# Approximation for Total Value Adjustment of American Put Options using Finite Difference Method.

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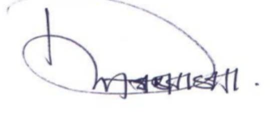
# Abstract

During the 2007-2008 financial crisis where we saw the bankruptcy of many financial institutions, the risk of counterparty is now considered when pricing derivatives. We consider the pricing of total value adjustments (XVA) to be added to the price of the derivative without counterparty. Since American options are governed by Linear Complementary Problem (LCP), we then formulate the LCP model. However, the LCP does not have analytic solution, so in this sense numerical method is adapted. The Linear PDE without counterparty risk is discretized in space using central difference method and for time Euler $\theta$ method is used. We then present the numerical simulations and conclusion.

## Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



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# 1. Introduction

The mismanagement of model risk can be drastic to many financial organizations and can result in a huge financial loss of funds in which we can see history repeating itself, like the 2008 global financial crisis. After this crisis we saw new development in pricing of derivatives and other contracts in the financial world (Arregui et al. (2017)). The introduction of valuation adjustments (XVAs) was one of such developments, in which many still find it difficulty to understand this concept (Lu (2016)). The XVA is a framework that governed the valuation adjustments needed when valuing the derivative (Lu (2016)). Arregui et al. (2017) revealed that the crisis was due to errors made in management of the risk which can be explained in two factors; the complexity of the financial derivatives and a low probability of default. According to Bozeat et al. (2015) the only challenge in the implementation of XVAs in the banking sector and other financial institutions is that, it requires a huge transformation in the operating model change to traditional front office trading operations and IT infrastructure, so that it can assist in the following three structures; Finance,Risk and Operations functions with the change.

The framework consists of the following components; Credit Valuation adjustment (CVA-captures the discount to the standard derivative value that a buyer would offer given the of counterparty default.), Debit Valuation adjustments (DVA-records gain and act as the bank's own credit risk deteriorates) (Bozeat et al. (2015)) and Funding Valuation adjustments(FVA-arises when Bank is in need of the cash). The literature from Bozeat et al. (2015) suggested that many people are not comfortable with the inclusion of DVA,CVA and FVA in the XVAs framework simply because they consider the inclusion as double counting in some cases.

Derivatives trading is becoming more popular in the world finance. Derivative is often defined as the secondary financial instrument because its price depend on the price of other securities such as stock, interest rate and etc. There are three popular contracts traded in the market namely; options, swaps and forwards. An option tend to give the holder with the right but not the obligation to sell (put) or buy (call) an asset or stock at a certain strike price on or before the expiry date (Hull (2012)). A swap is a financial contract between two parties to exchange cash flows in the future according to some prearranged format (Kwok (2008)). A forward(also called futures when traded on the exchange market) is an agreement between two parties that one party will purchase an asset from the counterparty on a certain date in the future for a predetermined price (Kwok (2008)). There are two main kinds of options; the European(it can only be exercised at the end of expiry date) and American(it can exercised any time until expiry date). The Black-Scholes Model is applicable when the pricing process is continuous and it can also estimate the value of any option (Hull (2012)). The model seems to work well on assets that do not pay dividends, meaning one can say it will produce good results when dealing with European options. In this case, pricing an American option won't be straight forward as European because we do not know the actual exercise day.

feng et al. (2009) suggested that pricing an American option is governed by Linear Complementary Problem which involves the Black-Scholes differential operator. In general, linear complementary problem is analytically not solvable but with the use of numerical methods, we can estimate the pricing of an American option.

In this essay we will consider pricing an American option using LCP and including the XVA's in our underlying model.

This project is organized as follows. In chapter 1 we introduce some concepts in probability theory, stochastic process, options, counterparty and valuation adjustment. Furthermore, we also going to derive the Black-Scholes PDE and linear complementary problem associated with American put option. In chapter 2 we will take into consider the concept of bilateral counterparty risk and derive the PDE to price total credit valuation to be added to the price of American option without counterparty risk. In chapter 3 central difference method will be used for space discretization and for time will use Euler $\theta$ method. Furthermore, present the numerical simulation. Finally, in chapter 4 we give conclusion based on our findings.

## 1.1 Basic Notion

We first introduce the concepts of **probability theory** and **stochastic calculus** because we think these two concepts are essential when defining some financial terms and theorems. Furthermore, we will derive the Black-Scholes model and linear **linear complementary problem** which will help in valuing American option.

## 1.2 Probability Theory

Probability is said to be the chance that something will happen. The main purpose of this section is give foundation to readers who might not be familiar with these concept.

**1.2.1 Definition** (Measurable)**.** (Knil (2017)) A map $X$ from a measure space$(\Omega, \mathcal{A})$ to an other measure space$(\delta, \mathcal{B})$ is called *measurable* if $X^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$.

**1.2.2 Definition** (Measurable space)**.** (Knil (2017)) Let $\Omega$ be an arbitrary set.

A set $\mathcal{A}$ of subsets $\Omega$ is called $\sigma$-algebra if the following three properties are satisfied:

(i) $\Omega \in \mathcal{A}$,

(ii) $\mathsf{A} \in \mathcal{A} \rightarrow \mathsf{A} = \Omega \setminus A \in \mathcal{A}$,

(iii) $\mathsf{A}_n \in \mathcal{A} \rightarrow \cup_{n\in N}\mathsf{A}_n \in \mathcal{A}$

A pair $(\Omega, \mathcal{A})$ for which $\mathcal{A}$ is a $\sigma$-algebra in $\Omega$ is called a *measurable space*.

**1.2.3 Definition** (Probability space)**.** (Knil (2017)) Let $\Omega, \mathcal{A}$ be a measurable space and the function $P : \mathcal{A} \Rightarrow \mathbf{R}$ be probability measure. Then $(\Omega, \mathcal{A}, P)$ is called a *probability space* if the following three properties are satisfied:

(i) $P(A) \geq 0$ for all $A \in \mathcal{A}$,

(ii) $P(\Omega) = 1$

(iii) $A_n \in \mathcal{A}$ disjoint $\Rightarrow P(\cup_n A_n) = \sum_n P(A_n)$

**1.2.4 Remark.** The above three properties are also called the Kolmogorov axioms of probability.

**1.2.5 Definition** (Conditional probability)**.** (Knil (2017)) Given a set $\mathsf{B} \in \mathcal{A}$ with $P(B) > 0$, we define

$$P(\mathsf{A} \setminus \mathsf{B}) = \frac{P(\mathsf{A} \cap \mathsf{B})}{P(\mathsf{B})},$$

the *conditional probability* of A with respect to B.

**1.2.6 Remark.** It is the probability of the event A, under the condition that the event B happens.

**1.2.7 Definition** (Random variable)**.** (Knil (2017)) A function $X : \Omega \to \mathbb{R}$ is called a random variable, if it is a measurable map from $(\Omega, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel $\sigma$-algebra of $\mathbb{R}$.

**1.2.8 Remark.** More generally, one can consider random variables taking values in a second measurable space (E,B). If E $= \mathbb{R}$, then random variable $X$ is called a random vector. For a random vector $X = (X_1, \cdots, X_d)$,each component $X_i$ is a random variable.

## 1.3 Stochastic Processes

The theory of **stochastic processes** is considered to be an important contribution to mathematical finance and it continues to be an active topic of research for both theoretical reasons and applications(Talagrand (2014)). Stochastic processes can be classified as **discrete-time stochastic process** is one where the value of the variable can change only at certain fixed points in time and **continuous-time stochastic process** is one where changes can take place at any time(Hull (2012)). In this section will focus on the **continuous-time stochastic process**. This concept will help the readers to understand the pricing of options and other derivatives.

**1.3.1 Definition** (Stochastic process)**.** (Knil (2017)) Let $(\Omega, \mathcal{A}, P)$ be a probability space and let T $\subset \mathbb{R}$ be time.

A collection of random variables $X_t, t \in$ T with values $\mathbb{R}$ is called a *stochastic process*.

**1.3.2 Definition** (Filtration)**.** (Knil (2017))

A filtration of a measurable space $(\Omega, \mathcal{A})$ is an increasing family $(\mathcal{A}_t)_{t\geq 0}$ of sub-$\sigma$-algebra of $\mathcal{A}$.

