On the Inverse Topology, with Applications to Function Rings

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Abstract

Let R be a commutative ring with unity. The collection of minimal prime ideals of R is denoted by $Min(R)$. The topology that is well studied on $Min(R)$ is the Zariski topology also known as the Hull-kernel topology. In this essay, we describe the inverse topology on $Min(R)$ denoted by $Min(R)^{-1}$ and show that it is always a compact T_1-space. We also characterise when it is a Hausdorff space.

In the second part of the essay, specialising to the ring $C(X)$ of continuous real-valued functions on a Tychonoff space X, we show that the inverse topology on $Min(C(X))$ is the Stone-Čech compactification of X if X is an F-space.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

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1. Introduction

The paper which paved way for the study of the space of minimal prime ideals of a commutative ring is [8]. The *Zariski topology* also known as the *Hull-kernel topology* on the space of minimal primes of a commutative ring with unity is one of the well studied topology and many rich properties have been established from studying this topology (see [8, 10]). On the other hand, the inverse topology on the space of minimal primes of a commutative ring with unity is another type of topology which has also been greatly studied by authors in [4, 12] amongst others. In this essay, we will mainly focus on the inverse topology and its properties.

More recently, the author in [3], studied two topologies on the space of minimal primes of L, considering L as an algebraic frame. The author in the paper studies the properties of the two topologies (Zariski and inverse) on the space of minimal prime elements of L, and were denoted by $Min(L)$ and $Min(L)^{-1}$ respectively. Our approach in this essay is along the lines of [12], working with the two topologies just as in [3], but our central focus will be on commutative rings instead of algebraic frames. We will put more emphasis on the inverse toplogy on the space of minimal primes of a commutative ring R, since it is the heart of our essay. Our goal is to then find conditions imposed on R for $Min(R)^{-1}$ to have various topological properties, like being Hausdorff, compact, zero dimensional (by zero-dimensional we mean a topological space with a base of clopen subsets) and extremally disconnected.

This is how the essay is structured. In Chapter 1, following the introduction we define terms and state results that we shall need, especially concerning commutative rings. We also recall how the Zariski topology is constructed. The gist of our essay begins in Chapter 2. Throughout we let R denote a commutative ring with unity.

In Chapter 2 we first construct the inverse topology on $Min(R)$. We then show that $Min(R)^{-1}$ is always a compact T_1-space. We further give a characterisation of when $Min(R)^{-1}$ is a Hausdorff space.

In Chapter 3 we specialise to the rings of real valued continuous functions on a Tychonoff space X, denoted by $C(X)$. We then investigate the effect on $C(X)$, of imposing some topological properties on X. Most importantly on $Min(C(X))$, where $Min(C(X))$ denotes the space of minimal prime ideals of $C(X)$.

1.1 General definitions and main tools

This section entails definition of terms, and theorems related to the general notion of topology which are fundamental to the entire work. We also consider some preliminary results concerning minimal prime ideals which play pivotal role in the subsequent sections. Some of the definitions and results used in this section can be found in [2, 8, 10, 12, 13, 18].

We begin by giving an overview regarding topological spaces and their properties.

If X is a space (see Definition 1.1.1), a subset S of X is said to be an *open set* if each point of S is an interior point. The notion of open sets generalises the idea of an open interval in the context of the real line.

With the knowledge of open sets in mind, we now give a formal definition of a topology on a set.

1.1.1 Definition. ([13, Definition A2.1]). Let X be a non-empty set. Let Λ be a non-empty indexing set and τ be a family of open sets of X, then, τ is a *topology* on X if:

(1) $X \in \tau$ and $\emptyset \in \tau$;

(2) $\{O_i\}_{i\in\Lambda} \subseteq \tau$ implies $\bigcup_{i\in\Lambda} O_i \subseteq \tau$;

(3) $\{O_i\}_{i\in\Lambda} \subseteq \tau$ implies $\bigcap_{i=1}^{n} O_i \subseteq \tau$.

We call the set (X, τ) a topological space.

Note that we shall denote a topological space by either (X, τ) or X.

1.1.2 Definition. ([13, Definition A9.4]). Given a topological space (X, τ) and a subset S of X, we define the *subspace topology* on S to be:

$$\tau_S = \{S \cap U : U \in \tau\}.$$

1.1.3 Definition. ([13, Definition A11.3]). Let (X, τ) be a topological and $\sim$ be an equivalence relation on X. The quotient space $Y = X/\sim$ is defined to be the set of equivalence classes of elements belonging to X:

$$Y = \{[x] : x \in X\} = \{\{v \in X : v \sim x\} : x \in X\},$$

endowed with the topology whose open sets are defined to be the sets belonging to the equivalence class, whose unions are open sets in X, that is,

$$\tau_Y = \{U \subseteq Y : \bigcup U = (\bigcup_{[a]\in U} [a]) \in \tau\}.$$

1.1.4 Definition. ([13, Definition A7.1]). Let (X, τ) be a topological space. Then, $\mathcal{B} \subseteq X$ is called a *basis (or base)* for the topology on X if $U \in \tau$ can be expressed as $U = \cup_{i\in\Lambda} V_i$, where $V_i \in \mathcal{B}$ and Λ is an indexing set.

1.1.5 Definition. ([13, Definition B1.6]). Let Λ be a non-empty indexing set. A class $\{O_i : i \in \Lambda\}$ of open sets in a topological space X is an *open cover* for X if

$$X \subseteq \bigcup_{i\in\Lambda} O_i.$$

1.1.6 Definition. ([13, Definition B1.2]). A topological space (X, τ) is *compact* if every open cover has a finite subcover.

1.1.7 Definition. ([13, Definition A3.6]). We call a topological space X *Hausdorff* (T_2-space) if for every distinct pair of points $x, y \in X$, there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $y \in V$.

1.1.8 Definition. ([13, Definition A4.10]). Let X and Y be two topological spaces. A function $f : X \longrightarrow Y$ is a *homeomorphism* if and only if f is:

(1) injective;

(2) surjective;

(3) continuous;

(4) the inverse exists, and it is continuous.

1.1.9 Definition. ([2, $p.3$]). Suppose X and Y are two topological spaces, then a continuous mapping $f : X \to Y$ is a (topological) embedding if the submapping $\phi : X \to f(X)$ is a homeomorphism.

1.1.10 Theorem. *([13, Lemma B3.2]). Every closed subspace of a compact space X is compact.*

1.1.11 Theorem. *([13, Lemma B3.4]). Every compact subspace of a Hausdorff space is closed.*

1.1.12 Definition. ([13, Definition A3.12]). We say a topological space X is *normal* (T_4 -space) if for every pair of disjoint closed sets $A, B \in X$, there are disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Now, we give the following theorem which classifies those Hausdorff spaces which satisfy the normality condition.

1.1.13 Theorem. *([20, Theorem 17.10]). Every compact Hausdorff space is normal.*

Proof. Let X be a compact Hausdorff space and A, B be two disjoint closed subsets of X. Our aim is to find a pair of disjoint open subsets $U, V \subseteq X$, such that $A \subseteq U$ and $B \subseteq V$.

Particular case: Suppose that B is a singleton, $B = \{b\}$.

We see that B is closed since X is a Hausdorff space. Using the fact that X is Hausdorff, let us take $a \in A$, so we can find open sets U_a and V_a containing a and b, respectively with $U_a \cap V_a = \emptyset$. Since A is closed, by Theorem 1.1.10 it is compact. On the other hand, we easily deduce that

$$A \subseteq \bigcup_{a \in A} U_a.$$

Now since A is compact, there exist $a_1, \ldots, a_n \in A$, such that

$$A \subseteq \bigcup_{i=1}^{m} U_{a_i} := U.$$

Then we are done by taking

$$U = U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_n} \qquad \text{and} \qquad V = V_{a_1} \cap V_{a_2} \cap \cdots \cap V_{a_n}.$$

Now, by applying the same reasoning we can see that the same result holds even when generalized. Thus X is normal. □

1.1.14 Definition. ([2, $p.6$]). A *compactification* of a topological space X is an ordered pair (K, f) where K is a compact Hausdorff space and f is an embedding of X as a dense (that is $\overline{S} = X$, for $S \subseteq X$. In other words, every nonempty open set in X contains a member of S) subset of K.

1.1.15 Example. We can identify the real line $\mathbb{R}$ with $(-1, 1)$ through a well-defined homeomorphism of $\mathbb{R}$ into $(-1, 1)$ given by

$$f : \mathbb{R} \to (-1, 1) \subset [-1, 1],$$

with

$$x \longmapsto \frac{x}{1 + |x|}.$$

Since $f(\mathbb{R}) = (-1, 1)$ and $\overline{(-1, 1)} = [-1, 1]$, then the pair $([-1, 1], f)$ is a compactification of $\mathbb{R}$.

We now give an overview of commutative rings and their minimal prime ideals.

1.1.16 Definition. For a ring R let $Spec(R), Min(R)$ and $Max(R)$ be the sets:

- $Spec(R) = \{P \subseteq R :$ P is a prime ideal of R$\}$;
- $Min(R) = \{P \subseteq R :$ P is a minimal prime ideal of R$\}$; and
- $Max(R) = \{P \subseteq R :$ P is a maximal prime ideal of R$\}$.

1.1.17 Definition. An element, $x \in R$, $x \neq 0$, is called *nilpotent* if for a positive integer n, $x^n = 0$.

1.1.18 Definition. We call an ideal *nilradical* if it consists of nilpotent elements and we denote it by $n(R)$ [12].

1.1.19 Definition. ([8, p.112]). We define an *annihilator* of a subset S of R by

$$Ann(S) = \{a \in R : as = 0 \text{ for } s \in S\}.$$

1.1.20 Definition. A ring R is called *reduced* if it has no nonzero nilpotent elements [12].

1.1.21 Definition. An ideal $I \subseteq R$ is called *dense* if $Ann(I) = 0$ [12].

1.1.22 Definition. We call an element of a ring R *regular* if it is not a zero divisor. The idea of regular elements generalises the notion of regular rings (that is rings whose elements are regular) [12].

1.1.23 Definition. Let R be a ring and $S \subseteq R$. Then, S is a *multiplicatively closed set* if:

(1) $1 \in S$, and

(2) for $x, y \in S$, $xy \in S$, that is to say S is closed under multiplication.

The minimal prime ideals of R and the minimal prime ideals of an arbitrary ideal I are of special interest [11], and in the next theorem we give the usefulness of these minimal prime ideals.

1.1.24 Theorem. *([11, Theorem 2.1]). Let $P \supseteq I$ be two ideals of a ring R, where P is a prime ideal. Then, the following statements are equivalent:*

(1) P is a minimal prime ideal of I;

(2) $R \setminus P$ is a multiplicatively closed set that is maximal with respect to missing I;

(3) For each $x \in P$ there is a $y \notin P$ and a nonnegative integer i such that $yx^i \in I$.

Proof. $(1) \Rightarrow (2)$. Expand $R \setminus P$ to a multiplicatively closed set S that is maximal with respect to missing I. If Q is an ideal containing I that is maximal with respect to being disjoint from S, then, Q is prime. Note that Q is disjoint from $R \setminus P$, which implies that $Q = P$. Thus $S = R \setminus P$.

$(2) \Rightarrow (3)$. Choose a nonzero $x \in P$ and let $S = \{yx^i : y \in R \setminus P, i = 0, 1, 2, \dots\}$. Then S is a multiplicatively closed set that properly contains $R \setminus P$. So there is some $y \in R \setminus P$ and a nonnegative integer i such that $yx^i \in I$.

$(3) \Rightarrow (1)$. Assume that $I \subset Q \subseteq P$, where Q is a prime ideal. If there exists some $x \in P \setminus Q$, then, there is a $y \notin P$ and a positive integer i such that $yx^i \in I \subset Q$, a contradiction. Therefore, $P = Q$. □

The following lemma provides the effective criterion for determining when a prime ideal is minimal.

1.1.25 Lemma. ([8, Lemma 1.1]). Let R be a ring and $P \in Spec(R)$. Then $P \in Min(R)$ if and only if for each $x \in P$ there exist an $r \in R \setminus P$ such that $xr \in n(R)$.

1.1.26 Corollary. ([11, Corollary 2.2]). A prime ideal P in a reduced ring R is minimal if and only if for each $x \in P$ there exists $y \in R \setminus P$ such that $xy = 0$.

1.1.27 Corollary. ([8, Corollary 1.2]). Let R be a ring. Every member of the minimal prime ideal $P \subseteq R$ is a zero divisor.

1.1.28 Lemma. ([11, Corollary 2.3]). Let R be a reduced ring and $P \in Min(R)$. For a finitely generated ideal $I \subseteq R$, $I \subseteq P$ if and only if $Ann_R(I) \nsubseteq P$.

