The topology of the nonabelian tensor square of groups

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Abstract

In this essay we first review some elementary notions on abelian groups. In addition, we discuss the concepts of divisibility and abelian tensor products. Finally, we study the nonabelian tensor products of projective limits of finite groups and describe their topology.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Mohammed Hamid Dodeh Younis, 18 May 2017

Contents

1. Introduction

The nonabelian tensor square $G \otimes G$ of a group G is a special case of the nonabelian tensor product $G \otimes H$ for a pair of groups G and H that was introduced by R. Brown and J.-L. Loday in 1987, and it arises from applications of a generalized Van Kampen theorem in the context of an application in homotopy theory, see (Brown and Loday, 1987). The nonabelian tensor squares are interesting in different context of algebraic topology, group theory, graph theory, and combinatorics. We will study the nonabelian tensor products of projective limits of finite groups and describe their topology. This topic has been recently investigated and there are a series of open questions. The structure of this essay is as follows:

In Chapter Two we introduce some elementary notions on abelian groups, they will be used for the main purpose of the present essay. For example, free abelian groups, which we will show later the tensor product $A \otimes B$ of abelian groups A and B is exactly the quotient group of the free abelian group over $A \times B$ under a specific bilinear map.

In Chapter Three we present the concepts of divisibility and tensor products of abelian groups. We discuss some important results on divisible groups, and connect the notion of divisible groups with that of abelian tensor products. Moreover, we consider the theory of abelian tensor products as a model for the nonabelian tensor products involved in the main chapter.

Chapter Four starts by introducing nonabelian tensor products of groups. In Section 4.2, we present some analogies with the abelian case. In Section 4.2, we introduce basic results of topological nature. Finally, in Section 4.4, we present nonabelian tensor products of topological groups and we will show what is the topology of projective limits.

2. Elementary notions on abelian groups

We recall some notions from Appendix 1 of (Hofmann and Morris, 2006). They will be used for the main purposes of the present thesis. With the symbol $\mathbb{Z}$, we denote the ring of the integers, which is an abelian group with respect to the usual sum and a commutative monoid with respect to the usual multiplication.

2.1 Free abelian groups

In general, if A is an abelian group, then we may endow A with the structure of a module over $\mathbb{Z}$, defining scalar multiplication via $\mathbb{Z} \times A \to A,\ (n, a) \mapsto n \cdot a$ and the sum as usual in A. The methods of linear algebra apply to A in several ways. One is the following, where we have linear combinations of elements of A with coefficients in $\mathbb{Z}$.

2.1.1 Definition. Let X be a subset of a nonzero abelian group A. If X satisfies one of the following conditions:

1. Every nonzero element $a \in A$ can be expressed uniquely in the form $a = \sum_{x \in X} n_x x$ for $n_x \neq 0$ in $\mathbb{Z}$ and for distinct $x \in X$,
2. Every element $a \in A$ can be written as linear combinations of elements of X, and $\sum_{x \in X} n_x x = 0$ iff $n_x = 0$ for all $x \in X$;

   we say that X is a $\mathbb{Z}$-*basis* for A.

The following proposition extends Remark A1.7. (i) in (Hofmann and Morris, 2006).

2.1.2 Proposition. The conditions in Definition 2.1.1 are equivalent.

Proof. Suppose that each nonzero element $a \in A$ can be expressed uniquely in the form $a = \sum_{x \in X} n_x x$. Assume that $\sum_{x \in X} n_x x = 0$ with some $n_x \neq 0$; by dropping terms with zero coefficients, there is no loss of generality in assuming all $n_x \neq 0$. Then

$$x_1 = x_1 + \underbrace{\sum_{x \in X} n_x x}_{=0} = (1 + n_{x_1})x_1 + \sum_{x \in X - \{x_1\}} n_x x,$$

gives two different ways of writing $x_1 \neq 0$ and this is a contradiction. Therefore $\sum_{x \in X} n_x x = 0$ iff $n_x = 0$ for all $x \in X$.

Now, suppose that each $a \in A$ can be written in the form $a = \sum_{x \in X} n_x x$. Assume a has another such expression in terms of elements of X, i.e.,

$$a = \sum_{x \in X} m_x x.$$

Subtracting, we obtain

$$0 = \sum_{x \in X} (n_x - m_x) x,$$

and thus for all $x \in X$ we have $n_x - m_x = 0$, and hence $n_x = m_x$. Thus the coefficients are unique.

□

Now we recall some classical notions on free abelian groups.

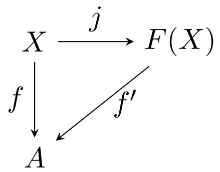
2.1.3 Definition. An abelian group A having a generating set X satisfying the conditions described in Definition 2.1.1 is called a free abelian group (or free $\mathbb{Z}$-*module*).

Every element a of a free abelian group A with a abasis X may be written uniquely in the form

$$a = \sum_{x \in X} n_x x,$$

where each coefficient $n_x \in \mathbb{Z} - \{0\}$, the elements x of X are distinct, and the sum has finitely many terms.

2.1.4 Definition. Let X denote a set and $\mathrm{F(X)}$ a group with a function $j : X \to \mathrm{F(X)}$. Then $\mathrm{F(X)}$ is said to be a free abelian group over X if for every abelian group A and every function $f : X \to A$ there is a unique morphism $f' : \mathrm{F(X)} \to A$ such that $f = f' \circ j$.



This definition is expressed in terms of the universal property, see ((Hofmann and Morris, 2006), Definition $\mathrm{A1.5}$) and has a very general nature.

From Definition 2.1.4, it is meaningful to say that an abelian group is free if it isomorphic to a free abelian group over some set. As usual, one give before a definition in terms of universal properties and then one wants to be sure that there are objects satisfying that property. The following proposition shows this.

2.1.5 Proposition. ((Hofmann and Morris, 2006), Proposition $\mathrm{A1.6}$.) The group $\mathbb{Z}^{(X)}$ is a free abelian group, where X is an arbitrary nonempty set and

$$\mathbb{Z}^{(X)} = \bigoplus_{x \in X} \mathbb{Z}$$

is the direct sum of $\mathbb{Z}$ over X.

Proof. We define

$$f : F(X) \ni a = \sum_{x \in X} n_x x \mapsto f(a) = (n_x)_{x \in X} \in \mathbb{Z}^{(X)}.$$

Note that f is well defined because each $a \in F(X)$ has a unique expression in the form $\sum_{x \in X} n_x x$, where each $0 \neq n_x \in \mathbb{Z}$ and distinct $x \in X$ (see Definition 2.1.1). Suppose $b = \sum_{x \in X} m_x x \in \mathrm{F(X)}$, then

$$\begin{aligned} f(a+b) &= f\left(\sum_{x \in X} n_x x + \sum_{x \in X} m_x x\right) \\ &= f\left(\sum_{x \in X} (n_x + m_x) x\right) \\ &= (n_x + m_x)_{x \in X} \\ &= (n_x)_{x \in X} + (m_x)_{x \in X} \\ &= f(a) + f(b). \end{aligned}$$

So f is a homomorphism. Now, we want to show that f is injective, i.e., if $f(a) = f(b)$, then $a = b$. If $f(a) = f(b)$, then $(n_x)_{x\in X} = (m_x)_{x\in X}$ implies that $n_x = m_x$ for all $x \in X$ therefore $a = b$. Thus f is injective. Clearly f is surjective because $\sum_{x\in X} n_x x$ is in $\mathrm{F(X)}$ for all n_x in $\mathbb{Z}$ and $x \in X$. Thus f is an isomorphism, i.e., $\mathrm{F(X)} \simeq \mathbb{Z}^{(X)}$. □

We denote the cyclic group of order m by $\mathbb{Z}(m) = \mathbb{Z}/m\mathbb{Z}$ (for any positive integer m). We recall this notion now, since we will use the case $m = 2$ in the following proposition.

2.1.6 Proposition. ((Hofmann and Morris, 2006), Remark $A1.7.(iii)$) Let X and Y be two non-empty sets, and $\mathrm{F(X)}$ and $\mathrm{F(Y)}$ free abelian groups on X and Y, respectively. Then $\mathrm{F(X)} \simeq \mathrm{F(Y)}$ if and only if $|X| = |Y|$.

Proof. Suppose that $\mathrm{F(X)} \simeq \mathrm{F(Y)}$. Let $\mathrm{Hom}(\mathrm{F(X)}, \mathbb{Z}(2))$ be the set of all homomorphisms between $\mathrm{F(X)}$ and $\mathbb{Z}(2)$, and $\mathrm{Hom}(\mathrm{F(Y)}, \mathbb{Z}(2))$ be the set of all homomorphisms between $\mathrm{F(Y)}$ and $\mathbb{Z}(2)$. Because $\mathrm{F(X)} \simeq \mathrm{F(Y)}$, $|\mathrm{Hom}(\mathrm{F(X)}, \mathbb{Z}(2))| = |\mathrm{Hom}(\mathrm{F(Y)}, \mathbb{Z}(2))|$. There are exactly $2^{|X|}$ homomorphisms between $\mathrm{F(X)}$ and $\mathbb{Z}(2)$, hence we conclude $2^{|X|} = |\mathrm{Hom}(\mathrm{F(X)}, \mathbb{Z}(2))| = |\mathrm{Hom}(\mathrm{F(Y)}, \mathbb{Z}(2))| = 2^{|Y|}$. Hence $|X| = |Y|$.

Now, suppose that $|X| = |Y|$. Since X and Y have the same cardinalities, there exists a bijection $f : X \to Y$ with inverse $f^{-1} : Y \to X$. Because $F(X)$ and $F(Y)$ are free groups, the universal property of Definition 2.1.4 ensures the existence of homomorphisms $\phi_1 : \mathrm{F(X)} \to \mathrm{F(Y)}$ and $\phi_2 : \mathrm{F(Y)} \to \mathrm{F(X)}$ of f and f^{-1}, respectively. Since ϕ_2 is related to f^{-1}, which is the inverse of f, we have that $\phi_2 \circ \phi_1 : F(X) \to F(X)$ is the identity function. Then we have both $\phi_2 \circ \phi_1 = id_{\mathrm{F(X)}}$ and $\phi_1 \circ \phi_2 = id_{\mathrm{F(Y)}}$. Thus ϕ_1 is isomorphism and so $\mathrm{F(X)} \simeq \mathrm{F(Y)}$.