A measurable space endowed with a filtration $(\mathcal{A})_{t\geq 0}$ is called **filtered space**. A process $X$ is called **adapted** to the filtration $\mathcal{A}_t$, if $X_t$ is $\mathcal{A}$ is measurable for all $t$.

**1.3.3 Definition** (Martingale)**.** (Knil (2017)) Given a filtration $\mathcal{A}_t$ of the probability space $(\Omega, \mathcal{A}, P)$.

A real-valued process $X_t$ which is $\mathcal{A}_t$ adapted is called a **submartingale**, if $E(X_t \setminus \mathcal{A}_t) \geq X_s$, it is called a **supermartingale**(if $X$ is a submartingale) and a **martingale**, if it is both a super and sub-martingale.

**1.3.4 Definition** (Brownian Motion)**.** (Klebaner (1998)) Brownian Motion $\{B(t)\}$ is a stochastic process with he following properties:

(i) For all $0 = t_0 < t_1 < \cdots < t_n$, the increments

$$B(t_1) - B(t_0), B(t_2) - B(t_1), \cdot, B(t_m) - B(t_{m-1})$$

are independent.

(ii) $B(t) - B(s)$ has normal distribution with mean zero and variance $t - s$.

(iii) $B(s), t \geq 0$ are continuous functions of $t$.

**1.3.5 Remark.** If the process is started at $x$, the $B(t)$ has the $N(x,t)$ distribution. This can be written as:

$$P_x(B(t) \in (a,b)) = \int_a^b \frac{1}{\sqrt{2\pi t}} \exp^{-\frac{(y-x)^2}{2t}} dy.$$

Since the Brownian motion can take on negative values, using it directly for modelling stock prices might be inappropriate.

So now, we introduce the **Geometric Brownian Motion**.

**1.3.6 Definition** (Geometric Brownian Motion)**.** (Klebaner (1998))

Let $S(t)$ be a non-negative of Brownian Motion given by;

$$S(t) = S_0 \exp^{X(t)} \tag{1.3.1}$$

where $X(t) = \sigma B(t) + \mu t$ is Brownian motion with drift and $S(0) = S_0 > 0$ is the initial value.

**1.3.7 Remark.** Taking logarithms of equation one we yields back the Brownian motion; $X(t) = \ln(\frac{S(t)}{S_0}) = \ln(S(t)) - ln(S_0)$. Meaning $\ln(S(t)) = \ln(S_0) + X_t$ is normal with mean $\mu t + \ln(S_0)$ and variance $\sigma^2 t$; thus, for each $t$, $S(t)$ has a log-normal distribution.

**1.3.8 Definition** (Ito's process)**.** (Jiang (2005)) The process $Y(t)$ is called an Ito's process if it can be represented as:

$$Y(t) = Y(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dB(s), 0 \leq t \leq T$$

or equivalently, it has a stochastic differential as:

$$dY(t) = \mu(t)dt + \sigma(t)dB(t)$$

where process $\mu(t)$ and $\sigma(t)$ satisfy conditions:

(i) $\mu(t)$ is adapted and $\int_0^T \mid \mu(t) \mid dt < \infty$ a.s

(ii) $\sigma(t)$ is predictable and $\int_0^T \sigma(t)ds < \infty$

**1.3.9 Remark.** Function $\mu$ is often called the drift coefficient and function $\sigma$ the diffusion coefficient. Notice that $\mu$ and $\sigma$ can depend on $Y(t)$ and $B(t)$.

**1.3.10 Definition** (Ito's Integral)**.** (Jiang (2005)) For any $T > 0$ and any stochastic process $f \in \mathcal{M}^2$, the stochastic integral of $f$ on $[0, T]$ is defined by

$$I_T = \lim_{n \to \infty} \sum_{j=0}^{n-1} f(t_j)(W(t_{j+1}) - W(t_j)),$$

where $(0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T)$ is any partition of $[0, T]$ with max$_j$ $\mid t_j - t_{j-1} \mid \to 0$ as n $\to \infty$. The function $I_T(f)$ is sometime written as:

$$I_T(f) = \int_0^T f(t)dW_t \text{ or } \int_0^T f dW.$$

**1.3.11 Theorem** (Ito's Lemma)**.** *(Jiang (2005)) Let $V$ be a function of (S,t). Then $V$ follows the Wiener process or Brownian motion which is given by:*

$$dV = \left( \frac{\partial V}{\partial S} \mu S + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial V}{\partial S} \sigma S dZ \tag{1.3.2}$$

*where dZ is the Wiener process.*

**1.3.12 Definition** (Stock)**.** (Webots1) A stock is a type of security that signifies ownership in a corporation and represent a claim on part of the corporation's assets and earnings.

**1.3.13 Definition** (Bond)**.** (Webots2) A bond is a debt security under which the issuer owes the holders a debt and is obliged to pay them interest at later date.

**1.3.14 Definition** (Over-The-Counter(OTC) Market)**.** (Siadat (2016)) OTC market is a telephone and computer-linked network of dealers and trades that are usually done over the phone.

**1.3.15 Definition** (Swap)**.** (Siadat (2016)) A Swap is an OTC agreement between two parties to exchange financial instruments in the future.

**1.3.16 Definition** (Repo Market)**.** (Siadat (2016)) A repurchase agreement is a form of transaction between two parties where one party sells an asset at a specified price and commits to repurchase the asset from the other party at a different price at a future date.

**1.3.17 Definition** (Self-financing portfolio)**.** (Kwok (2008)) An investment strategy is to be self-financing if no extra funds are added or withdrawn from the initial investment.

**1.3.18 Definition** (Hedging)**.** (Jiang (2005)) In simple terms hedging is to invest on both sides to avoid or reduce loss. In this case, investors enter the derivatives markets to shift or reduce the price risks in the underlying asset markets to secure anticipated profits.

## 1.4 Options

Today, stock options and futures are the most financial instruments that the investors tend to read about on the business news. Option trading is becoming the main tool for financial institution, managers and other people to generate more money. This concepts trading is believed to be risky if you don't have certain personalities like patience, perseverance and knowledge(Cohen (2013)).

Options differs from forward and futures contracts in this manner, in option contract you are not obligated to buy or sell whereas in forwards contract you are obligated to buy or sell. As mentioned in introduction, we have two types of options, put and call. A **put option** gives the holder the right to sell an asset by a certain date for a certain price (Hull (2012)). A **call option** gives the holder the right to buy an asset by a certain date for a certain price (Hull (2012)). The day in which the holder can exercise the right is called the **maturity date**. The price in which the holder can purchase the asset or is known as the **exercise price** or **strike price**.

We now consider two main styles of options, namely; the **European** and **American** option. The European option gives the right to buy or sell an underlying asset by a certain price at the maturity date, whereas an American option gives the right to buy or sell an underlying asset at a certain price at any time before the maturity date.

Every option contract has two sides namely; long and short positions. If $K$ is the strike price and $S_T$ is the final price of the underlying asset, the payoff from a long position in a **European call option** is:

$$\max(S_T - K, 0)$$

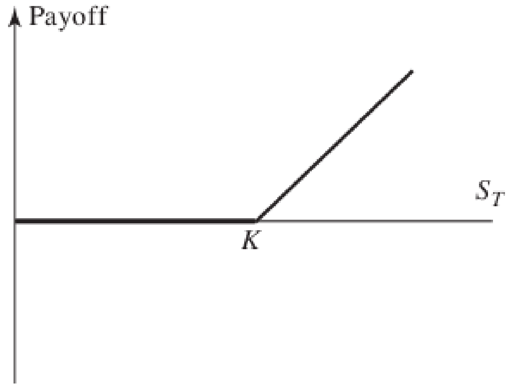


Figure 1.1: Call Option

This option will be exercised if $S_T > K$ and will not be exercised if $S_T \leq K$.

