hyman and Nicolaenko, 1986) they prove that Kuramoto-Sivashinsky equation behave as a finite dimensional dynamical system of ordinary differential equations. The method has also been applied in cosmology, in order to find asymptotic behaviour of nonlinear equations. The RG method has also been demonstrated as a powerful tool for resumming divergent perturbation series appearing in quantum mechanics by (Kikuchi et al., 2016) and discuss the quantum statistical effects, temperature dependence, and the viscous relaxation times using renormalization group method. (Pashko and Oono, 2000) they derived the Boltzmann equation by using RG equation and prove that Kinetic equations are slow motion equations.

In Chapter 1, we make a brief introduction on literature review and in the second chapter we define some basic definitions on asymptotic analysis. Background of study is in Chapter 3 , where we discourse the perturbation analysis using a straightforward expansion method and Poincaré-Lindstedt Method with linear damped mass-spring system (3.3.1). The two-terms approximate solution of the first method has a linear secular term, whereas the three-term approximation would have a quadratic secular term and so on. In Poincaré-Lindstedt Method , our attempt to expand the frequency in terms of powers ϵ has failed due to the presence of secular terms which make the expansion to breakdown at large ρ. Ideally this is why we proposed our method in order to treat those secular terms.

Chapter 4, is the heart of our essay where we review the renormalization group method using theory of envelopes of (Kunihiro, 1995). Following some literatures (O'Malley and Kirkinis, 2015) and the book of (Kuehn, 2015) we also proposed 3-Step RG method and illustrate its application with two examples one is in Section 4.4, and other is Rayleigh Equation in Section 4.5. Finally, conclusion remark and acknowledgements are in Chapter 5.

2. Asymptotic Methods

Asymptotic analysis is used to build numerical solutions to a differential equation (Bender and Orszag, 2013). The most common type of asymptotic expansion is a power series in either positive or negative powers. Methods of generating such expansions include Taylor series and Maclaurin series which is a form of Taylor series. Integral transforms such as the Laplace form and also repeated integration by parts will often lead to an asymptotic expansion. Typically, the best approximation is given when the series is truncated at the finite terms. This way of optimally truncating an asymptotic expansion is known as superasymptotics.

2.0.1 Asymptotic. A function $f(n)$ is said to be asymptotically equal to a function $g(n)$ (i.e. $f \sim g$ as $(n \longrightarrow \infty)$) if

$$\lim_{n \longrightarrow \infty} \frac{f(n)}{g(n)} = 1.$$

2.0.2 Asymptotic Notations. Asymptotic notations are useful in analysing the relationship between the asymptotic functions. Some of this notations includes big oh , little oh , big omega and big theta:

2.0.3 Big - O**.** Let $f(n)$ and $g(n)$ be two functions $f, g : N \longrightarrow R^+$. then we say that $f(n) = O(g(n))$ if for every $n \geq n_0$ and $n_0 \geq 1$ there exist $c \in R^+$, such that $f(n) \leq cg(n)$ for all $n \in N$. Which is nothing but $f(n)$ is smaller than $g(n)$.

2.0.4 Example. Let $f(n) = 3n+2$ and $g(n) = n$. Then can we say $f(n) = O(g(n))$ as $n \longrightarrow \infty$?

$$f(n) \leq cg(n), \quad c > 0, \quad n_0 \geq 1.$$

So, $3n+2 \leq cn$ if we choose $c = 4$ then implies $n \geq 2$. Which means that for every $n \geq 2$ and $c = 4$, $f(n) \leq cg(n)$. This implies that for $f(n) = 3n+2$ and $g(n) = n$, $f(n) = Og(n)$. Therefore, n is bounded the function $3n+2$. For any one of $g(n) = n^2$ or n^3 or n^4 or n^n or 2^n, $g(n)$ also bounded the function $3n+2$. But we are interesting on the list upper bound, which is n.

2.0.5 Big- Omega Ω**.** Let $f(n)$ and $g(n)$ be two functions $f, g : N \longrightarrow R^+$. We say that $f(n) = \Omega(g(n))$ if $\exists$ a constant $c \in R^+$ and an $n_0 \in N$ such that for every integer $n \geq n_0$ we have $f(n) \geq cg(n)$. The function f is asymptotically greater than or equal to g and Big-Omega gives an asymptotic lower bound.

2.0.6 Example. Consider the functions $f(n) = 3n+2$ and $g(n) = n$.

For $f(n) = \Omega g(n)$ then $f(n) \geq cg(n)$ $\quad 3n+2 \geq cn$ by choosing $c = 1$ and $n_0 \geq 1$. $3n+2 = \Omega(n)$ this means that $3n+2$ will be lower bound by n. Check: $3n+2 \quad =? \quad \Omega(n)$ (i.e we are not sure), $3n+2 \geq cn^2$ for some value n_0. But there is no value of c such that $3n+2$ can be lower bound by n^2 but it can be an upper bound. Now if we found out that $f(n) = \Omega(n)$ so any thing less then n can be the answer. $f(n)$ can be lower bound by $\log n$. i.e $f(n) = \Omega(n)$ where $\Omega(n)$ could be any one of $\log n$ or $\log(\log n)$ etc.

So what we mean to say is there is more than one answer for Big-O and Ω always. But it's better to take the closest one. The closest lower bound in case of Big-Ω and closest upper bound in case of Big-O.

2.0.7 Big- Θ. Let $f(n)$ and $g(n)$ be two functions $f, g : N \longrightarrow R^+$. Then the function $f(n) = \theta(g(n))$ if $f(n)$ is bounded by $g(n)$ to both lower and upper. i.e $\exists$ a constants $c_1, \ c_2 > 0, \ n \geq n_0, \ n_0 \geq 1$ such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$.

2.0.8 Example. Take $f(n) = 3n + 2$ and $g(n) = n$. Then $f(n) \leq c_2 g(n)$, for $c_2 = 4$, this implies $3n + 2 \leq 4n$ for all $n_0 \geq 1$, $g(n)$ can be an upper bound of $f(n)$. And $f(n) \geq c_1 g(n)$ for $c_1 = 1$, i.e $3n + 2 \geq n$, for all $n_0 \geq 1$. This implies $f(n) = 3n + 2$ is bounded by $g(n) = n$ both in the lower bound and upper bound for lower $c_1 = 1$ and for upper $c_2 = 4$ and for the both cases, $n_0 \geq 1$. Sometime the Θ is called asymptotically equal, which means that if the leading term is $3n$ then we can take as $3n + 2 = \Theta(n)$ that is always the deminating term can be taken to be in the function Θ e.g $3n^2 + n + 1 = \Theta(n^2)$ or $3n^3 + n^2 = \Theta(n^3)$.

2.0.9 Little- o. Let $g(n)$ be a function, then we define little oh as a set

$$o(g(n)) = \{f(n) : \text{ for constant } c \in R^+, \ \exists \ n_0 > 0 \ \ni \ 0 \leq f(n) \leq cg(n), \ \forall \ n \geq n_0\}.$$

This means that $f(n)$ becomes insignificant relative to $g(n)$ as $n \longrightarrow \infty$. That is

$$\lim_{n \longrightarrow \infty} \frac{f(n)}{g(n)} = 0.$$

2.0.10 Example. $2n = o(n^2)$ but $2n^2 \neq o(n^2)$.

2.0.11 Theorem. *Let $f, g : N \longrightarrow R^+$ be two functions from positive integers to the non-negative real numbers. Then $f(n) + g(n) = \Theta(\max\{f(n), g(n)\})$, for all $n \in N$.*

Proof:

To show $f(n)+g(n) = \Theta(\max\{f(n), g(n)\})$, we need to show $f(n)+g(n) = \Omega(\max\{f(n), g(n)\})$ and $f(n) + g(n) = O(\max\{f(n), g(n)\})$. First, since the functions are non-negetive, we have that $f(n) + g(n) \geq f(n)$ and $f(n)+g(n) \geq g(n)$ combining these, we get that $f(n)+g(n) \geq \max\{f(n), g(n)\})$ for all n; thus $f(n) + g(n) = \Omega(\max\{f(n), g(n)\})$. On the other hand, we also have that $f(n) + g(n) \leq 2\max\{f(n), g(n)\}$ for all n; thus $f(n) + g(n) = O(\max\{f(n), g(n)\})$. Hence, we get the require result.

2.0.12 Asymptotic Sequences. Let $\varphi_n : M \longrightarrow R$, $n \in N$ and a be a limit point of M. Let $\varphi_n(x) \neq 0$ in a neighborhood U_n of a. The sequence $\{\varphi_n\}$ is called asymptotic sequence at $x \longrightarrow a$, and $x \in M$ if for all $n \in N$

$$\varphi_{n+1}(x) = o(\varphi_n(x)).$$

2.0.13 Asymptotic Series. One of the importance of power series is a tool for solving differential equations.

2.0.14 Definition. Let $f : M \longrightarrow R$ and a be a limit point of M and $\{\varphi_n\}$ be an asymptotic sequence as $x \longrightarrow a$ $x \in M$. We say that the function f is expanded in an asymptotic series

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x), \quad \text{as } x \longrightarrow a \quad \text{and} \quad x \in M$$

where a_n are constants, if for all $N \geq 0$

$$R_N(x) \equiv f(x) - \sum_{n=0}^{N} a_n \varphi_n(x) = o(\varphi_N(x)) \quad \text{as } x \longrightarrow a, \quad x \in M.$$

This series is called asymptotic expansion of the function f with respect to the asymptotic sequence $\{\varphi_n\}$. Where $R_N(x)$ is called the rest term of the asymptotic series.

The main point of this chapter is to give an insight of what asymptotic, asymptotic sequence, series and big O is all about because we're going to make use of it in the next sections.

3. Background of Study

3.1 Perturbation Approach

Asymptotic and perturbation analysis has played a significant role in theoretical physics and applied mathematics. Perturbation theory is a mathematical tools used for finding an approximate solution to a problem, by starting from the exact solution of a related, simpler problem (O'Malley, 2014). The method breaks the problem into different powers of the small parameter (solvable and perturbation parts) for which the solution will be derived. This leads to an expression for the desired solution in terms of a formal power series in some small parameter usually called as naive perturbation series that quantifies the deviation from the exactly solvable problem.

