Rate-Induced Tipping

Mncedisi Comfort Ketwa (mncedisi@aims.ac.za)
African Institute for Mathematical Sciences (AIMS)

Supervised by: Dr. Michael Grinfeld
University of Strathclyde, Scotland

26 October 2017

Submitted in partial fulfillment of a structured masters degree at AIMS South Africa

Abstract

Rate-induced tipping is a newly discovered mechanism of collapse of various, e.g. ecological systems. The project is a review of the literature, and then will analyse in detail a simple deterministic model due to Ritchie and Sieber (2016). Then we will investigate the numerically a time-periodic version of the model.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Mncedisi Comfort Ketwa, 26 October 2017

Contents

1. Introduction to tipping phenomena

Tipping points are one of the studies that have recently caught the attention of researchers in the fields of climate science, ecology and financial market. This phenomenon have got attention from researchers because of its behaviour such as small change in a variable which dramatically affect the state of system at some time in the future. In this study we want to show that the tipping points are manifestation from deterministic systems and there are many debated about determinism as an ideology under the philosophy of mathematics and physics where they used a determinism to describe the deterministic systems which is the system that has no randomness involved in the development of the future state of it. This is to mean the state does not have to undergoes some large change immediately (Lamberson et al., 2012).

The deterministic in tipping points gives an ability to predict the future state. Ashwin et al. (2012) shows that recently the tipping points are related to a long-standing question in climate science.

Hasselmann (1976) tackled this question through simple climate model and in his work he argued that the climate system can be conceptually divided into a fast component which is the weather and a slow component which is the ocean, land vegetation and others. A system that have both components is called Fast/Slow Systems, and is mathematical represented as:

$$\dot{x} = f(x, \mu, \lambda(t)) \tag{1.0.1}$$

Since this study is focus only in slow components our system model is

$$\dot{x} = f(x, \lambda(t)) \tag{1.0.2}$$

where $x \in \Re^n$ is the state vector, with parameter $\mu \in \Re^k$ that do not vary over time. λ is the shifting parameter of the system. Tipping points are characterized by a sudden shift in the behaviour of this systems due to relatively small change in the input $\lambda(t)$. There are three tipping mechanism.

- Birfacation-induced tipping
- Noise-induced tipping
- Rate-induced tipping

Ritchie and Sieber (2016) tells us that in first two mechanism the is much research have done and both are behaving like an Ornstein-Uhlenbeck(OU) process. What make rate-induced tipping different from other tipping point mechanisms is that it fail to track the continuously changing quasi-steady state.

1.1 Tipping points

1.1.1 Definition. Tipping is the critical point in a system that has an unstoppable effect. It can be described into two forms: continuous and discrete. Let us use notation from the continuous case, since the discrete case is straightforward if you know continuous case. Let $x(t)$, $\lambda(t)$ and $y(t)$ be the dynamic processes such that future states of $x(t)$ are determined by the current states of $x(t)$, $\lambda(t)$ and $y(t)$ which belong to state spaces such as Ω_x, Ω_λ and Ω_y, respectively. Where we call the variable state $x(t)$ is the tipping variable and other variable $\lambda(t)$ is the parameter variable, and use $y(t)$ as a place holder for anything else that might affect the function value.

In generally tipping points occurs in some quantity(or quantities) over time, such as probability that the stock will collapse, the approval rating of the president, or the future collapse of peatland. The variable of interest is referred to as the state. So since the state is depended in time then we denote it as $x(t)$. The initial state is called the initial conditions and they are denoted as $x(0)$ where $t = 0$. Now we can let Ω_x be the state space that denote all possible states of the system. For purpose of classification in work of Ritchie and Sieber (2016), it will be important to distinguish between the variable of interest, $x(t)$, and other variables $\lambda(t)$.

- $\frac{\mathrm{d}x}{\mathrm{d}t}$ might be the model rate
- $\frac{\mathrm{d}\lambda}{\mathrm{d}t}$ might be the forcing rate

Note that the change in forcing rate can influence the model rate. Time can proceed continuously or in discrete steps.

- $\dot{x} = f(x, \lambda)$ is continuous
- $\dot{x}_{t+1} = F(x_t, \lambda_t)$ is discrete

This was the briefing by Lamberson et al. (2012), but in Ritchie and Sieber (2016) the focus is in the continuous case. Both equations are called equation of motion.

Tipping conditions.

- In No tipping conditions we mean, when the system is below some equilibrium point it tents to increase and when it is above that equilibrium, it tends to decrease. Thus, there is one and only one equilibrium is stable, so there is no possibility of tipping.
- In initial tipping condition exist when the equation of motion such as $\dot{x}(t) = f(x, \lambda)$, intersects the line $\dot{x}(t) = \bar{x}$ and is positive for all $x(t) \neq \bar{x}$ (and all values of $\lambda(t)$) then there is only one tipping point and that tipping point occurs at $\bar{x}$.
- The tipping existence condition describes situations in which tipping point is guaranteed to exist and that system has stable equilibrium point and unstable equilibrium points. Let assume that the conditions only motivation is to support the state most likely to tip and we take the current level of support for each state as a signal of the state tipping possibility. Let $x(t) = N_t - O_t$ denote the difference in support for N and O at time t. Let us again assume that sign function extracts the sign of a real number t, then we can defined the function as

$$sgn(t) = \begin{cases} -1 & \text{if } t < 0 \\ 0 & \text{if } t > 0 \\ 1 & \text{if } t = 0 \end{cases}$$