The payoff to the holder of a long position in a **European put option** is given by:

$$\max(K - S_T, 0)$$

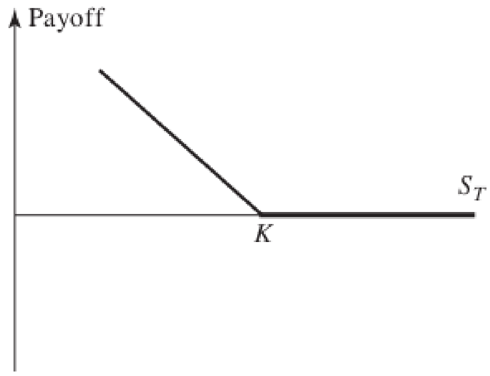


Figure 1.2: Put Option

Also, this option will be exercised if $K > S_T$ and will not be exercised if $K \leq S_T$.

Below is the analytic Black-Scholes formula for valuing call and put options when the underlying asset is ignoring the dividend payment, that is:

$$\boxed{C(S_t, t) = SN(d_1) - K\exp(-rt)N(d_2)} \tag{1.4.1}$$

and

$$\boxed{P(S_t, t) = K\exp(-rt)N(d_2) - SN(d_1)} \tag{1.4.2}$$

Where

$$d_1 = \frac{ln(\frac{S}{K}) + t(r + \frac{\sigma^2}{2})}{\sigma\sqrt{t}}, \quad d_2 = d_1 - \sigma\sqrt{t}$$

- $S$ = Stock price
- $K$ = Strike price
- $t$ = time remaining until expiration
- $r$ = Risk-free Interest rate
- $\sigma$ = Annual volatility of stock price
- $\ln$ = Natural logarithm
- $N(x)$ = Standard normal Cumulate distribution function
- $\exp$ = the exponetial function

## 1.5 The Black-Scholes Pricing Model

The theory of option pricing has been present since 1973, when the Black and Scholes published their paper providing some improvement in valuing dividend-protected European options(Webots3). The use of replicating portfolio made the following to be possible; the asset and the risk-free asset that had the same cash flows as the option being valued.

The Black-Scholes model it is the world 's most well-known model for calculating the premium of an option. The model was developed by three economists namely; Fischer Black, Myron Scholes and Robert Merton in which was first introduced in 1973 in a paper entitled, "The Pricing of Options and Corporate Liabilities"(Folger (2016)). The Black-Scholes model was meant to calculate the price of European put and call options when the underlying asset does not accept dividends.

The model is derived under the umbrella of the following assumptions Hull (2012):

- The stock prices follows the geometric brownian motion with the drift rate $\mu$ and with constant volatility $\sigma$.

$$dS = \mu S dt + \sigma S dz \tag{1.5.1}$$

  where $dz$ follows Weiner process.
- The short selling of securities with use of proceeds is permitted.
- There are no transactions cost or taxes.
- There are no dividends during the life of the derivative.
- There are no riskless arbitrage opportunities.
- Securities trading is continuous.
- The risk-free interest rate, $r$ is constant and the same for all maturities.

The above theory and assumptions gives us a bit of an understanding of the Black-Scholes model for pricing options. Then, next we start by deriving the equation.

The Equ 1.3.2 satisfies the value of an European option. That is;

We define a risk-less portfolio in this manner; a shot position on the option$(-V)$ and a long position on the number of shares$(\frac{\partial G}{\partial S})$(Hull (2012)). Then the value of the portfolio $\Pi$ is given by:

$$\Pi = -V + \frac{\partial V}{\partial S} S$$

then the change in the value of portfolio is;

$$d\Pi = -dV + \frac{\partial V}{\partial S} dS$$

and by the Weiner process we get;

$$\begin{aligned} d\Pi &= -\left[\left(\frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial V}{\partial S}\sigma S dZ\right] + \frac{\partial V}{\partial S}\left(\mu S dt + \sigma S dZ\right) \\ &= -\frac{\partial V}{\partial t}dt - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt \end{aligned}$$

We managed to eliminate the term with $dZ$, meaning the portfolio is risk-free. That is:

$$d\Pi = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)dt$$

Therefore a risk-free portfolio must earn a risk-free interest rate,$r$ at time $t$,

$$r\Pi dt = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)dt$$

Then, we substitute $\Pi = -V + \frac{\partial V}{\partial S}S$, we get;

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{\partial V}{\partial S} rS - rV = 0} \tag{1.5.2}$$

The Equ 1.5.2 is called the Black-Scholes Differential Equation.

## 1.6 The Linear Complementary Problem (LCP)

European and American options are more less the same, except that the American options can be exercised at any day prior to maturity, this is called **early exercise premium**. Then it is therefore clear that American options must worth at least as much as European options, that is:

$$C_t(S,T) \geq c_t(S,T)$$
$$Pt(S,T) \geq p_t(S,T)$$

Where $C_t(S,T)$ and $Pt(S,T)$ denote American call and put respectively and $c_t(S,T)$ and $p_t(S,T)$ denote European call and put options.

The Black-Scholes model was designed to value options that are being exercised at maturity date. So in this case, the European options tend to be the easy ones to be valued than the American options. From literature there are many suggestions on how to price an American put option. In this paper we will adopt one suggestion called the **Linear Complementary problems**. We are basically going to transform the Black-Scholes model in to linear form and add an XVA's term in our underlying model.

Most American option pricing model are governed by linear complementary problems which involves Partial differential operator(Huang and Pong (1998)). We start by stating some theorems first;

**1.6.1 Theorem.** *(Palczewski) The price of an American call option with expiry date $T$ and exercise price $K$ is equal to the price of a European call option with exercise price $K$ expiring at $T$ if the underlying asset does not pay dividends.*

**1.6.2 Definition** (American Instruments)**.** (Palczewski) American type instrument with maturity $T$ and pay-off function $f$ is a contingent claim that can be exercised at any moment up to $T$ with the payoff at $t$ being:

$$f(S_t, t).$$

**1.6.3 Definition** (The price of an American claim)**.** (Palczewski) The price of an American at time $t$ is given by:

$$V(S_t, t)$$

for some function $V : (0, \infty) \times [0, T] \to \mathbb{R}$.

**1.6.4 Remark.** Since $V(S_t, t)$ is the price of an instrument at $t$ and $f(S_t, t)$ is the pay-off of the instrument if it is exercised at $t$. $V(s,t) \geq f(s,t), \quad (s,t) \in (0,\infty) \times [0,T]$, so that we can prevent arbitrage from happening. If $V(s,t) > f(s,t)$ the holder should not exercise but sell the option so that he or she can make profit. And if $V(s,t) = f(s,t)$, now the holder must exercise the option immediately because is at optimal and avoid loosing the investment made for the option.

**1.6.5 The American Option PDE.** The price of an American option before the exercising day forms a value process of the replicating strategy(Webots).The value process is a martingale with respect to risk-neutral measure(Palczewski). We will follow the lecture notes from Palczewski to derive the LCP. By Ito's lemma, we have

$$dV = \left(\frac{\partial V(s,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 V(s,t)}{\partial s^2}\sigma^2 s^2 + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t)\right)dt + \sigma s\frac{\partial V(s,t)}{\partial s}dZ$$

The above process is a martingale if and only if the $dt$ term is zero. That is:

$$\frac{\partial V(s,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 V(s,t)}{\partial s^2}\sigma^2 s^2 + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) = 0$$

meaning we have $V(s,t) > f(s,t)$. But the optimal exercise is when $V(s,t) = f(s,t)$ and

$$\frac{\partial V(s,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 V(s,t)}{\partial s^2}\sigma^2 s^2 + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) \leq 0 \tag{1.6.1}$$

The inequality in Equ 1.6.1 arises from the fact that $V(s,t)$ may not represent the value process of some strategy after the optimal exercise point. So now, from the above equations we can now summarise the results in this manner;

$$\begin{cases} V(s,t) \geq f(s,t) \\ \frac{\partial V(s,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 V(s,t)}{\partial s^2}\sigma^2 S^2 + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) \leq 0 \\ V(s,t) = f(s,t) \text{ or } \frac{\partial V(s,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 V(s,t)}{\partial s^2}\sigma^2 s^2 + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) = 0 \end{cases}$$

The free-boundary set of points arises from the fact that $V(s,t) = f(s,t)$. Here are the boundary conditions for an American put option:

- Terminal: $V(s,t) = (K-S)^+$
- Left-boundary: as $s \to 0$, we have $\lim_{s\to 0} V(s,t) = K$
- Right-boundary: as $s \to \infty$, we have $\lim_{s\to\infty} V(s,t) = 0$