Perturbation methods was first used by astronomers to predict the effects of small disturbances in relates to the motions of celestial bodies but now become widely used as an analytical tools in virtually all branches of science (Minorsky, 1947). Due to the increasing accuracy of astronomical observations led to incremental demands in the accuracy of solutions to Newton's gravitational equations, which led several mathematicians, such as Lagrange and Laplace, to extend the methods of perturbation theory. These idea of perturbation analysis were adopted and used to solve many new problems arising during the development of quantum mechanics in 20th century (Gerjuoy and Thomas, 1974).

3.2 Singular Perturbation Method for BVP in ODEs

A differential equation (DE) with a small parameter ϵ is called a singular perturbation problem (SPP) if the order of differential equation reduces when the parameter is zero. A differential equation together with the boundary conditions is called boundary value problem (Smith, 1985).

We'll first discuss the regular perturbation expansion and make a review about the Poincar$\acute{e}$-Lindstedt method for a linear damped oscillator (LDO).

3.3 Straightforward Expansion Method

Consider the second order homogeneous differential equation for the linear damped mass-spring system without an external forces

$$\ddot{x} + 2\epsilon\dot{x} + x = 0 \tag{3.3.1}$$

with initial conditions $x(0) = 1, \;\; \dot{x}(0) = 0$ where $\epsilon \ll 1$.

When solving the Equation (3.3.1) together with the initial conditions $x(0) = 1, \;\; \dot{x}(0) = 0$, the exact solution is given by

$$x(t) = e^{-\epsilon t}\left(\cos\sqrt{1-\epsilon^2}t + \frac{\epsilon}{\sqrt{1-\epsilon^2}}\sin\sqrt{1-\epsilon^2}t\right). \tag{3.3.2}$$

But please observe that if the oscillation is undamped, that is, if $\epsilon = 0$, then we should have the exact solution as $x(t) = \cos t$ where both amplitude and phase of the oscillation remain constant. However, with the presence of damping, (3.3.2) shows that both amplitude and phase change with time.

Now, it is good to take our Equation (3.3.1) and solve by introducing the perturbation series as $\epsilon \longrightarrow 0$

$$x(t) \sim x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \epsilon^3 x_3(t) + \cdots \tag{3.3.3}$$

By substituting Equation (3.3.3) in to Equation (3.3.1) we get

$$\ddot{x}_0 + \epsilon \ddot{x}_1 + \epsilon^2 \ddot{x}_2 + \cdots + 2\epsilon(\dot{x}_0 + \epsilon \dot{x}_1 + \epsilon^2 \dot{x}_2 + \cdots +) + x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots = 0$$

collecting the coefficients of equal powers of ϵ we have:

$$\ddot{x}_0 + x_0 + \epsilon(\ddot{x}_1 + 2\dot{x}_0 + x_1) + \epsilon^2(\ddot{x}_2 + 2\dot{x}_1 + x_2) + \cdots = 0$$

Equating coefficients of like powers of ϵ to 0, gives a sequence of linear differential equations as follows

$$\ddot{x}_0 + x_0 = 0, \ \text{ with } \ x_0(0) = 1, \ \ \dot{x}_0(0) = 0 \tag{3.3.4}$$

$$\ddot{x}_1 + x_1 = -2\dot{x}_0, \ \text{ with } \ x_1(0) = 0, \ \ \dot{x}_1(0) = 0$$

$$\ddot{x}_2 + x_2 = -2\dot{x}_0, \ \text{ with } \ x_2(0) = 0, \ \ \dot{x}_2(0) = 0$$

Equation (3.3.4) is the unperturbed problem obtained by setting $\epsilon = 0$. It is the governing equation of a harmonic oscillator with angular frequency of unity. The solution is

$$x_0 = \cos t$$

Then we have

$$\ddot{x}_1 + x_1 = 2\sin t, \ \text{ with } \ x_1(0) = 0, \ \ \dot{x}_1(0) = 0 \tag{3.3.5}$$

The solution of the nonhomogeneous differential Equation (3.3.5) is given by

$$x_1 = x_c + x_p$$

But the right hand side of differential Equation (3.3.5) is of the same form as the general solution of the corresponding homogeneous equation. Then we make a trial of particular solution to be of the form

$$x_p = c_1 t \cos t + c_2 t \sin t$$

where the constants c_1 and c_2 can be found by the method of undetermined coefficients. Doing that, we find that

$$x_1 = A\cos t + B\sin t - t\cos t.$$

And using the initial conditions on x_1 gives $A = 0$ and $B = 1$. This implies that the solution of (3.3.5) is

$$x_1 = \sin t - t\cos t.$$

Hence, the two-term approximate solution of (3.3.1) takes the form

$$x(t) = \cos t + \epsilon(\sin t - t\cos t) + O(\epsilon^2) \tag{3.3.6}$$

The straightforward expansion is not valid when $t > O(\frac{1}{\epsilon})$ due to the presence of secular terms. It can be shown that the secular term become more compounded for higher order expansions (i.e., we'll get terms with $t^2, t^3, \cdots$). The two-term approximation has a linear secular term, whereas the three-term approximation would have a quadratic secular term and so on.

3.3.1 Remark. Expansion (3.3.6) can be constructed from the exact solution (3.3.2) by expanding the exponential, square root, and trigonometric functions. Non-uniformities are generated in forming the expansions of the exponential term $e^{-\epsilon t}$ and trigonometric functions $\cos\sqrt{1-\epsilon^2}t$ and $\sin\sqrt{1-\epsilon^2}t$ in powers of ϵ.

Next, we are going to apply the Poincaré-Lindstedt Method to the initial value problem (3.3.1) in order to see whether the method is capable of avoiding the secular term that ruined the approximation when a straightforward application of the regular perturbation method is used.

3.4 Poincaré-Lindstedt Method

To account for the fact that the frequency of the system is a function of ϵ, we let

$$\rho = \omega t$$

where ρ is called the strained coordinate and ω is a constant that depends on ϵ. We therefore, need to change the independent variable for t to ρ. Using the chain rule, we transform the derivatives according to

$$\frac{d}{dt} = \frac{d}{d\rho}\frac{d\rho}{dt} = \omega\frac{d}{d\rho} \quad \text{and} \quad \frac{d^2}{dt^2} = \omega\frac{d^2}{dtd\rho} = \omega\frac{d\rho}{dt}\frac{d^2}{d\rho^2} = \omega^2\frac{d^2}{d\rho^2}$$

Equation (3.3.1) now becomes

$$\omega^2\ddot{x} + 2\epsilon\omega\dot{x} + x = 0, \quad x(0) = 1, \ \dot{x}(0) = 0 \tag{3.4.1}$$

where $x = x(\rho)$ and $\dot{x}$ is the derivative of x with respect to ρ. Then we should expand x and ω in powers of ϵ as follows:

$$\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots, \tag{3.4.2}$$

$$x = x_0(\rho) + \epsilon x_1(\rho) + \epsilon^2 x_2(\rho) + \cdots. \tag{3.4.3}$$

The first term in Equation (3.4.2) is 1 which can easily be seen to be the unperturbed (undamped) frequency. Next we substitute Equation (3.4.2) in to the differential Equation (3.4.1) to get

$$(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)^2\ddot{x} + 2(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)\epsilon\dot{x} + x = 0 \tag{3.4.4}$$

again, using Equation (3.4.3) in (3.4.4), we obtain

$$(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)^2(\ddot{x}_0 + \epsilon\ddot{x}_1 + \epsilon^2\ddot{x}_2 + \cdots) + 2\epsilon(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)(\dot{x}_0 + \epsilon\dot{x}_1 + \epsilon^2\dot{x}_2 + \cdots)$$
$$+(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) = 0$$

with

$$x_0(0) + \epsilon x_1(0) + \epsilon^2 x_2(0) + \cdots = 1 \ \text{ and } \ \dot{x}_0(0) + \epsilon\dot{x}_1(0) + \epsilon^2\dot{x}_2(0) + \cdots = 0.$$

This can be written as

$$\ddot{x}_0 + x_0 + \epsilon(\ddot{x}_1 + 2\omega_1\ddot{x}_0 + 2\dot{x}_0 + x_1) + \epsilon^2(\ddot{x}_2 + x_2 + 2\omega_2\ddot{x}_0 + \omega_1^2\ddot{x}_0 + 2\omega_1\ddot{x}_1 + 2\omega_1\dot{x}_0 + 2\dot{x}_1) + \cdots = 0. \tag{3.4.5}$$

By equating the corresponding powers of ϵ, we get

$$\ddot{x}_0 + x_0 = 0, \quad x_0(0) = 1, \quad \dot{x}_1(0) = 0 \tag{3.4.6}$$

$$\ddot{x}_1 + x_1 = -2\omega_1 \ddot{x}_0 - 2\dot{x}_0, \quad x_1(0) = 0, \quad \dot{x}_1(0) = 0$$
$$\ddot{x}_2 + x_2 = -2\omega_2 \ddot{x}_0 - \omega_1^2 \ddot{x}_0 - 2\omega_1 \ddot{x}_1 - 2\omega_1 \dot{x}_0 - 2\dot{x}_1, \quad x_2(0) = 0, \quad \dot{x}_2(0) = 0.$$

By solving, $O(1)$ system of (3.4.6) has the solution

$$x_0(\rho) = \cos\rho. \tag{3.4.7}$$

This implies that $O(\epsilon)$ equation becomes

$$\ddot{x}_1 + x_1 = 2\omega_1 \cos\rho + 2\sin\rho, \quad x_1(0) = 0, \quad \dot{x}_1(0) = 0. \tag{3.4.8}$$

The solution of non-homogeneous differential Equation (3.4.8) is

$$x_1 = A\cos\rho + B\sin\rho - \rho\cos\rho + \omega_1\rho\sin\rho.$$

Using the initial conditions on x_1, we obtain $A = 0$ and $B = 1$. This implies that particular solution of Equation (3.4.8) is

$$x_1(\rho) = \sin\rho - \rho\cos\rho + \omega_1\rho\sin\rho \tag{3.4.9}$$

In the solution of x_1 there are two secular terms, which makes the expansion breakdown at large ρ. The secular term $\omega_1\rho\sin\rho$ can be eliminated by setting $\omega_1 = 0$, but, for $\rho\cos\rho$ cannot be eliminated as it does not contain any adjustable parameter. Therefore, approximate solution of Equation (3.3.1) becomes

$$x(\rho) = x_0(\rho) + \epsilon x_1(\rho), \quad \text{i.e.,} \tag{3.4.10}$$
$$x(\rho) = \cos\rho + \epsilon(\sin\rho - \rho\cos\rho) + O(\epsilon^2).$$

Observation: Of course, if we continue further by setting $\omega_1 = 0$ in system with x_2, it can be easily shown that the solution will provide a condition $\omega_2 = 0$. This method shows that our attempt to expand ω in terms of ϵ has failed and consequently, we get $\rho = t$. Thus, the Poincaré-Lindstedt method has failed to yield a perturbation approximation for the linear damped oscillator (3.3.1). This is the reason why we proposed our 3-Steps RG Method in order to get rid of those secular terms in the perturbation expansion and it was describe in the next section.