□

We sketch the proofs of some fundamental results.

2.1.7 Proposition. ((Hofmann and Morris, 2006), Proposition $A1.8$)

1. Every abelian group is a quotient group of a free abelian group.
2. Every countable abelian group is a quotient group of a countable free abelian group.

Proof.

1. Let A be an abelian group and $\mathrm{F(A)}$ a free abelian group over A with embedding $i : A \to \mathrm{F(A)}$. Consider the map $f : A \to A$. Since $\mathrm{F(A)}$ is free, the universal property of Definition 2.1.4 ensures a unique homomorphism $f' : \mathrm{F(A)} \to A$ with kernel K. From the first isomorphism theorem we have $A \simeq \mathrm{F(A)}/K$ and so the result follows.
2. This is a special case of (1).

□

The following remark summarizes a more general fact, which is valid even for nonabelian groups.

2.1.8 Remark. Every subgroup of a free abelian group is a free abelian group see ((Hofmann and Morris, 2006), Theorem $A1.9$). Unfortunately, the quotients of free belian groups are not free, e.g., $\mathbb{Z}$ is a free but $\mathbb{Z}(2)$ is not.

We end this section with a well known result on the structure of finitely generated abelian groups.

2.1.9 Theorem. *((Hofmann and Morris, 2006), Theorem* A1.11*) Let A be a finitely generated abelian group. Then A may be decomposed into the following form:*

$$A = \mathbb{Z}(m_1) \oplus \mathbb{Z}(m_2) \oplus \ldots \oplus \mathbb{Z}(m_d) \oplus \mathbb{Z}^{m_0},$$

where $m_0, m_1, m_2 \ldots, m_d$ are non-negative integers.

In particular, Theorem 2.1.9 implies that a finitely generated torsion-free group must have $m_1 = m_2 = \ldots = m_d = 0$ and so it is free abelian on a set of cardinality m_0. But we will study the properties of torsion abelian groups in the following section.

2.2 Torsion subgroups

We begin with the definition of torsion abelian group.

2.2.1 Definition. ((Hofmann and Morris, 2006), Definition A1.16.) Let A be an abelian group, then

$$\operatorname{tor}(A) = \left\{ a \in A \mid n \cdot a = 0 \quad \text{for some} \quad n \in \mathbb{N} \right\}$$

is called *torsion subgroup* of A. The group A is called *torsion-free*, if $\operatorname{tor}(A) = \{0\}$. We say that A is a *mixed group* if $\operatorname{tor}(A) \neq A$ and $\operatorname{tor}(A) \neq 0$.

If $\operatorname{tor}(A) = A$, then A is called *periodic (or torsion)*. Of course any finite abelian group E has $\operatorname{tor}(E) = E$. But we cannot characterize the finiteness of abelian groups in terms of torsion, since there are infinite abelian groups, which are torsion, e.g., the group $\mathbb{Q}/\mathbb{Z}$ is infinite and $\operatorname{tor}(\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$. In fact, if $q + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, where $q = \frac{m}{n} \in \mathbb{Q}, \quad m \in \mathbb{Z}$ and $n \in \mathbb{N}$, then $nq = m \in \mathbb{Z}$, hence $q \in \operatorname{tor}(\mathbb{Q}/\mathbb{Z})$, because $\mathbb{Z}$ is the neutral element of $\mathbb{Q}/\mathbb{Z}$.

2.2.2 Example. The groups $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are torsion-free. The group $\mathbb{Z} \oplus \mathbb{Z}(2)$ is mixed. Further examples of mixed groups can be found from Theorem 2.1.9.

The size of $\operatorname{tor}(A)$ is important in order to detect the presence of torsion-free elements in A. If $\operatorname{tor}(A)$ is very small, then A contains a lot of torsion-free elements. If $\operatorname{tor}(A)$ is big enough, then there are less torsion-free elements. Now one can re-phrase these intuitive considerations, looking at the quotient $A/\operatorname{tor}(A)$ instead of $\operatorname{tor}(A)$. In fact we will see that $\operatorname{tor}(A)$ is a fully characteristic subgroup of A, so we can form quotients.

2.2.3 Definition. A subgroup H of a (not necessarily abelian) group G is called *characteristic* if for every automorphism φ of G we have $\varphi(H) \subseteq H$, or equivalently, $\varphi(h) \in H$ for all $h \in H$, and is called *fully characteristic (or invariant)* if for every endomorphism φ of G we have $\varphi(H) \subseteq H$.

Of course, all fully characteristic sugroups are characteristic but, the inverse is not true.

2.2.4 Theorem. *((Hofmann and Morris, 2006), Remark A1.17.) Let A be an abelian group. Then*

1. $\mathrm{tor}(A)$ *is a subgroup of A.*
2. *The factor group $A/\mathrm{tor}(A)$ is torsion-free.*
3. *Let B be a torsion-free abelian group, and assume that $f : A \to B$ is a homomorphism. Then $\mathrm{tor}(A) \subseteq \ker(f)$ and there is a unique homomorphism $\phi : A/\mathrm{tor}(A) \to B$ such that $\phi(a + \mathrm{tor}(A)) = f(a)$.*
4. *The torsion subgroup is a fully characteristic subgroup.*

Proof.

1. Clearly $0 \in \mathrm{tor}(A)$. Suppose that $a \in \mathrm{tor}(A)$. There is an $n \in \mathbb{N}$ such that $n \cdot a = 0$. Then $n \cdot (-a) = 0$, and $-a \in \mathrm{tor}(A)$. It remains to show that $\mathrm{tor}(A)$ is closed with respect to the sum. If $a, b \in \mathrm{tor}(A)$, then there are some $n, m \in \mathbb{N}$, such that $m \cdot a = 0$ and $n \cdot b = 0$. This implies that $n(m \cdot a) = 0$ and $m(n \cdot b) = 0$. Since $mn(a + \cdot b) = n(m \cdot a) + m(n \cdot b) = 0$, we have $a + b \in \mathrm{tor}(A)$. Thus $\mathrm{tor}(A)$ is a subgroup.

2. Let $a + \mathrm{tor}(A) \in A/\mathrm{tor}(A)$. Suppose there exists an $n \in \mathbb{N}$, such that $n(a + \mathrm{tor}(A)) = na + \mathrm{tor}(A) = \mathrm{tor}(A)$. Hence $na \in \mathrm{tor}(A)$. Since $na \in \mathrm{tor}(A)$, there exists an $m \in \mathbb{N}$ such that $m(na) = 0$, so $(mn)a = 0$ and so $a \in \mathrm{tor}(A)$. Therefore $A/\mathrm{tor}(A)$ is torsion-free, because we proved that the only torsion element is the neutral element.

3. Let $f : A \to B$ be a homomorphism (where B is torsion-free abelian). Since any homomorphism from an abelian group into a torsion-free abelian group sends the torsion elements to zero, we have $\mathrm{tor}(A) \subseteq \ker f$. Define $\phi : A/\mathrm{tor}(A) \to B$ such that $\phi(a + \mathrm{tor}(A)) = f(a)$. We have to check that ϕ is well defined. Suppose that $a_1 + \mathrm{tor}(A) = a_2 + \mathrm{tor}(A)$. We need to check that $f(a_1) = f(a_2)$. Since $a_1 + \mathrm{tor}(A) = a_2 + \mathrm{tor}(A)$, it follows that $a_2 = a_1 + b$, for some $b \in \mathrm{tor}(A)$. In this case $f(a_2) = f(a_1 + b) = f(a_1) + f(b) = f(a_1)$ where we used the fact that $b \in \mathrm{tor}(A) \subseteq \ker f$. Thus ϕ is well defined.

   Now we check that ϕ is a homomorphism. Suppose that $x, y \in A/\mathrm{tor}(A)$. Then $x = a_1 + \mathrm{tor}(A)$ and $y = a_2 + \mathrm{tor}(A)$ for some $a_1, a_2 \in A$. Then

$$\begin{aligned}\phi(x+y) &= \phi\left((a_1 + \mathrm{tor}(A)) + (a_2 + \mathrm{tor}(A))\right) \\ &= \phi\left((a_1 + a_2) + \mathrm{tor}(A)\right) \\ &= f(a_1 + a_2) = f(a_1) + f(a_2) \\ &= \phi(a_1 + \mathrm{tor}(A)) + \phi(a_2 + \mathrm{tor}(A)) \\ &= \phi(x) + \phi(y).\end{aligned}$$

   Thus ϕ is a homomorphism.

   To check the uniqueness, let $\pi : A \to A/\mathrm{tor}(A)$, defined by $\pi(a) = a + \mathrm{tor}(A)$, be the natural homomorphism. Assume that there exists a homomorphism $\psi : A/\mathrm{tor}A \to B$ different from ϕ such that $\psi \circ \pi = f$. Let $a + \mathrm{tor}(A) \in A/\mathrm{tor}(A)$ such that $\psi(a + \mathrm{tor}(A)) \neq \phi(a + \mathrm{tor}(A))$. Then

$$f(a) = \psi \circ \pi(a) = \psi\left(\pi(a)\right) = \psi(a + \mathrm{tor}(A)) \neq \phi(a + \mathrm{tor}(A)) = f(a)$$

   is a contradiction.

4. Let $\varphi : A \ni a \mapsto \varphi(a) \in A$ is an endomorphism of A. If $b \in \operatorname{tor}(A)$, then there an $n \in \mathbb{N}$ such that $n \cdot b = 0$, but we have also

$$0 = \varphi(0) = \varphi(n \cdot b) = n \cdot \varphi(b),$$

and so $\varphi(b) \in \operatorname{tor}(A)$. Therefore $\varphi(\operatorname{tor}(A)) \subseteq \operatorname{tor}(A)$, because b was arbitrary in $\operatorname{tor}(A)$.

□

The Sylow theorems are particularly important for abelian groups. In fact, maximal *p-subgroups* of abelian groups have a special role in the theory of abelian groups; they are also called *p-primary* components.