Then the free-boundary problem, for $S > 0$ and $t \in [0,T]$ is given by:

$$\begin{cases} V(s,t) \geq (K-S)^+ \\ \frac{\partial V(s,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 V(s,t)}{\partial s^2}\sigma^2 s^2 + rs\frac{\partial V(s,t)}{\partial S} - rV(s,t) \leq 0 \\ V(s,t) = f(s,t) \text{ or } \frac{\partial V(s,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 V(s,t)}{\partial s^2}\sigma^2 s^2 + rS\frac{\partial V(s,t)}{\partial s} - rV(s,t) = 0 \\ V(s,t) = (K-S)^+ \\ \lim_{s\to 0} V(s,t) = K \\ \lim_{s\to\infty} V(s,t) = 0 \end{cases} \tag{1.6.2}$$

Next we compute the price of an American option by making use of our free-boundary problem and reconstruct the pay-off function $G(s,t)$. By making use of change of variables in our free boundary problem, we obtain the following:

$$\begin{cases} \left(\frac{\partial V(s,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 V(s,t)}{\partial s^2}\sigma^2 s^2 + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t)\right)\left(V(s,t) - G(s,t)\right) = 0 \\ V(s,t) - G(s,t) \geq 0, \\ \frac{\partial V(s,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 V(s,t)}{\partial s^2}\sigma^2 s^2 + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) \leq 0 \\ V(s,0) = G(s,0) \\ \lim_{s\to+\infty} V(s,t) = \lim_{s\to+\infty} G(s,t) \\ \lim_{s\to 0} V(s,t) = \lim_{s\to 0} G(s,t) \end{cases} \tag{1.6.3}$$

The above problem is called the **linear complementary problem** for the American instrument with the pay-off function $G(s,t)$.

## 1.7 The Counterparty Risk

Before 2007-2008 financial crisis , the market risk was measured using the volatility and interest rate only. There were two classes of risk namely; the non-specific and specific. Specific risk is the risk that can affect certain institution alone whereas non-specific risk can affect the whole market participants. The counterparty risk is a complex term in the world of finance which one need to first understand the following risk; market, credit, operational and liquidity, in order to have a clear picture. The presence of this risk left many financial institution worried about the future growth of derivative and market. It mainly occurs in the following financial products, namely; Over the counter(OTC) derivatives(example: FX forwards,credit defaults swaps and interest rate swaps) and securities financing transactions(example: securities borrowing and lending, repos and reverse repos).The OTC market seems to be the one which gives rise to this risk since it has many participants.

Counterparty risk is the risk that a counterparty in a derivatives transaction will default prior to expiration of a trade and will not therefore make the current and future payments required by the contract(Gregory (2010)). For example; for a fixed rate loan in which the borrower makes annual fixed payments to the lender for five years before repaying the loan principal(Green (2016)). The borrower will defaults if they fail to any of the interest payments(Gregory (2010)). The bankruptcies we witnessed in many financial institution during 2007-2008 was due to counterparty risk(Gregory (2010)). **Netting** and **collateralisation** can be useful methods in reducing the this risk of parties. Netting method is limited to certain derivatives contract, meaning it might not work well in some contracts, whereas collateralisation is not limited but it has some challenges in the execution of operation cost which gives rise to other risk like; liquidity and legal(Gregory (2010)). Usually the counterparty risk occurs when one counterparty is expose to the other(Green (2016)).

**1.7.1 Definition** (The exposure at default(EAD))**.** (Green (2016)) The exposure at default is the total amount owed by the defaulting party to the non-defaulting party, that is;

$$\text{EAD} = \max(V, 0)$$

where $V$ is the derivative value.

**1.7.2 Remark.** There is no exposure if the non-defaulting party owes the defaulting party money and this gives the max of EAD.

**1.7.3 Definition** (Expected positive exposure(EPE))**.** (Green (2016)) The expected positive exposure is the expected exposure of party A to their counterparty B at some future date with the expectation or average taken over all possible future outcomes on the date of interest, that is:

$$\text{EPE} = \mathcal{E}_t[\max(V, 0)|\mathcal{F}_t]$$

**1.7.4 Remark.** These exposure measures plays a crucial role in the calculation of credit valuation adjustment(CVA). Then the expected negative exposure is the expected exposure that party B has on party A, that is:

$$\text{ENE} = \mathcal{E}_t[\max(-V, 0)|\mathcal{F}_t]$$

ENE is necessary for the calculation of debit and funding valuation adjustments.

**1.7.5 Definition** (Unilateral Counterparty Risk). (Green (2016)) The risk-free value of derivative $V$ yields to $\hat{V}$ (the adjusted price of the derivative) when we consider the possibility of loss for the default of Counterparty. In the point of view of the seller the adjusted derivative price will be:

$$\hat{V} = V_s - \text{UCVA}$$

where UCVA is the unilateral CVA and $V_s$ is the derivative value calculated by the seller. For counterparty, the adjusted value of derivative is given by, $\hat{V}_C = V_C - \text{UCVA}_C$. Generally the unilateral credit adjustment will not be same with that one of the seller, so the two parties will not agree on the risky price of derivative.

**1.7.6 Definition** (Bilateral(CVA:DVA)). (Lu (2016)) The bilateral mechanism is when two credit entities enter into a trade with each other where they establish a relationship of borrowing or lending, meaning one entity will owe the other entity. for example, entity A trades a derivative with entity B, with the effect of A lending to B, meaning the effect of trade will create an asset on A's book and a liability on B's book. In general, the value of the trade will be:

$$\text{FairValue} = \text{RiskfreeValue} + \text{adjustment}(\text{creditRisk})$$

Specifically,

$$\text{Value A} = \text{Value A(credit risk free)} + \text{CVA(B's default risk)}$$

$$\text{Value B} = -\text{Value (A)} = \text{Value B(credit risk free)} + \text{DVA(B's credit risk)}$$

$$\text{CVA(B's credit risk)} - \text{DVA(B's credit risk)}.$$

## 1.8 Valuation Adjustment (the XVA Framework)

The main idea of Black-Scholes approach is that the seller can replicate the behaviour of derivative in two ways; buying and selling the underlying asset and lending and borrowing money at a riskless interest rate(Ruiz (2015)). In simple terms, you come up with a risk-less portfolio and making sure that the default chances are small. The whole Black-Scholes framework was working well back there but new developments in the market turned things around, for example the concept of "default-free" entity. In real world there is no such a thing called default-free entity. According to Ruiz (2015) the US government was downgraded from AAA to a AA rating in 2011(AAA rating means that it is most unlikely that, that institution will default) and another example was the London market(the most active and liquid market in the world) in which froze in 2008. So one can say that even though the crisis was so bad where we saw the bankruptcy of many financial institution(the likes of Lehman Brothers and others) but on the other side it was good in this manner; it made the researchers and financial institution to re-visit their models and books. The crisis made them realize how "wrong" the Black-Scholes framework for pricing derivatives and the need for new developments.

The above findings gives rise to a new technique for managing derivatives. The technique is called the XVA or valuation ajustments. As mentioned in the introduction XVAs consist of four components namely; credit valuation adjustment(CVA), debit valuation adjustment(DVA), funding valuation adjustment(FVA) and collateral valuation adjustment(CollVA). Next we define the above components and provide examples where possible.

**1.8.1 Definition** (Default risk)**.** (Ruiz (2015)) Default risk is the chance that companies or individuals will be unable to the required payments on their debt obligations.

- **CVA**(Ruiz (2015)) - is the price of the default risk we have in our book of OTC derivatives. The CVA is divided into two component namely; CVA$_{asset}$ - is the price of the default risk we are facing and CVA$_{liab}$ - is the price of the default risk our counterparties are facing from us also called the DVA.
- **DVA**(Ruiz (2015)) - is an accounting valuation technique related to how a company handles changes in its issued fixed income securities. Also DVA can be regarded as CVA$_{liab}$ - is the price of the default risk our counterparties are facing from us.
- **FVA**(Ruiz (2015)) - FVA was introduced after the 2007 crisis because back then, it was believed that bank will never default. Funding valuation adjustment is an adjustment to the value of derivative which is designed to make sure that a dealer recovers its average funding costs when it trades and hedges derivatives.