4. Application of RG Methods

Now we comes to the main section of this essay. The main idea for Renormalization Group (RG) methods is the removal of divergences from a perturbation series. We are now going to demonstrate that many singular perturbation methods (SPM) may be understood as renormalized perturbation theory. One of the advantages of the RG method is that at the beginning it's simply a straightforward by applying naive perturbation expansion in the given problem, for which very little a priori knowledge is required and then obtain a global asymptotic solution.

4.0.1 The 3-Steps RG Method. Following the paper of (O'Malley and Kirkinis, 2015) and the book of (Kuehn, 2015) , we proposed the following definition. Consider a perturbed vector initial value problem describe by differential equation

$$\dot{y} = \epsilon F(y, t, \epsilon), \quad y(t_0) = y_0 \tag{4.0.1}$$

where y_0 is the constant vector , t is a time independent with $t \geq t_0$ and we assume that the function F is smooth with a positive parameter $\epsilon \ll 1$. Then we define 3-Steps RG Method as follows:

4.0.2 Definition. Let V be a vector field that admits a regular perturbation expansion in powers of ϵ, for example the Equation (4.0.1), and suppose we have a space S of the asymptotic expansions that formally satisfy such equations. Then we may truncate the expansion at a finite order because the expansion is not asymptotically valid for all time due to the present of secular terms. The 3-Steps RG Method consist of the following:

Step 1: Map between vector field V and truncated asymptotic expansions S, this is done by introducing naive perturbation expansion into the given equation.

Step 2: Map from S to S, using renormalization transform which will absorb time-independent terms from naive perturbation expansion in to the initial conditions. The purpose of renormalization is to make each term in the perturbation series finite.

Step 3: A map from S back to V using Renormalization Group Condition define by

$$\left.\frac{\partial y}{\partial t_0}\right|_{t_0=t} = 0.$$

Justified by the fact that the solution does not depend on t_0 see (Chen et al., 1996). The idea of this step is to reduce the asymptotic expansion so that it will be easier to solve, i.e., by differentiating the expression of renormalized transform with respect to time t_0 and equate to zero to obtain the amplitude equation. We visualize this 3-Steps RG Method in Figure 4.1.

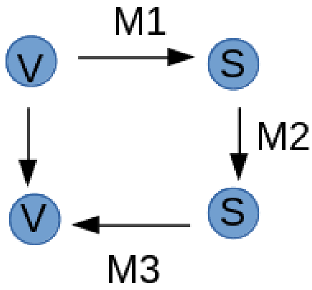

Figure 4.1: This is the 3-Steps RG Method diagram which shows maps M_1 from vector field V to asymptotic expansion S, then M_2 from S to S and finally M_3 from S back to V which will gives the desire solution. Applications of this 3-Steps Method ware illustrate in Sections 4.4 and 4.5.

4.1 Implementation of RG Methods

Here we review the simple linear damped oscillator. Consider the LDO

$$\frac{d^2y}{dt^2} + \epsilon\frac{dy}{dt} + y = 0, \text{ with, } y(0) = 3, \left.\frac{dy}{dt}\right|_{t=0} = 1 \text{ and } \epsilon \ll 1. \tag{4.1.1}$$

The general solution of this given equation is

$$y = e^{-\frac{\epsilon t}{2}}\left(C_1 \cos(\frac{1}{2}\sqrt{4-\epsilon^2})t + C_2 \sin(\frac{1}{2}\sqrt{4-\epsilon^2})t\right)$$

where the constants C_1 and C_2 can be obtain using the given boundary conditions. Now using the RG method, we assume the naive perturbation expansion of the form

$$y(t,\epsilon) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots \tag{4.1.2}$$

By substituting the expansion (4.1.2) into Equation (4.1.1) and using $\dot{y}$ for $\frac{dy}{dt}$ we obtain the following equations of different powers of epsilon

$$\ddot{y}_0 + y_0 = 0$$
$$\ddot{y}_n + y_n = -\dot{y}_{n-1}, \text{ for } n \geq 1$$

and the solutions of these are

$$y_0 = Ae^{it} + A^*e^{-it}, \tag{4.1.3}$$
$$y_1 = -\frac{1}{2}Ate^{it} + cc, \quad y_2 = \frac{1}{8}A(t^2 - it)e^{it} + cc,$$
$$y_3 = -\frac{1}{16}A\left(\frac{t^3}{3} - it^2\right)e^{it} + cc, \quad y_4 = \frac{1}{64}A\left(\frac{t^4}{6} - it^3 - \frac{t^2}{2} - \frac{i}{2}\right)e^{it} + cc, \text{ and so on}$$

where cc represent the complex conjugate the same with what (Matsuba and Nozaki, 1997) deed in their paper. Please observe that the polynomials in t that multiply the fundamental solution e^{it} are secular terms. Using the suitable notation define in (Matsuba and Nozaki, 1997) we obtain the following notation

$$y_{1A} = -\frac{1}{2}t, \quad y_{2A} = \frac{1}{8}\left(t^2 - it\right), \quad y_{3A} = -\frac{1}{16}\left(\frac{t^3}{3} - it^2\right), \quad y_{4A} = \frac{1}{64}\left(\frac{t^4}{6} - it^3 - \frac{t^2}{2} - \frac{i}{2}\right), \tag{4.1.4}$$

and with the corresponding A^* for those polynomials that multiply the exponent e^{-it} , we define the secular sequence $\{y_{iA}\}_{i=1}^{\infty}$. Equation (4.1.3) show that y_{0A} is 1 in the secular sequence. The renormalization group method replaces A in the $O(1)$ solution (4.1.3) with renormalized constant $\mathcal{A}(t,\epsilon)$ such that

$$\mathcal{A}(t,\epsilon) = A[1+\epsilon y_{1A}+\epsilon^2 y_{2A}+O(\epsilon^3)] \equiv Ay_A, \text{ or } A \equiv \mathcal{A}(t,\epsilon)y_A^{-1} \tag{4.1.5}$$

where those terms in the brackets are secular series, that is terms that grow without bound for large t

$$y_A = [1+\epsilon y_{1A}(A,t)+\epsilon^2 y_{2A}(A,t)+O(\epsilon^3)]. \tag{4.1.6}$$

Next we're going to determine $\mathcal{A}(t,\epsilon)$. Using the arguments in (Chen et al., 1996) and (Chen et al., 1994), we implicitly differentiate Equation (4.1.5) with respect to independent variable t and treating A as constant , we get

$$\frac{d\mathcal{A}}{dt} = \mathcal{A}y_A^{-1}\frac{dy_A}{dt}.$$

By substituting Equation (4.1.6) for y_A in the above relation and simplify, we get

$$\frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{dt} = \epsilon\frac{dy_{1A}}{dt}+\epsilon^2\left(\frac{dy_{2A}}{dt}-y_{1A}\frac{dy_{1A}}{dt}\right) \tag{4.1.7}$$

$$+\epsilon^3\left(\frac{dy_{3A}}{dt}-y_{1A}\frac{dy_{2A}}{dt}-y_{2A}\frac{dy_{1A}}{dt}+y_{1A}^2\frac{dy_{1A}}{dt}\right)+O(\epsilon^4).$$

If we integrate Equation (4.1.7) then we'll obtain

$$\ln\mathcal{A} = \ln\mathcal{A}(0)+\epsilon y_{1A}+\epsilon^2\left(y_{2A}-\frac{y_{1A}^2}{2}\right)+\epsilon^3\left(y_{3A}-y_{1A}y_{2A}+\frac{y_{1A}^3}{3}\right)+O(\epsilon^4). \tag{4.1.8}$$

The Equation (4.1.7) or (4.1.8) can be used in order to derive the amplitude equation, depending on the problem you're working with.

To find the final solution of Equation (4.1.1) we repeat the above process by replacing the constant A^* with renormalized constant $\mathcal{A}^*(t,\epsilon)$ and obtain the asymptotic solution of Equation (4.1.1) as given below

$$y(t,\epsilon) = \mathcal{A}(t,\epsilon)e^{it}+\mathcal{A}^*(t,\epsilon)e^{-it}. \tag{4.1.9}$$

This solution (4.1.9) is called the renormalized expansion (Goldenfeld et al., 1989; Goldenfeld, 1992). The idea of replacement of the constants in solution (4.1.3) by the renormalized constants as in (4.1.9) is justified because if Equation (4.1.9) is written by using the relation (4.1.5), one recovers that the regular perturbation expansion will make the secular terms in (4.1.3).