2.2.5 Definition. Let p be a prime number. A group A is called a p-group if for each element $a \in A$ there is $n \in \mathbb{N}$ such that $p^n \cdot a = 0$. For a group A we set

$$A_p = \Big\{ a \in A \mid p^n \cdot a = 0 \text{ for some } n \in \mathbb{N} \Big\}$$

and it is called the p-primary component of A (or p-Sylow subgroup of A).

The structure of the torsion subgroup of an abelian group may be further decomposed in its p-primary components and this is a well known result.

2.2.6 Theorem. *((Hofmann and Morris, 2006), Theorem* A1.19.*) For an abelian group A, the p-primary component A_p is fully characteristic, and $\operatorname{tor}(A) = \bigoplus_{p \in \mathbb{P}} A_p$, where $\mathbb{P}$ is the set of all prime numbers.*

Proof. In order to show that A_p is a fully characteristic subgroup of A, we will firstly show that A_p is a subgroup of A. Clearly A_p is closed under sum. Because if $a \in A_p$ and $b \in A_p$, then there exist $m, n \in \mathbb{N}$ such that $p^n \cdot a = 0$ and $p^m \cdot b = 0$, this implies that

$$p^m(p^n \cdot a) = p^m 0 = 0 \ \Rightarrow p^{n+m} \cdot a = 0 \text{ and } p^{n+m} \cdot b = 0.$$

Therefore

$$p^{n+m} \cdot a + p^{n+m} \cdot b = p^{n+m} \cdot (a + b) = 0.$$

Thus $a + b \in A_p$. Moreover, since $0 \in A_p$ and $p^n \cdot a = p^n \cdot (-a) = 0$, A_p is subgroup of A.

Now, we want to show that A_p is fully characteristic. As done in Theorem 2.2.4 (4), let $\varphi : A_p \ni a \mapsto \varphi(a) \in A_p$ be an endomorphism of A_p. If $b \in A_p$, then there is $p^n \in \mathbb{N}$ such that $p^n \cdot b = 0$, but we have also $0 = \varphi(0) = \varphi(p^n \cdot b) = p^n \cdot \varphi(b)$, and so $\varphi(b) \in A_p$. Therefore $\varphi(A_p) \subseteq A_p$, because b was arbitrary in A_p. Therefore A_p is fully characteristic.

In order to show $\operatorname{tor}(A) = \bigoplus_{p \in \mathbb{P}} A_p$, we must show $\operatorname{tor}(A) \subseteq \bigoplus_{p \in \mathbb{P}} A_p$ and $\operatorname{tor}(A) \supseteq \bigoplus_{p \in \mathbb{P}} A_p$.

We have $\operatorname{tor}(A) = \{a \in A \mid n \cdot a = 0 \text{ for some } n \in \mathbb{N}\}$ and $A_p = \{a \in A \mid p^n \cdot a = 0 \text{ for some } n \in \mathbb{N}\}$. It is obvious that $A_p \subset \operatorname{tor}(A)$ for each p, therefore $\operatorname{tor}(A) \supseteq \bigoplus_{p \in \mathbb{P}} A_p$.

Now, by using a well known fact of arithmetic, if $a \in \operatorname{tor}(A)$ has the order m, then $m = \prod_{i=1}^{r} p_i^{n_i}$ where p_i are distinct primes and $n_i \geq 0$. Let $m_i = \frac{m}{p_i^{n_i}}$ for $i = 1, \ldots, r$. Clearly $m_i \in \mathbb{N}$ and $m_i \mid m$, therefore the order of $m_i a$ is $\frac{m}{m_i} = p_i^{n_i}$, hence $m_i a \in A_p$. Since $\gcd(m_1, \dots, m_r) = 1$, there exist integers $c_1, \dots, c_r$ such that $\sum_{i=1}^r c_i m_i = 1$, and so

$$a = \sum_{i=1}^{r} c_i(m_i a) \in \bigoplus_{p_i} A_{p_i} \subseteq \bigoplus_{p \in \mathbb{P}} A_p.$$

Therefore $a \in \oplus_{p \in \mathbb{P}} A_p$, so $\mathrm{tor}(A) \subseteq \bigoplus_{p \in \mathbb{P}} A_p$. Thus $\mathrm{tor}(A) = \bigoplus_{p \in \mathbb{P}} A_p$. □

We will end this section with some important constructions, which will be heavily used in the rest of the thesis. They are located here, because we do appropriate considerations on torsion and torsion-free infinite abelian groups.

Following (Hofmann and Morris, 2006), (given a prime p) we denote by

$$\frac{1}{p^\infty}\mathbb{Z} = \{\frac{m}{p^n} \mid m \in \mathbb{Z} \ , \ n \in \mathbb{N}\}$$

the group of all rational numbers which can be written in the form $\frac{m}{p^n}$ for integer numbers m and natural numbers n. The group $\frac{1}{p^\infty}\mathbb{Z}$ satisfies the axioms of abelian group with respect to the usual sum and is involved in the construction of the so called *Prüfer group*. Details are below.

2.2.7 Example. Let p be a prime number and define the set

$$\mathbb{Z}(p^\infty) = \frac{1}{p^\infty}\mathbb{Z}/\mathbb{Z} = \left\{\frac{m}{p^n} + \mathbb{Z} \mid m \in \mathbb{Z} \ , \ n \in \mathbb{N}\right\}.$$

Since $\mathbb{Z}$ is a subgroup of $\frac{1}{p^\infty}\mathbb{Z}$ and $\frac{1}{p^\infty}\mathbb{Z}$ is an abelian group, one can check without difficulties that the quotient $\mathbb{Z}(p^\infty)$ is an abelian group with respect to the usual sum.

An alternative way to define $\mathbb{Z}(p^\infty)$ is the following. Consider the embeddings

$$\phi_n : \mathbb{Z}(p^n) \to \mathbb{Z}(p^{n+1}) \text{ defined by } z + p^n\mathbb{Z} \mapsto z + p^{n+1}\mathbb{Z}, \ n \in \mathbb{N}.$$

These embeddings involve injective homomorphisms ϕ_n and cyclic groups $\mathbb{Z}(p^n)$, which we may organize in the following way:

$$\mathbb{Z}(p) \xrightarrow{\phi_1} \mathbb{Z}(p^2) \xrightarrow{\phi_2} \mathbb{Z}(p^3) \xrightarrow{\phi_3} \mathbb{Z}(p^4) \xrightarrow{\phi_5} \mathbb{Z}(p^5) \to \dots$$

and by forming the union

$$\mathbb{Z}(p^\infty) = \bigcup_{n \in \mathbb{N}} \mathbb{Z}(p^n)$$

we get again the same group.

This last construction has more general nature and is related to the so called *direct limits of p-groups* (see (Hofmann and Morris, 2006) for details), but we don't spend further time in describing more methods and techniques.

3. Divisibility and abelian tensor product

3.1 Divisibility

In this section, we present the concept of divisibility for abelian groups and some important results on divisible groups. All the groups of this section are in fact abelian.

3.1.1 Definition. An abelian group A is called divisible if for every every $a \in A$ and every natural number $n \in \mathbb{N}$ there is an $x \in A$ such that $n \cdot x = a$, i.e., $A \subseteq \mathrm{Div}(A)$ where $\mathrm{Div}(A)$ is the set of all divisible element of A. If $\mathrm{Div}(A) = \{0\}$ we say that A is *reduced.*

The following proposition collects a series of facts on divisible groups.

3.1.2 Proposition. ((Hofmann and Morris, 2006), Exercise EA1.11.)

1. The additive group of a vector space V over $\mathbb{Q}$ is divisible.
2. The group $\frac{1}{p^\infty}\mathbb{Z}$ is not divisible.
3. Let A be a divisible group and $B < A$ be a proper subgroup. Then A/B is divisible.
4. The Prüfer group $\mathbb{Z}(p^\infty)$ is divisible.
5. If $\{A_j \mid j \in J\}$ is a family of divisible groups then $\prod_{j\in J} A_j$ and $\bigoplus_{j\in J} A_j$ are divisible.

Proof.

1. A vector space V over $\mathbb{Q}$ has its additive group of the form $V = \oplus_{i\in I} V_i$, where $V_i \simeq \mathbb{Q}$ for all $i \in I$. We know that $\mathbb{Q}$ is an abelian group with respect to the usual sum. Any element of $\mathbb{Q}$ is of the form $\dfrac{r}{s}$, where $r, s \in \mathbb{Z}$ and $s \neq 0$. These fractions have the property that $\dfrac{nr}{ns} = \dfrac{r}{s}$ for every $n \in \mathbb{N}$. Therefore for every $a = \dfrac{r}{s} \in \mathbb{Q}$ and every $n \in \mathbb{N}$, there is an $x = \dfrac{r}{ns} \in \mathbb{Q}$ such that $n\left(\dfrac{r}{ns}\right) = \dfrac{r}{s}$. Thus $\mathbb{Q}$ is divisible. Now we next prove that $V_1 \oplus V_2$ is divisible when $V_1 \simeq V_2 \simeq \mathbb{Q}$. Let $(v_1, v_2) \in V_1 \oplus V_2$, where $n \in \mathbb{N}$. Because $V_1 \simeq V_2 \simeq \mathbb{Q}$ are divisible, there is $u_1 \in V_1$ and $u_2 \in V_2$ such that $nu_1 = v_1$ and $nu_2 = v_2$, so $n \cdot (u_1, u_2) = (nu_1, nu_2) = (v_1, v_2)$. This means that $V_1 \oplus V_2$ is divisible. Now we can repeat the same argument for the general case of $V = \oplus_{i\in I} V_i$, with I eventually infinite.
2. We claim that $\frac{1}{p^\infty}\mathbb{Z}$ is not divisible. We will prove it by counter-example. Suppose $\frac{1}{p^\infty}\mathbb{Z} = \left\{\dfrac{m}{p^n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\right\}$ is divisible. Then for every $a \in \frac{1}{p^\infty}\mathbb{Z}$ and every $n \in \mathbb{N}$ there is an $x \in \frac{1}{p^\infty}\mathbb{Z}$ such that $a = n \cdot x$. Take $p = 2$, $a = \frac{1}{2}$ and $n = 3$ and search for the existence of x such that $\frac{1}{2} = 3x$. We should have $x = \frac{1}{6} \in \frac{1}{2^\infty}\mathbb{Z}$ and this is a contradiction. The argument may be repeated for any prime $p \geq 3$ and we conclude that $\frac{1}{p^\infty}\mathbb{Z}$ is not divisible.
3. Assume that A is divisible and $B < A$ is a proper subgroup. We want to show that the quotient A/B is divisible. Since B is proper, A/B is non-trivial, and because A is abelian, A/B is abelian. Let $a + B \in A/B$ and $n \in \mathbb{N}$. Because A is divisible there is $b \in B \subset A$ such that $n \cdot b = a$ and $n \cdot (b + B) = a + B$. Thus A/B is divisible.
4. We want to show $\mathbb{Z}(p^\infty)$ is divisible. There are several ways to prove this fact. We use a well

