A Categorical Approach to the Jordan-Hölder Theorem

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Abstract

The Jordan-Hölder theorem was proved for groups in the 19^{th} century. It has since been extended to other algebraic structures like rings and modules. Other ways of proving the theorem have also been written. This essay gives a generalized proof of Jordan-Hölder using concepts in category theory and Galois connections so that it can be applied to algebraic structures like the ones mentioned above. This essay also examines how Jordan-Hölder can be used to prove the fundamental theorem of arithmetic and how it reduces the problem of classification of finite groups to the extension problem.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Trevor Chilombo Chimpinde, 22 May, 2014.

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1. Introduction

The classical Jordan-Hölder theorem, proved by French mathematician Camille Jordan and German mathematician Otto Hölder, says that if a group has a composition series, then any two of its composition series are isomorphic. Later, Otto Schreier proved that every two normal series of an arbitrary group have isomorphic refinements. Other proofs and generalisations of the theorem have since been made, most notably by Hans Julius Zassenhaus, who in 1934, gave an elegant proof of Schreier refinement theorem using what has become known as the butterfly lemma.

This essay will give a general proof of Jordan-Hölder theorem using concepts in category theory, following the recent work of Zurab Janelidze (Janelidze (2013a), Janelidze (2013b), Janelidze (2013c)). The idea is to study "clusters" of an object in a category using a functor. A cluster of an object is, informally speaking, an object which sits inside another object in the same category. For example, if we consider a group to be an object of a category, then the clusters of the group are its subgroups. The functor will map an object to the collection of its clusters and a morphism to a Galois connection. A Galois connection from a partially ordered set O_1, to a partially ordered set O_2, consists of two order-preserving maps, one from O_1 to O_2 and the other from O_2 to O_1. A category equipped with such a functor is called a clustered category. The collection of clusters of an object will form a partially ordered set and Galois connections will enable us to investigate properties of clusters of an object in a category by going forward and backwards between collections of clusters. This leads us nicely to the isomorphism theorems. To arrive at Jordan-Hölder theorem, we will use Zassenhaus' 1934 approach to Jordan-Holder-Schreier's refinement theorem.

Although this essay gives a general proof of Jordan-Hölder theorem which could be applied to different algebraic structures, it is biased towards groups in the sense that we will be verifying that the concepts and axioms we introduce hold for groups. This will amount to showing that the functor from the category of groups to the category of Galois connections is a clustered category. It is for this reason that we have dedicated part of Chapter 2 to reviewing concepts in group theory that will be relevant to this essay. The other part of Chapter 2 covers necessary concepts in category theory, and Galois connections.

2. Preliminaries

This chapter is a review of the necessary pre-requisite concepts from group theory, category theory and Galois connections between partially ordered sets.

2.1 Groups

Definition 2.1.1 (Group)**.** A set G equipped with an associative binary operation "$\cdot$", is called a group if it satisfies the following axioms:

(i) There exists an element in G, which will be denoted by e, called the identity element of G and having the property $e \cdot g = g \cdot e = g$, for all $g \in G$.

(ii) For every $g \in G$, there exists an element in G, denoted by g^{-1}, having the property $g \cdot g^{-1} = g^{-1} \cdot g = e$. This element is called the inverse of g.

If g and h are elements in a group G, we will simply write gh for $g \cdot h$. The identity element of a group is unique. To verify this, suppose that e and e' are both identity elements of a group G. Since e is an identity element, $e\,e' = e'$ and since e' is an identity element $e'e = e$. Thus, $e = ee' = e'$. It can be easily shown that if g is an element of a group G, then the inverse g^{-1} of g, , is unique.

Proposition 2.1.2. Groups have the two-sided cancellation property. That is, if a, b, c are elements of a group G, then $ab = ac$ implies $b = c$ and $ba = ca$ implies $b = c$.

Proof. If $ab = ac$, then $b = a^{-1}ab = a^{-1}ac = c$ and if $ba = ca$, then $b = baa^{-1} = caa^{-1} = c$. □

Definition 2.1.3 (Subgroup)**.** A subset H of a group G is said to be a subgroup of G if H forms a group under the binary operation from G.

It follows from the uniqueness of the identity element that if e is the identity element of a group G and H is a subgroup of G, then $e \in H$.

In the rest of this chapter we will often be verifying whether a given subset of a group is a subgroup of that group. To do this, we will be using the theorem below.

Theorem 2.1.4 (Subgroup Criterion)**.** *Let G be a group. A subset H of G is a subgroup of G if and only if the identity element e, of G is in H and $\forall a, b \in H$, $ab^{-1} \in H$.*

Proof. Firstly, suppose H is a subgroup of G. Then $e \in H$. Let $a, b \in H$. Since H is a subgroup $b^{-1} \in H$ and consequently $ab^{-1} \in H$. Conversely, suppose that $e \in H$ and $\forall a, b \in H$, $ab^{-1} \in H$. The binary operation of G is associative in H since it is associative in G and $H \subseteq G$. Therefore, to prove that H is a subgroup of G, we only need to show that $\forall h \in H$, $h^{-1} \in H$. If $h \in H$, then $h^{-1} = eh \in H$. Hence, H is a subgroup of G. □

Proposition 2.1.5. If M and N are subgroups of a group G, then $M \cap N$ is a subgroup of G.

Proof. Let e be the identity element of G, then $e \in M$ and $e \in N$ since M and N are subgroups of a group G. Therefore, $e \in M \cap N$. Now, suppose $a, b \in M \cap N$. Then $a, b \in M$ and $a, b \in N$ so that $ab^{-1} \in M$ and $ab^{-1} \in N$. Therefore, $ab^{-1} \in M \cap N$ and hence $M \cap N$ is a subgroup of G. □

Let H and K be subgroups of a group G. Define

$$HK = \{hk \mid h \in H \text{ and } k \in K\}.$$

Proposition 2.1.6. If H and K are subgroups of a group G, then HK is a subgroup of G if and only if $HK = KH$.

Proof. Let H and K be subgroups of G and suppose HK is a subgroup of G. Let $x \in HK$. Then $x^{-1} \in HK$ so that $x^{-1} = hk$, for some elements $h \in H$ and $k \in K$. Thus, $x = \left(x^{-1}\right)^{-1} = (hk)^{-1} = k^{-1}h^{-1} \in KH$. Therefore, $HK \subset KH$. Now, suppose $y \in KH$. Then $y = kh$, for some elements $k \in K$ and $h \in H$. Therefore, $y = kh = \left(h^{-1}k^{-1}\right)^{-1} \in HK$ and so $KH \subseteq HK$. Hence, $HK = KH$. Conversely, suppose that $HK = KH$. We will show that HK is a subgroup of G. Since the identity element of G, e, is in H and K, we have that $e \in HK$. Now, let $x, y \in HK$. Then $x, y \in KH$, since $HK = KH$. Thus, there exists elements $h_1, h_2 \in H$ and $k_1, k_2 \in K$ such that $x = h_1k_1$ and $y = k_2h_2$. Furthermore, $k_1{h_2}^{-1} = hk$, for some elements $h \in H$ and $k \in K$, since $k_1{h_2}^{-1} \in KH$ and $HK = KH$. Thus, $xy^{-1} = (h_1k_1)(k_2h_2)^{-1} = h_1k_1{h_2}^{-1}{k_2}^{-1} = h_1hk{k_2}^{-1} \in HK$ and hence HK is a subgroup of G. □

Corollary 2.1.7. If H and K are subgroups of a group G and HK is a subgroup, then HK is the smallest subgroup containing $H \cup K$.

Proof. Now $H \cup K \subseteq HK$ and if S is any subgroup of G containing $H \cup K$, then $HK \subseteq S$. Thus, HK is the smallest subgroup containing $H \cup K$. □

Proposition 2.1.8. If H is a subgroup of a group G, then $HH = H$.