Now we can easily simplify Equation (4.1.4) to get the following

$$y_{1A} = -\frac{t}{2}, \quad y_{2A}-\frac{y_{1A}^2}{2} = \frac{1}{8}it, \quad y_{3A}-y_{1A}y_{2A}+\frac{y_{1A}^3}{3} = 0, \cdots$$

and to determine the renormalized constant $\mathcal{A}(t,\epsilon)$, we simplify one of the Equation (4.1.7) or (4.1.8) to obtain

$$\frac{d\mathcal{A}}{dt} = -\epsilon\frac{1}{2}\mathcal{A}+i\mathcal{A}\left(-\frac{\epsilon^2}{8}-\frac{\epsilon^4}{128}+O(\epsilon^6)\right). \tag{4.1.10}$$

By integrating Equation (4.1.10) we get

$$\mathcal{A} = \mathcal{A}(0)e^{\left(-\epsilon\frac{1}{2}-\epsilon^2\frac{i}{8}\right)t+O(\epsilon^4)}, \tag{4.1.11}$$

and the same way if we exponentiate (4.1.8). Therefore in any case, the renormalized expansion (4.1.9) for the Equation (4.1.1) is now becomes

$$y(t,\epsilon) = e^{-\frac{\epsilon t}{2}}\left[\mathcal{A}(0)e^{i(1-\frac{\epsilon^2}{8}-\frac{\epsilon^4}{128}+O(\epsilon^6))t} + \mathcal{A}^*(0)e^{-i(1-\frac{\epsilon^2}{8}-\frac{\epsilon^4}{128}+O(\epsilon^6))t}\right] \tag{4.1.12}$$

where frequency in the exponent is a convergent function of ϵ, for which the sum is $\frac{1}{2}\sqrt{4-\epsilon^2}$.

Hence, using the initial conditions $y(0) = 3, \ \ \dot{y}(0) = 1$ and frequency $w = 1 - \frac{\epsilon^2}{8} + \cdots$ we obtain the solution as

$$y(t) = e^{-\frac{\epsilon t}{2}}(3\cos wt + \frac{3\epsilon + 2}{2w}\sin wt) \tag{4.1.13}$$

and the graphical solution is plotted in Figure (4.2).

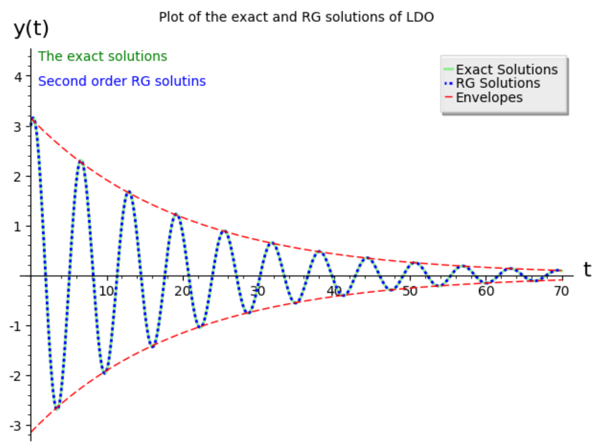

Figure 4.2: This is the plot of exact and RG solutions to linear damped oscillator for $\epsilon = \frac{1}{10}$. The envelope represent the slow scale of the leading exponential in (4.1.13). The solid curve is the exact solution while the dash represent RG method solution.

4.2 Theory of Envelopes

For this section to be clearer, let's review the theory of envelopes (Kunihiro, 1995).

Let $\{\mathcal{C}_\tau\}_\tau$ be a family of curves with parameter τ in the x, y plane, where $\mathcal{C}_\tau$ is represented by the equation

$$\mathbb{F}(x, y, \tau) = 0. \tag{4.2.1}$$

Now we assume that the family of the curves has an envelops $\mathbb{E}$ which is also in the form

$$\mathbb{G}(x, y) = 0. \tag{4.2.2}$$

Our aim is to obtain $\mathbb{G}(x, y)$ from $\mathbb{F}(x, y, \tau)$. Suppose that both $\mathbb{E}$ and a curve $\mathcal{C}_{\tau_0}$ have the same tangent line at $(x, y) = (x_0, y_0)$, i.e., (x_0, y_0) is the point of tangency. This implies that x_0 and y_0 are functions of τ_0 which can be express as $x_0 = \Phi(\tau_0), \quad y_0 = \Psi(\tau_0)$, and $\mathbb{G}(x_0, y_0) = 0$. Conversely, for every point on $\mathbb{E}$, say (x_0, y_0), there exits a parameter τ_0. Now we can get τ_0 as a function of (x_0, y_0) and $\mathbb{G}(x, y)$ can be express as

$$\mathbb{F}(x, y, \tau(x, y)) = \mathbb{G}(x, y).$$

Since x_0 and y_0 are both functions of τ_0, then $\tau_0(x_0, y_0)$ can be obtained by defining the tangent line of $\mathbb{E}$ at point (x_0, y_0) and that of $\mathcal{C}_{\tau_0}$ at the same point.

$$\text{For } \mathbb{E} \text{ at point } (x_0, y_0) \text{ is : } \quad \dot{\Psi}(\tau_0)(x - x_0) - \dot{\Phi}(\tau_0)(y - y_0) = 0 \text{ and}$$

$$\text{For } \mathcal{C}_{\tau_0} \text{ at point } (x_0, y_0) \text{ is : } \mathbb{F}_x(x_0, y_0, \tau_0)(x - x_0) + \mathbb{F}_y(x_0, y_0, \tau_0)(y - y_0) = 0,$$

where both $\mathbb{F}_x$ and $\mathbb{F}_y$ are partial derivatives of $\mathbb{F}$ with respect to x and y respectively (Kunihiro, 1995). But since they are at the same point, then the above equations must produce the same line as well. Therefore, we have

$$\mathbb{F}_x(x_0, y_0, \tau_0)\dot{\Phi}(\tau_0) + \mathbb{F}_y(x_0, y_0, \tau_0)\dot{\Psi}(\tau_0) = 0.$$

Similarly, if we differentiate the function $\mathbb{F}(x(\tau_0), y(\tau_0), \tau_0) = 0$ partially with respect to parameter τ_0, we get

$$\mathbb{F}_x(x_0, y_0, \tau_0)\dot{\Phi}(\tau_0) + \mathbb{F}_y(x_0, y_0, \tau_0)\dot{\Psi}(\tau_0) + \mathbb{F}_{\tau_0}(x_0, y_0, \tau_0) = 0,$$

then $\mathbb{F}_{\tau_0}(x_0, y_0, \tau_0)$ is identical to $\frac{\partial \mathbb{F}(x_0, y_0, \tau_0)}{\partial \tau_0} = 0,$ that is ,

$$\mathbb{F}_{\tau_0}(x_0, y_0, \tau_0) \equiv \frac{\partial \mathbb{F}(x_0, y_0, \tau_0)}{\partial \tau_0} = 0.$$

To get the relation between x_0 and y_0, we eliminate the parameter τ_0, and by transforming

$$(x_0, y_0) \longrightarrow (x, y)$$

we get

$$\mathbb{G}(x, y) = \mathbb{F}(x, y, \tau_0(x, y)) = 0.$$

4.2.1 Remark. If $x = g(y, \tau)$ is the family of curves, then $\frac{\partial \mathbb{F}(x_0, y_0, \tau_0)}{\partial \tau_0} = 0$ implies $\frac{\partial g}{\partial \tau_0} = 0,$ where the envelope is $x = g(y, \tau_0(y))$. Also, we can get both $\mathbb{E}$ and a set of singularities of the curves $\{\mathcal{C}_\tau\}_\tau$ from the equation $\mathbb{G}(x, y) = 0$ since $\frac{\partial \mathbb{F}}{\partial x} = \frac{\partial \mathbb{F}}{\partial y} = 0$ satisfy $\frac{\partial \mathbb{F}(x_0, y_0, \tau_0)}{\partial \tau_0} = 0.$

4.2.2 Application of the Envelope. In order to see what this envelope is all about, let's look at a function that is bounded local but not globally.

Consider the function

$$x = g(y, \tau) = e^{-\epsilon^2\tau}\Big(\epsilon^2(y - \tau) - 1\Big) + e^{-\epsilon y}.$$

Clearly this function is bounded locally i.e when $\epsilon = 0$. For $y - \tau \longrightarrow \infty$ then $g(y, \tau)$ is unbounded. We can now get the envelope $\mathbb{E}$ of the given curve $\mathcal{C}_\tau$ from the condition $\frac{\partial g}{\partial \tau} = 0$.

Now

$$\frac{\partial g}{\partial \tau} = x_\tau = -\epsilon^2 e^{-\epsilon^2\tau} - \epsilon^2 e^{-\epsilon^2\tau}\Big(\epsilon^2(y - \tau) - 1\Big) = 0, \text{ this implies that, } \quad y = \tau$$

where the parameter τ is on the y-coordinate of the point of tangency of the curves, $\mathcal{C}_\tau$ and envelope $\mathbb{E}$. This implies that $x = g(y, y) = e^{-\epsilon y} - e^{-\epsilon^2 y}$ and this envelope is bounded even for $y \longrightarrow \infty$. Hence, $g(y, y) = e^{-\epsilon y} - e^{-\epsilon^2 y}$ is an envelope with global nature derived from the curves, $g(y, \tau)$ that bounded only locally.

4.3 BVP using Theory of Envelopes

Let's consider the equation in (Kunihiro, 1995) which is the second order linear differential equation of the form

$$\epsilon\frac{d^2\xi}{dx^2}+\frac{d\xi}{dx}+\epsilon\frac{d\xi}{dx}+\xi=0 \tag{4.3.1}$$

equipped with boundary conditions $\xi(0)=0$ and $\xi(1)=1$.