known theorem of structure for $\mathbb{Q}/\mathbb{Z}$. This can be found in (Hofmann and Morris, 2006). In fact

$$\frac{\mathbb{Q}}{\mathbb{Z}} \simeq A_{p_1} \oplus A_{p_2} \oplus A_{p_3} \oplus \ldots \simeq \mathbb{Z}(p_1^{\infty}) \oplus \mathbb{Z}(p_2^{\infty}) \oplus \mathbb{Z}(p_3^{\infty}) \oplus \ldots,$$

where $A_{p_i} \simeq \mathbb{Z}(p_i^{\infty})$ for all $i \in I$. From 3 above, we may write the following quotient

$$A_{p_1} \simeq \frac{\mathbb{Q}/\mathbb{Z}}{A_{p_2} \oplus A_{p_3} \oplus \ldots}$$

which is divisible because it is a quotient of divisible groups.

5. Firstly we note that

$$\bigoplus_{j \in J} A_j = \left\{ (a_j)_{j \in J} \in \prod_{j \in J} A_j \mid a_j = 0_j \text{for all but finitely many } j \in J \right\}.$$

Now suppose that A_j are divisible for all $j \in J$. Let $(a_j)_{j \in J} \in \prod_{j \in J} A_j$, $n \in \mathbb{N}$. Because A_j are divisible for all $j \in J$, there is $b_j \in A_j$ such that $n \cdot b_j = a_j$ for all $j \in J$. Then $n \cdot (b_j)_{j \in J} = (n \cdot b_j)_{j \in J} = (a_j)_{j \in J}$. Thus $\prod_{j \in J} A_j$ is divisible. With a similar argument we can check that even $\bigoplus_{j \in J} A_j$ is divisible.

□

3.2 Abelian tensor product

In this section, we present the concept of abelian tensor product of groups. Most of the material is taken from (Hofmann and Morris, 2006). The theory of abelian tensor products will be used as a model for the theory of the nonabelian tensor products, involved in the rest of this thesis.

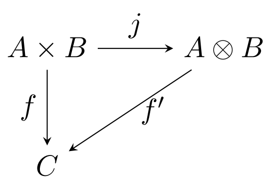
3.2.1 Definition. If A, B and C are abelian groups. A map $f : A \times B \to C$ is *bilinear*, if for every $a \in A$, and every $b \in B$, the maps $f(-, b) : A \to C$ given by $a \mapsto f(a, b)$ and $f(a, -) : B \to C$ defined by $b \mapsto f(a, b)$ are homomorphisms, i.e., f is *bilinear* if

$$f(a + a_1, b) = f(a, b) + f(a_1, b), \quad \text{and} \quad f(a, b + b_1) = f(a, b) + f(a, b_1)$$

for all $a, a_1 \in A$ and $b, b_1 \in B$.

Definition 3.2.1 is related to a universal construction, which we have seen already in Definition 2.1.4, for free abelian groups. Here we will formulate a similar universal construction, involving bilinear maps.

3.2.2 Proposition. ((Hofmann and Morris, 2006), Proposition $\mathrm{A1.44}\ (ii)$) For abelian groups A and B there is a group $A \otimes B$, with a function $j : A \times B \to A \otimes B$ defined by $j(a, b) = a \otimes b$, such that for every abelian group C and every bilinear map $f : A \times B \to C$ there is a unique homomorphism $f' : A \otimes B \to C$ such that $f'(a \otimes b) = f(a, b)$.



Proof. Let F be a free abelian group generated by $A \times B$ with a function $e : A \times B \to F$. Then by the universal property of Definition 2.1.4, for every function $f : A \times B \to C$ there is a unique homomorphism $\varphi : F \to C$ such that $f = \varphi \circ e$. Let U be a subgroup of a group F generated by the elements

$$e(a + a', b) - e(a, b) - e(a', b) \quad \text{and} \quad e(a, b + b') - e(a, b) - e(a, b') \;,$$

for all $a, a' \in A$ and $b, b' \in B$. Let $A \otimes B$ be the quotient group F/U, and set $a \otimes b = e(a, b) + U$. Let $\pi : F \to A \otimes B$ be the quotient homomorphism and $j : A \times B \to A \otimes B$ the composition map $\pi \circ e$. Then

$$\begin{aligned} j(a + a', b) - j(a, b) - j(a', b) &= \pi \circ e(a + a', b) - \pi \circ e(a, b) - \pi \circ e(a', b) \\ &= \pi\left(e(a + a_1, b)\right) - \pi\left(e(a, b)\right) - \pi\left(e(a', b)\right) \\ &= 0 \end{aligned}$$

therefore $j(a, b+b') = j(a, b)+j(a, b')$. Similarly $j(a, b+b') = j(a, b)+j(a, b')$. Thus $j : A\times B \to A\otimes B$ is bilinear.

Now, let $f : A \times B \to C$ be a bilinear map. Then there is a unique homomorphism $\varphi : F \to C$ such that $f = \varphi \circ e$. Hence

$$\begin{aligned} (\varphi \circ e)(a + a', b) - \varphi \circ e(a, b) - \varphi \circ e(a', b) &= \varphi\left((e(a + a', b)) - \varphi\left(e(a, b)\right) - \varphi\left(e(a', b)\right)\right. \\ &= f(a + a', b) - f(a, b) - f(a', b) \\ &= 0 \end{aligned}$$

therefore $\varphi(e(a + a', b) - e(a, b) - e(a', b)) = 0$ and similarly $\varphi(e(a, b + b') - e(a, b) - e(a, b')) = 0$. For all $a, a' \in A$ and $b, b' \in B$. Thus the subgroup U of the free a belian group F is generated by the elements of $\ker(\varphi)$, and so $U \subseteq \ker(\varphi)$. It follows that $\varphi : F \to C$ induced a unique homomorphism $f' : A \otimes B \to C$, $f'(a \otimes b) = f(a, b)$ such that $\varphi = f' \circ j$. then

$$f' \circ j = f' \circ \pi \circ e = \varphi \circ e = f.$$

Moreover if $\psi : A \times B \to C$ is any homomorphism satisfying $\psi \circ j = f$ then $\psi \circ \pi \circ e = f$. The uniqueness of the homomorphism $\psi : \mathrm{F} \to C$ ensures that $\psi \circ \pi = \varphi = f' \circ \pi$. But then $\psi = f'$, because the quotient homomorphism $\pi : \mathrm{F} \to A \otimes B$ is surjective. Thus the homomorphism f' is unique. □

We may connect the notion of divisible group with that of abelian tensor product. The following result shows an important relation.

3.2.3 Proposition. ((Hofmann and Morris, 2006), Proposition $A1.46$)

1. If D is divisible group and A is any abelian group, then $D \otimes A$ is divisible.
2. If D is divisible group and A is torsion group, then $D \otimes A = \{0\}$.

Proof.

1. Suppose that $d \otimes a \in D \otimes A$, where $d \in D$ and $a \in A$. Since D is divisible, for all $d \in D$, $n \in \mathbb{N}$ there is $x \in D$ such that $d = n \cdot x$. Therefore

   $$d \otimes a = (n \cdot x) \otimes a = x \otimes (n \cdot a) = n \cdot (x \otimes a),$$

   where $x \otimes a \in D \otimes A$. Since $d \otimes a \in D \otimes A$ is an arbitrary element, this implies that for all $d \otimes a \in D \otimes A$ and $n \in \mathbb{N}$ there is $x \otimes a \in D \otimes A$ such that $n \cdot (x \otimes a) = d \otimes a$, the result follows.

2. It suffices to show that every element $d \otimes a$, where $d \in D$ and $a \in A$, is the neutral element of $D \otimes A$. Since A is torsion for all $a \in A$ there exists $n \in \mathbb{N}$ such that $n \cdot a = 0$. Since D is divisible, we have for all $d \in D$ and for all $n \in \mathbb{N}$ there exists $x \in D$ such that $n \cdot x = d$. Therefore

$$d \otimes a = (n \cdot x) \otimes a = x \otimes (n \cdot a) = x \otimes 0 = 0$$

Since $d \otimes a \in D \otimes A$ is an arbitrary element, this implies that every element $d \otimes a$ equals the neutral element of $D \otimes A$. Thus $D \otimes A = \{0\}$.

□

The following example is instructive.

3.2.4 Example. Let n be a positive integer, compute $\mathbb{Z}(n) \otimes \mathbb{Q}$. Since $\mathbb{Z}(n)$ is torsion and $\mathbb{Q}$ is divisible, we claim that the tensor product is 0. It suffices to show that $a \otimes b$ in $\mathbb{Z}(n) \otimes \mathbb{Q}$ is 0. For any $a \in \mathbb{Z}(n)$ and $b \in \mathbb{Q}$, we have

$$a \otimes b = a \otimes \left(n \cdot \frac{y}{n}\right) = (n \cdot a) \otimes \frac{y}{n} = 0 \otimes \frac{y}{n} = 0.$$

Because $\mathbb{Z}(n)$ is torsion, for any $a \in \mathbb{Z}(n)$ we have $n \cdot a = 0$.

Another important relation between the notion of divisible group and of abelian tensor product is shown by the following result. This is reported without details from (Hofmann and Morris, 2006).

3.2.5 Proposition. Let A be an abelian group and $f : A \ni a \mapsto f(a) = 1 \otimes a \in \mathbb{Q} \otimes A$. Then

1. $\mathbb{Q} \otimes A = \langle q \otimes a \mid q \in \mathbb{Q}, a \in A \rangle$ and $(\mathbb{Q} \otimes A)/(1 \otimes A)$ is torsion;
2. $\mathbb{Q} \otimes A$ has the structure of a vector space over $\mathbb{Q}$;
3. $\mathbb{Q} \otimes A$ is a torsion-free divisible group and $\ker(f) = \operatorname{tor}(A)$;
4. $\mathbb{Q} \otimes A \simeq \mathbb{Q} \otimes (A/\operatorname{tor}(A))$.