Proof. Suppose H is a subgroup of a group H. To prove that $HH = H$, we will show that $HH \subseteq H$ and $H \subseteq HH$. Let $x \in HH$. Then $x = hh'$ for some elements $h, h' \in H$. Since H is a subgroup of G, $x = hh' \in H$ so that $HH \subseteq H$. Now let $h \in H$. The identity element of G, e, is in H. Thus, $h = he \in HH$ and so $H \subseteq HH$. Hence, $HH = H$. □

Let H be a subgroup of a group G. For $g \in G$ let

$$gH = \{gh \mid h \in H\}, \quad Hg = \{hg \mid h \in H\} \quad \text{and} \quad gHg^{-1} = \left\{ghg^{-1} \mid h \in H\right\}.$$

gH is called the left coset of H generated by g while Hg is called the right coset of H generated by g

Proposition 2.1.9. If H be a subgroup of a group G and g is an element in G, then gHg^{-1} is a subgroup of G.

Proof. Let e be the identity element of G. Then $e \in gHg^{-1}$, since $e \in H$ and $geg^{-1} = e$. Now let $a, b \in gHg^{-1}$. Then there exists elements $x, y \in H$ such that $a = gxg^{-1}$ and $b = gyg^{-1}$. Notice that $b^{-1} = gy^{-1}g^{-1}$ since

$$bb^{-1} = \left(gyg^{-1}\right)\left(gy^{-1}g^{-1}\right) = gyg^{-1}gy^{-1}g^{-1} = e$$

and

$$b^{-1}b = \left(gy^{-1}g^{-1}\right)\left(gyg^{-1}\right) = gy^{-1}g^{-1}gyg^{-1} = e.$$

Thus, $ab^{-1} = \left(gxg^{-1}\right)\left(gy^{-1}g^{-1}\right) = g\left(xy^{-1}\right)g^{-1} \in gHg^{-1}$. Hence, gHg^{-1} is a subgroup of G.

□

Definition 2.1.10. A subgroup N of a group G is called a normal subgroup if $\forall g \in G$, $gNg^{-1} \subseteq N$.

If G is a group with identity element e, then $\{e\}$ and G are normal subgroups of G.

Proposition 2.1.11. Let N be a subgroup of a group G. Then the following statements are equivalent:

(i) N is a normal subgroup of G.

(ii) $\forall g \in G$, $g^{-1}Ng \subseteq N$.

(iii) $\forall g \in G$, $gNg^{-1} = N$.

(iv) $\forall g \in G$, $Ng = gN$.

Proof. (i) ⇒ (ii) If N is a normal subgroup of a group G and $g \in G$, then $g^{-1}N\left(g^{-1}\right)^{-1} = g^{-1}Ng \subseteq N$ since $g^{-1} \in G$.

(ii) ⇒ (iii) If $\forall g \in G$, $g^{-1}Ng \subseteq N$, then

$N = g^{-1}\left(gNg^{-1}\right)g \subseteq gNg^{-1}$. We also have $gNg^{-1} = \left(g^{-1}\right)^{-1}Ng^{-1} \subseteq N$. Thus, $gNg^{-1} = N$.

(iii) ⇒ (iv) If $\forall g \in G$, $g^{-1}Ng = N$, then $gN = gg^{-1}Ng = eNg = Ng$.

(iv) ⇒ (i) If $\forall g \in G$, $Ng = gN$, then $gNg^{-1} = N$ and consequently $gNg^{-1} \subseteq N$. □

Proposition 2.1.12. Let H and K be subgroups of a group G.

(i) If H or K is normal then HK is a subgroup of G.

(ii) If H and K are normal then HK is a normal subgroup of G.

Proof. (i) Let K be a normal subgroup. To prove that HK is a subgroup of G, we will use Proposition 2.1.6 and show that $HK = KH$. Let $x \in HK$, then $x = h_1k_1$ for some elements $h_1 \in H$ and $k_1 \in K$. Since K is normal, $K = h_1Kh_1{}^{-1}$ so that $k' = h_1k_1h_1{}^{-1}$, for some $k' \in K$. Thus, $x = h_1k_1 = h_1k_1h_1{}^{-1}h_1 = k'h_1 \in KH$. Therefore, $HK \subseteq KH$. Now, let $y \in KH$, then $y = k_2h_2$ for some elements $k_2 \in K$ and $h_2 \in H$ so that $k'' = h_2{}^{-1}k_2h_2$, for some $k'' \in K$ since K is normal. Consequently, $y = k_2h_2 = h_2h_2{}^{-1}k_2h_2 = h_2k'' \in HK$. Therefore, $KH \subseteq HK$ and we have established that $HK = KH$. The same argument would hold if instead of supposing that K is a normal subgroup of G, we supposed that H was a normal subgroup of G.

(ii) If H and K are both normal then from (i), HK is a subgroup of G. Let $g \in G$, then $gHK = HgK = HKg$ and thus, HK is a normal subgroup of G.

□

Let H be a normal subgroup of a group G, then the collection of all left (or right) cosets of H forms a group. If xH and yH are left cosets of H generated by the elements $x \in G$ and $y \in G$, respectively, then

$$xHyH = xyHH = xyH.$$

The identity element of this group is the left coset $eH = H$. The inverse of the left coset xH is the coset $x^{-1}H$. We can verify this in the following: $xHx^{-1}H = xx^{-1}HH = H$ and $x^{-1}HxH = x^{-1}xHH = H$. This group is called the quotient or factor group of H and is denoted by G/H.

Definition 2.1.13. A group homomorphism f from a group G to a group G' is a map $f : G \to G'$ such that $\forall a, b \in G$, $f(ab) = f(a)f(b)$.

Let $f : G \to G'$ be a group homomorphism and let e and e' be the identity elements of G and G', respectively. Then $f(e) = f(ee) = f(e)f(e)$. That is $e'f(e) = f(e) = f(e)f(e)$, which implies that $f(e) = e'$. Also, if $g \in G$, then $e' = f(e) = f\left(gg^{-1}\right) = f\left(g\right)\left(g^{-1}\right)$ so that $f\left(g^{-1}\right) = f\left(g\right)^{-1}$.

Let H be a subgroup of a group G. The map $i : H \to G$, $h \mapsto h$, is a group homomorphism. It is normally referred to as the inclusion map from H to G.

For a group homomorphism $f : G \to G'$, define

$$\text{Ker} f = \{g \in G \mid f(g) = e'\} \quad \text{where } e' \text{ is the identity element in } G'$$

to be the kernel of f and

$$\text{Im} f = \{f(g) \in G' \mid g \in G\}$$

to be the image of f.

Proposition 2.1.14. If $f : G \to G'$ is a group homomorphism, then Ker$f$ is a normal subgroup of G while Imf is a subgroup of G'.

Proof. Since $f(e) = e'$ we have $e \in$ Ker$f$ and $e' \in$ Imf. If $a, b \in$ Ker$f$,then $f\left(ab^{-1}\right) = f\left(a\right)f\left(b\right)^{-1} = e'$, which implies that Ker$f$ is a subgroup of G. If $c, d \in$ Imf, then there exists $x, y \in G$ such that $f(x) = c$ and $f(y) = d$. Thus,

$$cd^{-1} = f(x)f(y)^{-1} = f(x)f\left(y^{-1}\right) = f\left(xy^{-1}\right)$$

and thus Imf is a subgroup of G'. Let $K =$ Ker$f$. To complete the proof we need to show that K is a normal subgroup of G. To do this will show that $\forall g \in G$, $gKg^{-1} \subseteq K$. For $g \in G$, let $x \in gKg^{-1}$. Then $x = gkg^{-1}$, for some $k \in K$. Now,

$$f(x) = f\left(gkg^{-1}\right) = f\left(g\right)f\left(k\right)f\left(g^{-1}\right) = f\left(g\right)f\left(g^{-1}\right) = f\left(gg^{-1}\right) = f(e) = e'.$$

That is $x \in K$. Thus, $gKg^{-1} \subseteq K$ so that K is a normal subgroup of G. □

Let N be a normal subgroup of a group G, then the map $f : G \to G/N$ which sends an element $g \in G$ to the coset gN,that is $f(g) = gN$, is a group homomorphism. To verify this let $g_1, g_2 \in G$ such that $g_1 = g_2$ then clearly $g_1N = g_2N$. This proves that f is a map. If $g_1, g_2 \in G$, then $f(g_1)f(g_2) =$

$g_1Ng_2N = g_1g_2N = f(g_1g_2)$. Therefore, f is a group homomorphism. This homomorphism is known as the quotient map.