This can be interpret as a boundary-layer problem. Using the (Kunihiro, 1995), we obtained the exact solution as

$$\xi(x)=\frac{e^{-x}-e^{-\frac{x}{\epsilon}}}{e^{-1}-e^{-\frac{1}{\epsilon}}}. \tag{4.3.2}$$

Next is to assign the variable $x=\epsilon X$ and $\xi(x)=Y(X)$, which when differentiate we get

$$\frac{1}{\epsilon}\frac{dY}{dX}=\frac{d\xi}{dx} \quad \text{and} \quad \frac{1}{\epsilon^2}\frac{d^2Y}{dX^2}=\frac{d^2\xi}{dx^2}$$

Substitute this relations in Equation (4.3.1) , then it becomes

$$\epsilon\left(\frac{1}{\epsilon^2}\frac{d^2Y}{dX^2}\right)+(1+\epsilon)\frac{1}{\epsilon}\frac{dY}{dX}+Y=0$$

which gives

$$\frac{d^2Y}{dX^2}+\frac{dY}{dX}=-\epsilon\left(\frac{dY}{dX}+Y\right). \tag{4.3.3}$$

Then we apply the naive perturbation expansion

$$Y(X)=Y_0(X)+\epsilon Y_1(X)+\epsilon^2 Y_2(X)+\cdots \tag{4.3.4}$$

to the Equation (4.3.3) , to obtain

$$\ddot{Y}_0+\epsilon\ddot{Y}_1+\cdots\dot{Y}_0+\epsilon\dot{Y}_1+\cdots=-\epsilon(\dot{Y}_0+\epsilon\dot{Y}_1+\cdots+Y_0+\epsilon Y_1+\cdots). \tag{4.3.5}$$

After equating the corresponding orders of ϵ, we get

$$O(1): \quad \ddot{Y}_0+\dot{Y}_0=0 \tag{4.3.6}$$

$$O(\epsilon): \quad \ddot{Y}_1+\dot{Y}_1=-(\dot{Y}_0+Y_0), \quad \text{and so on} \tag{4.3.7}$$

$$\text{with the boundary condition} \quad Y(X)=Y_0(X_0)=A_0 \tag{4.3.8}$$

for any arbitrary constant X_0 and A_0 as function of X_0. The solution of (4.3.6) is

$$Y_0(X)=A+Be^{-X}$$

and by applying the boundary conditions we have

$$Y_0(X)=A_0-B_0e^{-(X-X_0)}. \tag{4.3.9}$$

For the second Equation of (4.3.6), we have the solution

$$Y_1(X)=-A_0(X-X_0)-(B_0+C_0)(e^{-(X-X_0)}-1). \tag{4.3.10}$$

By substituting Equations (4.3.9) and (4.3.10) in (4.3.4), we obtain the curves describe as

$$Y(X, X_0) = A_0 - B_0 e^{-(X-X_0)} - \epsilon\Big(A_0(X - X_0) + (B_0 + C_0)(e^{-(X-X_0)} - 1)\Big) + O(\epsilon^2)$$

Now we are to define the renormalization constants A and B such that A_0 and B_0 would be absorb from the above curves. This can be seen as

$$A = A_0 + \epsilon(B_0 + C_0)$$

$$B = B_0 + \epsilon(B_0 + C_0)$$

this implies that

$$Y(X, X_0) = A - Be^{-(X-X_0)} - \epsilon A(X - X_0) + O(\epsilon^2). \tag{4.3.11}$$

By changing Equation (4.3.11) in to the given coordinate, we obtain

$$\xi(x, x_0) = A - Be^{-\frac{(x-x_0)}{\epsilon}} - A(x - x_0) + O(\epsilon^2),$$

because of the relation $x_0 = \frac{X_0}{\epsilon}$. Note that, the family of functions is $\{Y(X, X_0)\}_{X_0}$. Now we can derive the envelope $Y_E(X)$ from $\{Y(X, X_0)\}_{X_0}$ and both has the common tangent line at $X = X_0$. To do that, we use the condition derived in the previous section,

$$\frac{\partial Y}{\partial X_0}\bigg|_{X=X_0} = 0 \tag{4.3.12}$$

where $Y(X, X)$ will be the envelopes, $Y_E(X)$. Using condition (4.3.12) in the Equation (4.3.11), we get

$$\frac{\partial Y}{\partial X_0} = \frac{dA}{dX_0} - Be^{-(X-X_0)} - \frac{dB}{dX_0}e^{-(X-X_0)} + \epsilon A - \epsilon\frac{dA}{dX_0}(X - X_0)\bigg|_{X=X_0} = 0$$

$$\frac{dA}{dX} + \epsilon A = 0 \quad \text{and} \quad \frac{dB}{dX} + B = 0 \tag{4.3.13}$$

using separation of variables, yields

$$A(X) = \alpha e^{-\epsilon X} \quad \text{and} \quad B(X) = \beta e^{-\epsilon X}$$

where $\alpha = e^{c_1}$ and $\beta = e^{c_2}$ are constants. This implies that, Equation (4.3.11) becomes the required envelope in form

$$Y_E(X) = Y(X, X) = A(X) - B(X) = \alpha e^{-(\epsilon X)} - \beta e^{-X}. \tag{4.3.14}$$

And in terms of the given unknown the envelope becomes

$$\xi_E(x) = \alpha e^{-x} - \beta e^{-\frac{x}{\epsilon}} \tag{4.3.15}$$

with boundary conditions $\xi(0) = 0$ and $\xi(1) = 1$. To obtain α and β,

$$\xi(0) = \alpha e^{-(0)} - \beta e^{-\frac{0}{\epsilon}} = 0$$

$$\alpha = \beta \quad \text{and} \tag{4.3.16}$$

$$\xi(1) = \alpha e^{-(1)} - \beta e^{-\frac{1}{\epsilon}} = 1 \quad \text{and by using (4.3.16) we get} \quad \alpha e^{-(1)} - \alpha e^{-\frac{1}{\epsilon}} = 1, \quad \text{i.e.,}$$

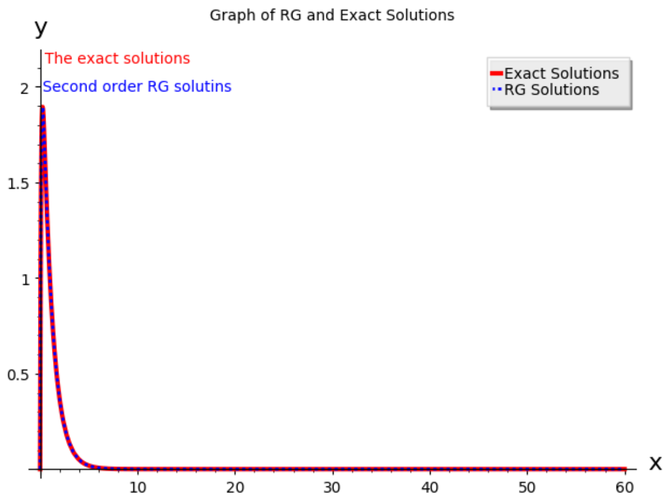

Figure 4.3: This is the graph solutions of exact (4.3.2) and RG method (4.3.15) using theory of envelope to second order linear differential equation (4.3.1) for small parameter, $\epsilon = \frac{1}{10}$.

$$\alpha = \frac{1}{e^{-1} - e^{-\frac{1}{\epsilon}}} \tag{4.3.17}$$

this implies

$$\alpha = \beta = \frac{1}{e^{-1} - e^{-\frac{1}{\epsilon}}}. \text{ Therefore, } \xi_E(x) = \frac{e^{-x} - e^{-\frac{x}{\epsilon}}}{e^{-1} - e^{-\frac{1}{\epsilon}}}.$$

Hence, the envelope $\xi_E(x)$ coincides with the exact solution $\xi(x)$ i.e., equation (4.3.2), and satisfy both the inner and outer boundary conditions together.

4.4 Solution of BVP using 3-Steps RG Method

Here we're going to apply the 3-Steps Renormalization Group method describe in first section to a boundary value problem (BVP) of the form

$$\epsilon\ddot{y} + x\ddot{y} + \dot{y} = 1, \tag{4.4.1}$$

with boundary condition $y(0) = 0$ and $y(1) = 2$.

Of course, this differential equation is boundary value problem (BVP) with boundary layer near $x = 0$ with thickness of $O(\epsilon)$. Because, the small parameter ϵ is coefficient of high order derivative. By rescaling the independent variable x to ϵt i.e., $t = \frac{x}{\epsilon}$, then we get the new BVP

$$\ddot{y} + t\ddot{y} + \dot{y} = \epsilon \tag{4.4.2}$$

with new boundary conditions $y_0(0) = 0$ and $y_0(\frac{1}{\epsilon}) = 2$.

Step 1: Apply the naive perturbation expansion (4.4.3) to the Equation (4.4.2):

$$y(t) = y_0(t) + \epsilon y_1(t) + O(\epsilon). \tag{4.4.3}$$

This will leads to get the hierarchy of equations of the corresponding orders of small parameter ϵ below

$$t(\ddot{y}_0 + \epsilon\ddot{y}_1 + \cdots) + \ddot{y}_0 + \epsilon\ddot{y}_1 + \cdots + \dot{y}_0 + \epsilon\dot{y}_1 + \cdots = \epsilon.$$

By equating the corresponding powers of ϵ , we obtain

$$\text{for order } O(1): \quad t\ddot{y}_0 + \ddot{y}_0 + \dot{y}_0 = 0,$$

this implies that

$$(t+1)\ddot{y}_0 + \dot{y}_0 = 0 \tag{4.4.4}$$

$$\text{for order } O(\epsilon): \quad t\ddot{y}_1 + \ddot{y}_1 + \dot{y}_1 = 1,$$

implies that

$$(t+1)\ddot{y}_1 + \dot{y}_1 = 1. \tag{4.4.5}$$

Equation (4.4.4) can be change to the system of first order differential equation as;

$$\dot{y}_0 = z_0 \tag{4.4.6}$$

$$(1+t)\dot{z}_0 + z_0 = 0, \quad \text{i.e.,} \quad \dot{z}_0 = \frac{-z_0}{1+t} \tag{4.4.7}$$