Then we will use the paper by Arregui et al. (2017) to define the total valuation adjustment but not taking into account the Collateral, that is;

$$\text{XVA} = \text{DVA} - \text{CVA} + \text{FVA}.$$

## 1.9 Objective of the Study

The aim of this essay is to apply the finite difference schemes to numerically solve the resulting American Options problem with total value adjustment (XVA).

## 1.10 Problem Statement

As mentioned in the above section that the Black-Scholes approach is that the seller can make his/her earnings by replicating the derivative's behaviour in the following; buy and sell the asset and lend the money at a risk-less rate. furthermore, that the market risk is based only on volatility and interest rate.

During the 2007-2008 financial crisis many people and financial institution realized how Black-Scholes assumptions were not well-constructed. This framework is based on European-style options, meaning those options that can be exercised only on the last trading day, so what about the "American ones". The formula in Equ 1.4.1 is used to value European call option at maturity. Furthermore,the assumptions of implied volatility on the asset will remain unchanged and no transactional cost during trading. All these above-mentioned assumptions are not well constructed and there is a need for change.

In this paper, we suggest using finite difference method to approximate or calculate the premium of an option but also taking into account the valuation adjustments. From the literature, the above-mentioned approach was done using the European option but in this project we consider a case for the American option.

In the next chapter, we start by deriving the pricing PDE in the presence of bilateral counterparty.

# 2. Pricing models for American Put option and Review of Literature

In this chapter we will start by reviewing literature and adopt the Burgard and Kjaer (2012) technique in deriving the model for pricing American options and taking into account the XVA. The self-financing portfolio and the arbitrage free assumptions will be applied for the derivation.

## 2.1 Literature Review

This section is designed to give motivation on the main title of this project and also on the problem statement.

The fact that American options can be exercise at any day before expiry day gives them more value over European options. The Black-Scholes Model is used to estimate any option that can be exercised at maturity. In this case, pricing an American option won't be striaghtforward. Magdon-Ismail (2013) stated why this is possible; the derivative cash flow function is not well defined because the actual execise day is unknown.

Long ago the only risk on financial derivatives was the market risk. Everyone believed that the possibility that a counterparty would default was seen impossible. There are several reasons why counterparty risk it was not considered as a risk. It was believed that most counterparties before the crisis their credit rating was in good state and the world economy was going through a phase of low defaults(Ruiz (2015)). During the 2008 financial crisis where we saw the Lehman Brothers defaulted, Citibank lossing $\$4.8$ million on CVA and $\$6.2$ million for Merrill Lynch. Checking the above figures of losses and the default of Lehman Brothers and also the reports revealing that two third of default was due to counterparty risk. Then in this case, financial institutions and regulators realized that they need to put much more emphasis in understanding, managing and controlling counterparty risk(check definition in chapter 1)(Ruiz (2015)). People tend to confuse the two terms namely; market and credit risk(also called default risk). Market risk measures how much investor/participate 's positions can move out of the money if the market moves against them, whereas credit risk deals with how much investor/participate can lose if one of his/her counterparties default and the markets moves in investor's favour(Ruiz (2015)).

After the crisis industries realized that it is matter of time to move on from Black-Scholes framework because the framework alone is not sufficient to manage the risks of derivatives. Furthermore, most assumptions of this framework are not well constructed. For example the **no arbitrage** that is; you can go to any market you like compare prices of certain product, you will then realize that indeed arbitrage opportunity exist. Again there is no such a thing as **frictionless market** that is; for example in the exchange market traders must pay brokers in order for them to trade at each other. the above-mentioned Black-Scholes assumptions are not the only ones that are not well constructed. The above literature is just revealing that the Black-Scholes pricing framework must be adjusted so that it can accommodate the reality we are facing in daily basis(Ruiz (2015)). The adjustments needed to adjust Black-Scholes model so it can "work" properly in reality are called XVA's. The XVA is a complex term which consist of many valuation adjustments, namely; credit valuation adjustment(CVA), debit valuation adjustment(DVA), funding valuation adjustment(FVA), collateral valuation adjustment, capital valuation adjustment and etc. These are adjustments used in Banks/financial institution to manage certain types of risks that might arose from their clients.

The above literature revealed that the Black-Scholes pricing model need to be adjusted by so called XVAs. In the next section, we present the full model for the adjusted Black-Scholes pricing model.

## 2.2 The Pricing Model for American Put Option with XVA

In this section we start by stating important component which will help us in deriving the PDE.

- Counterparty $D$ zero recovery bond price, $P_D$ with drift rate $\gamma_{P_D}$
- Counterparty $E$ zero recovery bond price, $P_E$ with drift rate $\gamma_{P_E}$
- With $S$, the free default risk asset

Then the above conditions can be modelled by the following stochastic processes satisfying this Stochastic Differential Equations:

$$\begin{aligned} dP_D &= \gamma_{P_D}(t)P_D - P_D dJ_D \\ dP_E &= \gamma_{P_E}(t)P_E - P_E dJ_E \\ dS &= \gamma_R(t)Sdt - \sigma(t)SdW \end{aligned} \tag{2.2.1}$$

where $dW$ is the Weiner process/Brownian motion, $J_D$ and $J_E$ are independent jump process(also called default states) that take two possible values 0 or 1 on default $D$ and $E$ respectively. In the situation of default in the parties D or E, the stock price S will not be affected by the change.The derivative value of the seller $D$ is given by, $\hat{V}(t, S, J_D, J_E)$ which depend on $S, t, J_D$ and $J_E$ and also for counterparty $E$ will follow from the seller. $\hat{V}(t, S, J_D, J_E)$ denote the value of defaultable derivative in which includes the following adjustments; CVA, DVA and FCA. $V(t, S)$ is counterparty risk free derivative.The following are the conditions associated with risk value for seller or the counterparty:

- $\hat{V}(t, S, 1, 0) = M^+(t, S) + R_D M^-(t, S)$:Seller $D$ defaults first here.
- $\hat{V}(t, S, 0, 1) = R_E M^+(t, S) + M^-(t, S)$: Counterparty $E$ defaults first here.

where $M$ is mark-to market value and $R_D \in [0, 1]$ and $R_E \in [0, 1]$ are the recovery rates on derivative of the seller $D$ and counterparty $E$.

Now, we define the content of self-financing portfolio $\Pi_t$ with the counterparty risk consisting of following:

- $\delta$ - the units of the underlying asset $S$
- $\alpha_D(t)$ - the units of $P_D$, default risky, zero coupon bond, zero recovery of $D$
- $\alpha_E(t)$ - the units of $P_E$, default risky, zero coupon bond, zero recovery of $E$
- $\beta(t)$ - funding cost, which is made up of the following; the cash needed to buy a position in $E$'s bond and a repo amount such that the portfolio hedges out derivative value of the seller at time $t$.

**2.2.1 Remark.** Note the following:

- $\gamma_P$ - cost of portfolio
- $\gamma_{P_D}$ - the cash needed to buy a position in $D$'s bond or cash obtained from selling $D$'s bond.

- $\gamma_F$ - the funding account: $\gamma_F = \gamma_P - \gamma_{PD}$.

Note that: $\gamma_{P_E}$ - cash needed to buy a position in $E$'s bond.

Then we define the self-financing portfolio $\Pi_t$, that is:

$$\Pi_t = \delta(t)S + \alpha_D(t)P_D + \alpha_E(t)P_E + \beta(t) \tag{2.2.2}$$

where $\delta(t)S$ is the stock position.