  This function can be presented diagrammatically as follows,

The mathematically expression for sign function is,

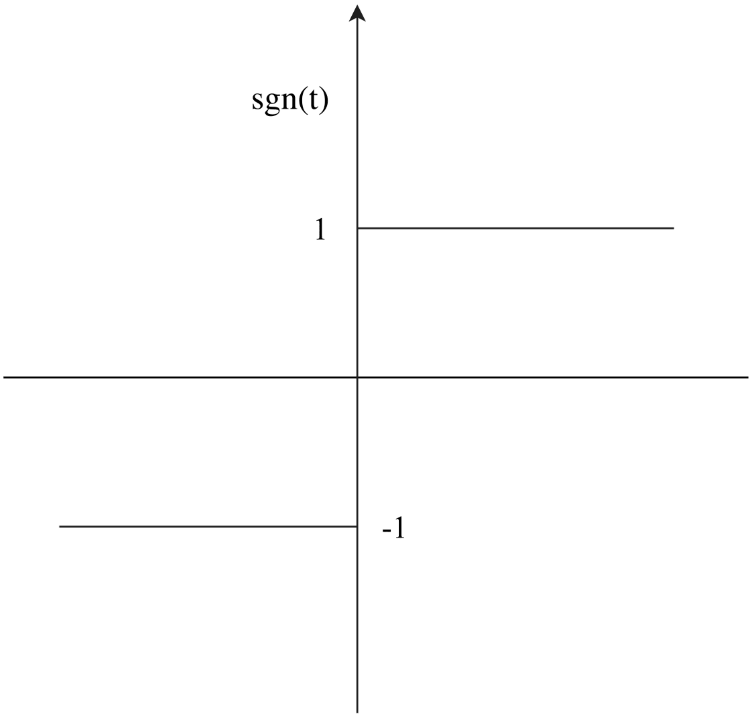

Figure 1.1: Dynamic of sign function

$$sgn(t) \quad = \quad \tanh(\frac{\lambda_{max}\epsilon t}{2}) \tag{1.1.1}$$

The relation between Sign function and Heaviside function is,

$$\begin{aligned} \lambda(t) &= \frac{\lambda_{max}}{2} + \frac{\lambda_{max}}{2} sgn(t) \\ &= \frac{\lambda_{max}}{2} + \frac{\lambda_{max}}{2} \tanh(\frac{\lambda_{max}\epsilon t}{2}) \end{aligned}$$

This tells us that the Heaviside function which is denote as $\lambda(t)$, is the rate of change in ramp functions. When $t = 0$, $\lambda(0) = \frac{\lambda_{max}}{2}$ because it has the rotational symmetry.

In this case of tipping phenomena we can generalising the sign function for real and complex expression as $csgn$, which is defined as follows.

$$csgn(t) = \begin{cases} 1 & \text{if } Re(z) > 0 \\ -1 & \text{if } Re(z) < 0 \\ sgn(Im(z)) & \text{if } Re(z) = 0 \end{cases} \tag{1.1.2}$$

Then when the tipping exist, now we categorize them into two binary distinctions as Lamberson et al. (2012) said, and the first one is called Direct tip while the second one is called Contextual tips. The difference between then is that a direct tip is caused by change in a state variable while the contextual tip is a change in the parameter of a system and these parameters denote the environment surrounding of the state. Note that a direct tip cause a system to move from one equilibrium point to another while a contextual tip cause the set of equilibria to change.

1.2 Bifurcation-induced tipping

The prediction of future climate change especially the possible collapse of the Atlantic Maridional Overturning Circulation(AMOC) due to the increasing freshwater input or the sudden release of carbon in peatlands due to an external temperature increase above a critical rate which is called the Compost bomb instability. A the conference at Copenhagen they debated that such prediction would be a sudden and irreversible abrupt change called tipping point (Thompson and Sieber, 2011). A bifurcation driven-tipping has been suggested as an important mechanism to predict those sudden shift of the systems. If we take (1.0.2) as our example where $\lambda(t)$ is in general a time-varying input and in a case where λ is a constant, we refer (1.2.1) as the parametric system with a parameter λ. We have also referred to its solution as the quasi-static attractor. from Ashwin et al. (2012) work, if $\lambda(t)$ passes through a bifurcation point of this parametric system where the quasi-static attractor loses stability. Intuitively, this is clear that the system will tip directly because of parameter variations. Although in same certain conditions the cause will be delayed because of well-documented slow passage through bifurcation effect (Ashwin et al., 2012).

This can be more clear if we consider a phase space as a bistable potential, under the influence of thermal noise in one dimension:

$$\dot{x} \quad = \quad -U^{'}(x) + \sqrt{2k_B T \beta} \tag{1.2.1}$$

where $U(x)$ is the potential, $U^{'}(x)$ is the first derivative of the potential and the second term is the thermal noise. Here a position of the system is a state and the variable of interest is to move between potential energy wells, where the energy wells are representing the state.

From Ritchie and Sieber (2016) work, the bifurcation-induced tipping is only focus in deterministic part. In bistable potential potential, when the system jump from the one well to another, we say there is the small oscillations. Ritchie and Sieber (2016) argue about this small oscillation as the small perturbation or disturbance that relax back to the equilibrium with a large decay rate. Diagrammatically, the bistable potential is as follows.