1.1.29 Definition. ([8, $p.111$]). Let $S \subseteq R$, then we denote by

$$h(S) = \{P \in Min(R) : S \subseteq P\}$$

the *hull* of a set S. Similarly we define the hull of an element $a \in R$ by

$$h(a) = \{P \in Min(R) : a \in P\}.$$

The *kernel* of a subset $\mathcal{W} \subseteq Min(R)$ is defined to be

$$Ker(\mathcal{W}) = \bigcap\{P : P \subseteq \mathcal{W}\}.$$

Throughout we use h to denote the hull of a set.

1.1.30 Remark. It is worth noting that the family $\{h(a) : a \in R\}$ of sets, forms a base for the closed sets. We will encounter this notion of hulls when we study the Zariski topology.

We end the section by giving definitions connecting finitely generated ideals to annihilators. This notion of annihilators is central to showing the level of disconnectedness of $Min(R)$. This idea will be made manifest in the next section regarding the Zariski topology.

1.1.31 Definition. ([12, Definition 2.3]). A ring R is said to satisfy *Property A* if whenever $I \subseteq R$ is a finitely generated ideal consisting of zero divisors, then $Ann(I) \neq 0$. Notice that this implies that any finitely generated ideal is dense if and only if it is regular. It is known that R satisfies Property A precisely when $q(R)$ satisfies Property A, and by $q(R)$ we mean the classical ring of quotients of R.

1.2 The Zariski topology on $Min(R)$

In this section we begin by showing that $Min(R)$ with respect to the Zariski topology is a Hausdorff space. We look at hulls and annihilators of elements in R, with the intention of showing that $Min(R)$

is compact. In conclusion we give conditions needed for $Min(R)$ to be basically and extremally disconnected. The main paper consulted regarding the Zariski topology on $Min(R)$ is [8]. The details entailed in this section are extracted from [8, 10, 12].

We recall from [12] the definition of the Zariski topology on $Min(R)$.

1.2.1 Definition. For an ideal $I \subseteq R$ we let

$$U(I) = \{P \in Min(R) : I \nsubseteq P\},$$

and

$$V(I) = \{P \in Min(R) : I \subseteq P\}.$$

However when $I = aR$ for some $a \in R$, we instead write $U(a)$, and note that $U(a) = \{P \in Min(R) : a \notin P\}$. The set-theoretic complement of $U(I)$ (respectively $U(a)$) will be denoted by $V(I)$ (respectively $V(a)$). Observe that $U(a) \cap U(b) = U(ab)$, and so the collection $\{U(a) : a \in R\}$ forms a base for the topology known as Zariski topology.

1.2.2 Lemma. ([4, Lemma 1.3]). Let R be a reduced ring, $a \in R$, and $I, J \subseteq R$ be finitely generated ideals.

Then:

(1) $V(I) \cup V(J) = V(IJ)$;

(2) $V(I) \cap V(J) = V(I + J)$;

(3) $V(a) = Min(R)$ if and only if $a = 0$;

(4) $V(a) = \emptyset$ if and only if a is not a zero divisor;

(5) $V(I) = \emptyset$ if and only if I is a dense ideal.

Proof. (1) For any two ideals I and J, the product IJ is the ideal generated by products xy where $x \in I$ and $y \in J$. Note that $IJ \subseteq I$ and $IJ \subseteq J$, therefore, $V(I) \cup V(J) \subseteq V(IJ)$. Note also that $V(IJ) \subseteq V(I) \cup V(J)$, since, if P is a prime ideal containing IJ, P contains either I or J (otherwise, you would have an element $xy \in IJ \subseteq P$ with $x \notin P$ and $y \notin p$).

Thus, $V(I) \cup V(J) = V(IJ)$.

(2) For any collection of ideals $I,\ J$, there is a smallest ideal, denoted by $I + J$ containing $I,\ J$. Since, $I + J$ contains each $I,\ J$, it follows that $V(I + J) \subseteq V(I) \cap V(J)$. On the other hand, if P is a prime ideal containing I and J, then P must contain $I + J$ since, $I + J$ is the smallest ideal containing I and J. So $V(I) \cap V(J) \subseteq V(I + J)$.

Therefore, $V(I + J) = V(I) \cap V(J)$.

(3) Since 0 is in every ideal, $V(0) = Min(R)$.

(4) By definition, no prime ideal contains 1, so $V(1) = \emptyset$.

(5) Follows directly from (4) and Definition 1.1.31.

□

1.2.3 Theorem. *([8, Theorem 2.1]). Let $I \subseteq R$ be an ideal which is not maximal. Then the map $\tau : h(I) \longrightarrow Min(R/I)$*

defined by:

$$P \longmapsto P/I,\ P \in h(I)$$

is a homeomorphism of $h(I) \subseteq Min(R)$ onto a subspace of $Min(R/I)$.

1.2.4 Remark. In general the above assumption is not true, that is, $\tau[h(I)]$ is not all of $Min(R/I)$, neither is it dense in $Min(R/I)$. So if I is a maximal, nonminimal, prime ideal in R then $h(I)$ is empty and $Min(R/I)$ consists of one point. However we can make some contentions particularly when $I = n(R)$, that $\tau[h(I)] = Min(R/I)$.

We know with homeomorphisms interesting characterisations can be made between two topological spaces and many topological properties are preserved. The following theorem establishes the notion of homeomorphisms between the spaces $Min(R)$ and $Min(R/n(R)$.

1.2.5 Theorem. *([8, Theorem 2.2]). $Min(R)$ and $Min(R/n(R))$ are homeomorphic.*

1.2.6 Remark. By virtue of the above theorem, no generality will be lost in studying topological properties of $Min(R)$ by imposing on R the condition that R be a reduced ring.

1.2.7 Theorem. *([8, Theorem 2.3]). Let R be a reduced ring. Then, $h(Ann(a)) = Min(R) \setminus h(a)$, for $a \in R$. In particular $h(a)$ and $h(Ann(a))$ are disjoint clopen sets.*

Proof. If $P \in h(a)$, then, according to Lemma 1.1.25, $Ann(a)$ is not contained in P, that is $P \notin h(Ann(a))$. Therefore, $h(a) \cap h(Ann(a)) = \emptyset$. On the other hand, if $P \in Min(R) \setminus h(a)$, then for any $x \in Ann(a)$, we have $ax = 0 \in P$. Since $a \notin P$ and P is prime, $x \in P$. Thus, $Ann(a) \subseteq P$, that is, $P \in h(Ann(a))$. Therefore, $h(Ann(a)) = Min(R) \setminus h(a)$. Both sets $h(a)$ and $h(Ann(a))$ are closed, and since they are complementary, they are also open. □

The following corollary asserts that the topology on $Min(R)$ is Hausdorff.

1.2.8 Corollary. ([8, Corollary 2.4]). $Min(R)$ is a Hausdorff space with a base of clopen sets.

Proof. Given $P \neq P' \in Min(R)$, let $a \in P \setminus P'$. Then, by Theorem 1.2.7, $h(a)$ and $h(Ann(a))$ are disjoint open sets containing P and P' respectively. Hence $Min(R)$ is a Hausdorff space. By Remark 1.1.30, the family $\{h(a) : a \in R\}$ forms a base for the closed sets, so that $\{h(Ann(a)) : a \in R\}$ forms also a base for the open sets. And each member of the latter base is closed. □

1.2.9 Remark. Although the family $\{h(a) : a \in R\}$ is a base for the closed sets in $Min(R)$ and each member is also open, however this family in general is not a base for the open sets.

1.2.10 Corollary. ([8, Corollary 2.5]). An element in a reduced ring belongs to some minimal prime ideal if and only if it is a divisor of zero.

We turn now to consider those class of ideals I for which the mapping τ of Theorem 1.2.3 is onto all of $Min(R/I)$. These are the ideals $Ann(a)$, for $a \in R$. We begin by developing some properties of ideals that are annihilators of subsets of R [8].

1.2.11 Lemma. ([8, Lemma 2.6]). Let R be a reduced ring. For any $S \subseteq R$, the residue class ring $R/Ann(S)$ has no nonzero nilpotent elements.

1.2.12 Theorem. *([8, Theorem 2.7]). Let R be a reduced ring. For any* $S \subseteq R$*,* $Ann(S) = Ker(h(Ann(S)))$*.*

The properties in the Lemma 1.2.13 below provides a connection between the hulls and annihilators of elements of a reduced ring R, as seen in Theorem 1.2.7.

1.2.13 Lemma. ([8, Lemma 3.1]). For all $x, y, z \in R$:

(i) $h(x) = h(Ann(Ann(x)))$;

(ii) $Ann(Ann(xy)) = Ann(Ann(x)) \cap Ann(Ann(y))$;

(iii) $Ann(z) = Ann(x) \cap Ann(y)$ if and only if $h(z) = h(x) \cap h(y)$;

(iv) $Ann(Ann(x)) = Ann(y)$ if and only if $h(x) = h(Ann(y))$.

The usual way of ensuring compactness of a space of ideals is with the presence of unity. In the case of minimal prime ideals, the above assertion does not hold, as we will see that having unity says nothing about compactness of $Min(R)$. The appropriate condition that ensures compactness of the space of minimal prime ideals is the presence of complements via annihilators, that is, $Ann(Ann(x')) = Ann(x)$. We will see that this condition is actually sufficient and necessary for compactness of $Min(R)$. But before we do that, we will give the following definition.

1.2.14 Definition. ([8, Definition 3.2]). A ring R satisfies the *annihilator condition* $(a.c)$ if for any $a, b \in R$ there is a $c \in R$ such that $Ann(a, b) = Ann(c)$. We call rings satisfying the annihilator condition property *a.c rings*.

1.2.15 Theorem. *([8, Theorem 3.4]). Let R be a reduced ring. Then the following conditions are equivalent:*

(1) $Min(R)$ is compact and R satisfies the annihilator condition;

(2) $Min(R)$ is compact and the family of sets $\{h(a) : a \in R\}$ is a base for the open sets of $Min(R)$;

(3) For each $x \in R$ there exists $x' \in R$ such that $Ann(Ann(x')) = Ann(x)$.

1.2.16 Remark. From (3) of Theorem 1.2.15 we regard the element x' as a (non unique) complement of x. Now, since $x' \in Ann(Ann(x')) = Ann(x)$, it follows that $xx' = 0$. Furthermore, $Ann(x') = Ann(Ann(Ann(x'))) = Ann(Ann(x))$, then x becomes a candidate for $(x')'$. The duality between the two elements x and x' can be be seen more clearly by characterising x': $xx' = 0$ and

$$Ann(x) \cap Ann(x') = (0).$$

Theorem 1.2.15 will be proved by means of a sequence of lemmas, and the first of which is a strengthened form of the statement (1) implies (3).

1.2.17 Lemma. ([8, Lemma 3.5]). Let R be a ring with the annihilator condition. If $x \in R$ and $h(x)$ is compact, then, there exist $x' \in Ann(x)$ such that $Ann(Ann(x')) = Ann(x)$.

The following lemma establishes a connection between (2) and (3) of Theorem 1.2.15.

1.2.18 Lemma. ([8, Lemma 3.6]). Suppose $\{h(a) : a \in R\}$ is a base for the open sets in $Min(R)$, and let $x \in R$ be such that $h(Ann(x))$ is compact. Then, there exists $x' \in Ann(x)$ such that $Ann(Ann(x')) = Ann(x)$.

1.2.19 Corollary. ([8, Corollary 3.7]). Under conditions (1) or (2) of Theorem 1.2.15, the ring R contains a nondivisor of zero.

1.2.20 Lemma. ([8, Lemma 3.8]). Let R satisfy condition (3) of Theorem 1.2.15. An ideal I of R is contained in some minimal prime ideal of R if and only if every member of I is a divisor of 0. In particular, a prime ideal in R is minimal if and only if each of its members is a divisor of 0.

1.2.21 Definition. ([8, Definition 4.1]). A reduced ring R satisfies the *countable annihilator condition* if for each sequence $\{x_n\} \in R$, there is an $x \in R$ such that $Ann(x) = \bigcap_{n=1}^{\infty} Ann(x_n)$. We call a topological space X *extremally disconnected* if the closure of every open set in X is open.

1.2.22 Definition. A topological space X is said to be *basically disconnected* if the closure of the complement of each zero-set (by zero-set we mean $\{x \in X : f(x) = 0\}$ for a continuous function f on X) is open in X [8].

1.2.23 Corollary. ([8, Corollary 3.9]). If R satisfies the annihilator condition and $Min(R)$ is compact. Then, every zero-set in $Min(R)$ is a countable intersection of hulls of members of R.

1.2.24 Lemma. ([8, Lemma 4.2]). Let R be a reduced ring. If $B \subseteq R$ and $\mathfrak{L} = \bigcup\{h(Ann(b)) : b \in B\}$ then, $\overline{\mathfrak{L}} = h(Ann(B))$.

The next lemma answers the question of when $Min(R)$ is extremally disconnected.

1.2.25 Lemma. ([8, Lemma 4.3]). Let R be a reduced ring. $Min(R)$ is extremally disconnected if and only if $h(Ann(B))$ is open for each $B \subseteq R$.