The quotient map is surjective because for every coset gN, $g \in G$ and $f(g) = gN$. The identity element of G/N is N. Therefore, the kernel of the quotient map $f : G \to G/N$ is

$$\begin{aligned}\text{Ker}f &= \{g \in G \mid f(g) = N\} \\ &= \{g \in G \mid gN = N\} \\ &= N.\end{aligned}$$

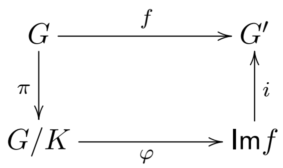
Corollary 2.1.15. A subgroup N of a group G is normal if and only if it the kernel of some group homomorphism $f : G \to G'$.

If a group homomorphism is bijective it is called an isomorphism. Two groups A and B are said to be isomorphic, written $A \approx B$, if there exists an isomorphism from A to B.

Theorem 2.1.16. *(First Isomorphism Theorem) If $f : G \to G'$ is a group homomorphism, then*

$$G/\text{Ker}f \approx \text{Im}f.$$

Proof. Let $K = \text{Ker}f$ and consider the diagram



where $\pi : G \to G/K$ is the quotient map, $\pi(g) = gK$. The map φ is given by $\varphi(gK) = f(g)$. We first check if φ is well-defined. Let $gK, hK \in G/K$ such that $gK = hK$. Then there exists $k \in K$ such that $g = hk$ so that

$$f(g) = f(hk) = f(h)f(k) = f(h)e' = f(h).$$

We now show that φ is a homomorphism. Let $gK, hK \in G/K$, then

$$\varphi(gKhK) = \varphi(ghK) = f(gh) = f(g)f(h) = \varphi(gK)\varphi(hK)$$

since f is a group homomorphism. Since $\varphi(gK) = f(g)$, $\text{Im}\varphi \subseteq \text{Im}f$. To establish $\text{Im}f \subseteq \text{Im}\varphi$, suppose $y \in \text{Im}f$. Then $y = f(x)$ for some $x \in G$. But $y = f(x) = \varphi(xK)$. Hence, $y \in \text{Im}\varphi$ and so $\text{Im}f \subseteq \text{Im}\varphi$. Therefore, $\text{Im}\varphi = \text{Im}f$ and hence φ is surjective. To show that φ is injective, let $gK, hK \in G/K$ and suppose that $\varphi(gK) = \varphi(hK)$. Then $f(g) = f(h)$, so that $e' = f(h)^{-1}f(g) = f\left(h^{-1}g\right)$. This implies that $h^{-1}g \in K \Rightarrow h^{-1}gK = K \Rightarrow gK = hK$. Therefore, φ is injective and so $\varphi : G/K \to \text{Im}f$ is an isomorphism. □

Corollary 2.1.17. Every homomorphism $f : G \to G'$ factors as $f = \phi \circ \pi$, where π the quotient map $\pi : G \to G/\text{Ker}f$ and ϕ is the injective homomorphism, $\phi : \text{Im}f \to G'$, $a \mapsto a$.

Proof. From the proof of the First Isomorphism Theorem, let $\phi = i \circ \varphi$. Then $f = \phi \circ \pi$. ϕ is injective since it is a composition of two injective homomorphisms. □

Let $f : G \to G'$ be a group homomorphism. If H is a subgroup of G, define $f(H) = \{f(h) \mid h \in H\}$.

Proposition 2.1.18. Let N be a normal subgroup of G. If $f : G \to G'$ is a homomorphism such that $f(N) = \{e'\}$, where e' is the identity element of G' , then there exist a unique homomorphism $\phi : G/N \to G'$ such that $f = \phi \circ \pi$, where $\pi : G \to G/N$ is the quotient map.

Proof. Let $\pi : G \longrightarrow G/N$ be the quotient map. Define a map

$$\phi : G/N \to G', \quad gN \mapsto f(g).$$

To show that ϕ is well-defined, suppose that g_1N and g_2N are left cosets of N such that $g_1N = g_2N$. Then there exists $n \in N$ such that $g_2 = g_1n$. Thus, $\phi(g_2N) = f(g_2) = f(g_1n) = f(g_1)f(n) = f(g_1) = \phi(g_1N)$, since $\forall n \in N,\ f(n) = e'$. Thus , ϕ is well-defined. Now, suppose that $g_1N, g_2N \in G/N$. Then

$$\phi(g_1Ng_2N) = \phi(g_1g_2N) = f(g_1g_2) = f(g_1)f(g_2) = \phi(g_1N)\,\phi(g_2N)$$

and so ϕ is a homomorphism. Let $g \in G$, then $(\phi\circ\pi)(g) = \phi(\pi(g)) = \phi(gN) = f(g)$, so that $f = \phi\circ\pi$ and clearly, ϕ is unique. □

2.2 Categories

Definition 2.2.1. A category $\mathbb{C}$ consists of the following structure:

- a class of objects, denoted by Ob($\mathbb{C}$);
- a class of morphisms (or arrows) between objects. This class is denoted by Mor($\mathbb{C}$) . For a morphism $f \in$ Mor($\mathbb{C}$), we write $f : X \to Y \in \mathbb{C}$ or $(X \xrightarrow{f} Y)$, where $X, Y \in$ Ob(C), and say f is a morphism from X to Y. we call X the domain of f and Y the codomain of f. Each morphism has a unique domain and codomain. For any two objects $X, Y \in$ Ob($\mathbb{C}$), we write hom(X, Y) for the set of all morphisms from X to Y;
- a binary operation hom(X, Y) $\times$ hom(Y, Z) $\to$ hom(X, Z),for every three objects $X, Y, Z \in$ Ob($\mathbb{C}$), called the composition of morphisms. The composite of $f : X \to Y$ and $g : Y \to Z$ will be written as $gf : X \to Z$;

This structure is required to satisfy:

(i) composition of morphisms is associative i.e for any three morphisms

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

we have $h(gf) = (hg)f$.

(ii) for each object $X \in$ Ob($\mathbb{C}$), there is a morphism $X \xrightarrow{1_X} X$ called the identity morphism such that for any morphism $W \xrightarrow{f} X$ and any morphism $X \xrightarrow{g} Y$ in $\mathbb{C}$,

$$1_X f = f \quad \text{and} \quad g1_X = g.$$

The identity morphism, 1_X, of each object X in a category is necessarily unique. For, if $1'_X$ is another identity of the object X, then we have $1_X = 1'_X 1_X = 1'_X$.

The collection of groups forms a category. The objects of this category are groups and the morphisms are group homomorphisms. The fact that composition of group homomorphisms is associative verifies the claim. This category is known as the category of groups and will be denoted as **Grp**.

A partially ordered set O with an order relation $\leqslant$ also forms a category. The objects are the elements of the set and for $x, y \in O$ we have a morphism $x \to y$ if and only if $x \leqslant y$. Also, if we consider partially ordered sets as objects and order-preserving maps between them, then we have a category called the category of partially ordered sets.

Definition 2.2.2. Let $\mathbb{C}$ be a category. The dual category, denoted by $\mathbb{C}^{\mathsf{op}}$, consists of

- objects and morphisms of $\mathbb{C}^{\mathsf{op}}$ are the objects and morphisms of $\mathbb{C}$;
- morphisms of $\mathbb{C}^{\mathsf{op}}$ are morphisms of $\mathbb{C}$. However, if $f : X \to Y$ is a morphism in $\mathbb{C}$, then $f : Y \to X$ is a morphism in $\mathbb{C}^{\mathsf{op}}$;
- if f and g are morphisms in $\mathbb{C}$ with composite gf, then the composite of f and g in $\mathbb{C}^{\mathsf{op}}$ is fg.

By definition, a dual category is indeed a category. Identity morphisms in the category $\mathbb{C}$ are the identity morphisms in $\mathbb{C}^{\mathsf{op}}$. Since composition of morphisms is associative in $\mathbb{C}$, it is also associative in $\mathbb{C}^{\mathsf{op}}$.

The concept of a dual category is very important in the sense that whatever statement, theorem , or definition we make in category theory, we can easily obtain its "dual" by simply reversing all the arrows.