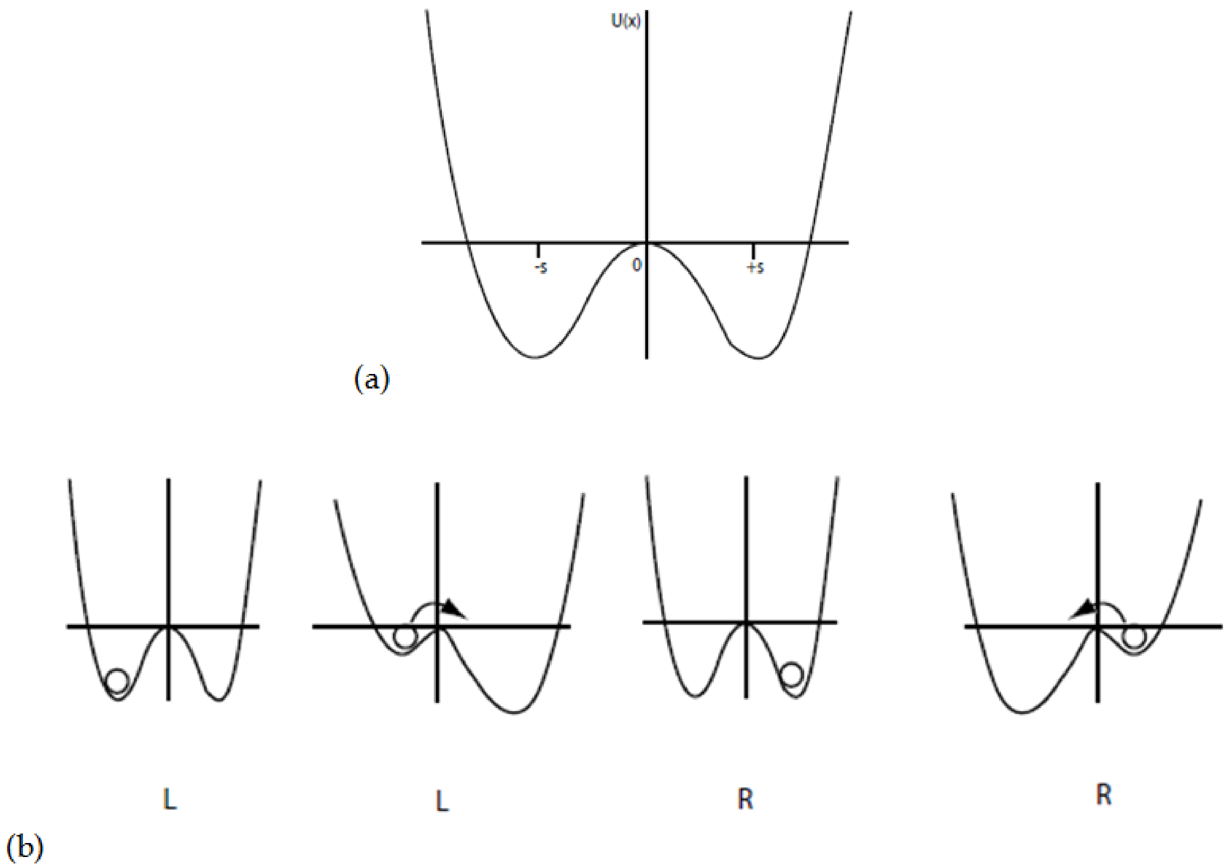

Figure 1.2: Bistable potential well. (a) is a normal well. (b) A system is in the left-well of bistable potential which is denoted as just L while the right-well is denoted as R

1.3 Noise-induced tipping

The explanation of climate tipping as a phenomenon purely induced by bifurcations has been called into question. For example, Ditlevsen and Johnsen (2010) suggested that the prediction techniques to forecast a forthcoming tipping point are not always reliable. Indeed, noise alone can drive a system to tipping without any bifurcation, that leads to what we call as the Noise-induced tipping.

This means noise-induced tipping is considering the second term in (1.2.1), which is to say it is both deterministic and stochastic. Both bifurcation- and noise-induced tipping are behaving like an Ornstein-Uhlenbeck (OU) process.

2. Technical material

In this chapter we focus in the material that we are going to need to the next chapter. It is consist of invariance manifold and Penichel theory. Here we are using a invariance manifold to show that a topological manifold is invariance under the action of the dynamical systems. We apply tangent space to the invariant manifold so that we can explain the trajectories by using vector fields. This will allows us to use vector calculus in the manifold, then manifold may be called differentiable manifold. Fenichel theory connect the manifolds and tangent bundle.

2.1 Invariance manifold

If the dynamical system contains a $<< \epsilon$ and we let $\epsilon \in 0$ then the system may become degenerate which means that it can not satisfy generic initial or periodicity conditions. In fact such system has two time scale: the fast variables x and the slow variables λ, where $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \lambda)$ and $\frac{\mathrm{d}\lambda}{\mathrm{d}t} = g(x, \lambda)$ respectively.

The idea behind singular perturbation theory is to take advantage of separable scales to obtain reduced problems that are simpler than the original full problem. We want to that problem by constructing an approximation of the full(difficult) solution of the singular problem in terms of solutions of the reduced problems in the basic element of the work in this field. Singular perturbation exist into two forms: the first one is algebraic and the second one is differential.

This study is based on the algebraic singular perturbation. Let us consider that $f(x, \lambda)$ is a quadratic equation. Then let us equate it to zero, therefore $f(x, \lambda) = 0$

Suppose that v_j is the eigenvector of A where its corresponding eigenvalue is λ_j. Then the set $Sp(v_j)$ is invariant under the flow that is generated by $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax$. Where A is the matrix.

If v_i is the second eigenvector, the set is $Sp(v_j, v_i)$ and it is invariant. A general case, is $Sp(v_1, ..., v_n)$ where the n-dimensional linear system with eigenvectors of A is $v_1, ..., v_n$ and is invariant. Note that for complex form, if λ_j is a complex eigenvalue, λ_j^*, is the complex conjugate of λ_j. Then the subspace is $Sp(Re(v_j), Im(v_j))$ and is invariant.

So if A is $n \times n$, we can decompose the space $\Re$ into a direct sum of three invariant subspaces:

$$E^s = \text{stable subspace} = Sp\{v_j | Re(\lambda_j) < 0\}$$
$$E^u = \text{unstable subspace} = Sp\{v_j | Re(\lambda_j) > 0\}$$
$$E^c = \text{centre subspace} = Sp\{v_j | Re(\lambda_j) = 0\}$$

Let dim $E^s = n^s$, dim $E^u = n^u$, and dim $E^c = n^c$. Clearly $n^s + n^u + n^c = n$

Now, considering $(1.2.1)$ is in $\Re^2$. It generates a flow ϕ *i.e.* $\phi(x_0, t)$ maps an initial condition $x(0) = x_0$ to the solution of $(1.2.1)$ is $x(x_0, t)$ at time $t \in \Re$.

2.1.1 Definition. $S \subset \Re^n$ is a local invariance manifold for (1.1.2) if for all $x_0 \in S$ we have that $\phi(x_0, t)$ for all $|t| < T$ for some number $T > 0$

Now let's consider an equilibrium point $\bar{x}$ and a little neighbourhood of it, V. We define:

$$W^s_{loc} \quad = \quad \{x \in V | \phi(x,t) \longrightarrow \bar{x} \; as \; t \longrightarrow \infty, \phi(x,t) \in V \; \forall t > 0\} \tag{2.1.1}$$

This is the local stable manifold of $\bar{x}$. The local unstable manifold is defined similarly:

$$W^u_{loc} \quad = \quad \{x \in V | \phi(x,t) \longrightarrow \bar{x} \; as \; t \longrightarrow -\infty, \phi(x,t) \in V \; \forall t < 0\} \tag{2.1.2}$$

The local stable and unstable manifolds of a equilibrium point $\bar{x}$ of the nonlinear system (1.1.2) are connect with stable and unstable subspaces of the origin corresponding linearised system,

$$\dot{y} \quad = \quad J(\bar{x}(t))y \tag{2.1.3}$$

2.1.2 Theorem. *(Stable manifold theorem). If the equilibrium point $\bar{x}$ of differential equation (1.1.2) is hyperbolic, then the local unstable and stable manifolds $W^u_{loc}(\bar{x})$ and $W^s_{loc}(\bar{x})$ have the same dimensions as the stable and unstable manifolds of the linearised system (2.1.3), E^s and E^u. Furthermore, W^s_{loc} is tangent to E^s and W^u_{loc} is a tangent to E^u at $\bar{x}$.*

Finally we define the global stable and unstable manifolds as

$$W^s(\bar{x}) = \bigcup_{t \leqslant 0} \phi(W^s_{loc}(\bar{x}), t)$$

,

$$W^u(\bar{x}) = \bigcup_{t \leqslant 0} \phi(W^u_{loc}(\bar{x}), t)$$

2.2 Fenichel theory

This section provides the proof of perturbed invariant manifolds due to Fenichel. In Jones (1995) work, we know that a theorem(Fenichel invariant manifold theorem) has proved by using two approaches. The first approach was given by Haramard and relies on the geometry that presents the splitting due to the decay rates. A second approach was given by Perron and is based on proving the existence of the invariant manifold as a fixed point of a certain integral equation. Fenichel had developed the work of Hadamard to come up with Fenichel theory while Parron work was developed to come up with Wazewski's principle.