Proof. Every open subset of $Min(R)$ is a union of hulls of annihilators of elements of R, that is, it is of the form $\mathfrak{L}$ of the preceding lemma. □

The following result gives a condition of when $Min(R)$ is basically disconnected.

1.2.26 Theorem. *([8, Theorem 4.5]). If R satisfies the countable annihilator condition and $Min(R)$ is compact. Then, $Min(R)$ is basically disconnected.*

2. On the inverse topology of $Min(R)$

In the previous chapter we described the Zariski topology on $Min(R)$ and showed that $Min(R)$ endowed with the Zariski topology is compact and Hausdorff (see Corollary 1.2.8 and Theorem 1.2.15). We end the chapter by showing that the Zariski topology on $Min(R)$ is basically and extremally disconnected (see Lemma 1.2.25 and Theorem 1.2.26) given some conditions. In this chapter we study the inverse topology on $Min(R)$ and prove some topological properties of the inverse topology. We use $(Min(R), Min(R)^{-1})$ to denote a topological space, with $Min(R)^{-1}$ denoting the inverse topology. Note that we will use $Min(R)^{-1}$ to denote both the topological space and the inverse topology.

2.1 The inverse topology on $Min(R)$

Our major task in this section lies in showing that the inverse topology on $Min(R)$ is always a compact T_1-space. We recall that by T_1-space we mean that whenever $A, B \in Min(R)$ distinct, there exist open sets $U, V \in Min(R)^{-1}$, such that $A \subseteq U$, $B \nsubseteq U$ and $B \subseteq V$, $A \nsubseteq V$. Some of the results and definitions used in this section are taken from [1, 4, 11, 12, 15].

2.1.1 Base Topology. Since we know that from Lemma 1.2.2, for $I, J \subseteq R$, $V(I) \cup V(J) = V(IJ)$ and $V(I) \cap V(J) = V(I+J)$. Then the collection

$$\{V(I) \ : \ I \subseteq R \text{ is a finitely generated ideal}\},$$

forms a base for the topology on $Min(R)$ (with the collection $\{V(a) : a \in R\}$ forming a subbase), known as the inverse topology (or cotopology on a structure space).

2.1.2 Lemma. ([16, Lemma 2.1]). The Zariski topology is always finer than the inverse topology on $Min(R)$.

The following theorem shows that the inverse topology on $Min(R)$ is always a compact T_1-space.

2.1.3 Theorem. *([12, Theorem 3.1]). For any commutative ring, the inverse topology on $Min(R)$ is always compact and T_1.*

Proof. We divide the proof of the theorem into two parts;

1. We first show that $Min(R)^{-1}$ is a T_1-space.

Let $P, Q \in Min(R)$ be distinct minimal prime ideals, and let $a \in P \setminus Q$. By Lemma 1.1.25, there is an $x \notin P$, such that, $ax \in n(R)$. It follows that $ax \in Q$, and so $x \in Q \setminus P$. Notice that $P \in V(a) \setminus V(x)$, and $Q \in V(x)$, so $V(x) \nsubseteq V(a)$. Hence, the inverse topology satisfies the T_1-separation axiom.

2. We next show that $Min(R)^{-1}$ is compact.

Let $\mathcal{S} = \{V(a) : a \in S\}$ be an arbitrary collection of subbasic open sets with the property that no finite union of elements covers $Min(R)$. We show that $\mathcal{S}$ is not an open cover of $Min(R)$. Since,

$V(a) \cup V(b) = V(ab)$, and $V(a) = Min(R)$, precisely when $a \in n(R)$, it follows that for any finite subset $\{a_1, \ldots, a_n\} \subseteq S$,

$$a_1 a_2 \cdots a_n \notin n(R).$$

Let $\hat{S}$ be totality of all finite products of elements belonging to S, and observe that $\hat{S}$ is a multiplicative system. The usual Zorn's Lemma argument yields the existence of some prime ideal, say P, for which $P \cap \hat{S} = \emptyset$. Without loss of generality, we assume that $P \in Min(R)$. The fact that P and S are disjoint implies that for all $a \in S$, $P \notin V(a)$. Therefore, the collection $\mathcal{S}$ is not an open cover of $Min(R)$. □

2.1.4 Example. Consider any field $\mathbb{F}$ it can be $\mathbb{R}, \mathbb{C},$ or $\mathbb{Q}$ for an example. Since a field is a commutative ring, we have that $Min(\mathbb{F})^{-1}$ is always a compact T_1-space.

The following proposition characterises when the two topologies (Zariski and Inverse topology) are equivalent. In the proof the set $V(a)$ for $a \in R$ denotes the basic open set of the Zariski topology on $Min(R)$.

2.1.5 Proposition. ([12, Proposition 3.2]). Let R be a reduced ring. Then following statements are equivalent:

(1) $Min(R) = Min(R)^{-1}$;

(2) $Min(R)$ is compact;

(3) For every $a \in R$, there is $I \leq Ann(a)$ (sub-ideal of $Ann(a)$) finitely generated ideal, such that $Ann(aR + I) = 0$.

Proof. $(1) \Rightarrow (2)$. Follows from the fact that $Min(R)^{-1}$ is compact, hence $Min(R)$ is also compact.

$(2) \Rightarrow (3)$. Let $a \in R$. Since $(Ann(a), a)$ contains no zero divisors, by Lemma 1.1.25 we have that $(Ann(a), a) \nsubseteq P$, where P is any minimal prime ideal and $(Ann(a), a)$ an ideal generated by $Ann(a)$ and a. Let

$$Min(R) = V(a) \bigcup \Big(\bigcup \{V(b) : b \in Ann(a)\} \Big),$$

be an open cover for $Min(R)$. Since $Min(R)$ is compact, there exist $b_i \in Ann(a)$, for $i = 1, \ldots n$, such that,

$$Min(R) = V(a) \cup V(b_1) \cup \cdots \cup V(b_n).$$

Let $I = (b_1, \ldots, b_n)$, and $I \leq Ann(a)$ be a finitely generated ideal, then, $Ann(aR + I) = 0$.

$(3) \Rightarrow (1)$. Suppose (3) holds. Since the Zariski topology is finer than the inverse topology, it suffices to show that any (Zariski) basic open set, $U(a)$ is of the form $V(I)$ for some finitely generated ideal of R. Given $a \in R$ let I be the ideal of (3). Since $Ann(a) \subseteq I$, we have $U(a) \subseteq V(I)$. Conversely, let $P \in V(I)$ and suppose that $P \notin U(a)$. Then, $aR + I \subseteq P$. Since $Ann(aR + I) = 0$, we have a contradiction to Lemma 1.1.28 which proves the result. □

Observe that if any of the equivalent conditions of Proposition 2.1.5 are satisfied, we have that $Min(R)^{-1}$ is a boolean space (compact, zero-dimensional and Hausdorff space) [12]. Furthermore we would like to know if the assertion made above holds without $Min(R)$ being compact, and also if we can characterise other topological properties possessed by the inverse topology on $Min(R)$. To do this, we need the following definitions.

2.1.6 Definition. We call a ring R *complemented* if whenever $a \in R$ there is a $b \in R$ such that $ab = 0$ and $a + b$ is a regular element [12].

2.1.7 Definition. ([12, Definition 2.6]). A ring R is *weakly complemented* if for $a, b \in R$, $ab = 0$, there exist finitely generated ideals $I, J \subseteq R$, such that $a \in I$, $b \in J$, $IJ = 0$, and $I + J$ is regular, that is, $I + J$ contains a regular element.

2.1.8 Definition. ([12, Definition 2.6]). We call a ring R *quasi-complemented* if whenever $a, b \in R$ and $ab = 0$, then, there exist finitely generated ideals $I, J \subseteq R$, such that $a \in I$, $b \in J$, $IJ = 0$, and $I + J$ is a dense ideal of R. Quasi-complemented rings are reduced rings, and they also generalise complemented rings.

Before we formally state the theorem which answers the above question. We begin with a lemma which characterises the clopen subsets of the inverse topology (assuming that R is reduced).

2.1.9 Lemma. ([12, Lemma 3.3]). Suppose R is a reduced ring and let $K \subseteq Min(R)^{-1}$. Then K is clopen under $Min(R)^{-1}$ if and only if there exist finitely generated ideals $I, J \subseteq R$, such that $V(I) = K$, $IJ = 0$, and $I + J$ is a dense ideal of R.

2.2 Inverse Hausdorff rings

In this section we concern ourselves with those properties imposed on the ring R to be an inverse Hausdorff ring. The property of the ring being inverse Hausdorff forms a base for our main characterisation of when $Min(R)^{-1}$ is Hausdorff. The definitions and results used in this section are taken from [1, 4, 12, 14, 19].

2.2.1 Definition. A ring R is called an *inverse Hausdorff* ring if $Min(R)^{-1}$ is a Hausdorff space [4].

2.2.2 Definition. ([12, Definition 2.10]). We define a ring R to be a *weak Baer ring* if for each $a \in R$, $Ann(a)$ is generated by an idempotent. A ring R is called *p.p ring* if every finitely generated ideal of R is a projective module.

2.2.3 Definition. A ring R is called a PF-ring if R satisfies the following equivalent conditions [12]:

(i) R is locally a domain [12];

(ii) Whenever $a, b \in R$ such that $ab = 0$, then, $Ann(a) + Ann(b) = R$;

(iii) For each $a \in R$, $Ann(a)$ is a pure ideal (an ideal $I \subseteq R$ is *pure*, if for every $x \in I$, there exists $y \in I$ such that $xy = x$).

In our next result we show that condition (ii) above, can be modified so as to give a necessary and sufficient condition on R so that $q(R)$ is a PF-ring[12]. We recall Definition 1.1.31 for $q(R)$.

2.2.4 Proposition. ([12, Proposition 2.11]). Let R be a ring. Then, $q(R)$ is a PF-ring if and only if given $a, b \in R$ such that $ab = 0$, we have $Ann_R(a) + Ann_R(b)$ is a regular ideal of R.

Proof. Observe that for any $\frac{a}{s}, \frac{b}{t} \in q(R)$, we have that, $\frac{a}{s}\frac{b}{t} = 0$ precisely when $ab = 0$. Furthermore, $Ann_{q(R)}(\frac{a}{s}) = Ann_{q(R)}(a)$ and $Ann_{q(R)}(a) \cap R = Ann_R(a)$. Therefore, $Ann_{q(R)}(\frac{a}{s}) + Ann_{q(R)}(\frac{b}{t}) = q(R)$ exactly when $Ann_R(a) + Ann_R(b)$ contains a regular element. □

2.2.5 Definition. ([12, Definition 2.12]). Our definition of weakly complemented leads us to define a *feebly Baer ring* as a ring R for which whenever $a, b \in R$ and $ab = 0$, there is an idempotent $e = e^2 \in R$, such that $a \in eR$ and $b \in (1 - e)R$. Clearly, a feebly Baer ring is weakly complemented, and hence reduced. One well known assertion concerning these types of rings is that a ring R is weak Baer if and only if it is a pp-ring. Al-Ezeh [1] called a ring R an *almost p.p. ring* if for every $a \in R$, $Ann(a)$ is generated by idempotents. Since an ideal which is generated by idempotents is a pure ideal, it follows that an almost p.p. ring is a PF-ring.

2.2.6 Lemma. ([12, Lemma 2.7]). If R is complemented, then, R is weakly complemented. If R is weakly complemented, then, R is quasi-complemented.

Proof. Suppose R is complemented and $ab = 0$. Choose $x \in R$ such that $ax = 0$ and $a + x$ is regular. Let $I = aR$ and $J = bR + xR$. Clearly, $IJ = 0$. Since $a + x \in I + J$, it follows that R is weakly complemented.

Suppose R is weakly complemented, then, by definition, for $a, b \in R$ such that $ab = 0$, there exist finitely generated ideals $I, J \subseteq R$, such that $a \in I$, $b \in J$, $IJ = 0$, and $I + J$ is a regular ideal. Since a regular ideal is dense we have that R is quasi-complemented. □

For use in the next result and for $P \in Spec(R)$, we define $\mathcal{O}(P)$ to be the intersection of all prime ideals contained in P, equivalently,

$$\mathcal{O}(P) = \{x \in P : xy = 0, \text{ for } y \in R \setminus P\}.$$

2.2.7 Lemma. ([14, Lemma, $p.160$]). Let R be a reduced ring. Then following statements are equivalent:

(1) Every prime ideal contains a unique minimal prime ideal;

(2) If $a, b \in R$ with $ab = 0$, then $Ann(a) + Ann(b) = R$;

(3) For every $a, b \in R$, $Ann(a) + Ann(b) = Ann(ab)$.

In the lemma below $\pi(R)$ denotes a subspace of $Min(R)$.