Proof. See ((Hofmann and Morris, 2006), Proposition A1.45). □

We give details on an interesting exercise of computation.

3.2.6 Proposition. If $\gcd(m, n)$ is the greatest common divisor of two positive integers m and n, then $\mathbb{Z}(m) \otimes \mathbb{Z}(n) \simeq \mathbb{Z}(\gcd(m, n))$.

Proof. Let $\gcd(m, n) = q = mx + ny$ for some $x, y \in \mathbb{Z}$. We want to show that $\mathbb{Z}(m) \otimes \mathbb{Z}(n) \simeq \mathbb{Z}(q)$. It suffices to show that $\mathbb{Z}(m) \otimes \mathbb{Z}(n)$ is cyclic group of order q. For any $r, s \in \mathbb{Z}$ we have

$$r \otimes s = r \otimes (s \cdot 1) = rs \otimes 1 = rs(1 \otimes 1).$$

So the tensor product $\mathbb{Z}(m) \otimes \mathbb{Z}(n)$ is generated by $1 \otimes 1$. Now since $m(1 \otimes 1) = m \otimes 1 = 0 \otimes 1 = 0$, and $n(1 \otimes 1) = 1 \otimes n = 1 \otimes 0 = 0$, then

$$q(1 \otimes 1) = mx(1 \otimes 1) + ny(1 \otimes 1) = x(m \otimes 1) + y(1 \otimes n) = 0.$$

Hence the generator $1 \otimes 1$ has order dividing q. To prove that the generator has order exactly q, let $j : \mathbb{Z}(m) \times \mathbb{Z}(n) \to \mathbb{Z}(m) \otimes \mathbb{Z}(n)$ be the map defined by $j(r, s) = r \otimes s$ for all $r \in \mathbb{Z}(m)$ and $s \in \mathbb{Z}(n)$, then construct a bilinear map

$$\varphi : \mathbb{Z}(m) \times \mathbb{Z}(n) \to \mathbb{Z}(q) \qquad \text{by} \quad \varphi(a + m\mathbb{Z}, b + n\mathbb{Z}) = ab + q\mathbb{Z}.$$

This is well-defined, since if $a' + m\mathbb{Z} = a + m\mathbb{Z}$ and $b' + n\mathbb{Z} = b + n\mathbb{Z}$, then $a' = a + mx$ and $b' = b + ny$ for some x, y and thus

$$a'b' + q\mathbb{Z} = (a + mx)(b + ny) = ab + (mxb + nya + mnxy) + q\mathbb{Z} = ab + q\mathbb{Z}$$

because $q|m$ and $q|n$. We need to check that φ is bilinear. Clearly

$$\begin{aligned}\varphi(a + m\mathbb{Z} + a' + m\mathbb{Z}, b + n\mathbb{Z}) &= \varphi((a + a') + m\mathbb{Z}, b + n\mathbb{Z})\\ &= (a + a')b + q\mathbb{Z}\\ &= (ab + q\mathbb{Z}) + (a'b + q\mathbb{Z})\\ &= \varphi(a + m\mathbb{Z}, b + n\mathbb{Z}) + \varphi((a' + m\mathbb{Z}, b + n\mathbb{Z}),\end{aligned}$$

and

$$\varphi(a + m\mathbb{Z}, (b + b') + n\mathbb{Z}) = \varphi(a + m\mathbb{Z}, b + n\mathbb{Z}) + \varphi(a + m\mathbb{Z}, b' + n\mathbb{Z})$$

for all $a + m\mathbb{Z}, a' + m\mathbb{Z} \in \mathbb{Z}(m)$ and $b + n\mathbb{Z}, b' + n\mathbb{Z} \in \mathbb{Z}(n)$. Therefore φ is bilinear. By the universal property of Proposition 3.2.2, there is a unique homomorphism $\hat{\varphi} : \mathbb{Z}(m) \otimes \mathbb{Z}(n) \to \mathbb{Z}(q)$ such that $\hat{\varphi}(1 \otimes 1) = 1 + q\mathbb{Z} = 1$. But the order of $1 \in \mathbb{Z}(q)$ is q, so that the order of $1 \otimes 1 \in \mathbb{Z}(m) \otimes \mathbb{Z}(n)$ must at least q. We have already proved that $1 \otimes 1$ has order at most q, hence $1 \otimes 1$ must be an element of order q. Thus $\mathbb{Z}(m) \otimes \mathbb{Z}(n)$ is a cyclic group of order q. □

From this property we conclude that if m and n are relatively prime, then $\mathbb{Z}(m) \otimes \mathbb{Z}(n) \simeq \{0\}$.

There are two final results which we want to mention since they show that the operator $\otimes$ is commutative. This fact is quite important from a computational point of view.

3.2.7 Proposition. ((Hofmann and Morris, 2006), Proposition A1.47.) The tensor product is commutative and distributive:

$$A \otimes B \cong B \otimes A \quad \text{and} \quad A \otimes (\bigoplus_{j \in J} B_j) \simeq \bigoplus_{j \in J} (A \otimes B_j).$$

Proof. Firstly, we shall show that $A \otimes B \cong B \otimes A$. Let $A \otimes B$ be abelian tensor product with a map $j : A \times B \ni (a, b) \mapsto a \otimes b \in A \otimes B$. Define a map $f : A \times B \to B \otimes A$ by $f(a, b) = b \otimes a$. Then

$$\begin{aligned}f(a + a', b) &= b \otimes (a + a')\\ &= b \otimes a + b \otimes a'\\ &= f(a, b) + f(a', b),\end{aligned}$$

and

$$\begin{aligned}f(a, b + b') &= (b + b') \otimes a\\ &= b \otimes a + b' \otimes a\\ &= f(a, b) + f(a, b')\end{aligned}$$

for all $a, a' \in A$ and $b, b' \in B$. Therefore f is a bilinear map. Thus by the universal property of Proposition 3.2.2, there is a unique homomorphism $\varphi : A \otimes B \to B \otimes A$ such that $\varphi(a \otimes b) = b \otimes a$. Similarly, there exists a unique homomorphism $\varphi' : B \otimes A \to A \otimes B$ such that $\varphi'(b \otimes a) = a \otimes b$. It remains to show that φ and φ' are inverse of each other, i.e., $\varphi \circ \varphi' = id_{B\otimes A}$ and $\varphi' \circ \varphi = id_{A\otimes B}$. Let $b \otimes a \in B \otimes A$, then

$$(\varphi \circ \varphi')(b \otimes a) = \varphi(\varphi'(b \otimes a)) = \varphi(a \otimes b) = b \otimes a.$$

Since $b \otimes a$ is an arbitrary element of $B \otimes A$, therefore $\varphi(\varphi'(x)) = x$ for all $x \in B \otimes A$. Thus $\varphi \circ \varphi' = id_{B\otimes A}$. Similarly, $\varphi' \circ \varphi = id_{A\otimes B}$. Hence φ and φ' are inverse of each other. Thus $A \otimes B \cong B \otimes A$.

Now, we shall show that $A \otimes (\bigoplus_{j\in J} B_j) \cong \bigoplus_{j\in J}(A \otimes B_j)$. The map

$$\gamma \ : A \times (\bigoplus_{j\in J} B_j) \ni (a, (b_j)_{j\in J}) \mapsto (a \otimes b_j)_{j\in J} \in \bigoplus_{j\in J}(A \otimes B_j)$$

is bilinear, because

$$\begin{aligned}\gamma(a + a', (b_j)_{j\in J}) &= ((a + a') \otimes b_j)_{j\in J} \\ &= (a \otimes b_j)_{j\in J} + (a' \otimes b_j)_{j\in J} \\ &= \gamma(a, (b_j)_{j\in J}) + \gamma(a', (b_j)_{j\in J}),\end{aligned}$$

and

$$\begin{aligned}\gamma(a, (b_j)_{j\in J} + (b'_j)_{j\in J}) &= a \otimes ((b_j)_{j\in J} + (b'_j)_{j\in J}) \\ &= a \otimes (b_j)_{j\in J} + a \otimes (b'_j)_{j\in J} \\ &= \gamma(a, (b_j)_{j\in J}) + \gamma(a, (b'_j)_{j\in J}).\end{aligned}$$

By Proposition 3.2.2, there exists a unique homomorphism of groups

$$\alpha \ : A \otimes (\bigoplus_{j\in J} B_j) \ni a \otimes (b_j)_{j\in J} \mapsto (a \otimes b_j)_{j\in J} \in \bigoplus_{j\in J}(A \otimes B_j)$$

and one can check that the following map

$$\beta \ : \ \bigoplus_{j\in J}(A \otimes B_j) \ni (a \otimes b_j)_{j\in J} \mapsto a \otimes (b_j)_{j\in J} \in A \otimes (\bigoplus_{j\in J} B_j)$$

is a homomorphism of groups such that $\alpha \circ \beta = id_H$, where $H = A \otimes (\bigoplus_{j\in J} B_j)$, and $\beta \circ \alpha = id_K$, where $K = \bigoplus_{j\in J}(A \otimes B_j)$. Then the result follows.

□

4. The nonabelian tensor product of groups

The present section recalls some notions from (Brown et al., 1987) and Brown and Loday (1987), which are two fundamental papers on the theory of nonabelian tensor products. We will see that most of the results of Sections 2 and 3 may be amplified and improved for larger classes of groups.

Let G and H be two groups. We will use the multiplicative notation from now on instead of the additive notation which we used in the previous sections, because this is more appropriate when we deal with nonabelian groups. For $g \in G$ and $h \in H$ we denote the conjugation by the left:

$$^{g}h = ghg^{-1}$$

and in such case we say G acts on H by conjugation. If $G = H$ and $^{g}g' = gg'g^{-1}$ for all $g, g' \in G$ we say G acts on itself by conjugation. For the elements $g, g' \in G$ we write $[g, g'] = gg'g^{-1}g'^{-1}$ for the commutator of g and g'. Finally, we set $G' = [G, G] = \langle [g, g'] \mid g, g' \in G \rangle$ for the group generated by all the commutators, and is called *commutator subgroup (or derived subgroup)*.