In Fenichel theory, the generalized Lyapunov-type numbers have introduced to control the flow in the tangent and normal directions to the manifolds(Kuehn, 2015). Follow we give the definition of flows in tangent space.

2.2.1 Definition. Let M be the compact connected C^r-manifold with boundary embedded in $\Re^N$. Let $\phi(.)$ denoted the flow defined by the vector field()$1.0.2$

- M is called an inflowing invariance manifold if for every $p \in \partial M$ the vector field is pointing strictly inward and for all $p \in M$, $\phi_t(p) \in M$ for all $t \geq 0$

- M is called an overflowing invariance manifold if for every $p \in \partial M$ the vector field is pointing strictly outward and for all $p \in M$, $\phi_t(p) \in M$ for all $t \leq 0$
- M is called an invariant manifold if for every $p \in M$ we have $\phi_t(t) \in M$ for all $t \in \Re$
- M is called a locally invariant manifold if for each $p \in M$ there exists a time interval $I_p = (t_1, t_2)$ such that $0 \in I_p$ and $\phi_t(p) \in M$ for all $t \in I_p$

To see this definition diagrammatically is illustrated in Kuehn (2015) works. Explaining M as a compact in detail we use the tangent bundle. Kuehn (2015) denoted the projection of a vector as $\pi : T\Re^N|_M \to \aleph$ onto the basepoint. Since there is two component which is tangent component and normal component. He come up with the linearisation of the tangent flow A_t and the linearisation of the flow in the normal direction of the flow in the normal direction B_t is,

$$A_t(p) \quad = \quad D_{\phi-t}(t)|_M : TpM \to T_{\phi-t(p)}M \tag{2.2.1}$$

$$B_t(p) \quad = \quad \mathit{\Pi} \circ D_{\phi_t(\phi_t(p))}|_M : \aleph_{\phi-t(p)} \to \aleph_p \tag{2.2.2}$$

where $\phi_{-t}(p)$ is the linearised flow at the point p, T_pM is the tangent space while $\aleph_p$ denotes the normal space.

Finally in the generalised lyapunov-type numbers by Kuehn (2015), if $\frac{1}{||w_{-t}||} \to 0$ as $t \to \infty$. This geometrically it corresponds to the contraction of vectors in M's normal direction. Where the w_{-t} is the linearised flow based at point $\phi_{-t}(p)$ and equal to $(\pi \circ D_{\phi_{-t}(p)})w_0$. lastly w_0 is its initial. Follow we are explaining the contraction.

2.2.2 Definition. (Contraction). A map $f : \Re^2 \to \Re^2$, where $(\Re^2, d)$ is a metric space, is a contraction if there exists $\epsilon < 1$ such that

$$d(f(x), f(\lambda)) \leq \epsilon d(x, \lambda) \tag{2.2.3}$$

This condition is called a Lipschitz condition, and where $\epsilon \geq 0$ is called the Lipschitz constant. Then contraction are Lipschitz maps with a Lipschitz constant that is smaller than 1.

2.2.3 Theorem. *(Contraction mapping theorem). Let $\Re^2$ be a contraction and let $f : \Re^2 \to \Re^2$ also be a contraction. Then f has a unique fixed point, and under the action of iterates of $f : \Re^2 \to \Re^2$, all the points converge with exponential rate to it.*

Proof. The iterating $d(f^t(x), f^t(\lambda)) \leq \epsilon^t d(x, \lambda)$ gives us

$$d(f^t(x), f^t(\lambda)) \leq \epsilon^t d(x, \lambda) \tag{2.2.4}$$

where $x, \lambda \in \Re^2$ and $t \in \aleph$, A time is denote as t and, x and λ are variables. Thus $(f^t(x))_{t \in \aleph}$ is a cauchy sequence, because with $s > t$ we will have

$$d(f^s(x), f^t(x)) \leq \sum_{i=0}^{s-t-1} d(f^{t+s+1}(x), f^{t+s}(x)) \tag{2.2.5}$$

$$\leq \sum_{i=0}^{s-t-1} \epsilon^{t+i} d(f(x), x) \tag{2.2.6}$$

$$\leq \frac{\epsilon^t}{1-\epsilon} d(f(x), x) \tag{2.2.7}$$

and $\epsilon^t \to 0$ and $t \to \infty$. In the above last step we used the fact that with $0 \le \epsilon < 1$

$$\sum_{i=0}^{s-t-1} \epsilon^i \le \sum_{i=0}^{\infty} \epsilon^i = \frac{1}{1-\epsilon} \tag{2.2.8}$$

the limit $\lim_{t\to\infty}$ exists because the Cauchy sequences converge in $\Re^2$. Let us denote the limits as x_0 When $(2.2.5)$ under the iteration by f all the points that are in $\Re^2$, they converge to the same point as $lim_{t\to\infty} d(f^t(x), f^t(x)) = 0.$ □

The contraction mapping is a Lipschitz continuous both are obeying this condition $0 \le \epsilon < 1$.