2.2.8 Theorem. *([14, Theorem, $p.160$]). Let R be a reduced ring. Then, the following statements are equivalent:*

(1) *R is a weakly Baer ring;*

(2) *Every prime ideal of R contains a unique minimal prime ideal and $\pi(R)$ is a compact space;*

(3) *$\pi(R)$ is a retract of $Spec(R)$, that is there exists a continuous function f of $Spec(R)$ onto $\pi(R)$, which is the identity on $\pi(R)$.*

2.3 Characterisations of the inverse topology

It is well known that the Zariski topology is both Hausdorff and zero-dimensional. Unlike the inverse topology on $Min(R)$, this is not always true. So we devote our attention in this section to giving a vivid characterisation of when $Min(R)^{-1}$ is Hausdorff. Some of the definitions and results we shall use in the section are taken from [4, 12].

We now give a characterisation of when $Min(R)^{-1}$ is zero-dimensional by using previously defined concepts.

2.3.1 Theorem. *([12, Theorem 3.4]). Suppose R is a reduced ring. $Min(R)^{-1}$ is a boolean space if and only if R is a quasi-complemented ring. In particular, a quasi-complemented ring is inverse Hausdorff.*

Proof. First suppose that R is quasi-complemented. To show that $Min(R)^{-1}$ has a base of clopen sets it is sufficient to show that given $P \in V(a)$ there is some clopen subset $K \subseteq Min(R)^{-1}$ for which $P \in K \subseteq V(a)$ (since the collection $\{V(a) : a \in R\}$ forms a subbase for the inverse topology). For $P \in V(a)$ we have $a \in P$. Since P is a minimal prime and R is reduced, by Corollary 1.1.26, there is some $b \in R \setminus P$, such that $ab = 0$. Since R is quasi-complemented there are finitely generated ideals $I, J \subseteq R$, such that $a \in I$, $b \in J$, $IJ = 0$ and $I + J$ is a dense ideal. By Lemma 2.1.9, we gather that $V(I)$ is a clopen subset of $Min(R)^{-1}$, and since $a \in I$, it follows that $V(I) \subseteq V(a)$. All that is left to show is that $P \in V(I)$. To that end, notice that if $P \notin V(I)$, then $P \in V(J)$, that is, $J \subseteq P$. But then $b \in P$, a contradiction. This forces $P \in V(I)$, and so the space has the desired properties.

Conversely, suppose that $Min(R)^{-1}$ is zero-dimensional and let $a, b \in R$ satisfy $ab = 0$. Since $Min(R)^{-1}$ is T_1 and zero-dimensional, it is a Hausdorff space. This means that disjoint closed subsets of $Min(R)^{-1}$ can be separated by a clopen set. Thus, since $U(a) \cap U(b) = U(ab) = U(0) = \emptyset$, and both $U(a)$ and $U(b)$ are disjoint closed subsets of $Min(R)^{-1}$, there is some clopen subset $K \subseteq Min(R)^{-1}$ such that $U(a) \subseteq K$ and $K \cap U(b) = \emptyset$. By Lemma 2.1.9, $K = V(J)$ and $Min(R) \setminus K = V(I)$ for some finitely generated ideals $I, J \subseteq R$. Next, $V(I) \subseteq V(a)$ and $V(J) \subseteq V(b)$. A quick check gives us that $V(I) = V(I + aR)$ and $V(J) = V(J + bR)$, so without loss of generality we assume that $a \in I$, $b \in J$. Also since $K \subseteq Min(R)^{-1}$ is clopen by Lemma 2.1.9, it follows that $IJ = 0$ and $I + J$ is a dense ideal of R.

□

Any $P \in Min(R)$ has the property that it does not contain any finitely generated dense ideals. We are interested in these prime ideals mainly because of the influence and role it has in our subsequent theorem. Now let

$$Spec_d(R) = \{P \in Spec(R) : P \text{ does not contain any dense finitley generated ideal}\}, \text{ and}$$

$$Spec_r(R) = \{P \in Spec(R) : P \text{ does not contain any nonzero divisor}\}.$$

2.3.2 Remark. Notice that $Min(R)$ is contained in both sets defined above, and also by applying Zorn's lemma we see that $Spec_d(R)$ and $Spec_r(R)$ each have maximal elements. We denote the set of all maximal elements of $Spec_d(R)$ and $Spec_r(R)$ by $Max_d(R)$ and $Max_r(R)$, respectively.

Before stating our main theorem we take a detour and state a lemma which we shall make use of in our main theorem.

2.3.3 Lemma. ([4, Lemma 2.5]). Suppose K is an ideal of R that does not contain a dense finitely generated ideal. Then, there exists a $P \in Spec_d(R)$, such that $K \subseteq P$. Furthermore, if K does not contain any nonzero divisor, then, there exists a $P \in Spec_r(R)$, such that $K \subseteq P$.

We call an ideal $I \subseteq R$ a semiprime ideal (or radical) if whenever $x^n \in I$, for some $n \in \mathbb{N}$, then $x \in I$. For an ideal I, its radical is denoted by

$$\sqrt{I} = \{x \in R : x^n \in I\}.$$

We can use Zorn's lemma to show that an ideal, say, $I \subseteq R$ is radical if and only if it is an intersection of prime ideals containing it. By a finitely generated radical ideal we mean a radical ideal I for which there is a finite set, say $\{a_1, \dots, a_n\}$, for which $I = \sqrt{a_1R + \dots + a_nR}$.

2.3.4 Remark. We make two observations: Let $I, J \subseteq R$.

(1) If $IJ = 0$ then, $\sqrt{I}\sqrt{J} = 0$.

(2) I is a dense ideal if and only if $\sqrt{I}$ is a dense ideal.

We will recall the definition of localization which we shall need in the theorem below.

We use the concept of a multiplicatively closed set (see Definition 1.1.23) to define localization. Let R be a ring and P be a prime ideal of R. Then the set defined by:

$$R_P = \left\{ \frac{a}{b} : a \in R \text{ and } b \in R \setminus P \right\}$$

with an equivalence relation, defined by $\frac{a}{b} = \frac{c}{d}$, is a *localization* of R at P if there is a $t \in R$ such that $t(ad - bc) = 0$.

Now we state the main theorem of this section. The theorem is a characterisation of when $Min(R)^{-1}$ is Hausdorff.

2.3.5 Theorem. *([4, Theorem 2.6]). Suppose R is a reduced ring. The following statements are equivalent:*

(1) $Min(R)^{-1}$ *is a Hausdorff space;*

(2) $Min(R)^{-1}$ *is a normal space;*

(3) *Whenever I, J are finitely generated ideals of R for which $IJ = 0$, there exist finitely generated ideals $I', J' \subseteq R$, such that $IJ' = 0 = I'J$, while $I' + J'$ is a dense ideal;*

(4) *Whenever S, T are finitely generated radical ideals of R for which $ST = 0$, there exist finitely generated radical ideals $S', T' \subseteq R$, such that $ST' = 0 = S'T$, while $S' + T'$ is a dense ideal;*

(5) *Whenever $a, b \in R$ and $ab = 0$, there exist finitely generated ideals $I, J \subseteq R$, such that $a \in I \subseteq Ann(b)$, $b \in J \subseteq Ann(a)$, and $I + J$ is a dense finitely generated ideal;*

(6) *If $ab = 0$, then, $Ann(a) + Ann(b)$ contains a dense finitely generated ideal;*

(7) *For every $M \in Max_d(R)$, R_M is an integral domain;*

(8) *For every $P \in Spec_d(R)$, R_P is an integral domain;*

(9) *For every $P \in Spec_d(R)$, $\mathcal{O}(P)$ is a prime ideal;*

(10) *For every $M \in Max_d(R)$, $\mathcal{O}(M)$ is a prime ideal.*

Proof. (1) $\Rightarrow$ (2). Suppose $Min(R)^{-1}$ is Hausdorff. Then, since $Min(R)^{-1}$ is always a compact T_1-space, by Theorem 1.1.13, it follows that $Min(R)^{-1}$ is a normal space.

(2) $\Rightarrow$ (3). Suppose I and J are finitely generated ideals for which $IJ = 0$. Then, $U(I)$ and $U(J)$ are disjoint closed subsets of $Min(R)^{-1}$. By normality it follows that there are disjoint open subsets, say $K_1, K_2 \subseteq Min(R)^{-1}$, such that $U(I) \subseteq K_1$ and $U(J) \subseteq K_2$. Since $Min(R)^{-1}$ is compact and T_1 , under the assumption of normality it is also Hausdorff. Therefore, K_1 and K_2 can be chosen to be basic open subsets of $Min(R)^{-1}$. Thus, there are finitely generated ideals $I', J' \subseteq R$, such that $K_1 = V(J')$ and $K_2 = V(I')$. It follows that $U(I) \cap U(J') = \emptyset = U(J) \cap U(I')$, whence $IJ' = 0 = JI'$. Finally, since $V(I + J) = V(I) \cap V(J) = \emptyset$, it follows that $I' + J'$ is a finitely generated ideal which

is not contained in any minimal prime ideal; Lemma 1.1.28 applies, and we conclude that $I' + J'$ is a dense ideal of R.

$(3) \Rightarrow (4)$. Suppose S and T are finitely generated radical ideals for which $ST = 0$. Let I and J be finitely generated ideals of R for which $S = \sqrt{I}$ and $T = \sqrt{J}$. Then, $IJ = 0$ and by (3) there are finitely generated ideals, say I' and J' , such that $IJ' = 0 = I'J$, while $I' + J'$ is a dense ideal. Set $S' = \sqrt{I'}$ and $T' = \sqrt{J'}$, finitely generated radical ideals. By the first observation in Remark 2.3.4 we conclude that $ST' = 0 = S'T$. Since $I' + J'$ is a dense ideal and $I' + J' \leq S' + T'$, we conclude that $S' + T'$ is a dense ideal.

$(4) \Rightarrow (3)$. Suppose I and J are finitely generated ideals of R for which $IJ = 0$. Set $S = \sqrt{I}$ and $T = \sqrt{J}$ finitely generated radical ideals. Then, $ST = 0$ and so by (4) there are finitely generated radical ideals S', T', such that $S'T = T'S = 0$, while $S' + T'$ is a dense ideal. Let I' and J' be finitely generated ideals of R for which $S' = \sqrt{I'}$ and $T' = \sqrt{J'}$. By the first observation in Remark 2.3.4 we conclude that $J'I = JI' = 0$. Finally by observation (2) of Remark 2.3.4 we conclude that $I' + J'$ is a dense ideal of R.

$(3) \Rightarrow (5)$. Suppose $a, b \in R$ for which $ab = 0$. By (3), there exist finitely generated ideals $I', J' \subseteq R$, such that $aJ' = bI' = 0$, with $I' + J'$ dense. Let $I = I' + aR$ and $J = J' + bR$, then, $I, J \subseteq R$ are also finitely generated ideals, thus, $a \in I \subseteq Ann(b)$ and $b \in J \subseteq Ann(a)$. Therefore, $I + J$ is a dense ideal.

$(5) \Rightarrow (1)$. Suppose $P, Q \in Min(R)$ are distinct minimal prime ideals of R. Choose $a \in P \setminus Q$. By Corollary 1.1.26 there is some $x \in R \setminus P$, such that $ax = 0$. It follows that $x \in Q$. By (5) there are finitely generated ideals I, J, such that $aI = xJ = 0$ and $I + J$ a dense ideal. Then, $Q \in V(I)$, $P \in V(J)$, and $V(I + J) = V(I) \cap V(J) = \emptyset$. Consequently $Min(R)^{-1}$ is a Hausdorff space.

At this point, we have shown that the first five conditions are equivalent.

$(3) \Rightarrow (6)$. Suppose $a, b \in R$ for which $ab = 0$. By (3), there exist finitely generated ideals $I', J' \subseteq R$, such that $aJ' = bI' = 0$ and $I' + J'$ is a dense ideal of R. Let $I = I' + aR$ and $J = J' + bR$, then, $I, J \subseteq R$ are also finitely generated ideals and $I + J$ is a dense ideal. Thus, $a \in I \subseteq Ann(b)$ and $b \in J \subseteq Ann(a)$.

$(6) \Rightarrow (2)$. Let $a, b \in R$ with $U(a) \cap U(b) = \emptyset$, which implies that $ab = 0$. By (6) there exist finitely generated ideals $I, J \subseteq R$, such that $I \subseteq Ann(b)$, $J \subseteq Ann(a)$, and $I + J$ a dense ideal of R. Since $I \subseteq Ann(b)$, it follows that $U(b) \subseteq V(I)$. Similarly, $U(a) \subseteq V(J)$. Finally, $I + J$ a dense ideal implies that $V(I + J) = V(I) \cap V(J) = \emptyset$. Therefore, any pair of disjoint subbasic closed sets are separated by disjoint open sets in the inverse topology. Consequently, $Min(R)^{-1}$ is a normal space.

At this point, we have shown that the first six conditions are equivalent.