The change in portfolio is given by:

$$d\Pi_t = \delta(t)dS + \alpha_D(t)dP_D + \alpha_E(t)dP_E + \beta(t)d \tag{2.2.3}$$

where $\beta(t) = (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_D} - r_R\gamma_R)$

Since there is no arbitrage opportunity, the hedging equation will be given by:

$$d\Pi_t + d\hat{V}_t \leq 0 \tag{2.2.4}$$

By applying Ito's lemma on the jump process, the change in the value of derivative is:

$$d\hat{V} = \frac{\partial \hat{V}}{\partial t}dt + \frac{\partial \hat{V}}{\partial S}dS + \frac{1}{2}\frac{\partial^2 \hat{V}}{\partial S^2}\sigma^2 S^2 + \Delta\hat{V}_D dJ_D + \Delta\hat{V}_E dJ_E \tag{2.2.5}$$

where

$$\Delta\hat{V}_D = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0), \quad \Delta\hat{V}_E = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0).$$

Now, substitute Equ 2.2.5 and 2.2.3 into 2.2.4, that is:

$$\delta(t)dS + \alpha_D(t)dP_D + \alpha_E(t)dP_E + \beta(t)dt \leq -\left(\frac{\partial \hat{V}}{\partial t}dt + \frac{\partial \hat{V}}{\partial S}dS + \frac{1}{2}\frac{\partial^2 \hat{V}}{\partial S^2}\sigma^2 S^2 + \Delta\hat{V}_D dJ_D + \Delta\hat{V}_E dJ_E\right) \tag{2.2.6}$$

From the first three equations. We observe Equ 2.2.6 changes to;

$$\begin{aligned} \delta(t)dS + \alpha_D(t)(\gamma_{P_D}(t)(P_D - P_D dJ_D) + \alpha_E(t)(\gamma_{P_E}(t)P_E - P_E dJ_E) + \beta(t)dt \\ \leq -\left(\frac{\partial \hat{V}}{\partial t}dt + \frac{\partial \hat{V}}{\partial S}dS + \frac{1}{2}\frac{\partial^2 \hat{V}}{\partial S^2}\sigma^2 S^2 + \Delta\hat{V}_D dJ_D + \Delta\hat{V}_E dJ_E\right) \end{aligned} \tag{2.2.7}$$

We now let the following to be our weights in order to remove all the risk from portfolio;

$$\delta(t) = -\frac{\partial \hat{V}}{\partial S}, \quad \alpha_D(t) = \frac{\Delta\hat{V}_D}{P_D}, \quad \alpha_E(t) = \frac{\Delta\hat{V}_E}{P_D}.$$

Now we substitute the above weights and into Equ 2.2.7, so that we can eliminate all risks in the portfolio $\Pi_t$, that is:

$$\alpha_D(t)\gamma_{P_D}(t)P_D dt + \alpha_E(t)\gamma_{P_E}(t)P_E dt + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_D} - r_R\gamma_R) + \frac{\partial \hat{V}}{\partial t}dt + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 \hat{V}}{\partial S^2}dt \leq 0$$

This gives us:

$$\alpha_D(t)\gamma_{P_D}(t)P_D + \alpha_E(t)\gamma_{P_E}(t)P_E + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_D} - r_R\gamma_R) + \frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 \hat{V}}{\partial S^2} \leq 0 \quad (2.2.8)$$

Next we simplify the funding cost term when the following are true $\gamma_{P_D} = \alpha_D P_D, \;\; \gamma_{P_E} = \alpha_E P_E, \;\; r_F = r + s_F$ and $\gamma_F = \gamma_P - \gamma_{P_D}$, then

$$\begin{aligned}\beta(t) &= \Big(r(\gamma_P - \gamma_{P_D})^+ + r_F(\gamma_{P_D})^- - r\alpha_E P_E - r_R\gamma_R + \alpha_D\gamma_{P_D}P_D + \alpha_E\gamma_{P_E}P_E\Big)\\ &= \alpha_D\gamma_{P_D}P_D + \alpha_E\gamma_{P_E}P_E + r(\gamma_P - \alpha_D P_D) + s_F(\gamma_P - \alpha_D P_D)^- - r\alpha_E P_E - r_R\gamma_R\end{aligned}$$

By repo account, $\gamma_R = \Delta S$, we have:

$$\begin{aligned}\beta(t) &= \alpha_D\gamma_{P_D}P_D + \alpha_E\gamma_{P_E} + r\gamma_P - r\alpha_D P_D + s_F\gamma_F^- - r\alpha_E P_E - r_R\Delta S\\ &= r\gamma_P + s_F\gamma_F^- + (\gamma_{P_D} - r)\alpha_D P_D + (\gamma_{P_E} - r)\alpha_E P_E - r_R\Delta S\end{aligned}$$

We let the hedging portfolio to be equals to the value of derivative so that we can eliminate the arbitrage opportunity, that is $\gamma_P = \hat{V}$ and since $P_D$ and $P_C$ are zero recovery bond, then $\lambda_D = r_{P_D} - r, \;\; \lambda_E = \gamma_{P_E} - r$, where $\lambda_D$ and $\lambda_E$ are called the default intensities. Then for $\gamma_P = \hat{V}$, the funding cost is given by:

$$\beta(t) = r\hat{V} + s_F\gamma_F^- + \lambda_D\alpha_D P_D + \lambda_E\alpha_E P_E - r_R\Delta S$$

According to mark-to-market value on the following addends $\alpha_D P_D$ and $\alpha_E P_E$ we obtain:

$$\beta(t) = -(r+\lambda_D+\lambda_E)\hat{V} + s_F\gamma_F^- + (M^+(t,S) + R_D M^-(t,S)) + \lambda_E(R_E(M^+(t,S) + M^-(t,S)) - r_R\Delta S$$

From Equ 2.2.8, we get:

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - r\hat{V} \leq (\lambda_D + \lambda_E)\hat{V} + s_F M(t,S)^- + (M^+(t,S) + R_D M^-(t,S)) + \lambda_E(R_E(M^+(t,S) + M^-(t,S)) \quad (2.2.9)$$

Below is the PDE problem model for pricing **American option with counterparty risk.**

$$\begin{cases} \mathcal{L}(\hat{V}) = \frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - r\hat{V} - (\lambda_D + \lambda_E)\hat{V} - s_F M(t,S)^- - (M^+(t,S) + R_D M^-(t,S)) \\ -\lambda_E(R_E(M^+(t,S) + M^-(t,S)) \leq 0 \\ \hat{V}(t,S) \geq G(S) \\ \mathcal{L}(\hat{V})(\hat{V} - G) = 0 \\ \hat{V}(T,S) = G(S) \end{cases}$$

where $G$ is the pay-off function of the value of derivative and $\mathcal{A}$ is called the differential operator which is given by;

$\mathcal{A}V = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S}$

Using mark-to-market value, we obtain two obstacle problems;

For $M = \hat{V}$, we have the non-linear partial differential equation that is;

$$\begin{cases} \mathcal{L}(\hat{V}) = \frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - r\hat{V} - (1 - R_D)\lambda_D \hat{V}^- - s_F \hat{V}^+ - (1 - R_E)\lambda_E \hat{V}^+ \leq 0 \\ \hat{V}(t,S) \geq G(S) \\ \mathcal{L}(\hat{V})(\hat{V} - G) = 0 \\ \hat{V}(T,S) = G(S) \end{cases}$$

For $M = V$, we have linear partial differential equation;

$$\begin{cases} \mathcal{L}(\hat{V}) = \frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - (r + \lambda_D + \lambda_E)\hat{V} + V^-(R_D\lambda_D - \lambda_E) + (R_E\lambda_E + \lambda_D)V^+ - s_F V^+ \leq 0 \\ \hat{V}(t,S) \geq G(S) \\ \mathcal{L}(\hat{V})(\hat{V} - G) = 0 \\ \hat{V}(T,S) = G(S) \end{cases} \tag{2.2.10}$$

We transform the linear PDE model in this manner $U = \hat{V} - V$ in order to find the valuation adjustments(XVA), U.

# 3. Numerical Method

In this chapter we start by introducing the concept of discretization and finite difference method in order to solve the above model with valuation adjustments. The main purpose of this chapter is to give the readers some light on how to transform continuous PDE into discrete dimension using finite difference method.

## 3.1 Discretization Schemes

Discretization is a technique used to transform continuous partial differential equation into stable, consistent and accurate form so to produce numerical solutions. There are three methods which uses this technique namely; finite volume(FV), finite element(FE) and finite difference(FD). Then the discretized equation can be represented by this model, that is;

$$\mathcal{L}(u) = f \rightarrow AX = b \tag{3.1.1}$$

Form Equ 3.1.1, one can say this process will result in loss of information during the transformation. This is not the case in discretization schemes because of some use of generic concept called the reference discretization scheme. Below we discuss briefly the definitions of three methods mentioned in the beginning of this chapter.