3. Rate induced tipping

3.1 Tipping points with an non-autonomous system

The following scalar ordinary differential is a representative of prototype model which was firstly introduced by Ashwin et al. (2012). This model is represented by two differential equations which have state variable x and forcing parameter λ increasing with rate ϵ.

$$\dot{x} = f(x, \lambda) = (x + \lambda)^2 - 1$$

From singular perturbation, let $\dot{x} = 0$ then this equation will becomes ,

$$x^2 + 2\lambda x + \lambda^2 - 1 = 0$$

This implies that $x = -\lambda \pm 1$ is becoming two components $x_1 = -\lambda + 1$ and $x_2 = -\lambda - 1$ which are called two λ-dependent families of equilibria or equilibrium solution, where s and u are stable and unstable equilibria, respectively. From Section 2.2, we can give $\lambda(t)$ as

$$\lambda(t) = \frac{\lambda_{max}}{2}(\tanh(\lambda_{max}\epsilon t) + 1) \tag{3.1.1}$$

which is the Heaviside function. It's derivation of $\lambda(t)$, gives

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\lambda_{max}}{2}(\cosh^2(\lambda_{max}\epsilon t))^{-1} = \epsilon\lambda(\lambda_{max} - \lambda) \tag{3.1.2}$$

Therefore,

$$\dot{\lambda} = g(x, \lambda) = \epsilon\lambda(\lambda_{max} - \lambda)$$

Let $\dot{\lambda}$ and (3.1.2) becomes,

$$\epsilon\lambda(\lambda_{max} - \lambda) = 0$$

Which gives us the $\lambda_1 = \lambda_{max}$ and $\lambda_2 = 0$

Then the system's model is,

$$\dot{x} = f(x, \lambda) = (x + \lambda)^2 - 1 \tag{3.1.3}$$

$$\dot{\lambda} = g(x, \lambda) = \epsilon\lambda(\lambda_{max} - \lambda) \tag{3.1.4}$$

Let us assume that for every fixed value of λ in some domain, the system has a stable equilibrium, $\tilde{x}(\lambda)$. We can also assume that the stable state depends continuously on λ, so that equilibrium of the system can be written as $\tilde{x}(\lambda)$. But in a representative form for Stable and Unstable manifolds we will write $\tilde{x}(\lambda)$ as two quasi-static equilibrium(QSE), $W_0(s)$ and $W_0(u)$, respectively.

$$W_0^{(s)} = \{(\lambda, x) \in \Re^2 : \lambda = -x_1 + 1\} \quad (3.1.5)$$

$$W_0^{u} = \{(\lambda, x) \in \Re^2 : \lambda = -x_2 - 1\} \quad (3.1.6)$$

Both have a negative slope.

This system has four equilibrium points. In line $\lambda = \lambda_{max}$, a point $(\lambda_1, x_1) = (\lambda_{max}, -\lambda_{max} + 1)$ which is called U_+ and $(\lambda_1, x_2) = (\lambda_{max}, -\lambda_{max} - 1)$ which is called S_+. In line $\lambda = 0$ there is $(\lambda_2, x_1) = (0, 1)$ which is called U_- and $(\lambda_2, x_2) = (0, -1)$ which is called S_-

Here, we are linearise a non-linear system by perturbing x to $\delta x = x - x_0$ and $\delta\lambda = \lambda - \lambda_0$ then this implies $x = \delta x + x_0$ and $\lambda = \delta\lambda + \lambda_0$. This can be expressed in 2- dimensions by using Jacobian matrix,

$$J = \begin{pmatrix} f_\lambda & f_x \\ g_\lambda & g_x \end{pmatrix} = \begin{pmatrix} \epsilon\lambda - 2\epsilon\lambda & 0 \\ 2x + 2\lambda & 2x + 2\lambda \end{pmatrix} \quad (3.1.7)$$

where $f_\lambda = \epsilon\lambda\lambda_{max} - \epsilon\lambda^2$ and $f_x = 0$ are the entities of a row $g_\lambda = 2x + 2\lambda$ and $g_x = 2x + 2\lambda$ are the entities of a second row.

At equilibrium point$(\lambda_{max}, -\lambda_{max} + 1)$

$$J_{(\lambda_{max}, -\lambda_{max}+1)} \begin{pmatrix} -\epsilon\lambda_{max} & 0 \\ 2 & 2 \end{pmatrix} \quad (3.1.8)$$

Now consider the determinant $|J - \xi I| = 0$. In the 2×2 matrix case $|J - \xi I| = \xi^2 - (2 + \epsilon\lambda_{max})\xi - 2\epsilon\lambda_{max} = 0$, which is the characteristic polynomial of the matrix $J_{(\lambda_{max}, -\lambda_{max}+1)}$ with $\sum c_i \xi^i$ and the eigenvalues are $\xi_1 = -\epsilon\lambda_{max}$ and $\xi_2 = 2$. Since the eigenvalue have different signs, this equilibrium point is a saddle point and the eigenvectors that corresponding to $\xi_1 = -\epsilon\lambda_{max}$ is

$$v_1 = x \begin{pmatrix} -\frac{(2+\epsilon\lambda_{max})}{2} \\ 1 \end{pmatrix} \quad (3.1.9)$$

At equilibrium point $(\lambda_{max}, -\lambda_{max} + 1)$

$$J_{(\lambda_{max}, -\lambda_{max}-1)} = \begin{pmatrix} -\epsilon\lambda & 0 \\ -2 & -2 \end{pmatrix} \quad (3.1.10)$$

Likewise, the determinant $|J - \xi I| = 0$. Now, in this case of 2×2 matrix, the characteristic polynomial of the matrix $J_{(\lambda_{max}, -\lambda_{max}-1)}$ is $|J - \xi I| = \xi^2 + (2 + \epsilon\lambda_{max})\xi + 2\epsilon\lambda_{max} = 0$ with $\sum c_i \xi^i$ and eigenvalues are $\xi_3 = -\epsilon\lambda_{max}$ and $\xi_4 = -2$. Since both eigenvalues are negatives then this is a stable node. The eigenvectors corresponding to the ξ_3 are

$$v_3 = x \begin{pmatrix} -\frac{(2-\epsilon\lambda_{max})}{2} \\ 1 \end{pmatrix} \quad (3.1.11)$$

At equilibrium point $(0, 1)$ the Jacobian matrix is

$$J_{(0,1)} = \begin{pmatrix} \epsilon\lambda_{max} & 0 \\ 2 & 2 \end{pmatrix} \tag{3.1.12}$$

eigenvalues are $\xi_5 = \epsilon\lambda_{max}$ and $\xi_6 = 2$. Both eigenvalue signs are positives which means that the equilibrium point is an unstable node. Then the eigenvectors corresponding to $\xi_5 = \epsilon\lambda_{max}$ is

$$v_5 = \begin{pmatrix} -\frac{(2+\epsilon\lambda_{max})}{2} \\ 1 \end{pmatrix} \tag{3.1.13}$$