$(6) \Rightarrow (7)$. Let $a, b \in R$ and $m_1, m_2 \notin M$ such that $\frac{a}{m_1}\frac{b}{m_2} = \frac{0}{1}$. There exists some $s \notin M$ such that $(sa)b = s(ab) = 0$. By (6), $Ann(sa) + Ann(b)$ contains a dense finitely generated ideal. Since $M \in Max_d(R)$, it follows that $Ann(sa) + Ann(b) \nsubseteq M$. So there exist $x_1 \in Ann(sa)$ and $x_2 \in Ann(b)$ such that $x_1 + x_2 \notin M$, which means either $x_1 \notin M$ or $x_2 \notin M$. If $x_1 \notin M$, then $x_1 s \notin M$ and $(x_1 s)a = 0$ implies that $\frac{a}{m_1} = \frac{0}{1}$. On the other hand if $x_2 \notin P$, then, $x_2 b = 0$, which implies that $\frac{b}{m_2} = \frac{0}{1}$. Therefore, R_P is an integral domain.

$(7) \Rightarrow (8)$. Since each $P \in Spec_d(R)$ is contained in some M, $M \in Max_d(R)$, and the localization R_P can be obtained as an appropriate localization of R_M, it follows that each R_P is an integral domain.

$(8) \Rightarrow (9)$. Suppose $a, b \in R$ with $ab \in \mathcal{O}(P)$, then, $ab \in P$ and there exists some $x \in R \setminus P$, such that $x(ab) = 0$. It follows that $\frac{a}{1}\frac{b}{1} = \frac{0}{1}$. Since R_P is an integral domain $\frac{a}{1} = \frac{0}{1}$ or $\frac{b}{1} = \frac{0}{1}$. Without loss of generality, suppose that $\frac{a}{1} = \frac{0}{1}$, then, there exists $z \in R \setminus P$, such that $za = 0$. Therefore, $a \in \mathcal{O}(P)$, proving that $\mathcal{O}(P)$ is a prime ideal.

$(9) \Rightarrow (10)$. Let $M \in Max_d(R)$. Then, $M \in Spec_d(R)$ since $Spec_d(R) \supseteq Max_d(R)$. Therefore, by (9), $\mathcal{O}(M)$ is a prime ideal.

$(10) \Rightarrow (6)$. Suppose that for each $M \in Max_d(R)$, $\mathcal{O}(M)$ is a prime ideal, and let $ab = 0$. Suppose, by means of contradiction, that $Ann(a) + Ann(b)$ does not contain a dense finitely generated ideal. By Lemma 2.3.3 there is $M \in Spec_d(R)$, such that $Ann(a) + Ann(b) \subseteq M$. Without any loss of generality, we can assume that $M \in Max_d(R)$. By hypothesis, either $a \in \mathcal{O}(M)$ or $b \in \mathcal{O}(M)$. Without loss of generality, we assume $a \in \mathcal{O}(M)$. This means there is an $x \notin M$, such that $ax = 0$. But then $x \in Ann(a) \subseteq Ann(a) + Ann(b) \subseteq M$, which is a contradiction. We are forced to conclude that $Ann(a) + Ann(b)$ contains a dense finitely generated ideal.

□

Observe that in a reduced ring, an ideal $I \subseteq R$ contains a zero divisor precisely when $\sqrt{I}$ contains a zero divisor.

2.3.6 Theorem. *([4, Theorem 2.7]). Suppose R is a reduced ring. The following statements are equivalent:*

(1) *$q(R)$ is a PF-ring;*

(2) *If $ab = 0$, then, there exist $x, y \in R$, such that $ay = 0 = bx$, while $x + y$ is a nonzero divisor;*

(3) *If $ab = 0$, then, $Ann_R(a) + Ann_R(b)$ contains a nonzero divisor;*

(4) *Whenever I, J are finitely generated ideals of R for which $IJ = 0$, there exist finitely generated ideals $I', J' \subseteq R$, such that $IJ' = 0 = I'J$, while $I' + J'$ contains a nonzero divisor;*

(5) *Whenever S, T are finitely generated radical ideals of R for which $ST = 0$, there exist finitely generated radical ideals $S', T' \subseteq R$, such that $ST' = 0 = S'T$, while $S' + T'$ contains a nonzero divisor;*

(6) *Whenever $a, b \in R$ and $ab = 0$, there exist finitely generated ideals $I, J \subseteq R$, such that $a \in I \subseteq Ann_R(b)$, $b \in J \subseteq Ann_R(a)$, and $I + J$ contains a nonzero divisor;*

(7) *For every $M \in Max_r(R)$, R_M is an integral domain;*

(8) *For every $P \in Spec_r(R)$, R_P is an integral domain;*

(9) *For every $P \in Spec_r(R)$, $\mathcal{O}(P)$ is a prime ideal;*

(10) *For every $M \in Max_r(R)$, $\mathcal{O}(M)$ is a prime ideal.*

3. The inverse topology on $Min(C(X))$

We saw in the previous chapter the role played by compact Hausdorff spaces in realising our main results in the chapter. In this chapter we specialise to the space of minimal prime ideals of $C(X)$, denoted by $Min(C(X))$, where $C(X)$ is the ring of real valued continuous functions on a Tychonoff space X. We then show that the inverse topology on $Min(C(X))$ is the Stone-Čech compactification of X if X is an F-space. All topological spaces in this chapter will be Tychonoff spaces unless stated otherwise.

3.1 The ring $C(X)$

In this section we study the ring $C(X)$ on a Tychonoff space X, and find topological conditions on X which characterise $C(X)$ as an inverse Hausdorff ring. But before we do that, we give some preliminary definitions. Some of the details in this section are taken from [1, 4, 6, 7, 8].

3.1.1 Definition. ([4, $p.106$]). A topological space X is called a *Tychonoff* space, if it is both completely regular and Hausdorff.

3.1.2 Remark. ([13, Definition A3.12]). We call a space X completely regular if for each closed set $F \subseteq X$ and $x \notin F$ there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 0$ and $f[F] = 1$.

3.1.3 Definition. For a Tychonoff space X, we define $C^*(X)$ to be the collection of all bounded continuous real-valued functions on X. It should be noted that $C^*(X)$ is a subring of $C(X)$ [4].

Notice that $C(X)$ is actually a subring of $\mathbb{R}^X$ under coordinate-wise operations. If X has more than one point, then $C(X)$ becomes a ring with zero divisors. The standard reference for rings of continuous functions is [7].

3.1.4 Definition. [12]. Let $f \in C(X)$ then, the set $Z(f) = \{x \in X : f(x) = 0\}$ is called the zero set of f.Similarly we define the set $X \setminus Z(f)$ to be a cozero set of f denoted by $coz(f)$. The units of $C(X)$ are precisely those $f \in C(X)$ for which $Z(f) = \emptyset$.

In the following definition $Z(X)$ denotes the family $Z[C(X)]$ of all zero sets in X.

3.1.5 Definition. ([7, Definition 2.2]). A nonempty family $\mathfrak{F}$ of $Z(X)$ is a *z-filter* on X if:

(i) $\emptyset \notin \mathfrak{F}$;

(ii) if $Z_1, Z_2 \in \mathfrak{F}$, then, $Z_1 \cap Z_2 \in \mathfrak{F}$; and

(iii) if $Z_1 \in \mathfrak{F}$, $Z_2 \in Z(X)$, and $Z_2 \supseteq Z_1$, then, $Z_2 \in \mathfrak{F}$.

3.1.6 Theorem. *([7, Theorem 2.3(a)]). If $I \subseteq C(X)$ is an ideal, then, the family $Z[I] = \{Z(f) : f \in I\}$ is a* z-filter *on X.*

3.1.7 Theorem. *([7, Theorem 2.3(b)]). If $\mathfrak{F}$ is a* z-filter *on X, then, the family $Z^{\leftarrow}[\mathfrak{F}] = \{f : Z(f) \in \mathfrak{F}\}$ is an ideal in $C(X)$.*

3.1.8 Definition. An ideal $I \in C(X)$ is called a *z-ideal* if $Z(f) \in Z[I]$ implies $f \in I$, that is to say, $I = Z^{\leftarrow}[Z[I]]$ [7].

3.1.9 Definition. Let I be an ideal in $C(X)$ or $C^*(X)$. If $\bigcap Z[I] \neq \emptyset$, then, I is called a fixed ideal. If $\bigcap Z[I] = \emptyset$, then, I is called a free ideal [7].

3.1.10 Remark. We similarly define z-filters to be free or fixed according to their intersections.

3.1.11 Definition. If I is a fixed ideal in $C(X)$, then, the set $S = \bigcap Z[I] \neq \emptyset$, and the set

$$I' = \{f \in C(X) : f[S] = \{0\}\}$$

is a fixed maximal ideal that contains I [7].

Our ability to relate X, $C(X)$ and $C^*(X)$, relies upon an analysis of the set of all maximal ideals. The fixed maximal ideals are easy to describe as opposed to those maximal ideals which are free.

3.1.12 Theorem. *([7, Theorem 4.6]). The fixed maximal ideals in $C(X)$ are precisely the sets*

$$M_p = \{f \in C(X) : f(p) = 0\} \qquad\qquad (p \in X).$$

The ideals M_p are distinct for distinct p. For each p, $C(X)/M_p$ is isomorphic with the real line $\mathbb{R}$; in fact, the mapping $M_p(f) \to f(p)$ is the unique isomorphism of $C(X)/M_p$ onto $\mathbb{R}$.

Proof. M_p is the kernel of the homomorphism $f \to f(p)$ of $C(X)$ onto $\mathbb{R}$. Since $\mathbf{r}(p) = r$ ($\mathbf{r}$ denotes the constant function) for each real r, the homomorphism is onto the field $\mathbb{R}$. Hence its kernel M_p is a maximal ideal. Uniqueness of p is an immediate consequence of the complete regularity of X. On the other hand, if M is any fixed ideal in $C(X)$, there exists a point p in $\bigcap Z[M]$. Evidently, M is contained in M_p, which has just been shown to be a (proper) ideal. Hence if M is maximal, we must have $M = M_p$.

Since M_p is the kernel of the homomorphism onto $\mathbb{R}$, $C(X)/M_p$ is isomorphic with $\mathbb{R}$; and the isomorphism is unique, because the only automorphism (by automorphism we mean an isomorphism from a mathematical object to itself.) of $\mathbb{R}$ is the identity [see 6, Corollary 0.23]. □

With reference to Theorem 3.1.12, we define the set

$$O_p = \{f \in C(X) : p \in Int(Z(f))\}$$

we also note that O_p is the set of minimal prime ideals of $C(X)$ contained in M_p.

The following theorem gives a condition of when ideals in $C(X)$ are fixed.

3.1.13 Theorem. *([7, Theorem 4.8]). If X is compact, then every ideal in $C(X)$ is fixed.*

Proof. $Z[I]$ is a family of closed sets with finite intersection property. □

In light of Theorem 3.1.6 we have that;

If X is compact, then the correspondence given by $p \to M_p$ is one-one from X onto the set of all maximal ideals in $C(X)$ [7].

3.1.14 Definition. We define an *algebraic invariant* to be a quantity such as a polynomial discriminant which remains unchanged under a given class of algebraic transformations.

3.1.15 Remark. [7]. Since maximal ideals are algebraic invariants, the points of a compact space can be recovered from the algebraic structure of the ring. Now, the zero-sets in X form a base for the closed sets, and the relation $p \in Z(f)$ is equivalent to the purely algebraice relation $f \in M_p$. Thus, the topology of X can be recovered from $C(X)$.

3.1.16 Theorem. *([7, Theorem, $p.57$]). Let X and Y be compact spaces. Then, X and Y are* homeomorphic *if and only if $C(X)$ and $C(Y)$ are isomorphic.*

3.1.17 Lemma. ([7, Lemma, $p.58$]). A zero-set Z is compact if and only if it belongs to no free z-filters.

Proof. Necessity is clear. Conversely let $\mathfrak{B}$ be any family of closed subsets of Z with the finite intersection property. The members of $\mathfrak{B}$ are closed in X. The collection of all zero-sets in X that contain finite intersection of members of $\mathfrak{B}$ is a z-filter $\mathfrak{F}$; and, of course, $Z \in \mathfrak{F}$. Since the zero-sets in the completely regular space X form a base for the closed sets, $\cap\mathfrak{B} \supseteq \cap\mathfrak{F}$. But this latter intersection is nonempty, by hypothesis; so $\cap\mathfrak{B} \neq \emptyset$. Thus, Z is compact. □

Next, we state the theorem which characterises a compact space X in terms of ideals of $C(X)$.