- **Finite Difference Method(FDM)**(Webots4) - the method uses a finite difference approximation for the differential operator and is simple to implement.Another important aspect about FDM, it gives an optimal solution to a different problem than the default model but is limited to structured grids.
- **Finite Volume Method(FVM)**(Webots4) - the method is based on the approximation of conservation laws directly in its formulation and is therefore flux conserving by construction. Also the topological laws and time stepping procedures can be integrated easily.
- **Finite Element Method(FEM)**(Webots4) - FEM is considered as a remarkably flexible and general method for solving partial differential equations.

In this chapter we will only focus on the **Finite Differential Method**.

## 3.2 The Finite Difference Method

The finite difference method are powerful schemes to numerical PDEs. The discrete approximation obtained using taylor expansion are being applied to PDE's partial derivatives to get simple ODE.

Next we start by introducing the space and time discretization of derivatives. furthermore we present the Euler $\theta$ methods for solving ODEs.

## 3.3 Central Difference Method for Space Derivatives

Here we present the reader with three space discretization methods for PDEs. Then later will convert the discretized PDE to ODE using Euler method. We consider the domain $I \times [0, T]$, for $I = (0, S)$. We divide the interval $I$ into $N \in \mathbb{N}$ sub-intervals $I_j = (s_j, s_{j+1}),\ \ j = 0, 1, \cdots, N-1$ with $0 = s_0 < s_1 < s_2 < \cdots < s_N = S$.

- **Explicit Method(forward difference)** - for this method we consider time at $t_m$, then for space derivative at position $x_j$, we have:

$$\frac{\partial U}{\partial x} = \frac{U_{j+1}^m - U_j^m}{\Delta x}, \quad \frac{\partial^2 U}{\partial x^2} = \frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2}$$

- **Implicit Method(backward difference)** - for backward difference at time $t_{m+1}$,then for the space derivative at position $x_j$, we have:

$$\frac{\partial U}{\partial x} = \frac{U_j^m - U_{j-1}^m}{\Delta x}, \quad \frac{\partial^2 U}{\partial x^2} = \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2}$$

- **Central Difference** - for this method at time $t_{m+1/2}$ the space derivative at position $x_j$ will be given by:

$$\frac{\partial U}{\partial x} = \frac{U_{j+1}^m - U_{j-1}^m}{2\Delta x}, \quad \frac{\partial^2 U}{\partial x^2} = \frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2}$$

## 3.4 Euler Method for Time Derivatives

In this section we present $\theta$ Euler method for transforming PDE to ODEs. In this paper will consider the case when $\theta = 0.5$. We start by letting $\tau = T - t$ and considering the subdivision of the time interval $[0, \mathrm{T}]$ given $0 = \tau_0 < \tau_1 < \cdots < \tau_m = \mathrm{T}$ and the time step $\Delta\tau = \tau_{m+1} - \tau_m,\ \ \forall m \in \{0, \cdots, M\}$.

- **The Explicit Euler** is given by:

$$\frac{\partial U}{\partial t}|_{t=t_{m+1}} = \frac{U_j^{m+1} - U_{j-1}^m}{\Delta t} + O(\Delta t)$$

The above is also called the Euler method for Ordinary Differential Equations. Then matrix formula of this method is:

$$\mathbf{U}^{m+1} = (\mathbf{I} + \Delta t\mathbf{B}) \cdot \mathbf{U}^m \tag{3.4.1}$$

Where **A** is the tri-diagonal matrix, **I** the identity matrix and **U** is a vector matrix. The entries of matrix **A** are derived from the model.

- **The Implicit Euler** is given by:

$$\frac{\partial U}{\partial t}|_{t=t_{m+1}} = \frac{U_j^m - U_j^{m-1}}{\Delta t} + O(\Delta t)$$

This above lead to Euler Implicit, that is:

$$\mathbf{U}^m = (\mathbf{I} - \Delta t \mathbf{B})^{-1} \cdot \mathbf{U}^{m-1} \quad (3.4.2)$$

- **Euler - $\theta$ - Method** - the method is composed by linear combination of the Implicit and Explicit formula with parameter $\theta$, that is:

$$\theta\textbf{Explicit} + (1-\theta)\textbf{Implicit} \approx \frac{\partial U}{\partial t}|_{t=t_{m+1}} + O(\Delta t^p) \quad (3.4.3)$$

Substitute Equ 3.4.1 and 3.4.2 into 3.4.3, that is:

$$\frac{U^{m+1} - U^m}{\Delta t} = \theta \mathbf{B}\mathbf{U}^{m+1} + (1-\theta)\mathbf{B}\mathbf{U}^m$$

We now, solve for $U^{m+1}$, that is:

$$\mathbf{U}^{m+1} = (\mathbf{I} - \theta\Delta t \mathbf{B})^{-1}(\mathbf{I} + (1-\theta)\Delta t \mathbf{B})\mathbf{U}^m = \mathbf{D}_h \cdot \mathbf{U}^m. \quad (3.4.4)$$

## 3.5 Discretization of the Models

In this section, we are going to discretize two PDEs, namely the one in Equ 1.6.3 and the other one in Equ 2.2.10 using central difference method and later convert the discretized PDEs into ODEs using Euler-$\theta$-method. After discretization, we then use this technique $U = \hat{V} - V$ to find the XVA value.

**3.5.1 PDE without Counterparty Risk.** Below we discretize PDE without counterparty risk.

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2 s^2 + rs\frac{\partial V}{\partial s} - rV \leq 0 \\ \frac{\partial V}{\partial t} + \frac{\sigma^2 s_i^2}{2}\left[\frac{V_{i+1} - 2V_i + V_{i-1}}{h^2}\right] + rs_i\left[\frac{V_{i+1} - V_{i-1}}{2h}\right] - rV_i \leq 0 \\ \frac{\partial V}{\partial t} + \frac{\sigma^2 s_i^2}{2h^2}V_{i+1} - \frac{\sigma^2 s_i^2}{h^2}V_i + \frac{\sigma^2 s_i^2}{2h^2}V_{i-1} + \frac{rs_i}{2h}V_{i+1} - \frac{rs_i}{2h}V_{i-1} - rV_i \leq 0 \\ \frac{dV_i}{dt} + \left[\frac{\sigma^2 s_i^2}{2h^2} + \frac{rs_i}{2h}\right]V_{i+1} - \left[\frac{\sigma^2 s_i^2}{h^2} + r\right]V_i + \left[\frac{\sigma^2 s_i^2}{2h^2} - \frac{rs_i}{2h}\right]V_{i-1} \leq 0 \\ \frac{dV_i}{dt} + a_i V_{i+1} + b_i V_i + c_i V_{i-1} \leq 0 \\ \frac{dV_h}{dt} + A_h V_h \leq 0 \end{cases}$$

Where

$$A_h = \begin{pmatrix} b_1 & a_1 & & 0 \\ c_2 & \ddots & \ddots & \\ & \ddots & \ddots & a_{N-2} \\ 0 & & c_{N-1} & b_{N-1} \end{pmatrix} \quad \text{and} \quad V_h = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_{N-1} \end{pmatrix}.$$

By using the results in Equ 3.4.4, we get;

$$\frac{V_h^{m+1} - V_h^m}{\Delta t} + \theta A_h V_h^{m+1} + (1-\theta) V_h^m \leq 0 \tag{3.5.1}$$

Finally, we have this:

$$\boxed{V^{m+1} \leq (\mathbf{I} + \theta \Delta t A_h)^{-1} \left[ \Big( \mathbf{I} - (1-\theta)\Delta t A_h \Big) V^m \right]}$$

**3.5.2 PDE with Counterparty Risk.** Now using this argument: $\boxed{V = V^+ + V^-}$ we have;

$$\partial_t \hat{V} + \mathcal{A}\hat{V} - (r + \lambda_D + \lambda_E)\hat{V} + (R_D\lambda_D)V^- + (R_E\lambda_E - s_F)V^+ \leq 0 \tag{3.5.2}$$

Where $M(\hat{V}) = (R_D\lambda_D)V^- + (R_E\lambda_E - s_F)V^+$.