4.1 The construction

Following (Brown et al., 1987), if the groups G and H act on themselves by conjugation and each of them acts upon the other in a *compatible way*, that is, satisfying the following conditions:

$$^{(^{g}h)}g' = {}^{ghg^{-1}}g', \quad ^{(^{h}g)}h' = {}^{hgh^{-1}}h', \tag{4.1.1}$$

for all $g, g' \in G$ and $h, h' \in H$, we may define the *nonabelian tensor product* $G \otimes H$ as the group generated by the symbols $g \otimes h$, satisfying the following relations:

$$gg' \otimes h = (^{g}g' \otimes {}^{g}h)(g \otimes h), \tag{4.1.2}$$

$$g \otimes hh' = (g \otimes h)(^{h}g \otimes {}^{h}h'), \tag{4.1.3}$$

for all $g, g' \in G$ and $h, h' \in H$. Moreover, if $G = H$, then $G \otimes G$ is known as *nonabelian tensor square* of G.

As one may expect, the standard theory of abelian tensor products can be found as a special case of the theory of the nonabelian tensor products. In fact if G and H are abelian and act trivially on each other (and by conjugation on themseleves), then $G \otimes H$ is the usual abelian tensor product, which we have discussed in the previous sections.

There are a series of important facts, which can be observed in (Brown and Loday, 1987; Brown et al., 1987; Russo, 2016). For instance, the usual tensor product of two abelian groups is always an abelian group (see Proposition 3.2.2), while the tensor product of two nonabelian groups is sometimes abelian and sometimes not. The following result is well known and describes the behaviour of the nonabelian tensor square of dihedral groups.

4.1.1 Proposition. ((Brown et al., 1987) , Proposition 14) Let D_m be the dihedral group of order $2m$ with presentation

$$D_m = \langle a, b \mid a^m = b^2 = 1, bab^{-1} = a^{-1} \rangle$$

with $m \geq 1$.

1. If m is odd, then $D_m \otimes D_m \simeq \mathbb{Z}(2) \times \mathbb{Z}(m)$.
2. If m is even, then $D_m \otimes D_m \simeq \mathbb{Z}(2) \times \mathbb{Z}(m) \times \mathbb{Z}(2) \times \mathbb{Z}(2)$

Thanks to the previous proposition, whose proof is not reported here because is long and contains many computational difficulties, we may do some important considerations.

Given an arbitrary group G, one can check that the commutator subgroup G' is always a normal subgroup of G and the quotient G/G' turns out to be abelian. This means that in any group, we have always an abelian quotient, that is, G/G' and so we may always form the usual abelian tensor square $G/G' \otimes G/G'$.

Now the following example shows that in $G = D_4$ we have that $D_4 \otimes D_4$ is different from $D_4/D_4' \otimes D_4/D_4'$.

4.1.2 Example. Let D_4 be the dihedral group of order 8. We want to find $D_4/D_4' \otimes D_4/D_4'$. We have

$$D_4' = \langle [a, b] \mid a, b \in D_4 \rangle = \langle aba^{-1}b^{-1} \mid a, b \in D_4 \rangle = \langle a^2 \rangle = \{1, a^2\} \simeq \mathbb{Z}(2)$$

is cyclic of order 2. Clearly $D_4/D_4' \simeq \mathbb{Z}(2) \times \mathbb{Z}(2)$. Applying Propositions 3.2.6 and 3.2.7

$$\begin{aligned} D_4/D_4' \otimes D_4/D_4' &\simeq (\mathbb{Z}(2) \times \mathbb{Z}(2)) \otimes (\mathbb{Z}(2) \times \mathbb{Z}(2)) \\ &\simeq ((\mathbb{Z}(2) \times \mathbb{Z}(2)) \otimes \mathbb{Z}(2)) \times ((\mathbb{Z}(2) \times \mathbb{Z}(2)) \otimes \mathbb{Z}(2)) \\ &\simeq (\mathbb{Z}(2) \otimes \mathbb{Z}(2)) \times (\mathbb{Z}(2) \otimes \mathbb{Z}(2)) \times (\mathbb{Z}(2) \otimes \mathbb{Z}(2)) \times (\mathbb{Z}(2) \otimes \mathbb{Z}(2)) \\ &\simeq \mathbb{Z}(2) \times \mathbb{Z}(2) \times \mathbb{Z}(2) \times \mathbb{Z}(2) \end{aligned}$$

On the other hand, Proposition 4.1.1 shows that $D_4 \otimes D_4 \simeq \mathbb{Z}(2) \times \mathbb{Z}(4) \times \mathbb{Z}(2) \times \mathbb{Z}(2)$ and this is different, as we claimed.

The following notion is what we need in order to generalize the notion of "bilinear map" which we have seen in the previous sections for abelian groups.

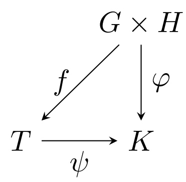
4.1.3 Definition. ((Brown et al., 1987), Remark 3) Let G, H and K be groups. A map $\varphi : G \times H \to K$ is said to be a *crossed pairing* if for all $g, g' \in G$ and $h, h' \in H$,

$$\begin{aligned} \varphi(gg', h) &= \varphi({}^g g', {}^g h)\varphi(g, h), \\ \varphi(g, hh') &= \varphi(g, h)\varphi({}^h g, {}^h h'). \end{aligned}$$

If G and H are two abelian groups, the actions are by conjugations on themselves and trivially on each other, we have already noted that $G \otimes H$ is the usual abelian tensor product. Now consider a third abelian group K and use the additive notation. Then the crossed pairing of Definition 4.1.3 specializes to the notion of bilinear map in Definition 3.2.1.

Now it is possible to see that a universal property like Proposition 3.2.2 holds in the context of nonabelian tensor products. In fact the following result describes the universal property for crossed pairings.

4.1.4 Proposition. ((Russo, 2016), Proposition 2.3) Let G, H and T be groups, and a map $f : G \times H \to T$ be a crossed pairing. Assuming that G and H act in such a way that $G \otimes H$ can be defined, then $T \simeq G \otimes H$ if and only if for every group K and every crossed pairing $\varphi : G \times H \to K$ there exists a unique homomorphism $\psi : T \to K$ making commutative the following diagram



i.e., $(\psi \circ f)(g, h) = \varphi(g, h)$ for all $g \in G$ and $h \in H$.

In order to be confident with notations and nonabelian tensor products, we give details on Proposition 1 (i) of (Brown et al., 1987), which was left as exercise for the reader.

4.1.5 Proposition. ((Brown et al., 1987), Proposition 1 (i)) Suppose the groups G and H act compatibly on each other. Then G and H act on $G \otimes H$ so that

$${}^g(g' \otimes h) = {}^gg' \otimes {}^gh, \quad {}^h(g \otimes h') = {}^hg \otimes {}^gh'$$

for all $g, g' \in G$ and $h, h' \in H$.

Proof. We must check that the actions defined above preserve the relations (4.1.2) and (4.1.3). Let p be an arbitrary element of G or H. Then

$$\begin{aligned}
{}^p(gg' \otimes h) &= {}^p(gg') \otimes {}^ph \\
&= {}^pg\, {}^pg' \otimes {}^ph \\
&= \left({}^{({}^pg)}({}^pg') \otimes {}^{({}^pg)}({}^ph)\right)({}^pg \otimes {}^ph) \quad \text{using (4.1.2)} \\
&= \left({}^{(pgp^{-1})}({}^pg') \otimes {}^{(pgp^{-1})}({}^ph)\right)({}^pg \otimes {}^ph) \\
&= \left({}^{pgp^{-1}p}(g') \otimes {}^{pgp^{-1}p}(h)\right)({}^pg \otimes {}^ph) \\
&= ({}^{pg}g' \otimes {}^{pg}h)({}^pg \otimes {}^ph) \\
&= {}^p\,({}^gg' \otimes {}^gh){}^p\,(g \otimes h) \\
&= {}^p\big[({}^gg' \otimes {}^gh)(g \otimes h)\big].
\end{aligned}$$

Similarly,

$$\begin{aligned}
{}^p(g \otimes hh') &= {}^pg \otimes {}^p(hh') \\
&= {}^pg \otimes {}^ph\, {}^ph' \\
&= ({}^pg \otimes {}^ph)\left({}^{({}^ph)}({}^ph) \otimes {}^{({}^ph)}({}^ph')\right) \quad \text{using (4.1.3)} \\
&= ({}^pg \otimes {}^ph)\left({}^{(php^{-1})}({}^pg) \otimes {}^{(php^{-1})}({}^ph')\right) \\
&= ({}^pg \otimes {}^ph)\left({}^{php^{-1}p}(g) \otimes {}^{php^{-1}p}(h')\right) \\
&= ({}^pg \otimes {}^ph)\left({}^{ph}g \otimes {}^{ph}h'\right) \\
&= {}^p\,(g \otimes h)\; {}^p\left({}^hg \otimes {}^hh'\right) \\
&= {}^p\Big[(g \otimes h)\left({}^hg \otimes {}^hh'\right)\Big].
\end{aligned}$$

Therefore the relations (4.1.2) and (4.1.3) are preserved.

□

4.2 Some analogies with the abelian case

In order to show that the nonabelian tensor product is commutative, we will show a result which generalizes Proposition 3.2.7. This result is rewritten with more details and comments in the proof below.

4.2.1 Proposition. ((Brown et al., 1987), Proposition 1 (iii)) Suppose the groups G and H act compatibly on each other in such a way that $G \otimes H$ may be defined. Then there is an isomorphism

$$\tau : G \otimes H \to H \otimes G$$

such that $\tau(g \otimes h) = (h \otimes g)^{-1}$, for all $g \in G, h \in H$.