Definition 2.2.3. A morphism $f : X \to Y$ in a category $\mathbb{C}$ is called an isomorphism if there exists a morphism $g : Y \to X \in \mathbb{C}$ such that

$$gf = 1_X \quad \text{and} \quad fg = 1_Y.$$

Observe that when the morphism g exists, it is an isomorphism and unique. The fact that it is also an isomorphism follows from $gf = 1_X$ and $fg = 1_Y$. To verify uniqueness, suppose g and g' are morphisms such that $gf = 1_X$,$fg = 1_Y$ and $g'f = 1_X$,$fg' = 1_Y$ then

$$g' = g'1_Y = g'(fg) = (g'f)g = 1_X g = g.$$

The above definition leads to the notion of isomorphic objects. If X and Y are two objects in a category, we say that X and Y are isomorphic if there is an isomorphism $f : X \to Y$.

Proposition 2.2.4. The composite of two isomorphisms is an isomorphism.

Proof. Suppose $f : X \to Y$ and $m : Y \to Z$ are isomorphisms. We wish to show that the composite $mf : X \to Z$ is an isomorphisms. Since f and m are isomorphism, there exist morphisms $g : Y \to X$ and $n : Z \to Y$ such that $gf = 1_X$, $fg = 1_Y$, $nm = 1_Y$ and $mn = 1_Z$. Therefore,

$$\begin{aligned} (mf)(gn) &= m(fg)n = (m1_Y)n = mn = 1_Z \quad \text{and} \\ (gn)(mf) &= g(nm)f = (g1_Y)f = gf = 1_X. \end{aligned}$$

Hence, gf is an isomorphism. □

Definition 2.2.5. Let $\mathbb{C}$ and $\mathbb{D}$ be categories. A functor $F : \mathbb{C} \to \mathbb{D}$ consists of two separate maps, $\mathrm{Ob}(\mathbb{C}) \to \mathrm{Ob}(\mathbb{D})$ called the object function and $\mathrm{Mor}(\mathbb{C}) \to \mathrm{Mor}(\mathbb{D})$ called the morphism function with the following properties:

(i) If $X \xrightarrow{f} Y$ is a morphism in the category $\mathbb{C}$, then $F(X) \xrightarrow{F(f)} F(Y)$ is a morphism in $\mathbb{D}$.

(ii) For any two composable morphisms f and g in $\mathbb{C}$, we have $F(gf) = F(g)F(f)$.

(iii) For any object X in $\mathbb{C}$, we have $F(1_X) = 1_{F(X)}$.

2.3 Galois Connections

Definition 2.3.1 (Galois Connection)**.** A Galois connection f from a partially ordered set $O_1 = (O_1, \leqslant_1)$ to a partially ordered set $O_2 = (O_2, \leqslant_2)$ consists of two order-preserving maps

$$\begin{aligned} O_1 \to O_2, \quad x \mapsto fx \quad \text{(left action)}, \\ O_2 \to O_1, \quad y \mapsto yf \quad \text{(right action)}, \end{aligned}$$

such that

$$\forall x \in O_1, x \leqslant_1 (fx)f \quad \text{and} \quad \forall y \in O_2, f(yf) \leqslant_2 y.$$

Proposition 2.3.2. A pair of left and right action maps $f : O_1 \to O_2$ is a Galois connection if and only if $\forall x \in O_1$ and $\forall y \in O_2$,

$$fx \leqslant_2 y \Leftrightarrow x \leqslant_1 yf.$$

Proof. Firstly, suppose $f : O_1 \to O_2$ is a Galois connection. If $x \in O_1$ and $y \in O_2$ such that $fx \leqslant_2 y$, then

$$x \leqslant_1 (fx)f \leqslant_1 yf$$

and if $x \leqslant_1 yf$ then

$$fx \leqslant_2 f(yf) \leqslant_2 y$$

since the right and left actions of f are order preserving. Conversely, suppose $\forall x \in O_1$ and $\forall y \in O_2$, $fx \leqslant_2 y \Leftrightarrow x \leqslant_1 yf$. We first show that the left and right actions by f are order-preserving . Let $x_1, x_2 \in O_1$ such that $x_1 \leqslant_1 x_2$. Then $fx_2 \leqslant_2 fx_2$ implies $x_2 \leqslant_1 (fx_2)f$. Since $x_1 \leqslant_1 x_2$, by transitivity we have $x_1 \leqslant (fx_2)f$ so that $fx_1 \leqslant_2 fx_2$. Hence, the left action by f is order preserving. To prove that the right action by f is order-preserving, let $y_1, y_2 \in O_2$ such that $y_1 \leq y_2$. Then $y_1 f \leqslant_1 y_1 f$ implies that $f(y_1 f) \leqslant_2 y_1 \leqslant_2 y_2$. Therefore, $y_1 f \leqslant_1 y_2 f$ and hence right action by f is order-preserving. What remains is to show that $\forall x \in O_1, x \leqslant_1 (fx)f$ and $\forall y \in O_2, f(yf) \leqslant_2 y$. This follows easily since if $x \in O_1$, then $fx \leqslant_2 fx$ which implies $x \leqslant_1 (fx)f$ and if $y \in O_2$, then $yf \leqslant_1 yf$ so that $f(yf) \leqslant_2 y$. □

Partially ordered sets and Galois connections between them form a category, denoted by **Gls**. Composition of Galois connections is obtained as follows: Let $O_1 \xrightarrow{f} O_2 \xrightarrow{g} O_3$ be an arrangement of Galois connections, then the composite $O_1 \xrightarrow{gf} O_3$ of f and g is defined as follows:

$$\forall x \in O_1, \quad (gf)x = g(fx), \quad \text{and} \quad \forall z \in O_3, \quad z(gf) = (zg)f.$$

Let $x \in O_1$ and $z \in O_3$. Since g is a Galois connection,$gfx \leqslant_3 z$ is equivalent to $fx \leqslant_2 zg$. The latter is also equivalent to $x \leqslant_1 zgf$, since f is Galois connection. Thus, the composite gf defined above is also a Galois connection. Let O be an object in **Gls**, the identity Galois connection $1_O : O \to O$ of O, is a the Galois connection whose left and right action are identity maps.

Now, if $f : O_1 \to O_2$ is a Galois connection in **Gls**, then $f : O_2 \to O_1$ is a Galois connection in the dual category, **Gls**$^{\text{op}}$, so that the right action of a Galois connection in **Gls** is the left action of that Galois connection in **Gls**$^{\text{op}}$. Also, the left action of a Galois connection in **Gls** is the right action of that Galois connection in the dual category of **Gls**$^{\text{op}}$.

Proposition 2.3.3. If $f : O_1 \to O_2$ is a Galois connection, then

(i) $f((fx)f) = fx, \quad \forall x \in O_1,$

(ii) $(f(yf))f = yf, \quad \forall y \in O_2.$

Proof. (i) If $x \in O_1$ then $(fx)f \leqslant_1 (fx)f$ implies $f((fx)f) \leqslant_2 fx$ and $x \leqslant_1 (fx)f$ implies $fx \leqslant_2 f((fx)f)$. Therefore $f((fx)f) = fx$.

(ii) Follows from (i), by duality.

□

If G is a group, then we denote by $(\mathsf{Sub}(G), \subseteq)$ the partially ordered set containing all the subgroups of G with order relation $\subseteq$. We will now show that a homomorphism $f : G \to G'$ from a group G to a group G' gives rise to a Galois connection from $(\mathsf{Sub}(G), \subseteq)$ to $(\mathsf{Sub}(G'), \subseteq)$.

Let $A \in (\mathsf{Sub}(G), \subseteq), B \in (\mathsf{Sub}(G'), \subseteq)$. We define

$$fA = f(A) = \{f(a)|a \in A\} \quad \text{and} \quad Bf = f^{-1}(B) = \{g \in G|f(g) \in B\}.$$

We will first show that $fA \subseteq B \implies A \subseteq Bf$. $x \in A \implies f(x) \in fA \implies f(x) \in B$. Thus, $x \in Bf$. We now show that $A \subseteq Bf \implies fA \subseteq B$. If $y \in fA$, $\exists x \in A$ such that $f(x) = y$. Thus, $x \in Bf \implies y \in B$.

There is a functor from **Grp** to **Gls**. The object function of this functor sends a group G to the partially ordered set of subgroups $\mathsf{Sub}(G)$ and the morphism function sends a group homomorphism to the corresponding Galois connection.

Definition 2.3.4. Let a and b be elements of an partially ordered set O with the order relation $\leqslant$. An element $c \in O$ is said to be the meet of a and b if it satisfies the following conditions:

(i) $c \leqslant a$ and $c \leqslant b$.