Next we solve the system (4.4.6) and (4.4.7), with arbitrary initial values $y_0(t_0)$ and $z_0(t_0)$. Since Equation (4.4.6) is in terms of Equation (4.4.7), so we should solve (4.4.7) first in order to solve (4.4.6). Equation (4.4.7) with $z(t_0) = z_0(t_0)$ gives

$$z_0(t) = z_0(t_0)\frac{(1+t_0)}{(1+t)}. \tag{4.4.8}$$

Substituting Equation (4.4.8) in (4.4.6), we get

$$\dot{y}_0 = z_0(t_0)\frac{(1+t_0)}{1+t}, \quad \text{by separating variables, we get}$$

$$dy_0(t) = z_0(t_0)\frac{(1+t_0)}{1+t}dt,$$

integrating both side and considering $z_0(t_0)(1+t_0)$ as constant, we get

$$y_0(t) = y_0(t_0) + z_0(t_0)(1+t_0)\log\left(\frac{1+t}{1+t_0}\right). \tag{4.4.9}$$

Therefore, the solution of order $O(1)$ is

$$y_0(t) = y_0(t_0) + z_0(t_0)(1+t_0)\log\left(\frac{1+t}{1+t_0}\right), \tag{4.4.10}$$

$$z_0(t) = z_0(t_0)\frac{(1+t_0)}{(1+t)}.$$

From Equation (4.4.5), the single second order differential equation becomes the system of first order differential equations of the form

$$\dot{y}_1 = z_1 \tag{4.4.11}$$

$$\dot{z}_1 = \frac{1}{1+t} - \frac{z_1}{1+t}. \tag{4.4.12}$$

The solution of these system of equations with zero initial conditions can be written as follows

$$y_1(t) = (t-t_0) - (1+t_0)\log\left(\frac{1+t}{1+t_0}\right) \tag{4.4.13}$$

$$z_1(t) = \frac{(t-t_0)}{(1+t)}. \tag{4.4.14}$$

This implies that, naive perturbation expansion i.e., Equation (4.4.2), becomes

$$y(t,\epsilon) = y_0(t_0) + z_0(t_0)(1+t_0)\log\left(\frac{1+t}{1+t_0}\right) + \epsilon\Big((t-t_0) - (1+t_0))\log\left(\frac{1+t}{1+t_0}\right)\Big) + O(\epsilon^2) \tag{4.4.15}$$

$$z(t,\epsilon) = z_0(t_0)\left(\frac{1+t_0}{1+t}\right) + \epsilon\left(\frac{t-t_0}{1+t}\right) + O(\epsilon^2), \tag{4.4.16}$$

and the governing equations at order $O(\epsilon^n)$ are given by

$$\dot{y}_n = z_n, \quad \dot{z}_n = -\frac{z_n}{1+t},$$

with zero initial conditions. This implies that for every positive integer $n \geq 2$, $y_n \equiv 0$. Equations (4.4.15) and (4.4.16), contains secular terms and this implies that terms that are proportional to $t-t_0$ will diverge on the asymptotically large interval of time.

Step 2: make a change of coordinates, called renormalization transform:

To make change of coordinates, we introduce a near-identity transformation that will renormalize the initial conditions $y_0(t_0)$ and $z_0(t_0)$ with new constants define by

$$y_0(t_0) = \sum_{i=0}^{\infty}\epsilon^i Y_i(t_0), \tag{4.4.17}$$

$$z_0(t_0) = \sum_{i=0}^{\infty} \epsilon^i Z_i(t_0). \tag{4.4.18}$$

By substituting Equations (4.4.17) and (4.4.18) in to Equations (4.4.15) and (4.4.16), we get

$$y(t,\epsilon) = Y_0(t_0) + Z_0(t_0)(1+t_0)\log\left(\frac{1+t}{1+t_0}\right) + \epsilon\Big(Y_1(t_0) + Z_1(t_0)(1+t_0)\log\left(\frac{1+t}{1+t_0}\right) \tag{4.4.19}$$

$$+(t-t_0) - (1+t_0))\log\left(\frac{1+t}{1+t_0}\right)\Big) + O(\epsilon^2),$$

$$z(t,\epsilon) = Z_0(t_0)\left(\frac{1+t_0}{1+t}\right) + \epsilon\left(Z_1(t_0)\left(\frac{1+t_0}{1+t}\right) + \frac{t-t_0}{1+t}\right) + O(\epsilon^2). \tag{4.4.20}$$

The reason why these $Y_1(t_0)$ and $Z_1(t_0)$ were introduced is to remove the instances of t_0 which are on secular terms by change of coordinates. This change of coordinates is as follows;

$$Y_1(t_0) = 0, \quad Z_1(t_0) = 1 \quad \text{while} \quad Y_i(t_0) \equiv 0 \quad \text{and} \quad Z_i(t_0) \equiv 0 \quad \text{for} \quad i \geq 2. \tag{4.4.21}$$

Now by substituting Equation (4.4.21) into (4.4.19) and (4.4.20), we obtain the following renormalized expansion

$$y(t,\epsilon) = Y_0(t_0) + Z_0(t_0)(1+t_0)\log\left(\frac{1+t}{1+t_0}\right) + \epsilon(t-t_0), \tag{4.4.22}$$

$$z(t,\epsilon) = Z_0(t_0)\left(\frac{1+t_0}{1+t}\right) + \epsilon. \tag{4.4.23}$$

Step 3: next we apply the RG condition describe in the first section:

$$\frac{\partial}{\partial t_0}\begin{pmatrix} y(t,\epsilon) \\ z(t,\epsilon) \end{pmatrix}\Bigg|_{t_0=t} = 0. \tag{4.4.24}$$

Using Equations (4.4.24) in (4.4.22) and (4.4.23), we get

$$\frac{\partial z(t,\epsilon)}{\partial t_0} = Z_0(t_0)\left(\frac{1}{1+t}\right) + \left(\frac{1+t_0}{1+t}\right)\frac{dZ_0}{dt_0} = 0, \quad \text{at} \quad t_0 = t$$

this implies that

$$\frac{dZ_0}{dt} = -\frac{Z_0}{1+t} \tag{4.4.25}$$

and for $y(t,\epsilon)$ we get

$$\frac{dY_0}{dt} = Z_0 + \epsilon \tag{4.4.26}$$

Equations (4.4.25) and (4.4.26) are called Renormalization Group equations. These contains only resonant terms. Next is to solve these equations.

For Equation (4.4.25), we have

$$Z_0 = \frac{k_1}{1+t}. \tag{4.4.27}$$

For Equation (4.4.26), we substitute (4.4.27) into (4.4.26) to get

$$\frac{dY_0}{dt} = \frac{k_1}{1+t} + \epsilon.$$

We integrate both sides and obtain,

$$\int dY_0 = k_1 \int \frac{1}{1+t} dt + \epsilon \int dt, \text{ which gives}$$

$$Y_0(t) = k_1 \log(1+t) + \epsilon t + k_2. \tag{4.4.28}$$

Now we substitute Equations (4.4.27) and (4.4.28) in to (4.4.22) and (4.4.23) to obtain

$$y(t,\epsilon) = k_1 \log(1+t) + k_2 + k_1 \log\left(\frac{1+t}{1+t}\right) + \epsilon t, \text{ but } \log\left(\frac{1+t}{1+t}\right) = 0, \text{ this simplify to}$$

$$y(t,\epsilon) = k_1 \log(1+t) + k_2 + \epsilon t \tag{4.4.29}$$

$$z(t,\epsilon) = \frac{k_1}{1+t} + \epsilon. \tag{4.4.30}$$

Using the boundary conditions $y(0) = 0$ and $y(\frac{1}{\epsilon}) = 2$ on equation above, we find that $k_2 = 0$ and $k_1 = \frac{1}{\log(1+(\frac{1}{\epsilon}))}$. Therefore, the solution of Equation (4.4.29) is

$$y(t,\epsilon) = \frac{\log(1+t)}{\log(1+\frac{1}{\epsilon})} + \epsilon t. \tag{4.4.31}$$

In terms of $x = \epsilon t$, the solution is

$$y(x) = \frac{\log(1+\frac{x}{\epsilon})}{\log(1+\frac{1}{\epsilon})} + x,$$

so $y(0) = 0$ and $y(1) = 2$ and this expression is indeed the exact solution.

Observation: Using our proposed 3-Steps RG method to Equation (4.4.1), we observe that all the terms at $O(\epsilon^2)$ and higher order vanish to zero. The remaining expansion which is just order(1) and linear terms represents an exact solution. This illustrate the application of method where in this example the method naturally identifies the exact solution.

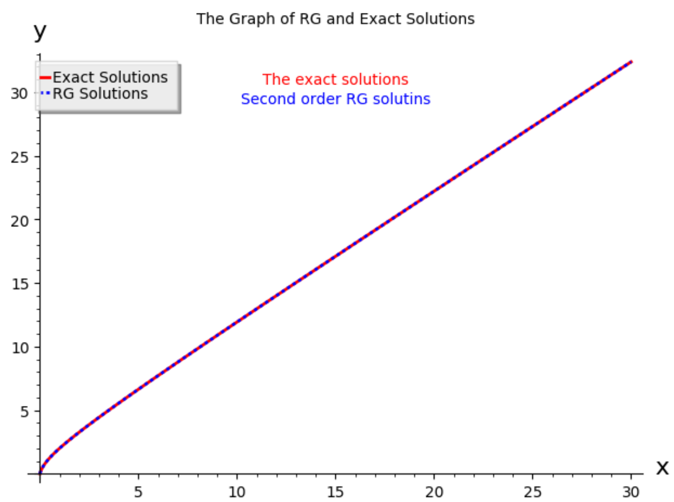

Figure 4.4: This is the graphical solutions of Equation (4.4.1) for both the exact and RG method (4.4.31) with small parameter, $\epsilon = \frac{1}{10}$, where the solid graph is for exact solution and dash is for our proposed 3-steps RG method.