Proof. Define a map $\varphi : G \times H \to H \otimes G$ by $\varphi(g, h) = (h \otimes g)^{-1}$. Then

$$\begin{aligned}\varphi(gg', h) &= (h \otimes gg')^{-1}\\ &= [(h \otimes g)({}^{g}h \otimes {}^{g}g')]^{-1}\\ &= ({}^{g}h \otimes {}^{g}g')^{-1}(h \otimes g)^{-1}\\ &= \varphi({}^{g}g', {}^{g}h)\varphi(g, h),\end{aligned}$$

and

$$\begin{aligned}\varphi(g, hh') &= (hh' \otimes g)^{-1}\\ &= [({}^{h}h' \otimes {}^{h}g)(h \otimes g)]^{-1}\\ &= (h \otimes g)^{-1}({}^{h}h' \otimes {}^{h}g)^{-1}\\ &= \varphi(g, h)\varphi({}^{h}g, {}^{h}h')\end{aligned}$$

for all $g, g' \in G$ and $h, h' \in H$. Therefore φ is a crossed pairing. Hence by the universal property of Proposition 4.1.4, there is a unique homomorphism $\tau : G \otimes H \to H \otimes G$ such that $\tau(g \otimes h) = (h \otimes g)^{-1}$. Similarly, define a map $\varphi' : H \times G \to G \otimes H$ by $\varphi'(h, g) = (g \otimes h)^{-1}$. clearly φ' is a crossed pairing. Hence there is a unique homomorphism $\tau' : H \otimes G \to G \otimes H$ by $\tau'(h \otimes g) = (g, \otimes h)^{-1}$. It remains to show that τ and τ' are inverse of each other, i.e., $\tau \circ \tau' = id_{H \otimes G}$ and $\tau' \circ \tau = id_{G \otimes H}$. Let $h \otimes g$ be an element of $H \otimes G$, then

$$\begin{aligned}(\tau \circ \tau')(h \otimes g) &= \tau(\tau'(h \otimes g))\\ &= \tau((g \otimes h)^{-1})\\ &= (h \otimes g).\end{aligned}$$

Since $h \otimes g$ is an arbitrary element of $H \otimes G$, therefore $\tau(\tau'(a)) = a$ for all $a \in H \otimes G$. Thus $\tau \circ \tau' = id_{H \otimes G}$. Similarly, $\tau' \circ \tau = id_{G \otimes H}$. Hence τ and τ' are inverse of each other. Thus the map $\tau : G \otimes H \to H \otimes G$ such that $\tau(g \otimes h) = (h \otimes g)^{-1}$ for all $g \in G$ and $h \in H$ is an isomorphism. □

We report a series of technical rules of computation in the nonabelian tensor products, because this shows why several authors use GAP (Russo, 2012) when they want to get new results in this topic.

4.2.2 Proposition. ((Brown et al., 1987), Proposition 3) Let G and H be groups acting upon each other compatibly and on themselves by conjugation, the following relations hold for all $g, g' \in G$ and

$h, h' \in H$:

$$^{g}(g^{-1} \otimes h) = (g \otimes h)^{-1} = \ ^{h}(g \otimes h^{-1}), \tag{4.2.1}$$

$$(g \otimes h)(g' \otimes h')(g \otimes h)^{-1} = \ ^{[g,h]}(g' \otimes h'), \tag{4.2.2}$$

$$(g\ ^{h}g^{-1}) \otimes h' = (g \otimes h)\ ^{h'}(g \otimes h)^{-1}, \tag{4.2.3}$$

$$g' \otimes (^{g}hh^{-1}) = \ ^{g'}(g \otimes h)(g \otimes h)^{-1}, \tag{4.2.4}$$

$$[g \otimes h, g' \otimes h'] = (g^{h}g^{-1}) \otimes (^{g'}h'\ h'^{-1}) \tag{4.2.5}$$

4.3 Basic results of topological nature

In order to present some new relations with the theory of the nonabelian tensor products and that of topological groups, we recall some basic definitions of general topology.

4.3.1 Definition.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces and $f : X \to Y$ is a map. Then:

1. f is *continuous* if for each $U \in \mathcal{T}'$, $f^{-1}(U) \in \mathcal{T}$.
2. X is *Hausdorff (or $\mathbf{T}_2$)*, if for every distinct points $x, y \in X$ there are open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.
3. X is *compact* if for any covering of X by open sets $X = \bigcup_{\alpha \in A} U_\alpha$ there is a finite subset $\{\alpha_1, \ldots, \alpha_r\}$ of A such that $X \subseteq \bigcup_{i=1}^{r} U_{\alpha_i}$.

As usual, if we refer to X as topological space, we may omit the notation $(X, \mathcal{T})$, writing briefly X.

On a group we may have both an algebraic structure and a topological structure. This motivates the notion of topological group, which we report below from (Hofmann and Morris, 2006).

4.3.2 Definition. ((Hofmann and Morris, 2006), Definition 1.1.) A topological group is a group G which is also a topological space, such that multiplication $G \times G \ni (g, h) \mapsto gh \in G$ and inversion $G \ni g \mapsto g^{-1} \in G$ are continuous maps. A compact group is a topological group whose topology is compact Hausdorff.

A *locally compact* group is a topological group such that the topology is Hausdorff and the identity has a compact neighborhood. There are some useful examples of topological groups, listed below (see (Hofmann and Morris, 2006)).

4.3.3 Example.

1. The additive group $\mathbb{R}$ with the usual topology is a topological group.
2. The multiplicative group $\mathbb{R}^{\times}$ with the induced topology is a topological group.
3. The *general linear group*

$$\mathrm{Gl}(n, \mathbb{R}) = \{g \text{ is an } n \times n \text{ matrix with real coefficients} \mid \det(g) \neq 0\}$$

   of degree n is a topological group.

4. In $\mathbb{R}^n$ with the standard scalar product $(x \mid y) = x_1y_1 + \ldots + x_ny_n$,

$$\mathrm{Sl}(n, \mathbb{R}) = \{g \in \mathrm{Gl}(n, \mathbb{R}) \mid \det(g) = 1\}$$

is a topological group, called *special linear group* of degree n over $\mathbb{R}$,

$$\mathrm{O}(n, \mathbb{R}) = \{g \in \mathrm{Gl}(n, \mathbb{R}) \mid (gx \mid gx) = (x \mid x), \forall x \in \mathbb{R}^n\}$$

denotes the *orthogonal group* of degree n and

$$\mathrm{SO}(n, \mathbb{R}) = \mathrm{O}(n, \mathbb{R}) \cap \mathrm{Sl}(n, \mathbb{R})$$

denotes the *special orthogonal group* of degree n. Replacing $\mathbb{R}$ with $\mathbb{C}$ and $(x \mid y)$ with $(x \mid \overline{y})$, we get the *unitary group* $\mathrm{U}(n, \mathbb{C})$ instead of $\mathrm{O}(n, \mathbb{R})$ and the *special unitary group* $\mathrm{SU}(n, \mathbb{C})$ instead of $\mathrm{SO}(n, \mathbb{R})$.

5. $\mathrm{O}(n, \mathbb{R})$, $\mathrm{SO}(n, \mathbb{R})$, $\mathrm{U}(n, \mathbb{C})$ and $\mathrm{SU}(n, \mathbb{C})$ are compact groups for all $n \geq 1$.

6. The group additive abelian group $\mathbb{R}/\mathbb{Z} = \mathbb{T}$ is called **torus** and is compact.

7. Finite groups are compact when we consider the discrete topology.

The following proposition shows some facts about topological groups.

4.3.4 Proposition. ((Hofmann and Morris, 2006), pp. $2 - 10$)

1. Subgroups of a topological group are topological groups with respect to the induced topology.

2. Quotients of topological groups are topological groups with respect to the quotient topology.

3. The cartesian product of topological groups is a topological group with respect to the product topology.

Proof.

1. Let G be a topological group and H a subgroup of G. Let $\mathcal{T}_H$ be the induced topology on H in the space $(G, \mathcal{T})$, i.e., $\mathcal{T}_H = \{U \cap H \mid U \in \mathcal{T}\}$. Since the multiplication $G \times G \ni (x, y) \mapsto x \cdot y \in G$ is continuous, the restriction of this map $H \times H \to H$ is continuous. By continuity of G the inversion $G \ni x \mapsto x^{-1} \in G$ is continuous, the restriction of this map $H \to H$ is continuous. Then the result follows, because of the continuity of this operations.

2. Let G be a topological group and H be a normal subgroup of G. We want to show that G/H is a topological group.

   Firstly, we will show that the multiplication map $G/H \times G/H \ni (xH, yH) \mapsto (xH)(yH) = xyH \in G/H$ is continuous. Let UH be a neighborhood of xyH, where U is a neighborhood of xy . Since the multiplication is continuous in G, there is a neighborhood U_1 of x and neighbourhood U_2 of y such that $U_1U_2 \subset U$. Clearly, U_1H is a neighborhood of xH and U_2H is a neighborhood of yH. We have $U_1HU_2H = U_1U_2H \subset UH$ as $U_1U_2 \subset U$. Thus the multiplication map is map is continuous. It remains to show that the inversion $G/H \ni xH \mapsto (xH)^{-1} \in G/H$ is continuous. Let VH be a neighborhood of $(xH)^{-1} = x^{-1}H$, where V is a neighborhood of x^{-1}. Since the inversion is continuous in G, there is a neighborhood V_1 of x such that $V_1^{-1} \subset V$. Clearly, V_1H is a neighborhood of xH and $(V_1H)^{-1} = V_1^{-1}H \subset VH$. So the inversion is continuous. Thus G/H is a topological group.

3. Let G_i be a topological group for each $i \in I$. We want to show that $\prod_{i\in I} G_i$ is topological group. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} G_i$ and $\mathcal{N}$ be a neighborhood of $(x_i)_{i\in I}(y_i)_{i\in I}$. Then, by the definition of the product topology, there are open subsets U_i of G_i such that $x_i y_i \in U_i$ for all $i \in I$, and only finite number of U_i are different from G_i, where $\prod_{i\in I} U_i \subset \mathcal{N}$. Now, for $i \in I$ assume that V_i is a neighborhood of x_i and W_i is a neighborhood of y_i, in G_i such that $V_i W_i \subset U_i$. Let $V = \prod_{i\in i} V_i$ and $W = \prod_{i\in I} W_i$. Clearly, V is a neighborhood of $(x_i)_{i\in I}$ and W is a neighborhood of $(y_i)_{i\in I}$, and $VW \subset \prod_{i\in I} U_i \subset \mathcal{N}$.