(ii) If $d \in O$ is such that $d \leqslant a$ and $d \leqslant b$, then $d \leqslant c$.

Suppose that c and c' are meets of a and b. Since c is a meet of a and b, $c \leqslant c'$. Likewise, since c' is a meet of a and b, $c' \leqslant c$ and we obtain $c = c'$. Hence, the meet is unique.

When the meet of two elements a and b exists, it is denoted by $a \wedge b$.

The dual notion of meet is join.

Definition 2.3.5. Let a and b be elements of an partially ordered set O with the order relation $\leqslant$. An element $c \in O$ is said to be the join of a and b if it satisfies the following conditions:

(i) $a \leqslant c$ and $b \leqslant c$.

(ii) If $d \in (O, \geqslant)$ is such that $a \leqslant d$ and $b \leqslant d$, then $c \leqslant d$.

The join of two elements of a partially ordered set is unique. The join of two elements a and b, when it exists, will be denoted by $a \vee b$.

Proposition 2.3.6. If $f : O_1 \to O_2$ is a Galois connection, then

(i) $f(x_1 \vee x_2) = fx_1 \vee fx_2, \quad \forall x_1, x_2 \in O_1$.

(ii) $(y_1 \wedge y_2)f = y_1 f \wedge y_2 f, \quad \forall y_1, y_2 \in O_2$.

Proof. (i) Let $x_1, x_2 \in O_1$, then $x_1 \leqslant_1 x_1 \vee x_2$ and $x_2 \leqslant_1 x_1 \vee x_2$ implies $fx_1 \leqslant_2 f(x_1 \vee x_2)$ and $fx_2 \leqslant_2 f(x_1 \vee x_2)$. Therefore, $fx_1 \vee fx_1 \leqslant_2 f(x_1 \vee x_2)$. Proving $f(x_1 \vee x_2) \leqslant_2 fx_1 \vee fx_2$ is the same as proving $x_1 \vee x_2 \leqslant_2 (fx_1 \vee fx_2)f$. Now,

$$x_1 \leqslant_1 (fx_1)f \leqslant_1 (fx_1 \vee fx_2)f \quad \text{and} \quad x_2 \leqslant_1 (fx_2)f \leqslant_1 (fx_1 \vee fx_2)f.$$

Therefore, $x_1 \vee x_1 \leqslant_2 (fx_1 \vee fx_2)f$ and we have $f(x_1 \vee x_2) = fx_1 \vee fx_2$.

(ii) Follows from (i), by duality.

□

Let G be a group and let $A, B \in \mathsf{Sub}(G)$. $A \cap B \subseteq B$ and $A \cap B \subseteq A$. Moreover, if $S \in \mathsf{Sub}(G)$ such that $S \subseteq A$ and $S \subseteq B$, then $S \subseteq A \cap B$. Thus, $A \wedge B = A \cap B$. Since the union of two subgroups is not always a subgroup, the join of A and B is the smallest subgroup containing $A \cup B$.

The definition of meet and join can be easily generalised to a set. The meet of a subset $S \subseteq (O, \leqslant)$ denoted $\wedge S$ is an element $c \in O$ such that $\forall s \in S$, $c \leqslant s$, and for any $d \in O$ we have: $(\forall s \in S,\ d \leqslant s)$ $\Rightarrow d \leqslant c$. The join of a subset $S \subseteq (O, \leqslant)$ denoted $\vee S$ is an element $c \in O$ such that $\forall s \in S$, $s \leqslant c$, and for any $d \in O$ we have $(\forall s \in S,\ s \leqslant d) \Rightarrow c \leqslant d$.

For a partially ordered set $(O, \leqslant)$, we denote the meet and join of O by $\bigwedge O = 0$ and $\bigvee O = 1$, when they exist. A bounded lattice is a partially ordered set where meets and joins of finite subsets exist.

For the rest of this chapter, partially ordered sets are bounded lattices.

Proposition 2.3.7. If $f : O_1 \to O_2$ is a Galois connection, then $f0 = 0$ and $1f = 1$.

Proof. Clearly, $0 \leqslant f0$. Also, $0 \leqslant 0f$ implies $f0 \leqslant 0$. Thus, $f0 = 0$. Dually, $1f = 1$. □

Proposition 2.3.8. Let f, g, h be Galois connections such that $f = gh$. Then,

(i) $g1 = 1$ if $f1 = 1$,

(ii) $0h = 0$ if $0f = 0$.

Proof. (i) $f1 = 1$ implies $gh \cdot 1 = g \cdot h1 = 1$. In particular, $1 \leqslant g \cdot h1$. Now, $h1 \leqslant 1$ and since h is order-preserving $1 \leqslant g \cdot h1 \leqslant g1$. We already know that $g1 \leqslant 1$, thus $g1 = 1$.

(ii) Follows from (i), by duality.

□

Definition 2.3.9. A Galois connection $f : O_1 \to O_2$ is said to be cartesian if

$$0f \leqslant x \Rightarrow (fx)f = x \quad \text{and} \quad y \leqslant f1 \Rightarrow y = f(yf)$$

for all $x \in O_1$ and $y \in O_2$.

Proposition 2.3.10. A Galois connection $f : O_1 \to O_2$ is cartesian if and only if $\forall x \in O_1$ and $\forall y \in O_2$,

$$(fx)f = x \vee 0f \quad \text{and} \quad f(yf) = y \wedge f1.$$

Proof. Firstly, suppose the Galois connection f is cartesian and let $x \in O_1$. Then, $x \leqslant (fx)f$ and since $0 \leqslant fx$ we have $0f \leqslant (fx)f$. Consequently, $x \vee 0f \leqslant (fx)f$. Since $0 \leqslant f1$ and $0f \leqslant x \vee 0f$, we have $f(0f) = 0$ and $(f(x \vee 0f))\, f = x \vee 0f$. Therefore,

$$fx \leqslant fx \vee 0 \Rightarrow fx \leqslant fx \vee f(0f) \Rightarrow fx \leqslant f\,(x \vee 0f) \Rightarrow (fx)f \leqslant (f(x \vee 0f))\, f = x \vee 0f.$$

Hence, $(fx)f = x \vee 0f$. Dually, $f(yf) = y \wedge f1$. Conversely, suppose $\forall x \in O_1$, $(fx)f = x \vee 0f$ and $\forall y \in O_2$, $f(yf) = y \wedge f1$. Then $x \in O_1$, $0f \leqslant x$ implies $(fx)f = x \vee 0f = x$ and $y \in O_2$, $y \leqslant f1$ implies $f(yf) = y \wedge f1 = y$. □

Proposition 2.3.11. The left action of a cartesian Galois connection $f : O_1 \to O_2$ is injective if and only if $0f = 0$.

Proof. Suppose the left action of f is injective and let $0f = x$. We wish to show that $x = 0$. $0f = x$ implies $x \leqslant 0f$ which further implies $fx \leqslant 0 = f0$. Since $f0 = 0 \leqslant fx$, we obtain $fx = f0$. Since the left action is injective, we have $x = 0$. Conversely, suppose that $0f = 0$. Let $x_1, x_2 \in O_1$ such that $fx_1 = fx_2$, then $(fx_1)f = (fx_2)f$. Since $0 = 0f \leqslant x_1$ and $0 = 0f \leqslant x_2$, $x_1 = (fx_1)f = (fx_2)f = x_2$. □

Proposition 2.3.12. The left action of a cartesian Galois connection $f : O_1 \to O_2$ is surjective if and only if $f1 = 1$.

Proof. If the left action of f is surjective, then $fx = 1$, for some $x \in O_1$. Now $x \leqslant 1 \Rightarrow 1 = fx \leqslant f1$. Since, $f1 \leqslant 1$ we can conclude that $f1 = 1$. Conversely, suppose $f1 = 1$ and let $y \in O_2$. Since $y \leqslant 1 = f1$, $f(yf) = y$ and so the left action of f is surjective. □

3. Jordan-Hölder in a Clustered Category

3.1 Clustered Category

Let $\mathbb{C}$ be a category. If $\varphi : \mathbb{C} \to$ **Gls** is a functor, then the pair $(\mathbb{C}, \varphi)$ is said to be a clustered category. For each object $X \in \mathbb{C}$, the elements of the partially ordered set $\varphi(X)$ are called the clusters of the object X while objects and morphisms of the category $\mathbb{C}$ are called the objects and morphisms of the clustered category.