This method shows that the change of coordinate help to removes nonresonant terms from the asymptotic expansions. Precisely, it generates good asymptotic for many other classes of systems see (Goldenfeld et al., 1989). Furthermore , it is simpler in identify the divergent (resonant) terms from the naive asymptotic expansions. From this point therefore , we think that the RG Method will be useful as a technique for generating global information from the given differential equation.

4.5 Rayleigh Equation using 3-Steps RG Method

Consider the Project given by (Kuehn, 2015) on page 283 for Rayleigh equation

$$\frac{d^2x}{dt^2} + x = \delta\Big(\frac{dx}{dt} - \frac{1}{3}\Big(\frac{dx}{dt}\Big)^3\Big). \tag{4.5.1}$$

Finds it asymptotic expansion free of secular terms up to and including first order in delta.

Step 1: Apply the naive perturbation expansion to the Equation (4.5.1):

The naive perturbation expansion is

$$x(t) = x_0(t) + \delta x_1(t) + \delta^2 x_2(t) + \cdots . \tag{4.5.2}$$

We substitute regular expansion (4.5.2) into the given Equation (4.5.1), to get

$$\ddot{x}_0 + \delta\ddot{x}_1 + \delta^2\ddot{x}_2 + \cdots + x_0 + \delta x_1 + \delta^2 x_2 + \cdots = \delta\Big(\dot{x}_0 + \delta\dot{x}_1 + \delta^2\dot{x}_2 + \cdots - \frac{1}{3}(\dot{x}_0 + \delta\dot{x}_1 + \delta^2\dot{x}_2 + \cdots)^3\Big)$$

the second series on the right hand side implies

$\ddot{x}_0 + \delta\ddot{x}_1 + \delta^2\ddot{x}_2 + \cdots + x_0 + \delta x_1 + \delta^2 x_2 + \cdots = \delta\Big(\dot{x}_0 + \delta\dot{x}_1 + \delta^2\dot{x}_2 + \cdots - \frac{1}{3}(\dot{x}_0^3 + 3\delta\dot{x}_0^2\dot{x}_1 + 3\delta^2\dot{x}_0^2\dot{x}_2 + 3\delta^2\dot{x}_0\dot{x}_1^2 + \cdots)\Big)$. After equating the corresponding order of δ, we obtain;

$$\text{for order } O(1): \quad \ddot{x}_0 + x_0 = 0 \tag{4.5.3}$$

$$\text{for order } O(\delta): \quad \ddot{x}_1 + x_1 = \dot{x}_0 - \frac{1}{3}\dot{x}_0^3 \quad \text{and so on.} \tag{4.5.4}$$

The solution for $O(1)$ i.e., Equation (4.5.3) is

$$x_0(t) = Ae^{i(t-t_0)} + c.c. \tag{4.5.5}$$

where $c.c.$ represent the complex conjugate in order to make the calculation easier. The right hand side of Equation (4.5.4) is in terms of x_0 which is acting as a forcing term in the equation. And we want to make the homogeneous solution (the solution of equation when the right hand side is zero) of x_1 , $x_2, \cdots$ to be equal to zero at time t_0 (i.e., $x_1(t) = x_2(t) = \cdots = 0$). Then to find the particular solution of x_1, we use either method of undetermined coefficients or variation of parameters. Now for method of undetermined coefficients,

$$x_0(t) = Ae^{i(t-t_0)} + A^* e^{-i(t-t_0)} \implies \dot{x}_0(t) = iAe^{i(t-t_0)} - iA^* e^{-i(t-t_0)}$$

and

$$\begin{aligned}\dot{x_0}^3 &= (iAe^{i(t-t_0)} - iA^*e^{-i(t-t_0)})^3\\ &= (iAe^{i(t-t_0)} - iA^*e^{-i(t-t_0)})(iAe^{i(t-t_0)} - iA^*e^{-i(t-t_0)})(iAe^{i(t-t_0)} - iA^*e^{-i(t-t_0)})\\ &= (-A^2e^{2i(t-t_0)} + 2AA^* - A^{*2}e^{-2i(t-t_0)})(iAe^{i(t-t_0)} - iA^*e^{-i(t-t_0)})\\ &= -iA^3e^{3i(t-t_0)} + 3iA^2A^*e^{i(t-t_0)} - 3iAA^{*2}e^{-i(t-t_0)} + iA^{*3}e^{-3i(t-t_0)}.\end{aligned}$$

Also,

$$\dot{x}_0 - \frac{1}{3}\dot{x}_0^3 = iAe^{i(t-t_0)} - iA^*e^{-i(t-t_0)} + \frac{1}{3}iA^3e^{3i(t-t_0)} - iA^2A^*e^{i(t-t_0)} - iAA^{*2}e^{-i(t-t_0)} - \frac{1}{3}iA^{*3}e^{-3i(t-t_0)}$$

that is

$$\dot{x}_0 - \frac{1}{3}\dot{x}_0^3 = i(A - A^2A^*)e^{i(t-t_0)} + i\left(\frac{1}{3}A^3 - \frac{1}{3}A^{*3}\right)e^{3i(t-t_0)} + c.c..$$

The above equation is a forcing terms of the Equation (4.5.4), therefore we guess the particular solution to be

$$x_{1p} = c_1te^{i(t-t_0)} + c_2e^{3i(t-t_0)}.$$

Now,

$$\dot{x}_{1p} = ic_1te^{i(t-t_0)} + c_1e^{i(t-t_0)} + 3ic_2e^{3i(t-t_0)}$$

and

$$\ddot{x_{1p}} = -c_1te^{i(t-t_0)} + 2ic_1e^{i(t-t_0)} - 9c_2e^{3i(t-t_0)}.$$

This implies that the Equation (4.5.4), becomes

$$\begin{aligned}-c_1te^{i(t-t_0)} + 2ic_1e^{i(t-t_0)} - 9c_2e^{3i(t-t_0)} + c_1te^{i(t-t_0)} + c_2e^{3i(t-t_0)} &= i(A - A^2A^*)e^{i(t-t_0)}\\ &+ i\left(\frac{1}{3}A^3 - \frac{1}{3}A^{*3}\right)e^{3i(t-t_0)}.\end{aligned}$$

Equating the corresponding coefficients of $e^{i(t-t_0)}$, $te^{i(t-t_0)}$, and $e^{3i(t-t_0)}$, we gets

$$2ic_1 = i(A - A^2A^*) \implies c_1 = \frac{1}{2}(A - A^2A^*), \quad \text{and}$$

$$-9c_2 + c_2 = -\frac{1}{3}iA^{*3} + \frac{1}{3}iA^3 \implies c_2 = i\frac{(A^{*3} - A^3)}{24}.$$

Substituting c_1 and c_2 into our guess equation, we obtain

$$x_{1p} = \frac{1}{2}(A - A^2A^*)te^{i(t-t_0)} + \left(i\frac{A^{*3}}{24} - i\frac{A^3}{24}\right)e^{3i(t-t_0)}.$$

Therefore, solution of Equation (4.5.4) is

$$x_1(t) = \frac{1}{2}(A - A^2A^*)(t - t_0)e^{i(t-t_0)} + \frac{i}{24}A^{*3}e^{3i(t-t_0)} - \frac{i}{24}A^3e^{3i(t-t_0)}$$

that is

$$x_1(t) = \frac{i}{24}A^{*3}e^{3i(t-t_0)} + \frac{1}{2}(A - A^2A^*)(t - t_0)e^{i(t-t_0)} - \frac{i}{24}A^3e^{3i(t-t_0)} + c.c. \ . \tag{4.5.6}$$

Now we have

$$x(t) = x_0(t) + \delta x_1(t) + O(\delta^2), \text{ that is}$$

$$x(t) = Ae^{i(t-t_0)} + \delta\Big(\frac{i}{24}A^{*3}e^{3i(t-t_0)} + \frac{1}{2}(A - A^2A^*)(t-t_0)e^{i(t-t_0)} - \frac{i}{24}A^3e^{3i(t-t_0)}\Big) + O(\delta^2).$$

The first and third term inside the brackets are from homogeneous solution and second term is a secular term because it's a multiple of polynomial t. The next step, is to apply a renormalization transform.

Step 2: make a change of coordinates, called renormalization transform:

In this case, the idea is to absorb the homogeneous parts of solution by renormalize the constant of integration A and define a new integral constant as a function of t_0, say, $\mathcal{A}$ by

$$A = \mathcal{A} + \delta a_1 + O(\delta^2) \tag{4.5.7}$$

where a_1 is the coefficient that will absorb the homogeneous part of solution $x_1(t)$ for $O(\delta^2)$. From Equation (4.5.6), first and third terms of right hand side is from homogeneous and by setting $t = t_0$ we find out that

$$a_1 = \frac{i}{24}A^{*3} - \frac{i}{24}A^3.$$

Equation (4.5.5), becomes

$$x_0(t) = \mathcal{A}e^{i(t-t_0)},$$

and that of (4.5.6) yields

$$x_1(t) = \frac{1}{2}(\mathcal{A} - \mathcal{A}^2\mathcal{A}^*)(t-t_0)e^{i(t-t_0)} + c.c.\ .$$

This implies that Equation (4.5.7), becomes

$$A = \mathcal{A} + \delta\Big(\frac{i}{24}A^{*3} - \frac{i}{24}A^3\Big) + O(\delta^2) \tag{4.5.8}$$

and is the total renormalization transform.

After substituting the above x_0 and x_1 in the naive perturbation expansion, we obtain

$$x(t) = \mathcal{A}e^{i(t-t_0)} + \delta\Big(\frac{1}{2}(\mathcal{A} - \mathcal{A}^2\mathcal{A}^*)(t-t_0)e^{i(t-t_0)}\Big) + O(\delta^2) \tag{4.5.9}$$

But please observe that, the asymptotic expansion (4.5.9) contains the secular terms (i.e., the terms that grows unbound for large t). This would make $x(t)$ to diverge as $t \longrightarrow \infty$. The next step is to apply the RG condition which was already describe in the previous sections.