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}\hat{V} - (r + \lambda_D + \lambda_E)\hat{V} + (R_D\lambda_D)V^- + (R_E\lambda_E - s_F)V^+ \leq 0 \\ \frac{d\hat{V}}{dt} + \left[ \frac{1}{2}\sigma^2 s_i^2 \left( \frac{\hat{V}_{i+1} - 2U_i + \hat{V}_{i-1}}{h^2} \right) + r s_i \left( \frac{\hat{V}_{i+1} - \hat{V}_{i-1}}{2h} \right) \right] - \Big( r + \lambda_D + \lambda_E \Big) \hat{V}_i + (R_D\lambda_D)V_i^- + (R_E\lambda_E - s_F)V_i^+ \leq 0 \\ \frac{d\hat{V}_i}{dt} + \left[ \frac{\sigma^2 s_i^2}{2h^2} + \frac{r s_i}{2h} \right] \hat{V}_{i+1} + \left[ \frac{-\sigma^2 s_i^2}{h^2} (r + \lambda_D + \lambda_E) \right] \hat{V}_i + \left[ \frac{\sigma^2 s_i^2}{2h^2} - \frac{r s_i}{2h} \right] \hat{V}_{i-1} + (R_D\lambda_D)V_i^- + (R_E\lambda_E - s_F)V_i^+ \leq 0 \\ \frac{d\hat{V}_i}{dt} + R_i\hat{V}_{i+1} + F_i\hat{V}_i + H_i\hat{V}_{i-1} + M(\hat{V}) \leq 0 \\ \frac{d\hat{V}_h}{dt} + A_h\hat{V}_h + M(\hat{V}) \leq 0 \end{cases}$$

Where

$$D_h = \begin{pmatrix} F_1 & R_1 & & 0 \\ H_2 & \ddots & \ddots & \\ & \ddots & \ddots & R_{N-2} \\ 0 & & H_{N-1} & F_{N-1} \end{pmatrix}, \quad M(\hat{V}) = \begin{pmatrix} M_1(\hat{V}) \\ M_2(\hat{V}) \\ \vdots \\ M_{N-1}(\hat{V}) \end{pmatrix} \quad \text{and} \quad \hat{V}_h = \begin{pmatrix} \hat{V}_1 \\ \hat{V}_2 \\ \vdots \\ \hat{V}_{N-1} \end{pmatrix}.$$

By using the results in Equ 3.4.4, we have;

$$\frac{\hat{V}_h^{m+1} - \hat{V}_h^m}{\Delta t} + \theta A_h \hat{V}_h^{m+1} + (1-\theta)\hat{V}_h^m + M(\hat{V}) \leq 0 \tag{3.5.3}$$

Finally we get this:

$$\boxed{\hat{V}^{m+1} \leq (\mathbf{I} + \theta \Delta t D_h)^{-1}\left[\left(\mathbf{I} - (1-\theta)\Delta t D_h\right)\hat{V}^m - M(\hat{V})\right]}$$

## 3.6 Numerical Results

In this section, we present the results of our experiments, so that we can demonstrate how useful XVA is, on Black-Scholes pricing model. The numerical simulations are run on 2 GB RAM ACER Laptop.

For the analysis, we consider the following parameters for American put option, that is; $K = 10,\ \ r = 0.03,\ \ \sigma = 0.25,\ \ T = 0.5,\ \ \lambda_D = 0.3,\ \ \lambda_E = 0.3,\ \ R_D = 0.4,\ \ R_E = 0.4,\ \ s_F = (1 - R_D)\lambda_D$ and $S_{\text{max}} = 2 * K$. We choose the space and time intervals to be $(0, 30)$ and $(0, 0.5)$ respectively. They are subdivided into $N = 200$ and $M = 1000$ subintervals respectively.

Our analysis will be as follows; first we going to plot the models with and without counterparty risk, so that we can see whether there is an effect when counterparty risk is considered. Finally, present the plot of total value adjustment(XVA) for American put option.

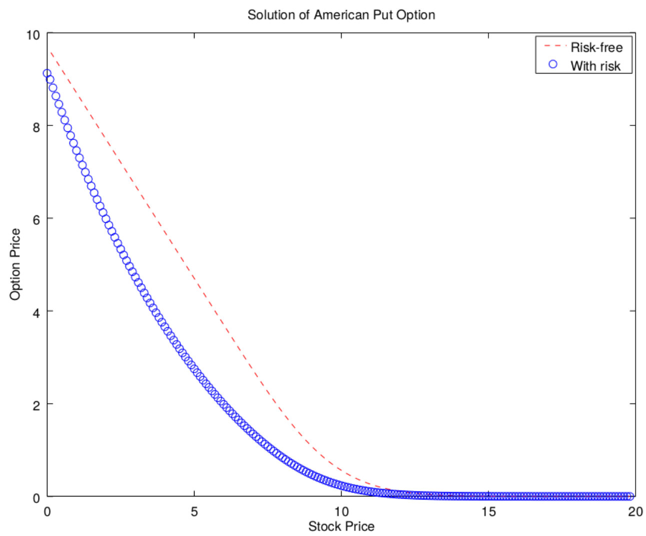


Figure 3.1: American put option value for Risky and Risk-free Derivative

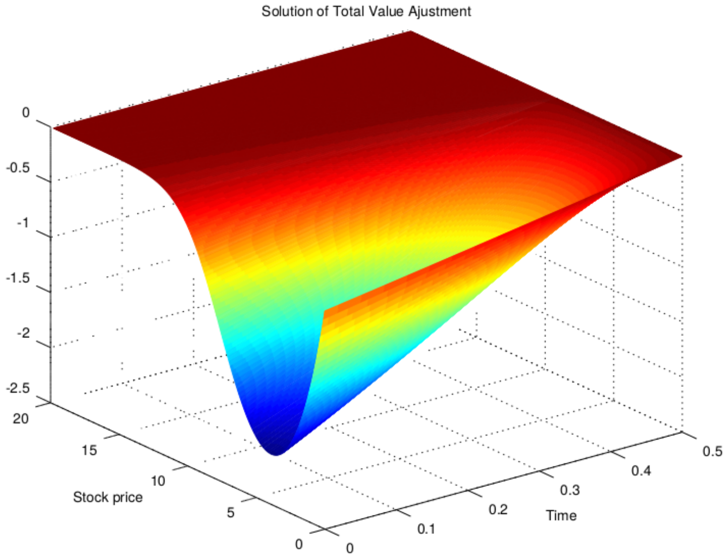


Figure 3.2: Total value adjustment surface for American put option

As expected in Figure 3.1, the model without counterparty risk yields more option value than the one with counterparty risk. This means that, if it happens that counterparty $D$ decide to buy a put option, the price that he/she has to pay by risk-free derivatives is higher than the amount that has to be paid for an option if default risk is considered. In this case, if one wishes to be in the derivative contract where by the risk of counterparty is considered, then he/she will pay less premium value as compared to the one in derivative without risk.

Figure 3.2 can be interpreted as follows; checking plot properly, it is clear that the XVA is negative and the reason being is because it represent the discount value upon risk-free value, due to the risk exposure of counterparty $D$. Furthermore, the value of XVA is zero at maturity, because the counterparty is not exposed to any risk, meaning the risky derivative is equal to risk-free derivative.

# 4. Conclusion

In this project, we started with different PDE problems formulation which is associated to the pricing of total value adjustment(XVA) to be added to the price of the derivatives without counterparty. From literature review it was stated that the XVA framework offers an optimal set-up for trade valuation and risk management. Furthermore, we used the concept of mark-to-market to simplify our underlying PDE's in which yielded to two obstacle problems namely; linear and non-linear. After solving the linear PDE, we then presented numerical plots to illustrate and discuss the behaviour of the models when the counterparty risk is present and XVA adjustments that have been obtained in the project.

We observed that the derivatives with counterparty yields less option value than the one without risk. This findings gives the Banks and other financial institution a valid reason that they must not worry about their clients failing to pay their loans or other contracts because all the downfalls are recorded in the adjustment. However, also the market participate must consider this risk when trading with their counterparties. Also, we observed that the XVA is negative because it deals with discount upon risk-free value whereas in the Black-Scholes framework it was considered as zero by default. The above results reveals that it is advisable to add the adjustments to Black-Scholes pricing model.

For future work, we will consider the above problem using Finite Volume and Finite Element Method.

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