If $f : X \to Y$ is a morphism in a clustered category $(\mathbb{C}, \varphi)$, then $\varphi(f) : \varphi(X) \to \varphi(Y)$ is a Galois connection. For a cluster $C \in \varphi(X)$, we write fC to mean the left action by $\varphi(f)$ on C $(\varphi(f)C)$ and simply say the left action by f on C. Similarly, if $D \in \varphi(Y)$ is a cluster, we write Df for $D\varphi(f)$ and say the right action by f on D.

Definition 3.1.1 (Right Universaliser)**.** Let $\varphi : \mathbb{C} \to$ **Gls** be a clustered category and let C be a cluster in $\varphi(X)$. A right universalizer of C is a morphism $r : X \to Y$ in $\mathbb{C}$ having the property $rC = 0$, and if $r' : X \to Y'$ any morphism such that $r'C = 0$, then there exists a unique morphism $y : Y \to Y'$ such $yr = r'$.

It follows from Proposition 2.1.18 that if H is a subgroup of a group G, we can regard H as a cluster of the object G, then the quotient map $G \to G/N$, where N the smallest subgroup containing H, is a right universalizer of H in the clustered category φ : **Grp** $\to$ **Gls**.

Proposition 3.1.2. The identity morphism of an object is a right universaliser of 0.

Proof. Let 1_X be the identity morphism of an object X. Then $1_X\,0 = 0$ and if $f : X \to Y$ is another morphism such that $f0 = 0$, then f itself is the unique morphism such that $f1_X = f$. □

Proposition 3.1.3. Let f, g, h be morphisms of the clustered category $\varphi : \mathbb{C} \to$ **Gls** such that $f = gh$. If h is a right universalizer of a cluster C and g is an isomorphism, then f is a right universalizer of C.

Proof. Since g is an isomorphism, there exists $g^{-1} \in \mathbb{C}$ such that $gg^{-1} = 1$ and $g^{-1}g = 1$. Thus, $h = 1h = g^{-1}gh = g^{-1}f$. Since h is a right universalizer of C, $h \cdot C = 0$. Also,

$$f \cdot C = gh \cdot C = g \cdot hC = g0 = 0.$$

Suppose f' is another morphism such that $f'C = 0$. Since h is a right universalizer of C, there exists a unique morphism y such that $yh = f'$. Therefore,

$$f' = yh = y(g^{-1}f) = (yg^{-1})f.$$

Uniqueness of y and g^{-1} implies yg^{-1} is unique. Hence, f is a right universalizer of C. □

The dual notion of a right universalizer is a left universalizer and is defined below.

Definition 3.1.4 (Left Universaliser)**.** Let $\varphi : \mathbb{C} \to$ **Gls** be a clustered category and let C be a cluster in $\varphi(X)$. A left universalizer of C is a morphism $l : W \to X$ in $\mathbb{C}$ having the property $Cl = 1$, and if $l' : W' \to X$ is any morphism in $\mathbb{C}$ such that $Cl' = 1$, then there exists a unique morphism $w : W' \to W$ such that $lw = l'$.

Let H be a subgroup of G, then the subgroup inclusion $i : H \to G$ is a left universalizer of H, regarded as a cluster in $\varphi(G)$ of the object G in the clustered category φ : **Grp** $\to$ **Gls**.

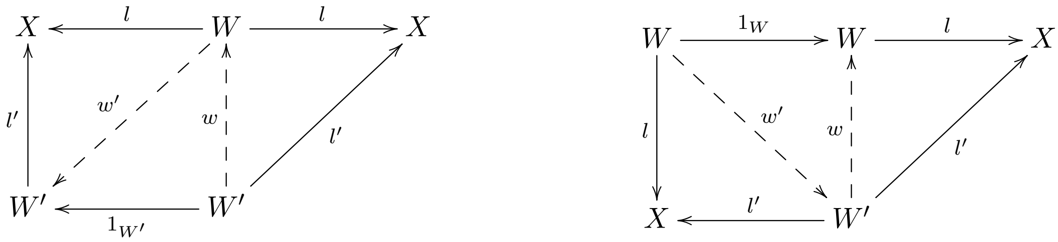
Proposition 3.1.5. The identity map of any object in a clustered category is a left universaliser of 1.

Proof. This follows from Proposition 3.1.2 by duality. □

Proposition 3.1.6. If l and l' are both left universalizers of the same cluster C, then $lw = l'$ for a unique isomorphism w.

Proof. Suppose $l : W \to X$ and $l' : W' \to X$ are both left universalizers of the same cluster $C \in \varphi(X)$.

Since l is a left universalizer of C and l' is a morphism having the property $Cl' = 1$, there exists a unique morphism $w : W' \to W$ such that $lw = l'$. Also, since l' is a left universalizer of C and l is a morphism having the property $Cl = 1$, there exists a unique morphism $w' : W \to W'$ such that $l'w' = l$.



Since w and w' are unique we have that $ww' = 1_{W'}$ and $w'w = 1_W$.

Therefore, $w : W' \to W$ is a unique isomorphism. □

The dual result of the above proposition is: if r and r' are both right universalizers of the same cluster C, then $yr = r'$ for a unique isomorphism y.

A left universalizer of a cluster $C \in \varphi(X)$ will be written as

$$\mathsf{lun}(C) : \mathsf{Lun}(C) \to X$$

while a right universalizer will be written as

$$\mathsf{run}(C) : X \to \mathsf{Run}(C).$$

Proposition 3.1.7. run(0) is an isomorphism and dually lun(1) is an isomorphism.

Proof. As shown in Proposition 3.1.2, the identity morphism is a right universalizer of 0. The dual result of Proposition 3.1.6 implies that there is an isomorphism, say f such that $f = f1_X = \mathsf{run}(0)$. Thus, $\mathsf{run}(0)$ is an isomorphism. Dually, $\mathsf{lun}(1)$ is an isomorphism. □

Definition 3.1.8. A cluster $C \in \varphi(X)$ is said to be normal if there exists a morphism $f : X \to Y$ such that $C = 0f$ and is said to be conormal if there exists a morphism $g : W \to X$ such that $C = g1$.

Observe that the definition of normal and conormal clusters are dual, that is, if a cluster is normal in a category $\mathbb{C}$, then the same cluster will be conormal in the dual category $\mathbb{C}^{\mathsf{op}}$.

Proposition 3.1.9. In a clustered category, the right action by a morphism preserves normal clusters and dually, the left action by a morphism preserves conormal clusters.

Proof. If A is a normal cluster in $\varphi(X)$, then there is a morphism $g : X \to Y \in \mathbb{C}$ such that $A = 0g$. Thus, for any morphism $f : W \to X \in \mathbb{C}$, we have $Af = (0g)f = 0(gf)$ so that Af is normal. Dually, the left action by a morphism preserves conormal clusters. $\square$

We conclude from corollary 2.1.15, that normal clusters in the clustered category φ : **Grp** $\to$ **Gls** are the normal subgroups. If H be a subgroup of G, then the image of the subgroup inclusion map $i : H \to G$ is H. It follow from this that the conormal clusters in φ : **Grp** $\to$ **Gls** are the subgroups.

Definition 3.1.10. A cartesian clustered category is a clustered category $\varphi : \mathbb{C} \to$ **Gls** in which the following axioms hold:

Axiom 1 For each object $X \in \mathbb{C}$, $\varphi(X)$ is a bounded lattice.

Axiom 2 Any cluster has a left universalizer and a right universalizer.

Axiom 3 Any morphism is cartesian. Formally, for any morphism $f : X \to Y$

i) $(fC)f = C \vee 0f$ for any cluster $C \in \varphi(X)$,

ii) $f(Df) = D \wedge f1$ for any cluster $D \in \varphi(Y)$.

Axiom 4 Any morphism $f : X \to Y$ decomposes as $f = me$, where e is a surjective right universalizer of the cluster $0f$ and m is an injective left universalizer of the cluster $f1$.

Axiom 5 The join of two normal clusters is normal and the meet of two conormal clusters is conormal.

The clustered category (**Grp**, φ) is cartesian.