□

In the context of compact groups we should reformulate Proposition 4.3.4 as following.

4.3.5 Proposition.

1. If H is a closed subgroup of a compact group G, then H is a compact group.
2. If N is a closed normal subgroup of a compact group G, then G/N is a compact group.
3. If G_i is a compact group for each $i \in I$, then $\prod_{i\in I} G_i$ is a compact group.

Proof.

1. Let H be a closed subgroup of a compact group G. From Proposition 4.3.4 we have a subgroup H of a topological group G is topological group with respect to the induced topology. It remains to show that H is compact. Let $\bigcup_{\alpha\in I} U_\alpha$ for some index set I be and open cover of H where each U_α is an open subset of G. Since H is closed, the complement of H is an open and $\bigcup_{\alpha\in I} U_\alpha \bigcup (G \setminus H)$ is an open cover of G. Since G is compact, there is finite subcover of G such that
$$G = \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) \bigcup (G \setminus H)\,,$$
this is implies that $H \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$, which is finite subcover H. Hence H is a compact.
2. Let G be a compact group and N a closed normal subgroup of G. Let $\varphi : G \ni x \mapsto [x] = xN \in G/N$ be the natural homomorphism onto quotient topology G/N. Let $\{U_\alpha N \mid \alpha \in I\}$ be a collection of open subsets of G/N such that $G/N = \bigcup_{\alpha\in I} U_\alpha N$. Then, given any $x \in G$, there is some α such that $\varphi(x) = [x] = xN \in U_\alpha N$, this means, there is some α such that $x \in \varphi^{-1}(U_\alpha N)$. As φ is continuous, therefore the open subsets $\varphi^{-1}(U_\alpha N)$ form an open cover of G. Since G is a compact, there is a finite subcover of G such that $G = \bigcup_{i=1}^{n} \varphi^{-1}(U_{\alpha_i} N)$, this implies that
$$\varphi(G) = G/N = \varphi\left(\bigcup_{i=1}^{n} \varphi^{-1}(U_{\alpha_i} N)\right) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} N,$$
which is a finite subcover of G/N. Thus G/N is a compact.
3. See ((Hofmann and Morris, 2006), Proposition 1.14).

□

We have already constructed (see Example 2.2.7) a group which is the union of some of its subgroups, satisfying some particular properties. Now we will see another important example, which may be constructed as union of its subgroups, but this will lead to a different notion which we formalize below.

4.3.6 Definition. ((Hofmann and Morris, 2006), Definition 1.25.) Let I be a directed set (or the set of ordered pair). Then, a *projective system of topological groups* over I is a family of morphisms $\{f_{ij} : G_j \to G_i \mid (i,j) \in I \times I, i \leq j\}$, where $G_i, i \in I$ are topological groups, satisfying the following conditions

1. $f_{ii} = id_{G_i}$ for all $i \in I$
2. $f_{ij} \circ f_{jk} = f_{ik}$ for all $i, j, k \in I$ with $i \leq j \leq k$.

When we have a projective system of topological groups, it is possible to construct a new group, called *projective limit*. This new group (see (Hofmann and Morris, 2006), Lemma 1.26) is placed in a suitable cartesian product.

4.3.7 Definition. ((Hofmann and Morris, 2006), Definitions 1.27.) If $\mathcal{P} = \{f_{ij} : G_j \to G_i \mid (i,j) \in I \times I, i \leq j\}$ is a projective system of topological groups, then the group

$$G = \left\{(g_i)_{i\in I} \in \prod_{i\in I} G_i \mid (\forall\ i, j \in I)\ i \leq j \Rightarrow f_{ij}(g_j) = g_i\right\}$$

is called its *projective limit* and is written $G = \lim_{i\in I} G_i$.

The morphisms $f_i : G \to G_i$ are called *limit maps*.

The morphisms $f_{ij} : G_j \to G_i$ are called *bonding maps*.

The reason why we introduced the previous notions, is due to the next important example, which is described in (Hofmann and Morris, 2006), Example 1.28. (i).

4.3.8 Example. Choose a prime number p and for $n \in \mathbb{N}$ set $\mathbb{Z}(p^n) = \mathbb{Z}/p^n\mathbb{Z}$. We have $\mathbb{Z}(p^n)$ with the discrete topology is a compact abelian group. Define $\varphi_n : \mathbb{Z}(p^{n+1}) \to \mathbb{Z}(p^n)$ by $\varphi_n(z + p^{n+1}\mathbb{Z}) = z + p^n\mathbb{Z}$:

$$\mathbb{Z}(p) \xleftarrow{\varphi_1} \mathbb{Z}(p^2) \xleftarrow{\varphi_2} \mathbb{Z}(p^3) \xleftarrow{\varphi_3} \mathbb{Z}(p^4) \xleftarrow{\varphi_4} \mathbb{Z}(p^5) \xleftarrow{\varphi_5} \dots$$

The compact abelian groups $\mathbb{Z}(p^n)$ and the continuous groups homomorphisms φ_n are inverse system of topological compact abelian groups. The projective limit of this system is a compact abelian group denoted by $\mathbb{Z}_p$, called the additive group of p-*adic* integers, and is given by:

$$\mathbb{Z}_p = \lim_{n\in\mathbb{N}} \mathbb{Z}(p^n) = \left\{(z + p^{n+1}\mathbb{Z})_{n\in\mathbb{N}} \mid (\forall n \in \mathbb{N})\ \varphi_n(z + p^{n+1}\mathbb{Z}) = z + p^n\mathbb{Z}\right\}.$$

4.4 Nonabelian tensor products of topological groups

Compact groups can be found always in the cartesian product of groups of matrices, but the cartesian product possesses a very rich structure, containing not only projective limits but many other groups different from the factors. The following result gives an idea of how compact groups are done and is a classical result on the theory of compact groups (see (Hofmann and Morris, 2006)). Its proof involves some technical notions and we don't report the details.

4.4.1 Theorem (Peter and Weyl, 1927)**.** *Let G be a compact group and $1 \neq x \in G$.*

1. *There exist an $n \in \mathbb{N}$ and a continuous homomorphism $f : G \to \mathrm{U}(n, \mathbb{C})$ such that $f(x) \neq 1$.*

2. *G can always be embedded in $\prod_{n\in\mathbb{N}} \mathrm{U}(n,\mathbb{C})$.*

Here $\mathrm{U}(n,\mathbb{C})$ denotes the unitary group.

In fact we can say something more in the abelian case, specializing the Theorem of Peter and Weyl (see (Hofmann and Morris, 2006)) :

4.4.2 Theorem (Structure of Compact Abelian Groups)**.** *Any compact abelian group is always the projective limit of groups which are extensions of $\mathbb{R}/\mathbb{Z}$ by finite abelian groups.*

Even the proof the theorem above requires some techniques which we haven't mentioned and so we just mention it, without details of proof.

The reason why we mention Theorems 4.4.1 and 4.4.2 is because of the presence of projective limits and cartesian products. These theorems illustrate that the role of projective limits and cartesian products is fundamental for the general description of compact groups.

In particular, a $pro\text{-}p\text{-}group$ (p prime) is a compact Hausdorff group in which closed subgroups are those of p-power index. This means that

$$\mathcal{P}(G) = \{N = \overline{N} \triangleleft G \mid G/N \text{ is a finite } p - \text{group}\},$$

and $G = \lim_{N\in\mathcal{P}(G)} G/N$, that is G is a projective limit of finite p–groups. Of course, we may replace $\mathcal{P}(G)$ with something more general like

$$\mathcal{F}(G) = \{N = \overline{N} \triangleleft G \mid G/N \text{ is a finite group}\},$$

and if $G = \lim_{N\in\mathcal{F}(G)} G/N$, the we say that G is a *profinite group*.

One could formulate the notions of crossing pairs and of nonabelian tensor product for projective limits of groups and ask what is the behaviour of the operator $\otimes$ of nonabelian tensor product with respect to the operator $\lim$ of projective limit. The answer has been found recently in (Russo, 2016) and is reported below for pro-p-groups.

4.4.3 Theorem. *Let $G = \lim_{N\in\mathcal{P}(G)} G/N$ and $H = \lim_{M\in\mathcal{P}(H)} H/M$ be pro-p-groups. Then there exists a natural isomorphism of pro-p-groups such that*

$$\lim_{(N,M)\in\mathcal{P}(G)\times\mathcal{P}(H)} G/N \otimes H/M \simeq \lim_{N\in\mathcal{P}(G)} G/N \otimes \lim_{M\in\mathcal{P}(H)} H/M.$$

The same theorem has been proved in (Russo (2016), Theorem 1.1), replacing $\mathcal{P}(G)$ with $\mathcal{F}(G)$ and further generalizations seem to be possible. In fact this theorem allows us to conclude that the nonabelian tensor product of the pro–p–groups G and H has the topology which is the projective limit of the topologies of each factor with respect to the operator $\otimes$ of the nonabelian tensor product.

5. Conclusion

We introduced some elementary notions on abelian groups. Furthermore, we showed the link between divisible groups and abelian tensor products. Finally, we studied the nonabelian tensor products of projective limits, and described their topology which is the projective limit of the topologies of each factor with respect to the operator $\otimes$ of nonabelian tensor products.

The theory of pro–p–groups and profinite groups may be related under this new perspective of the theory of the nonabelian tensor products. Several authors are recently interested in this perspective of research. In fact in recent years, these relations are involving a series of new concepts from graph theory and combinatorics, which is not possible to describe with details here. This will be the subject of the future studies.

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References

R. Brown and J.-L. Loday. Van kampen theorems for diagrams of spaces. *J. Topology*, 111:311–335, 1987.

R. Brown, D. Johnson, and E. Robertson. Some computations of non-abelian tensor products of groups. *J. Algebra*, 111:177–202, 1987.

K. Hofmann and S. Morris. *The Structure of Compact Groups*. de Gruyter, Berlin, 2006.

F. Russo. Problems of connectivity between the sylow graph, the prime graph and the non-commuting graph of a group. *Advances in Pure Mathematics*, 6:391–396, 2012.

F. G. Russo. On the topology of the nonabelian tensor product of profinite groups. *Bull. Korean Math. Soc.*, 53:751–763, 2016.