3.1.18 Theorem. *([7, Theorem, $p.58$]). For a space X the following statements are equivalent:*

(1) *X is compact;*

(2) *Every ideal in $C(X)$ is fixed, that is, every z-filter is fixed;*

(2^*) *Every ideal in $C^*(X)$ is fixed;*

(3) *Every maximal ideal in $C(X)$ is fixed, that is, every z-ultrafilter is fixed;*

(3^*) *Every maximal ideal in $C^*(X)$ is fixed.*

Proof. The equivalence of (1) with (2) is the special case $Z = X$ of the Lemma 3.1.17. Likewise, (1) implies (2^*), because $C^*(X) = C(X)$ when X is compact. Next, if I is a free ideal in $C(X)$, then, $I \cap C^*(X)$ is a free ideal in $C^*(X)$; therefore, (2^*) implies (2). Finally (2) is equivalent with (3), and (2^*) with (3^*), because every free ideal is contained in a free maximal ideal. □

We now show that $Min(C^*(X))$ is homeomorphic to $Min(C(X))$. But before we do that, we shall state a theorem which we shall use in our result.

3.1.19 Theorem. *([8, Theorem 5.1]). Let S be a subring of a ring R without nonzero nilpotents elements, and suppose that for each $b \in R$, there exists $a_b \in S$, and $u_b \in R$, such that $b = a_b u_b$, and $Ann(u_b) = (0)$. Then, the mapping δ defined by*

$$\delta(P) = P \cap S, \qquad P \in Min(R),$$

is a homeomorphism of $Min(R)$ onto $Min(S)$.

3.1.20 Corollary. ([7, Corollary 5.2]). For any space X, $Min(C(X))$ and $Min(C^*(X))$ are homeomorphic.

Proof. Apply Theorem 3.1.19, with $S = C^*(X)$, $R = C(X)$, $a_b = (-1) \wedge (b \vee 1)$, and $u_b = |b| \wedge 1$. □

3.1.21 Definition. ([6, $p.369$]). Two sets $U, V \subseteq X$ are *completely separated* if there exists a continuous map $\pi : X \to \mathbb{R}$, such that $\pi(U) = 0$ and $\pi(V) = 1$ (whence $\overline{U}$ and $\overline{V}$ are completely separated). We also recall Definition 2.2.3 (iii), which we shall need in the proof of the following theorem.

Next, we give an important characterisation of when $C(X)$ is a PF-ring.

3.1.22 Theorem. *([1, Theorem 4]). The ring $C(X)$ is a PF-ring if and only if for any nonzero $f, g \in C(X)$, $X \setminus Z(f)$ and $X \setminus Z(g)$ are completely separated whenever $fg = 0$.*

Proof. Assume $C(X)$ is a PF-ring. Let f and g be any two nonzero elements in $C(X)$ such that $fg = 0$. Then, $g \in Ann(f)$. But $Ann(f)$ is pure. So there exists $k \in Ann(f)$, such that $gk = g$. Hence,

$$k(x) = \begin{cases} 0, & \text{if } x \in X \setminus Z(f) \\ 1, & \text{if } x \in X \setminus Z(g). \end{cases}$$

So $X \setminus Z(f)$ and $X \setminus Z(g)$ are completely separated.

Conversely, let $g \in C(X)$. We want to show that $Ann(f)$ is pure. If $f = 0$, then, $Ann(f) = C(X)$ which is pure, so we can assume $f \neq 0$. Let $g \in Ann(f)$. If $g = 0$, then, there exists $g = 0 \in Ann(f)$, such that $gg = g = 0$. So we can assume $g \neq 0$, because $fg = 0$, $X \setminus Z(f)$ and $X \setminus Z(g)$ are completely separated. So there exists $k \in C(X)$, such that:

$$k(x) = \begin{cases} 0, & \text{if } x \in X \setminus Z(f) \\ 1, & \text{if } x \in X \setminus Z(g). \end{cases}$$

Thus, $fk = 0$ and $gk = g$. Hence $Ann(f)$ is pure. □

3.1.23 Lemma. ([1, Lemma 6]). For any $f, g \in C(X)$, $Int(Z(f) \cap Z(g)) = \emptyset$, if and only if $Ann(f^2 + g^2) = \{0\}$.

We conclude the section by giving the following result which characterise when the inverse topology on $Min(C(X))$ is Hausdorff.

3.1.24 Theorem. *([4, Theorem 3.1]). Let X be a topological space. Then, the following statements are equivalent:*

(a) *$Min(C(X))^{-1}$ is a Hausdorff space;*

(b) *$Min(C(X))^{-1}$ is a Normal space;*

(c) *For each pair $C, D \subseteq X$ of disjoint cozero sets, there exist a pair of cozero sets C', $D' \subseteq X$, with $C \subseteq C'$, $D \subseteq D'$, $C \cap D' = D \cap C' = \emptyset$ and $C' \cup D'$ is dense in X;*

(d) *For each pair of disjoint cozero sets C, $D \subseteq X$, there exist zero sets Z_1, $Z_2 \subseteq X$, such that $C \subseteq Z_1$, $D \subseteq Z_2$, and $Int(Z_1) \cap Int(Z_2) = \emptyset$;*

(e) *$q(X)$ is a PF-ring.*

Proof. The proof follows in two steps. First, that (a), (b), and (e) are equivalent follows from Theorems 2.3.5 and Theorem 2.3.6. To see that (c) implies (d), let C and D be disjoint cozero sets. By (c), there are cozero sets, say $coz(f)$, $coz(g)$, such that $C \subseteq coz(f)$, $D \subseteq coz(g)$, $C \cap coz(g) = \emptyset$, $D \cap coz(f) = \emptyset$, and $coz(f) \cup coz(g)$ is a dense subset of X. Set $Z_1 = Z(g)$ and $Z_2 = Z(f)$.To show that Z_1 and

Z_2 satisfy the condition in (d). Note that $Z_1 = X \setminus coz(g)$ and $Z_2 = X \setminus coz(f)$. Applying condition (c) to Z_1 and Z_2 we have that, Z_1 and Z_2 are disjoint. Therefore, $Int(Z_1) \cap Int(Z_2) = \emptyset$.

Finally, that (e) and (c) are equivalent follows from the equivalence of (1) and (2) of Theorem 2.3.6, together with the following observations. For $f, g \in C(X)$

$$fg = 0 \text{ if and only if } coz(f) \cap coz(g) = \emptyset,$$

and as mentioned before f is a nonzero-divisor if and only if $coz(f)$ is a dense subset of X.

□

3.2 On F-*spaces*

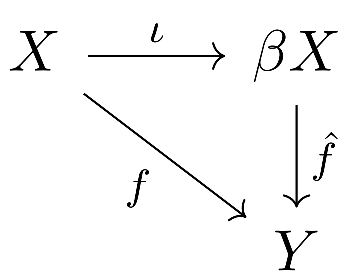
The theory of F-spaces has many applications in Mathematics, especially in Functional and Real analysis. This section serves as a way of introducing the notion of F-spaces and finding several characterisations. In particular, we characterise them in terms of their rings of continuous functions. These characterisations serve as a build up for our study concerning the Stone-Čech compactifications of F-spaces in the next section. Some of the results used in this section are taken from [6, 7, 8, 9].

The function $|f|$ defined as $f \vee -f$ for $f \in C(X)$ satisfies,

$$|f|(x) = |f(x)|.$$

3.2.1 Definition. ([9, $p.60$]). A topological space X is called an F-*space* if every finitely generated ideal in $C(X)$ is principal; equivalently X is an F-space if whenever $f \in C(X)$ there is a $k \in C(X)$, such that $f = k|f|$, that is, f is a multiple of $|f|$. We can also characterise F-spaces in terms of sets. In particular, a space X is an F-space if and only if whenever $S_1, S_2 \subseteq X$ are disjoint cozero sets, there is a $g \in C(X)$ such that, g separates S_1 and S_2.

3.2.2 Definition. ([18, Definition 1]). A *Stone-Čech compactification* of a topological space X is an embedding $\iota : X \to \beta X$, where βX is a compact Hausdorff space, such that for every continuous map from X to a compact Hausdorff space Y, say, $f : X \to Y$, there is a unique continuous map $\hat{f} : \beta X \to Y$, such that the following diagram commutes:



With this interpretation, $\hat{f}$ is merely an extension of f; and the defining property of the Stone-Čech compactification is that, the correspondence $f \mapsto \hat{f}$ constitutes a bijection

$$C(X, Y) \to C(\beta X, Y),$$

where $C(X, Y)$ is the set of all continuous functions from $X \to Y$.

3.2.3 Definition. ([6, Definition 1.1]). Let $f \in C(X)$, we define:

$$P(f) = \{x \in X : f(x) > 0\},$$
$$N(f) = \{x \in X : f(x) < 0\}.$$

Throughout we will use the symbol A^β to denote the closure in βX (the Stone-Čech compactification of X) of any subset A of βX, (while reserving $\overline{A}$ to denote the closure of a subset $A \subseteq X$ in X).

We recall from [8], that every prime ideal in $C(X)$ is contained in a unique maximal ideal, and that, the maximal ideals are in one-one correspondence with the points in βX, given by the following formula:

$$M^p = \{f \in C(X) : p \in Z(f)^\beta\}.$$

We then define a map $\psi : Min(C(X)) \to \beta X$, given by $\psi(P) = p$ if $P \subseteq M^p$. We can similarly use the notion of minimal prime ideals to define the above mapping as follows;

$$O^p = \{f \in C(X) : p \in Int(Z(f)^\beta)\}.$$

That is to say, every prime ideal contain exactly one ideal of the form O^p. For $P \in Min(C(X))$, $\psi(P)$ is the unique $p \in \beta X$ such that $O^p \subseteq P$.

3.2.4 Remark. Just as we did with non maximal ideals in $C(X)$, a maximal ideal M^p is free or fixed provided that $p \in X$ or $p \in \beta X \setminus X$.

3.2.5 Definition. Let p be any point of βX. The set of all $f \in C(X)$ for which there exists a neighbourhood Ω of p such that $f(\Omega \cap X) = 0$, is easily seen to constitute an ideal of $C(X)$; we denote this ideal by N^p. When $p \in X$, and when this fact deserves emphasis, we write N_p in place of N^p [6].

We now give the following lemmas for which we will make use of in the proof of the following theorem.

3.2.6 Lemma. ([6, Theorem 1.5)]). Let X be any completely regular space. Then to every $f \in C(X)$, there correspond $f^*, f_0 \in C(X)$, such that

(i) $|f^*(x)| \leq 1$, for all $x \in X$, and $f^*(x) = f(x)$, whenever $|f(x)| \leq 1$,

(ii) f_0 is everywhere positive, and

(iii) $f = f^* f_0$, whence $f^* = (\frac{1}{f_0})f$, so that f and f^* belong to the same ideals of $C(X)$.

3.2.7 Lemma. ([6, Theorem 1.6)]). Let X be any completely regular space, consider any function $\phi \in C(\beta X)$, and let f denote the restriction of ϕ to X. Then, $P(f)^\beta = \overline{P}(f)^\beta = P(\phi)^\beta$ (and $N(f)^\beta = \overline{N}(f)^\beta = N(\phi)^\beta$).

We give the following algebraic observations regarding $|f|$.

(1) $|f|$ is the unique element g such that, $g^2 = f^2$.

(2) $g + u^2$ is a unit for every unit u (uniqueness follows from the fact that any such g must be non-negative).

The following result gives a characterisation of when a topological space is an F-space.

3.2.8 Theorem. *([6, Theorem 2.3)]). Let X be a topological space. Then the following statements are equivalent:*

(a) *X is an F-space, that is every finitely generated ideal of the ring $C(X)$ is a principal ideal;*

(a^*) *βX is an F-space, that is every finitely generated ideal of the ring $C(\beta X)($ or $C^*(X))$ is a principal ideal;*

(b) *For every $f, g \in C(X)$, the ideal (f, g) is the principal ideal $(|f| + |g|)$;*

(b^*) *For every $\phi,\ \Psi \in C(\beta X)$, the ideal $(\phi,\ \Psi)$ is the principal ideal $(|\phi| + |\Psi|)$;*

(c) *For every $f \in C(X)$, the sets $P(f), N(f)$ are completely separated;*

(c^*) *For every $\phi \in C(\beta X)$, the sets $P(\phi), N(\phi)$ are completely separated;*

(d) *For every $f \in C(X)$, f is a multiple of $|f|$, that is, $|f| = kf$ and $f = k|f|$, for some $k \in C(X)$;*

(d^*) *For every $\phi \in C(\beta X)$, ϕ is a multiple of $|\phi|$, that is, $|\phi| = \Psi\phi$ and $\phi = \Psi|\phi|$, for some $\Psi \in C(X)$;*

(e) *For every $f \in C(X)$, the ideal $(f, |f|)$ is principal;*

(e^*) *For every $\phi \in C(\beta X)$, the ideal $(\phi, |\phi|)$ is principal.*

We divide the proof of Theorem 3.2.8 into 3 parts as follows:

- In the first part we show the chain of implications: $(c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (c)$. Applying this to the space βX we obtain the chain: $(c^*) \Rightarrow (d^*) \Rightarrow (e^*) \Rightarrow (c^*)$.
- In the second part we show the chain of implications: $(b^*) \Rightarrow (b) \Rightarrow (a) \Rightarrow (c) \Rightarrow (c^*)$. Applying this to the space βX we obtain the chain: $(b^*) \Rightarrow (a^*) \Rightarrow (c^*)$.