1. Let G be a group in **Grp**. Sub(G) is a bounded lattice since

$$\bigwedge \text{Sub}(G) = \{e\} \quad \text{and} \quad \bigvee \text{Sub}(G) = G.$$

2. If H is a subgroup of a group G, then by Proposition 2.1.18 the quotient map $G/N \to N$, where N is the smallest subgroup containing H, is a right universalizer of H. The subgroup inclusion map $H \to G$ is a left universalizer of H.

3. Let $f : G \to G'$ be a group homomorphism and let D be a subgroup of G'. We wish to show that

   (a) $f(Df) = D \wedge f1$. In the case of groups this becomes $f(f^{-1}(D)) = D \cap f(G)$

   Let $y \in f(f^{-1}(D))$, then there exists $x \in f^{-1}(D) \subseteq G$ such that $f(x) = y$. Now $x \in f^{-1}(D) \Rightarrow y = f(x) \in D$ and $x \in G \Rightarrow y = f(x) \in f(G)$. Therefore, $y \in D \cap f(G)$ and hence $f(f^{-1}(D)) \subseteq D \cap f(G)$.

   Let $y \in D \cap f(G)$. Then $y \in D \subseteq G'$ and $y \in f(G)$. Since $y \in f(G)$, there exists $x \in G$ such that $f(x) = y$. Thus, $y = f(x) \in D \Rightarrow x \in f^{-1}(D) \Rightarrow y = f(x) \in f(f^{-1}(D))$.

   (b) $(fC)f = C \vee 0f$. In the case of groups this becomes $f^{-1}(f(C)) = Cf^{-1}\{e'\}$, where e' is the identity element of G'.

   Let $x \in f^{-1}(f(C))$. Then $f(x) \in f(C)$. Thus, there exists $c \in C$ such that $f(c) = f(x)$. Therefore, $e' = f\left(c^{-1}x\right)$. This implies that $c^{-1}x \in f^{-1}\{e'\}$. Hence, $x = cc^{-1}x \in Cf^{-1}\{e'\}$ and so $f^{-1}(f(C)) \subseteq Cf^{-1}\{e'\}$.

Let $x \in Cf^{-1}\{e'\}$, then $x = yz$ for some $y \in C$ and some $z \in f^{-1}\{e'\}$. Thus,

$$f(x) = f(yz) = f(y)f(z) = f(y)e' = f(y) \in f(C)$$

since $y \in C$. Now, $f(x) \in f(C) \Rightarrow x \in f^{-1}(f(C))$. Therefore, $Cf^{-1}\{e'\} \subseteq f^{-1}(f(C))$ and so $f^{-1}(f(C)) = Cf^{-1}\{e'\}$.

4. Corollary 2.1.17 ensures that every group homomorphism factors as required in Axiom 4.
5. The join of two normal subgroups is normal (Proposition 2.1.12) and the meet (intersection) of two subgroups is also a subgroup (Proposition 2.1.5).

Definition 3.1.11. In a cartesian clustered category, a morphism is said to be injective if the left action by that morphism is injective and is said to be surjective if the left action by that morphism is surjective.

Let X be a object of the clustered category $(\mathbb{C}, \varphi)$. Since the left action of the identity Galois connection is injective and surjective and $\varphi(1_X)$ is the identity Galois connection because φ is a functor, the identity morphism 1_X is injective and surjective.

Proposition 3.1.12. An isomorphism is injective and surjective.

Proof. Suppose $f : X \to Y$ is an isomorphism, then there exists $g : Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$. Since 1_X is injective and 1_Y is surjective, it follows from Proposition 2.3.8 that f is injective and surjective. □

Theorem 3.1.13. *In a cartesian clustered category* $(\mathbb{C}, \varphi)$*:*

a) If the left action by a morphism f is a bijective map, then f is an isomorphism.

Dual result: If the right action by a morphism f is a bijective map, then f is an isomorphism.

b) A morphism f is a right universalizer if and only if $f1 = 1$.

Dual result: A morphism f is a left universalizer if and only if $0f = 0$.

c) A cluster $C \in (\mathbb{C}, \varphi)$ is normal if and only if $0\mathit{run}(C) = C$.

Dual result: A cluster $C \in (\mathbb{C}, \varphi)$ is normal if and only if $\mathit{lun}(C)1 = C$.

d) Let X be an object in $\mathbb{C}$. If $C \in \varphi(X)$ is a normal cluster then $A \vee C = (\mathit{run}(C) \cdot A) \cdot \mathit{run}(C)$, where A is any cluster in $\varphi(X)$.

Dual result: If $C \in \varphi(X)$ is a conormal cluster then $A \wedge C = \mathit{lun}(C) \cdot (A \cdot \mathit{lun}(C))$, where A is any cluster in $\varphi(X)$.

e) (Modular law) For any clusters A, B, C in $\varphi(X)$, where $X \in \mathbb{C}$ then

$$A \leqslant C \Rightarrow A \vee (B \wedge C) = (A \vee B) \wedge C$$

if

i) B is normal and C is conormal, or

ii) A is normal and B is conormal.

f) (Frobenius reciprocity law) Let X be an object in $\mathbb{C}$. If $A \in \varphi(X)$ is conormal then

$$f(A \wedge Bf) = fA \wedge B$$

for any cluster $B \in \varphi(X)$.

Dual result: If X be an object in $\mathbb{C}$ and $A \in \varphi(X)$ is normal, then $(A \vee fB)f = Af \vee B$, for any cluster $B \in \varphi(X)$.

g) Let C be a cluster in $(\mathbb{C}, \varphi)$. Then $r\,C$ is normal if C is normal and r is a right universalizer.

Dual result: Let C be a cluster in $(\mathbb{C}, \varphi)$. Then $C\,l$ is conormal if C is conormal and l is a left universalizer.

h) Let X be an object in $\mathbb{C}$. If $C, D, S \in \varphi(X)$ such that $C \leqslant S$ and $D \leqslant S$, then

$$(C \vee D)\mathit{lun}(S) = C\mathit{lun}(S) \vee D\mathit{lun}(S).$$

Dual result: Let X be an object in $\mathbb{C}$. If $C, D, S \in \varphi(X)$ such that $S \leqslant C$ and $S \leqslant D$, then

$$\mathit{run}(S)\,(C \wedge D) = \mathit{run}(S)C \wedge \mathit{run}(S)D.$$

Proof. a) Suppose the left action of $f : X \to Y$ is bijective. Since the left action of f is injective, $0f = 0$ and since the left action of f is surjective we have $f1 = 1$. Therefore, $f = \mathsf{lun}(f1) \circ \mathsf{run}(0f)$ implies $f = \mathsf{lun}(1) \circ \mathsf{run}(0)$. It follows from Proposition 3.1.7 and that composite of two isomorphism is an isomorphism that f is an isomorphism.

b) Let f be a morphism in $\mathbb{C}$ and suppose $f1 = 1$. Decompose f as $f = me$, where e is a surjective right universalizer of $0f$ and m is an injective left universalizer of $f1$. Since $f1 = 1$, m is a left universalizer of 1. By Proposition 3.1.7, m is an isomorphism. Proposition 3.1.3 implies that f is a right universalizer.

Conversely, suppose that f is a right universalizer. Then it is a right universalizer of $0f$. Again, decompose f as $f = me$, where e is a surjective right universalizer of $0f$ and m is an injective left universalizer of $f1$. Then

$$e = \mathsf{run}(0e) = \mathsf{run}(0me) = \mathsf{run}(0f).$$

That is e is a right universalizer of $0f$. The dual result of Proposition 3.1.6 implies that m is an isomorphism. Consequently, $m1 = 1$. Therefore,

$$f1 = me1 = m1 = 1.$$

c) If $C = 0\mathsf{run}(C)$, then object C is normal. Conversely, suppose the object C is normal then there exists a morphism f such that $C = 0f$. Now $f = \mathsf{lun}(f1) \circ \mathsf{run}(0f)$ from axiom 4. 3.1.13 b implies $0\mathsf{lun}(f1) = 0$. Thus,

$$C = 0f = 0\,(\mathsf{lun}(f1) \circ \mathsf{run}(0f)) = (0\mathsf{lun}(f1))\,\mathsf{run}(0f) = 0\mathsf{run}(0f) = 0\mathsf{run}(C).$$

d) Since C is normal, $C = 0\mathsf{run}(C)$ and by axiom 3 we have

$$A \vee C = A \vee 0\mathsf{run}(C) = (\mathsf{run}(C) \cdot A) \cdot \mathsf{run}(C).$$

e) i) Suppose B is normal, C is conormal and $A \leqslant C$.