The Jacobian matrix at equilibrium point $(0, -1)$

$$J_{(0,-1)} = \begin{pmatrix} \epsilon\lambda_{max} & 0 \\ -2 & -2 \end{pmatrix} \tag{3.1.14}$$

and the eigenvalues are $\xi_7 = \epsilon\lambda_{max}$ and $\xi_8 = -2$.This is a Saddle point since the eigenvalues have different signs. Then the eigenvectors that are corresponding to $\xi_7 = \epsilon\lambda_{max}$ is

$$v_7 = x\begin{pmatrix} -\frac{(2-\epsilon\lambda_{max})}{2} \\ 1 \end{pmatrix} \tag{3.1.15}$$

3.2 Analysis for small epsilon

Consider these two differential equations $\frac{dx}{dt} = f(x, \lambda)$ and $\frac{d\lambda}{dt} = g(x, \lambda)$. Since both are divided by dt, we can called them parametric curves. If we think of parametric curves, as being traced out by a moving particle, then $\frac{dx}{dt}$ and $\frac{d\lambda}{dt}$ are the vertical and horizontal velocities of the particle and following equation says that the slope of tangent is the ratio of these velocities.

$$\frac{dx}{d\lambda} = \frac{\frac{dx}{dt}}{\frac{d\lambda}{dt}} \iff \frac{d\lambda}{dt} \neq 0 \tag{3.2.1}$$

Since the derivative is the slope of the tangent line in Figure 3.1, we interpret $\frac{dx}{d\lambda}$ geometrically to mean that any point (λ, x) is the plane, the tangent line must have slope, $h(x, \lambda)$ where $\frac{dx}{d\lambda} = h(x, \lambda)$. From the numerical experiment, a slope field in phase plane, slows with arrows indicate the slope at the grid of points.

The solution to $\frac{dx}{d\lambda}$ is a curve that is tangent to the arrows of the slope field when ϵ is two small. Since differential equation (3.2.1) is solved by using integrating, we call such a curve an integral curve. There are a lot of possible integral curves, infinitely many solutions to (3.2.1). To specify a particular integral curve, we specify a point, then the rest of the curves are determined by following the arrows. This corresponds to finding a particular4 solution by specifying an initial values.

From the limits, we can check the continuity of $f(x, \lambda)$ and $g(x, \lambda)$ at each equilibrium point. For U_- where (x, λ) is $(1, 0)$

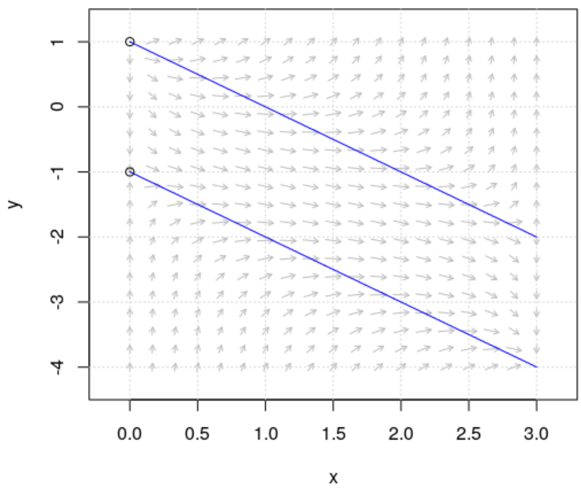

Figure 3.1: A phase plane for $<< \epsilon$. The upper line is the unstable QSE which is $W_0^{(u)}$ while the lower is stable QSE, $W_0^{(s)}$

$$\lim_{(x,\lambda)\to(1,0)} (x^2 + 2x\lambda + \lambda^2 - 1) = 0 \tag{3.2.2}$$

and

$$\lim_{(x,\lambda)\to(1,0)} (\epsilon\lambda_{max}\lambda - \epsilon\lambda^2) = 0 \tag{3.2.3}$$

At equilibrium point U_+ where (x, λ) is $(-2, 3)$,

$$\lim_{(x,\lambda)\to(-2,3)} (x^2 + 2x\lambda + \lambda^2 - 1) = 0 \tag{3.2.4}$$

and

$$\lim_{(x,\lambda)\to(-2,3)} (\epsilon\lambda_{max}\lambda - \epsilon\lambda^2) = 0 \tag{3.2.5}$$

This means that $f(x, \lambda)$ and $g(x, \lambda)$ is continuous. By using the deductive argument we can say,

- The manifold is bounded
- The bounded manifold has ϵ that is less than critical value.
- The manifold has its critical value. Therefore the critical value can be denoted as ϵ_c

Now we can say when $\epsilon < \epsilon_c$, and to produce Figure 3.1 we let ϵ to be less than 1. This agree with the contraction mapping theorem or Lipschitz continuous, this means that there is contraction mapping.

The unstable manifold of S_- goes to the ($\lambda = \lambda_{max}$)-plane below S_+ then all trajectories started in a neighbourhood of S_- will attract and converge to the stable node S_+ , due to Fenichel's theory.

A proof of existence of solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 + 2x\lambda(t) + \lambda^2 - 1 \tag{3.2.6}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} - 2x\lambda(t) = x^2 + \lambda^2 - 1 \tag{3.2.7}$$

The integrating factor is,

$$I(t) = e^{-\int 2\lambda(t)\mathrm{d}t} = e^{-2\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}}$$

If we multiply by integrating factor both sides in (3.2.7). We get $x(t)$ as the results.

$$x(t) = x_0 + \int e^{-2\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}}(x^2 + \lambda^2 - 1)\mathrm{d}t \tag{3.2.8}$$

This (3.2.8) is called Picard iteration.

$$x(t) - x_0 = \int e^{-2\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}}(x^2 + \lambda^2 - 1)\mathrm{d}t \tag{3.2.9}$$

Therefore, by using the contraction mapping theorem we can say,

$$|x(t) - x_0| = |\int e^{-2\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}}(x^2 + \lambda^2 - 1)\mathrm{d}t| \tag{3.2.10}$$

So, the L.H.S is the same what Ashwin et al. (2012) shows as the tipping radius.

$$|x(t) - x_0| < R \tag{3.2.11}$$

Now we can call the trajectories as the curvature in tangent space of the invariant manifolds. This tell us that the manifold is compact.

3.3 Analysis for large epsilon

If slope field in Figure 3.2 is represented by arrows and each arrow has a point. The solution to $\frac{dx}{d\lambda}$ is a curve that is tangent to the arrows of the slope field, when ϵ is too large. Since differential equation eqn:curvature01 is solved by using integrating, we call such a curve an integral curve. There are a lot of possible integral curves, infinitely many solutions to (3.2.1). To specify a particular integral curve, we specify a point, then the rest of the curves are determined by following the arrows. This corresponds to finding a particular4 solution by specifying an initial values.