  The parenthetical remarks in the statements (a^*) and (b^*) are justified by the fact that $C(\beta X)$ and $C^*(X)$ are isomorphic.
- Finally we establish the implication $(c^*) \Rightarrow (b^*)$.

  On now combining all our results, we obtain the theorem.

We now give the first chain of implications. The following implications are local, that is, they hold for any one function in $C(X)$.

Proof. $(c) \Rightarrow (d)$: By hypothesis there exists a continuous function $k \in C(X)$, that is 1 everywhere on $P(f)$, and -1 everywhere on $N(f)$. Hence, $f = k|f|$.

$(d) \Rightarrow (e)$: Follows from the fact that f and $|f|$ are multiples of each other, hence $f = |f|$. Therefore, $(f, |f|)$ is a principal ideal.

$(e) \Rightarrow (c)$: Since $(f, |f|)$ is principal, there exists $d \in C(X)$, such that $(f, |f|) = (d)$. So $f = gd$ and $|f| = kd$, and $d = mf + n|f|$. Then, $d = (mg + nk)d$. Therefore, since d has no zeroes on $P(f) \cup N(f)$, we have that $mg + nk = 1$ thereon. Next, $g = k$ on $P(f)$ and $g = -k$ on $N(f)$. Hence, if we put,

$$a_1 = mg + ng, \qquad a_2 = mk + nk,$$
$$b_1 = mg - ng, \qquad b_2 = -mk + nk,$$

then, we have that,

$$a_1a_2 = 1 \text{ on } P(f), \qquad a_1a_2 \leq 0 \text{ on } N(f),$$
$$b_1b_2 \leq 0 \text{ on } P(f), \qquad b_1b_2 = 1 \text{ on } N(f).$$

Define

$$\phi = max\{a_1a_2, 0\} - max\{b_1b_2, 0\},$$

then, $\phi = 1$ on $P(f)$, and $\phi = -1$ on $N(f)$. Therefore, $P(f)$ and $N(f)$ are completely separated. □

We next show the chain of implications: $(b^*) \Rightarrow (b) \Rightarrow (a) \Rightarrow (c) \Rightarrow (c^*)$.

Proof. $(b^*) \Rightarrow (b)$. Consider any $f, g \in C(X)$. By Lemma 3.2.6, there exist $f^*, g^* \in C^*(X)$, and everywhere positive functions $f_0, g_0 \in C(X)$, such that, $f = f^*f_0$, and $g = g^*g_0$. By hypothesis, the ideal (f^*, g^*), of $C^*(X)$ is generated by the element $|f^*| + |g^*|$ of $C^*(X)$. Evidently the ideal (f, g) is also generated by this same element of the ring $C(X)$. To show that this ideal is generated by the element $|f| + |g|$, it suffices to show that $|f| + |g|, |f^*| + |g^*|$ are multiples of one another (in $C(X)$). Let $m \in C^*(X)$, satisfy $|f^*| = m(|f^*| + |g^*|)$. We may certainly suppose that $0 \leq m \leq 1$ everywhere. Then, the element $u = f_0m + g_0(1-m)$ is everywhere positive, hence is a unit of $C(X)$. The observation that if $|f| + |g| = u(|f^*| + |g^*|)$ now completes the proof.

$(b) \Rightarrow (a)$. Since for every $f, g \in C(X)$, the ideal (f, g) is a principal ideal, we get that X is an F-space, since $C(X)$ consists of finitely generated ideals.

$(a) \Rightarrow (c)$. From the completely regularity of X we have that, there exist $k \in C(X)$, which separates $P(f)$ and $N(f)$.

$(c) \Rightarrow (c^*)$. Consider any function $\phi \in C(\beta X)$. Let f denote the restriction of ϕ to X. By hypothesis, the sets $P(f), N(f)$ are completely separated. Now, since completely separated subsets of X have completely separated closures in $C(\beta X)$ (see Definition 3.1.21). We have by Lemma 3.2.7, that the sets $P(\phi)^\beta, N(\phi)^\beta$ are completely separated. □

Finally we establish the implication $(c^*) \Rightarrow (b^*)$.

Proof. Consider any $\phi, \Psi \in C(\beta X)$; we want to show that $(\phi, \Psi) = (|\phi| + |\Psi|)$. Now as previously observed, our hypothesis (c^*) implies (d^*). From this latter, it is clear that $(\phi, \Psi) = (|\phi|, |\Psi|)$. We may assume throughout the remainder of the proof that both ϕ and Ψ are non-negative. Define $\theta = \phi + \Psi$. Then, $\theta \in (\phi, \Psi)$, so that, $(\theta) \subseteq (\phi, \Psi)$. It remains, to show that $(\theta) \supseteq (\phi, \Psi)$. To this end, we construct a function $\phi_1 \in C(\beta X)$, such that, $\phi = \phi_1\theta$ (for then, $\Psi = (1 - \phi_1)\theta$). Since, $\phi \geqq 0$ and $\Psi \geqq 0$, we have that, $\theta \geqq 0$, and $\theta(x) = 0$ if and only if $\phi(x) = \Psi(x) = 0$.

Define

(1)

$$\phi_1 = \frac{\phi}{\theta} \text{ on } P(\theta).$$

Then, ϕ_1 is continuous on $P(\theta)$. We shall first extend ϕ_1 to all of $P(\theta)^\beta$. To that end, consider any fixed $p \in P(\theta)^\beta \setminus P(\theta)$. Then, $\theta(p) = 0$. For every real r, define a function $\mu_r \in C(\beta X)$ by:

(2)

$$\mu_r(x) = \phi(x) - r\theta(x).$$

If $r > s$, then, $\mu_r(x) \leqq \mu_s(x)$, for every $x \in \beta X$ (since, $\theta(x) \geqq 0$). Furthermore, $\mu_r(p) = 0$ for every real r.

For all $x \in \beta X$, we have that, $\mu_0 = \phi(x) \geqq 0$; and for all $x \in P(\theta)$, and every real $\epsilon > 0$, we also have that, $\mu_{1+\epsilon}(x) \leqq -\epsilon\theta(x) < 0$. Therefore, since every neighbourhood of p meets $P(\theta)$, we may put,

(3)

$$\phi_1(p) = \sup\{r : \mu_r(x) \geqq 0 \text{ throughout some neighbourhood } p\}$$

$$(p \in P(\theta)^\beta \setminus P(\theta)).$$

The function ϕ_1 is now defined on all of $P(\theta)^\beta$.

To establish continuity of ϕ on $P(\theta)^\beta$, it is enough to establish its continuity at any point $p \in P(\theta)^\beta \setminus P(\theta)$. We write $\alpha = \phi_1(p)$. By (3), for every $r > \alpha$, and for every neighbourhood U of p, there is a $x \in U$, such that, $\mu_r(x) < 0$. Since, the hypothesis (c^*) applies to the function $\mu_r \in C(\beta X)$, the sets $P(\mu_r)^\beta, N(\mu_r)^\beta$ are disjoint. Consequently, since $\mu_r(p) = 0$, there is a neighbourhood V of p such that, $\mu_r(x) \leqq 0$, for all $x \in V$. On the other hand, by (3), for every $s < \alpha$, there is a neighbourhood W of p such that, $\mu_s(x) \geqq 0$, for all $x \in W$. Thus, for every $\epsilon > 0$, there is a neighbourhood U of p such that,

$$\mu_{\alpha+\epsilon}(x) \leqq 0 \leqq \mu_{\alpha-\epsilon}(x) \qquad \text{for all } x \in U.$$

With the substitution (2), this reads:

$$\phi(x) - (\alpha+\epsilon)\theta(x) \leqq 0 \leqq \phi(x) - (\alpha-\epsilon)\theta(x) \qquad \text{for all } x \in U.$$

If we further restrict x to lie in $P(\theta)$, then, on applying (1), this last reduces to:

$$|\phi_1(p) - \phi(x)| \leqq \epsilon \qquad \text{for all } x \in U \cap P(\theta).$$

From this, it further follows that, $|\phi(p) - \phi(q)| \leqq 2\epsilon$ for all $q \in U \setminus P(\theta)$. We then conclude that ϕ_1 is continuous on $P(\theta)^\beta$. Clearly, $\phi = \phi_1\theta$ from here onwards. Finally, by normality of βX and the fact that $P(\theta)^\beta$ is closed, ϕ_1 can be extended continuously over all of βX. Since, $\theta \geqq 0$, we have that, $\beta X \setminus P(\theta)^\beta \subseteq Z(\theta) \subseteq Z(\phi)$. Therefore, $\phi = \phi_1\theta$ everywhere on βX.

□

We end the section with the following theorem, which gives characterisation between maximal ideals of $C(X)$ and an F-spaces X.

3.2.9 Theorem. *([6, Theorem 2.5)]). A completely regular space X is an F-space if and only if, for every maximal ideal M of $C(X)$, the intersection of all the prime ideals contained in M is a prime ideal. In other words, if and only if, for every point $p \in \beta X$, the ideal N^p of $C(X)$ (see Definition 3.2.5) is a prime ideal.*

Proof. Obviously, an ideal I of $C(X)$ is prime if and only if $I \cap C^*(X)$ is a prime ideal of $C^*(X)$ (Lemma 3.2.6). Accordingly, in view of Theorem 3.2.8 (a, a^*), there is no loss of generality in supposing that the space X is compact. Assume, first, that X is not an F-space. Then, there exists an $f \in C(X)$ such that the sets $\overline{P}(f), \overline{N}(f)$ are not completely separated (Theorem 3.2.8 (a, c)). Since X is normal, this means that the two sets are not disjoint. There accordingly exists a point $p \in Z(f)$ such that f changes sign on every neighbourhood of p. Define $g = max\{f, 0\}$, $h = min\{f, 0\}$. Then, $g \notin N^p$ and $k \notin N^p$, while $gk = 0 \in N^p$. Therefore, the ideal N^p is not prime.

Conversely, suppose that there is a point $p \in X$ for which the ideal N^p is not prime. Then there exist $g, k \in C(X)$, and a neighbourhood U of p, such that gk vanishes identically on U, while neither g nor k vanishes identically on any neighbourhood of p. Hence if V is any neighbourhood of p that is contained in U, there must exist $x, y \in V$, such that $g(x) \neq 0$, $k(y) \neq 0$. But then, $k(x) = g(y) = 0$. The function $f = |g| - |k|$ $(\in C(X))$, therefore changes sign on V. Thus, the sets $\overline{P}(f), \overline{N}(f)$ are not disjoint (hence not completely separated). Therefore, X is not an F-space (Theorem 3.2.8 (a, c)). □

3.3 On the Stone-Čech compactification of F-*spaces*

There are many reasons why topologists study compactifications of non-compact topological spaces. Firstly, it is often conceptually easier to have a non-compact topological space as a subspace of a compact space, thus, allowing the use of all machineries available in the compactness package, like (boundedness, existence of limit points, etc).

Secondly, the importance of compactness in applications abounds, for example in existence proofs. This is realised by constructing a sequence and then showing that the limit points satisfy the given requirements. Another use of compactification is in measuring how "congenial" a function is.

Another reason to study compactifications, is that, often times we seek structural extensions of topological spaces to a corresponding structure on its compactification. Consider as an example the natural numbers $\mathbb{N}$. For any given space, there is always a "largest" compactification, called the Stone-Čech Compactification (for the natural numbers, and this is denoted by $\beta\mathbb{N}$).

Our main goal in this section is to show that if X is an F-space, then $Min(C(X))^{-1}$ is the Stone-Čech compactification of X. To do this, we will recall some results from previous sections and from the following sources [4, 5, 12, 15, 17].

We begin the section by giving preliminary definitions which we shall need. Recall that in Chapter 1, we gave a general definition of what a compactification and embedding are (see Definition 1.1.9, and Definition 1.1.14).

Now, let X and Y be topological spaces. The map $\phi : X \to Y$ is a *dense embedding* if $\phi : X \to \phi(X)$ is a homeomorphism, and $\phi(X) \subseteq Y$ is dense. In this case, we usually identify X with its image $\phi(X)$. The reason why X is embedded in spaces with some desirable properties (such as completeness or compactness) which X itself might lack, is because in order to study the most general compactifications we need our spaces to be densely embeddable, that is, to be Tychonoff spaces.

The Stone-Čech compactification

We recall the definition of a Stone-Čech compactification of a topological space X from Definition 3.2.2 as an embedding.

The next result gives the properties of the map (denoted by δ) between $Min(C(X))$ and βX. It also gives the characterisations of the map.