Step 3: We apply the RG condition describe in the first section:

This will cancel out the resonant terms in the solution and is given by

$$\left.\frac{\partial x}{\partial t_0}\right|_{t_0=t} = 0. \tag{4.5.10}$$

Then we simplify the Equation (4.5.10) together with (4.5.9) , as follows

$$\frac{\partial x(t)}{\partial t_0} = -i\mathcal{A}e^{i(t-t_0)} + \frac{\partial \mathcal{A}}{\partial t_0}e^{i(t-t_0)} + \delta\Big(-\frac{1}{2}i(\mathcal{A}-\mathcal{A}^2\mathcal{A}^*)(t-t_0)e^{i(t-t_0)} - \frac{1}{2}(\mathcal{A}-\mathcal{A}^2\mathcal{A}^*)e^{i(t-t_0)} + \frac{1}{2}\Big(\frac{\partial \mathcal{A}}{\partial t_0} - 2\mathcal{A}\mathcal{A}^*\frac{\partial \mathcal{A}}{\partial t_0}\Big)(t-t_0)e^{i(t-t_0)}\Big)\Big|_{t_0=t} = 0$$

this implies that

$$-i\mathcal{A} + \frac{\partial \mathcal{A}}{\partial t} + \delta\Big(-\frac{1}{2}(\mathcal{A}-\mathcal{A}^2\mathcal{A}^*)\Big) = 0\ .$$

Therefore , we get

$$\frac{\partial \mathcal{A}}{\partial t} = i\mathcal{A} - \delta\frac{1}{2}\mathcal{A}(\mathcal{A}\mathcal{A}^*-1) + O(\delta^2)$$

correct to the order $O(\delta^2)$. If we let $\mathcal{A} = \frac{R}{2}e^{i(t+\theta)}$, then we will get the corresponding amplitude and phase flow equations to the order $O(\delta^2)$ as we'll be describe below.

$$\frac{\partial \mathcal{A}}{\partial t} = \mathcal{A}i - \delta\frac{1}{2}\mathcal{A}(\mathcal{A}\mathcal{A}^*-1), \quad \text{for} \quad \mathcal{A} = \frac{R}{2}e^{i(t+\theta)},$$

Since we have only one independent variable, this implies that

$$\frac{d(\frac{R}{2}e^{i(t-\theta)})}{dt} = \frac{R}{2}e^{i(t-\theta)}\Big(i - \frac{1}{2}\delta\Big(\frac{R^2}{4}-1\Big)\Big).$$

Differentiating the left hand side, we get

$$\frac{R}{2}ie^{i(t-\theta)} + \frac{1}{2}e^{i(t-\theta)}\frac{dR}{dt} = \frac{R}{2}e^{i(t-\theta)}\Big(i - \frac{1}{2}\delta\Big(\frac{R^2}{4}-1\Big)\Big).$$

This implies

$$\frac{R}{2}i + \frac{1}{2}\frac{dR}{dt} = \frac{R}{2}\Big(i - \frac{1}{2}\delta\Big(\frac{R^2}{4}-1\Big)\Big), \text{ that is } \quad Ri + \frac{dR}{dt} = R\Big(i - \frac{1}{2}\delta\Big(\frac{R^2}{4}-1\Big)\Big).$$

By rearranging, we get

$$Ri + \frac{dR}{dt} = Ri + \frac{1}{2}\delta R\Big(1 - \frac{R^2}{4}\Big).$$

Since there is no θ in the above equation, implies that it's constant. Therefore, we have

$$\frac{dR}{dt} = \frac{1}{2}\delta R\Big(1-\frac{R^2}{4}\Big) \text{ and } \tag{4.5.11}$$

$$\frac{d\theta}{dt} = 0. \tag{4.5.12}$$

The amplitude and phase flow equations we derive is the same results with multiple time scales used in MS by Chen , Goldenfeld , and Y. Oono using slow variables $T_1 = \epsilon t$, $T_2 = \epsilon^2 t, \cdots$ in their paper

(Chen et al., 1996) and in our method the slow variables appear naturally. Hence, the solution of the Rayleigh equation to order $O(\delta^2)$ by (Chen et al., 1996) is

$$x(t) = R(t)\sin(t+\theta) + \delta\frac{1}{96}R(t)^3\Big(\cos 3(t+\theta) - \cos(t+\theta)\Big) + O(\delta^2),$$

where both $R(t)$ and $\theta(t)$ are obtain from Equations (4.5.11) and (4.5.12). Because, Equation (4.5.11) can be solve as follows;

$$\frac{dR}{dt} = \frac{1}{2}\delta R\Big(1 - \frac{R^2}{4}\Big), \text{ implies } \frac{dR}{dt} - \frac{1}{2}\delta R = \frac{-R^3\delta}{8}$$

is a Bernoulli equation which can be express as

$$R^{-3}\frac{dR}{dt} - \frac{1}{2}\delta R^{-2} = \frac{\delta}{8}.$$

Let $v = R^{-2}$ then $\frac{dv}{dt} = -2R^{-3}\frac{dR}{dt}$ i.e., $R^{-3}\frac{dR}{dt} = -\frac{1}{2}\frac{dv}{dt}$ therefore $-\frac{1}{2}\frac{dv}{dt} - \frac{1}{2}\delta v = -\frac{\delta}{8}$ that is

$$\frac{dv}{dt} + \delta v = \frac{\delta}{4}.$$

This equation becomes linear ODE with integral factor $e^{\delta t}$. By solving the linear equation and substitute $v = R^{-2}$ then we get

$$R(t) = R(0)\Big[e^{-\delta t} + \frac{R(0)^2(1 - e^{-\delta t})}{4}\Big]^{-\frac{1}{2}}.$$

Equation (4.5.12) implies $\theta = 0$, since $\frac{d\theta}{dt}$ = constant , at order $O(\delta^2)$. Finally , for $t \longrightarrow \infty$, the result approaches a limit circle of radius closed 2.

Observation: From this example we observe that; the RG calculation shows that our assumption of perturbative renormalizability is consistent and we got the results of multiple scales method from renormalized perturbation theory for what (Chen et al., 1996) deed in their paper. Secondly, the RG equations obtained by using RG condition describes the long time scale motion for both amplitude and phase plane.

5. Conclusion

In this essay, we discussed how the Renormalization Group methods works in approximating the solution of differential equation and we show logically how the secular terms that arise in the naive perturbation expansion can be eliminate using " renormalization transform ". By this method of RG, we obtained detailed analytical results for a different variety of singularly perturbed problems and compare then with the exact solution of the same problem.

In Section 4.4, we used our proposed 3-Steps RG Method to the boundary value problem (4.4.1) and rescales the independent variable x to ϵt where we obtain the new boundary value problem (4.4.2). This process is called formulation of inner equation. The asymptotic expansion of (4.4.2) is

$$y(t,\epsilon) = y_0(t_0) + z_0(t_0)(1+t_0)\log\left(\frac{1+t}{1+t_0}\right) + \epsilon\Big((t-t_0) - (1+t_0))\log\left(\frac{1+t}{1+t_0}\right)\Big) \tag{5.0.1}$$

$$z(t,\epsilon) = z_0(t_0)\left(\frac{1+t_0}{1+t}\right) + \epsilon\left(\frac{t-t_0}{1+t}\right). \tag{5.0.2}$$

This shows that their is an existence of secular terms, so terms that are proportional to $t-t_0$ will diverge on the asymptotically large interval of time. Using " renormalization transform " (4.4.17) , (4.4.18) we get (4.4.22) and (4.4.23). After applying RG condition (4.4.24), finally we obtained the approximate solution as (4.4.31).

We observe that all the terms at $O(\epsilon^2)$ and higher order vanish to zero. The remaining expansion which is just order $O(1)$ and linear terms represents an exact solution. This illustrate the application of 3-Steps RG method where in this example the method naturally identifies the exact solution at order $O(\epsilon)$.

In Section 4.5 (solution of Rayleigh Equation) , we observe that; the RG calculation shows that our assumption of perturbative renormalizability is consistent because, we got the results of multiple scales method from renormalized perturbation theory for what (Chen et al., 1996) deed in their paper. Secondly, the amplitude and phase flow equations we derive is the same results with multiple time scales used in MS by Chen , Goldenfeld , and Y. Oono using slow variables $T_1 = \epsilon t$, $T_2 = \epsilon^2 t, \cdots$ in (Chen et al., 1996) while in our method the slow variables appear naturally. Furthermore, the RG equation obtained by using RG condition describes the long time scale motion for both amplitude and phase plane. Thirdly, the solution (4.5.6) show that the naive perturbation expansion will breakdown when $\delta(t-t_0) > 1$ because of the secular terms and for $R = 2$ the amplitude equation (4.5.11) will reduces to

$$\frac{dR}{dt} = 0 + O(\delta^2) \quad \text{and} \quad \frac{d\theta}{dt} = 0 + O(\delta^2).$$

From the review of theory of envelopes and our proposed 3-Steps RG methods, we notice that there is a similarity between them. The basic equation of the theory of envelopes

$$F_{\tau_0}(x_0, y_0, \tau_0) \equiv \frac{\partial F(x_0, y_0, \tau_0)}{\partial \tau_0} = 0$$

is similar with RG condition define in the first section. Both of the two methods they use naive perturbation expansion and finally the solutions are concise with the exact solution.

To this end, we'll like to suggest that base on the above observations RG method is a more straightforward method in extracting global information from the perturbation expansion than any other methods of multiple scales and matched asymptotic. Because in asymptotic matching the solution is derive by calculating the separated outer and inner solutions and then matching them across intermediate scale solutions.

Renormalization Group Method is clear in theory but difficult in practice. It is no doubt that the proposed method can be applied to many linear and nonlinear differential equations. For the future work, we want to propose how to apply this method in wireless market intelligence as it's describe in (Simkin and Olness, 2001). Finally here is a question left to a reader. Can you reformulate the undamped nonlinear oscillator described by Duffing's equation

$$\frac{d^2x(t)}{dt^2} + x(t) + \epsilon x(t)^3 = 0$$

using Renormalization Group methods up to including order $O(\epsilon)$ and what can you deduce?.

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