Figure 3.2 is obtained by using $\epsilon > \epsilon_c$, where $\epsilon = 1.9$.

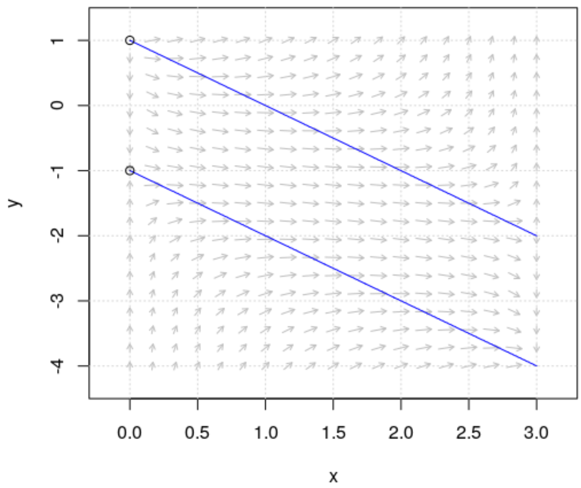

Figure 3.2: A phase plane for $>> \epsilon$ with W_0^u and W_0^s

By looking in Figure 3.2 the unstable manifold of the saddle on the left will cross the upper quasi-static equilibrium W_0^u. Then unstable manifold $W^u(S_-)$ no longer converges to the stable node S_+ such that trajectories from initial conditions close to S_- with $\lambda > 0$ diverge $(x(t) \longrightarrow +\infty)$. In this case the parameter λ is too large for the unstable manifold $W^u(S_-)$ to track the quasi-static equilibrium, $W_0^{(s)}$.

By using the deductive argument, we say,

- The manifold is unbounded
- The unbounded manifold has ϵ that is greater than critical value.
- The manifold has its critical value.

This deductive argument can be explained better by using Wazewski's principle (Onuchic, 1961). According to his principle somewhere in the middle there must be a critical value when the unstable manifold of the saddle point on th left must coincide with the stable manifold of the saddle at $\lambda = \lambda_{max}$. The existence of such a value $\epsilon = \epsilon_c$

Then a homotopy between quasi-static equilibrium and trajectories is the solution of ordinary differential equations $f(x, \lambda)$ and $g(x, \lambda)$, is denoted as $H(x, \lambda)$. A solution $x(t)$ is called asymptotic to $x \equiv 0$, if it well defined in $[0, \infty]$ and it satisfies $x(t) \to 0$ and $t \to \infty$. Then since the space in $\Re^3$ we can decomposed it as $\Re^2 \times \Re$ which means $\dot{x} = f(x, \lambda(t))$ and $\dot{\lambda} = g(x, \lambda(t))$. Let us consider the set:

$$\Omega_t := \{(x, \lambda) : ||x|| < \varphi(t), |\lambda| < \psi(t)\} \tag{3.3.1}$$

where $\varphi, \psi : [0, \infty] \to \Re$ are C^1 decreasing functions with $\varphi(t) \to 0$, $\psi(t) \to 0$ as $t \to 0$. In this we can assume that some trajectories would remain in inside the compost forever. If all the trajectories would escape the boundary of vibrating molecules of compost, they should do that through the lateral boundary and which is impossible. This outside the unit ball Lipschitz continuous. So this agrees with the Ashwin et al. (2012).

$$|X(t) - x_0| > R \tag{3.3.2}$$

3.4 Existence of a tipping transition

The time profile is used to find the relationship between parameter variable $\lambda(t)$ and time t. The aim is to see the transition period.

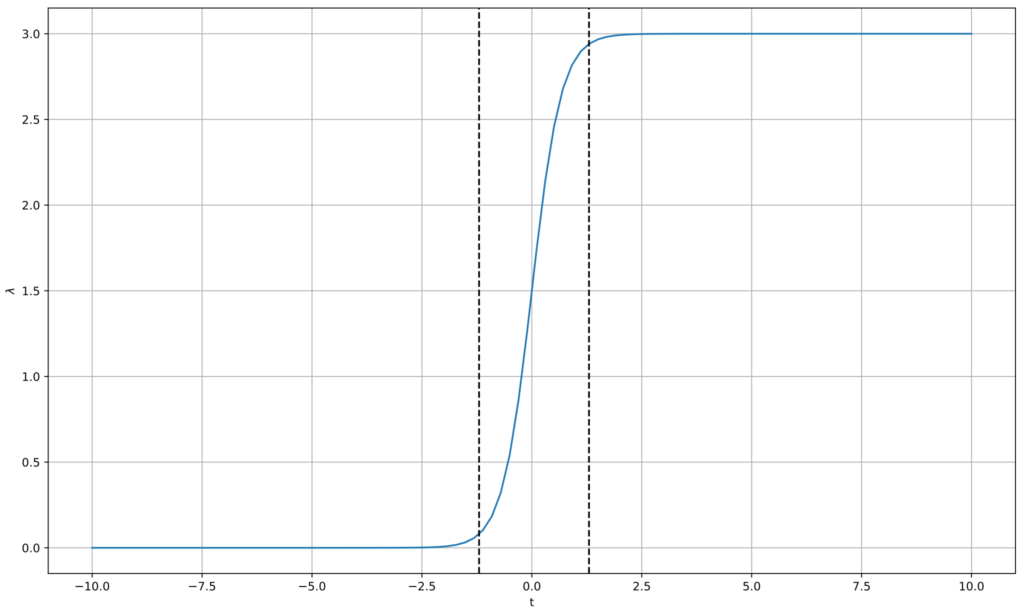

Figure 3.3: The phase transition

Let the origin be critical points for both differential equation of the model and let also assume that the starting and ending point of the Heaviside graph are following the circumference of circle. Then the solution of $f(x, \lambda)$ and $g(x, \lambda)$ are even since they hold for all x and $-x$ in the domain of $\varphi(t)$ and $\psi(t)$. If we denoted the solution of $f(x, \lambda)$ and $g(x, \lambda)$, respectively:

$$\varphi(x, \lambda) \quad = \quad \varphi(-x, -\lambda), \tag{3.4.1}$$

or

$$\varphi(x, \lambda) - \varphi(-x, -\lambda) \quad = \quad 0 \tag{3.4.2}$$

Geometrically, if we rotate Heaviside graphs by 180^0, it generate rotations about an origin.

$$\lim_{\epsilon \to 0} \lambda(t) = \lim_{\epsilon \to 0} (\frac{\lambda_{max}}{2} \tanh(\lambda_{max} \epsilon t) + \frac{\lambda_{max}}{2}) = \frac{\lambda_{max}}{2} \tag{3.4.3}$$

Since we plot this

This means that at $\frac{\lambda_{max}}{2}$

$$\begin{bmatrix} x - x_c \\ \lambda - \lambda_c \end{bmatrix} \to \begin{bmatrix} x_c - x \\ \lambda_c - \lambda \end{bmatrix} \tag{3.4.4}$$

So, we can say the Heaviside functions are odd functions and they are rotational symmetry with matrix $T = (T_x, T_\lambda)$. If we zoom in the near identity of an origin.