3.3.1 Theorem. *([8, Theorem 5.3]). Let X be a topological space:*

- (a) *δ is a continuous mapping of $Min(C(X))$ onto βX;*
- (b) *δ maps no proper closed subset of $Min(C(X))$ onto βX;*
- (c) *δ is one-one if and only if O^p is a prime ideal for each $p \in \beta X$, that is X is an F-space;*
- (d) *δ is a homeomorphism if and only if the interior of every zero-set in X is closed, that is X is basically disconnected;*
- (e) *If X is an F-space, then, $Min(C(X))$ is compact if and only if X is basically disconnected.*

Proof. (a). Given an open neighbourhood U of a point p in βX, there is a function $g \in C(\beta X)$ that vanishes on some neighbourhood of p and is equal to 1 outside of U. We claim that if f is the restriction of g to X, then,

$$\delta^{-1}(p) \subseteq h(f) \subseteq \delta^{-1}[U],$$

and since $h(f)$ is an open set in $Min(C(X))$, this implies that δ is continuous. To prove our assertions about f, note first that $p \in Int(Z(f)^{\beta})$ so that $f \in O^p$, whence $h(f) \supseteq \delta^{-1}(p)$. Secondly, if $f \in M^q$ for some $q \in \beta X$, then, $q \in Z(f)^{\beta}$ so that $g(q) = 0$. Thus, $q \in U$. This means that $\delta[h(f)] \subseteq U$.

(b). Every proper closed set in $Min(C(X))$ is contained in a set $h(f)$ for some nonzero $f \in C(X)$, because such sets form a base for the closed sets. Let $p \in X$ with $f(p) \neq 0$. Then, $f \notin M^p$, and this implies $p \notin \delta[h(f)]$.

(c). O^p is the intersection of all the minimal prime ideals that are contained in M^p.

Conversely, since X is an F-space then, the mapping δ from $Min(C(X))$ to βX is one to one.

(d). As we observed in [7, Problem 6M, $p.96$], X is basically disconnected if and only if βX is. Therefore, by Corollary 3.1.20, we may assume that $X = \beta X$, that is, X is compact. Now, if δ is one-one, then, the minimal prime ideals of $C(X)$ are precisely the ideals O^p, $p \in X$. Moreover, for any $f \in C(X)$,

$$\delta[h(f)] = \{p \in X : f \in O^p\} = \{p \in Int(\overline{Z(f)})\} = Int(Z(f)),$$

and if δ is a homeomorphism, then, $\delta[h(f)]$ is a closed set.

Suppose, conversely, that X is compact and basically disconnected. Since every basically disconnected space is an F-space, the mapping δ is one-one, and it is a closed mapping, because $\delta[h(f)] = Int(Z(f))$ for all $f \in C(X)$.

(e). This is an immediate consequence of (c) and (d).

□

3.3.2 Example. As a consequence, the space $Min(C(X))$ is always countably compact, however it is not always compact. According to (e) of Theorem 3.3.1, if X is an F-space that is not basically disconnected, then, $Min(C(X))$ is not compact. The conditions which then guarantees sufficiency and necessity for compactness of $Min(C(X))$, without X being as in Theorem 3.3.1 are;

(1) $Z(f) \cup Z(f') = X$, and

(2) $Int[Z(f) \cap Z(f')] = \emptyset$, where $f, f' \in C(X)$.

3.3.3 Definition. ([12, Definition 5.2]). Let X be a topological space and Y a subspace of X. Then, Y is said to be C-*embedded* in X if for every $f \in C(Y)$, there is $g \in C(X)$, such that the restriction of g to Y is f. Notationally, we say $g|_Y = f$. We similarly define Y to be a C^*-*embedded* subspace of X if for every $f \in C^*(Y)$, there is $g \in C^*(X)$, such that $g|_Y = f$.

We give some results concerning the Stone-Čech compactifications which have been extracted from [6]. Actually, in [6, $p.367$] the authors asserted that every completely regular space can be embedded in a compact space βX (the Stone-Čech compactification) characterised by the following three properties:

(1) βX is compact;

(2) X is homeomorphic with a dense subspace of βX;

(3) Every function in $C^*(X)$ has a unique extension over all of βX.

By (3) we have that $C^*(X)$ is isomorphic to $C(\beta X)(= C^*(\beta X))$.

3.3.4 Definition. ([12, Definition 5.6]). A topological space X is *strongly zero dimensional* if any two zero sets in X can be separated by a clopen set. Equivalently a topological space X is strongly zero dimensional if βX is zero dimensional.

From this equivalence we have that a strongly zero dimensional space is zero dimensional, but the converse is not true.

Next, we give a characterisation of when the set of minimal prime ideals of $C(X)$ is zero-dimensional. But before we do that we will give some definitions.

3.3.5 Definition. ([12, Definition 5.3]). We say a topological space X is *cozero complemented* if for each disjoint cozero set $C \subseteq X$, there is a disjoint cozero set $D \subseteq X$, with $C \cup D$ dense in X. A topological space X is *weakly cozero complemented*, if whenever $C_1, C_2 \subseteq X$ are disjoint cozero sets, there are disjoint cozero sets D_1, D_2 such that $C_1 \subseteq D_1, C_2 \subseteq D_2$, and $D_1 \cup D_2$ is dense in X.

The following example shows that a strongly zero dimensional F-space is a feebly baer ring.

3.3.6 Example. Since $\beta\mathbb{N} \setminus \mathbb{N}$ is a strongly zero dimensional F-space, then $\beta\mathbb{N} \setminus \mathbb{N}$ is a feebly baer ring.

3.3.7 Example. ([4, Example 3.2]). If X is an F-space which is not strongly zero-dimensional, then $C(X)$ is an inverse Hausdorff ring which is not quasi-complemented. For such an example, consider $\mathbb{H} = \beta\mathbb{R}^+ \setminus \mathbb{R}^+$, an F-space which is not zero-dimensional. Thus, $C(\mathbb{H})$ is an inverse Hausdorff ring which is not quasi-complemented.

3.3.8 Example. The space $\beta\mathbb{R} \setminus \mathbb{R}$ is an example of a weakly cozero complemented space which is not cozero complemented. It follows that if X is weakly cozero complemented, then $C(X)$ is inverse

Hausdorff.

3.3.9 Definition. Let $S \subseteq R$ be a subring of R. Then S is a *rigid extension* of R if for each $s \in S$ there is an $a \in R$ such that $Ann_S(s) = Ann_S(a)$ [5].

3.3.10 Proposition. ([15, Proposition 7.2]) For a space X the following are equivalent:

(1) $Min(C(X))$ is zero-dimensional with respect to the inverse topology;

(2) X is weakly cozero complemented;

(3) βX is weakly cozero complemented.

Proof. We first show that $C^*(X) \leq C(X)$ is a rigid extension.

The contraction map that takes $P \in Min(C(X))$ to $P^* = P \cap C^*(X)$ is always well-defined between $Spec(C(X))$ and $Spec(C^*(X))$. To check that P^* is a minimal prime ideal involves showing that for all $f \in P^*$ there is a g not in P^* which annihilates it. Such a g exists in $C(X)$ as P is a minimal prime ideal. Then take $|g| \wedge 1$ which is a bounded function with the same zero-set as g.

Take $f \in C(X)$. Notice that f and $|f|$ have the same annihilator. Then take $f' = |f| \wedge 1$ and observe that clearly $f' \in C^*(X)$. Also, they have the same annihilator. This is a rigid extension as $Ann(f) = f^{\perp}$ in $C(X)$.

Now, since $C^*(X) \leq C(X)$ is a rigid extension, it follows that $Min(C^*(X)) \cong Min(C(X))$ with respect to the inverse topologies [15]. □

We are now in a position to state the main result of this section concerning the Stone-Čech compactification of F-spaces.

3.3.11 Proposition. ([15, Proposition 7.3]). Let X be an F-space. Then βX is homeomorphic to $Min(C(X))$ under the inverse topology.

Proof. By Proposition 3.3.10, $Min(C(X)) \cong Min(C^*(X))$. Therefore, it is enough to show that the proposition is true for compact F-spaces X. Let X be a compact F-space. Then, every minimal prime ideal of $C(X)$ is of the form O_p for some $p \in X$, and so there is an obvious bijection between X and $Min(C(X))$. Now, for $f \in C(X)$, we recall that

$$\begin{aligned} N(f) &= \{O_p : f \in O_p\} \\ &= \{O_p : p \in Int(Z(f))\}, \end{aligned}$$

and therefore, the inverse topology on $Min(C(X))$ is homeomorphic to the topology on X generated by basic sets of the form $Int(Z(f))$ for an arbitrary $f \in C(X)$. To show that this latter topology is equal to the original topology on X, we recall that the base for the original topology on X is of the form $\{coz(f) : f \in C(X)\}$. Notice that

$$\begin{aligned} Int(X \setminus coz(f)) &= X \setminus \overline{coz(f)} \\ &= X \setminus Z(f) \\ &= coz(f). \end{aligned}$$

Which proves that the two topologies are the same. □

The following theorem gives a characterisation of when an F-space X is weakly cozero complemented.

3.3.12 Theorem. *([15, Proposition 7.4]) Let X be an F-space. Then the following are equivalent:*

(i) *X is weakly cozero complemented;*

(ii) *βX is weakly cozero complemented;*

(iii) *X is strongly zero-dimensional;*

(iv) *βX is zero-dimensional.*

Proof. This follows directly from Proposition 3.3.11. Since X is an F-space, $Min(C(X))$ and βX are homeomorphic. Therefore, (ii) and (iv) are equivalent [15].

$(i) \Rightarrow (iii)$. Since X is weakly cozero complemented we have that $Min(C(X))$ is zero dimensional since $C(X)$ is an inverse Hausdorff ring, and hence βX is zero dimensional (because $Min(C(X))$ and βX are homeomorphic), thus, X is strongly zero dimensional, by Definition 3.3.4.

$(iii) \Rightarrow (i)$. By (iii), we have that $C(X)$ is an inverse Hausdorff ring which is quasi-complemented by Example 3.3.7. Thus, we have that $C(X)$ is weakly complemented ring. Therefore X is weakly cozero complemented.

$(ii) \Leftrightarrow (iii)$ Suppose X is weakly cozero complemented. Since $C^*(X) \cong C(\beta X)$, it follows that $C(X) \cong C(\beta X)$ because $Min(C(X))$ is homeomorphic to $Min(C^*(X))$, by Corollary 3.1.20. Thus, $C(X) \cong C(\beta X)$ is weakly complemented, therefore, βX is weakly cozero complemented.

Conversely, suppose βX is weakly cozero complemented, then using the same arguments as above, it then follows that, $C(\beta X) \cong C(X)$ is weakly complemented. therefore, X is weakly cozero complemented.

$(iv) \Rightarrow (iii)$. Follows from Definition 3.3.4.

$(i) \Leftrightarrow (iv)$. Suppose first that X is weakly cozero complemented, we have that $Min(C(X))$ is zero dimensional since $C(X)$ is an inverse Hausdorff ring, and hence βX is zero dimensional (because $Min(C(X))$ and βX are homeomorphic).

Conversely, if βX is zero dimensional, it follows from Proposition 3.3.11, that $Min(C(X))$ is also zero dimensional. Therefore $C(X)$ is quasi-complemented by Theorem 2.3.1. Thus, X is weakly cozero complemented, since $C(X)$ is weakly complemented. □

We will end this section with the following remark on F-spaces.

3.3.13 Remark. ([15, Example 7.5]). It is known that an F-space X is cozero complemented precisely when X is basically disconnected (recall Definition 1.2.22). It follows by Theorem 3.3.12, that if X is a compact zero-dimensional F-space which is not basically disconnected, e.g. $\beta\mathbb{N} \setminus \mathbb{N}$, then $C(X)$ is a feebly Baer ring, and hence a weakly complemented ring with stranded primes (that is, prime ideals in $C(X)$ form a disjoint union of maximal chains) which is not complemented.

4. Conclusion

In this essay, the main study was on the inverse topology on the space of minimal prime ideals of a ring R. We gave some useful results emanating from definitions and theorems that play crucial roles in this work. In particular, we gave characterisations of when the inverse topology on $Min(R)$ is Hausdorff. This is recorded in Theorem 2.3.5 which is sourced from ([4, Theorem 2.6]). We also showed that $Min(R)$ endowed with the inverse topology is always a compact T_1-space (see Theorem 2.1.3) which is sourced from ([12, Theorem 3.1]).

We then studied the structure space of the ring $C(X)$ of continuous real-valued functions defined on a Tychonoff space X, with the intention of studying the inverse topology on $Min(C(X))$. The main result in this essay, which is Proposition 3.3.11 was showing that the inverse topology on the space of minimal prime ideals of $C(X)$ is the Stone-Čech compactification of X if X is an F-space (see [15, Proposition 7.3]).

The possibility of future work in this research area is great, in the sense that, this work can be extended to pointfree topology. That is, it can be shown that if a completely regular frame L is an F-frame, then the frame $Min(\mathcal{R}L)^{-1}$ is precisely the Stone-Čech compactification of L, where $\mathcal{R}L$ is the ring of all real-valued continuous functions on a completely regular frame L.

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