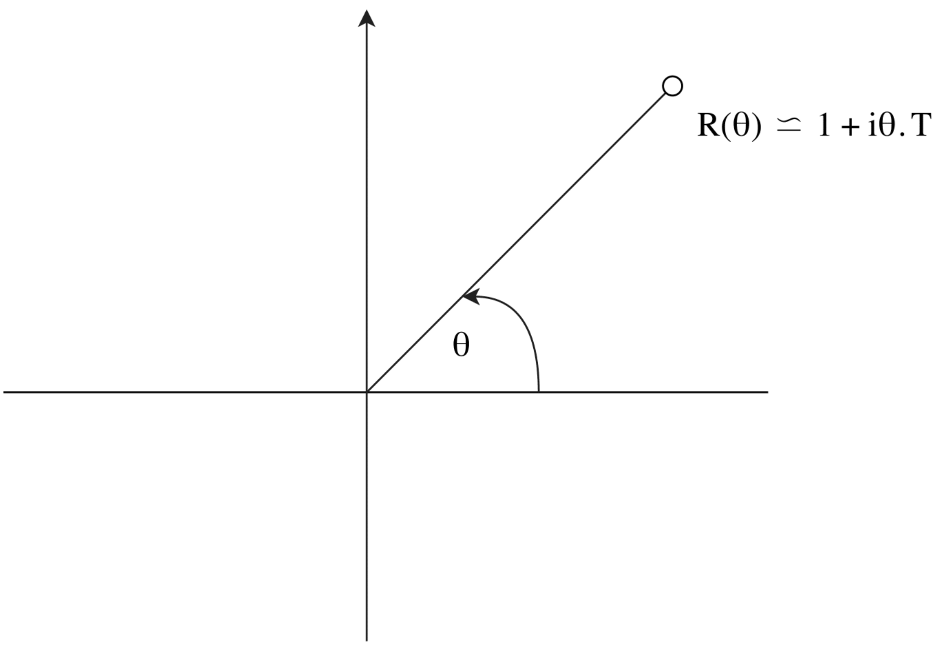

Figure 3.4: Radius of Tipping point of deterministic system

This means $R(\theta)$ is the tipping radius, as shown in (3.2.10) that $|x(t) - x_0|$ can be equal or less or greater than a tipping radius R The,

$$R(\theta) \backsimeq 1 + i\theta.T = |x(t) - x_0| \tag{3.4.5}$$

Since both are equal to tipping radius which is to mean quasi-static equalibrium x_0(which is W_0^s and W_0^u) is traced by the general solution $x(t)$.

4. Conclusion

When the $\epsilon = 0.1$ the manifold is compact or is less than the unit ball, but when $\epsilon = 1.9$ the manifold is not compact. This means the initial state was inside the unit ball then, as $t \to \infty$ the system escapes the unit ball and it never come back according Figure 3.3 after one phase transition. Ritchie and Sieber (2016) in this work show that the rate-induced tipping is the failure in tracking the continuous changes in quasi-states. Which is to say the rate-induced tipping has one transition that is between -1 and 1 according Figure 3.3. A practical application of this behaviour is in climate science, where the action is irreversible or has one transition. This is exactly what is happening when the Carbon dioxide CO_2 escape from compost in peatland.

4.1 Future work

Heaviside function as shown in Figure 3.3 tells us that the population density will increases slowly initially, then sudden increases rapidly. This we call it fast transition or canard. In future we will like to study this canards of compost or compost bomb instability.

Acknowledgements

Foremost, I would like to express my sincere gratitude to my supervisor Dr. Michael Grinfeld for the continuous support of my essay, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped and writing of this essay. I would not have imagined a better supervisor and mentor for my essay.

Besides my supervisor, I would like to say thanks to Hasinjatovo Mandanirina Nambinintsoa Roméoh for tutoring and mentoring me in writing the essay.

My sincere thanks to Prof. Jeff Sanders and Dr Eddy Kimba Phongi for assisting and discussing maths part.

The last thanks is going to Jan Groenewald and his team.

References

P. Ashwin, S. Wieczorek, R. Vitolo, and P. Cox. Tipping points in open systems: bifurcation, noise-induced and rate-dependent examples in the climate system. *Phil. Trans. R. Soc. A*, 370(1962): 1166–1184, 2012.

N. Berglund. Geometrical theory of dynamical systems. *arXiv preprint math/0111177*, 2001.

S. Dineen. Complex analysis, functional analysis and approximation theory, J. Mujica (editor). *Complex analysis, functional analysis and approximation theory*, 125:67, 1986.

P. D. Ditlevsen and S. J. Johnsen. Tipping points: early warning and wishful thinking. *Geophysical Research Letters*, 37(19), 2010.

K. Hasselmann. Stochastic climate models part i. theory. *Tellus*, 28(6):473–485, 1976.

C. K. Jones. Geometric singular perturbation theory. In *Dynamical systems*, pages 44–118. Springer, 1995.

C. Kuehn. General Fenichel theory. In *Multiple Time Scale Dynamics*, pages 19–51. Springer, 2015.

P. Lamberson, S. E. Page, et al. Tipping points. *Quarterly Journal of Political Science*, 7(2):175–208, 2012.

N. Onuchic. Applications of the topological method of Ważewski to certain problems of asymptotic behavior in ordinary differential equations. *Pacific Journal of Mathematics*, 11(4):1511–1527, 1961.

C. G. Perryman. How fast is too fast? Rate-induced bifurcations in multiple time-scale systems. 2015.

R. M. Redheffer and D. Port. *Differential equations: theory and applications*. Jones & Bartlett Learning, 1991.

P. Ritchie and J. Sieber. Early-warning indicators for rate-induced tipping. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 26(9):093116, 2016.

J. M. T. Thompson and J. Sieber. Predicting climate tipping as a noisy bifurcation: a review. *International Journal of Bifurcation and Chaos*, 21(02):399–423, 2011.