$$\begin{aligned} A \vee (B \wedge C) &= (A \wedge C) \vee (B \wedge C) \quad (\text{since} \quad A \leqslant C) \\ &= \mathsf{lun}(C) \cdot A\mathsf{lun}(C) \vee \mathsf{lun}(C) \cdot B\mathsf{lun}(C) \\ &= \mathsf{lun}(C) \cdot (A\mathsf{lun}(C) \vee B\mathsf{lun}(C)) \quad (\text{left action preserves joins}) \\ &= \mathsf{lun}(C) \cdot (A\mathsf{lun}(C) \vee 0\mathsf{run}(B)\mathsf{lun}(C)) \\ &= \mathsf{lun}(C) \cdot ((\mathsf{run}(B)\mathsf{lun}(C) \cdot A\mathsf{lun}(C)) \cdot \mathsf{run}(B)\mathsf{lun}(C)) \\ &= \mathsf{lun}(C) \cdot ((\mathsf{run}(B)\,(A \wedge C)) \cdot \mathsf{run}(B)\mathsf{lun}(C)) \\ &= \mathsf{lun}(C) \cdot (\mathsf{run}(B)A \cdot \mathsf{run}(B)\mathsf{lun}(C)) \quad (\text{since} \quad A \leqslant C) \\ &= \mathsf{lun}(C) \cdot ((A \vee B)\,\mathsf{lun}(C)) \\ &= (A \vee B) \wedge C. \end{aligned}$$

ii) Dually,

$$C \leqslant A \Rightarrow A \wedge (B \vee C) = (A \wedge B) \vee C \quad \text{if } B \text{ is conormal and } C \text{ is normal.}$$

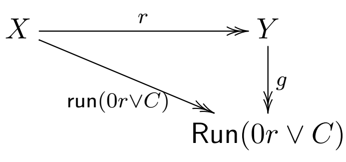
Changing A to C and C to A and interchanging the two sides of the equality above we get

$$A \leqslant C \Rightarrow A \vee (B \wedge C) = (A \vee B) \wedge C \quad \text{if } A \text{ is normal and } B \text{ is conormal.}$$

f) Since $fA \wedge B \leqslant fA \leqslant f1$, we have $f((fA \wedge B)f) = fA \wedge B$. Suppose A is conormal.

$$\begin{aligned} f((fA \wedge B)f) &= f((fA)f \wedge Bf) \\ &= f((0f \vee A) \wedge Bf) \\ &= f(0f \vee (A \wedge Bf)) \quad \text{by the modular law} \\ &= f((f(A \wedge Bf))\,f) \\ &= f(A \wedge Bf). \end{aligned}$$

g) Let $r : X \to Y$ be a right universalizer and let C be a normal cluster. Then r is a right universalizer of $0r$. Now consider $\mathsf{run}(0r \vee C)$. Since $\mathsf{run}(0r \vee C) \cdot 0r = 0$ there exists a unique morphism g such that $gr = \mathsf{run}(0r \vee C)$ as shown in the diagram below.



Since $C \leqslant 0r \vee C$, $\mathsf{run}(0r \vee C) \cdot C = 0$ which implies $g \cdot rC = 0$.

$$g \cdot rC = 0 \Rightarrow g \cdot rC \leqslant 0 \Rightarrow rC \leqslant 0g.$$

To show that $0g \leqslant rC$, observe that $0\mathsf{run}(0r \vee C) = 0r \vee C$ since $0r \vee C$ is normal. Thus, $0 \cdot gr = 0r \vee C$ so that $0r \vee C \leqslant 0 \cdot gr$ implies $r\,(0r \vee C) \leqslant 0g$. But $r\,(0r \vee C) = r(0r) \vee rC = 0 \vee rC = rC$. Hence, $rC = 0g$ and thus rC is normal.

h) $C \leqslant C \vee D$ and $D \leqslant C \vee D$ implies $C\mathsf{lun}(S) \leqslant (C \vee D)\mathsf{lun}(S)$ and $D\mathsf{lun}(S) \leqslant (C \vee D)\mathsf{lun}(S)$ so that $C\mathsf{lun}(S) \vee D\mathsf{lun}(S) \leqslant (C \vee D)\mathsf{lun}(S)$. Let $X = C\mathsf{lun}(S) \vee D\mathsf{lun}(S)$, then

$$\begin{aligned}\mathsf{lun}(S)X &= \mathsf{lun}(S)\,(C\mathsf{lun}(S) \vee D\mathsf{lun}(S)) \\ &= \mathsf{lun}(S)(C\mathsf{lun}(S)) \vee \mathsf{lun}(S)(D\mathsf{lun}(S)) \\ &= (C \wedge S) \vee (D \wedge S) \\ &= C \vee D.\end{aligned}$$

And we have $C \vee D \leqslant \mathsf{lun}(S)X$. Thus,

$$(C \vee D)\mathsf{lun}(S) \leqslant (\mathsf{lun}(S)X)\,\mathsf{lun}(S) = X \vee 0 \cdot \mathsf{lun}(S) = X \vee 0 = X.$$

Hence, $(C \vee D)\mathsf{lun}(S) = C\mathsf{lun}(S) \vee D\mathsf{lun}(S)$.

□

3.2 Isomorphism Theorems

For the rest of this essay, we will be working in cartesian clustered category $(\mathbb{C}, \varphi)$.

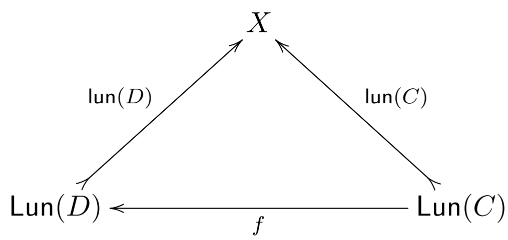
Definition 3.2.1. Let X be an object in $\mathbb{C}$. A cluster $C \in \varphi(X)$ is said to be normal to a cluster $D \in \varphi(X)$ when $C \leqslant D$ and $C \cdot \mathsf{lun}(D)$ is a normal cluster in $\mathsf{Lun}(D)$. Dually, a cluster $D \in \varphi(X)$ is said to be conormal to a cluster $C \in \varphi(X)$ when $C \leqslant D$ and $\mathsf{run}(C) \cdot D$ is a conormal in $\mathsf{Run}(C)$.

If C is normal to D we write, $C \lhd D$, and if D is conormal to C we write, $C \subset D$.

Let H and K be subgroups of a group G and suppose H is a normal subgroup of K. Then $H \subset K$ and if $i : K \to G$ is the subgroup inclusion map, then $i^{-1}(H) = H$ is normal in K. Therefore, if G is an object in $(\mathbf{Grp}, \varphi)$ and if we consider H and K to be clusters in $\varphi(G)$, then $H \lhd K$ obtains its usual meaning.

Lemma 3.2.2 (Cancelation property of normality)**.** Let B, C, D be clusters of an object X such that $B \leqslant C \leqslant D$. If $B \lhd D$ then $B \lhd C$.

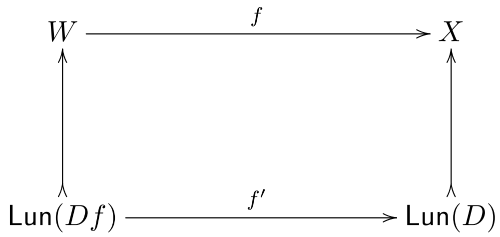
Proof.



$C\mathsf{lun}(C) = 1$ and $C \leqslant D \Rightarrow D\mathsf{lun}(C) = 1$. Also, $D\mathsf{lun}(D) = 1$. By the universal mapping property of $\mathsf{lun}(D)$ it follows that there exists a unique f such that $\mathsf{lun}(D)f = \mathsf{lun}(C)$. Since $B \lhd D$, $B\mathsf{lun}(D)$ in normal in $\mathsf{Lun}(D)$. Since right action preserves normal clusters, we have that $B \cdot \mathsf{lun}(D)f$ is normal in $\mathsf{Lun}(C)$. But $B\mathsf{lun}(D)f = B\mathsf{lun}(C)$. Thus, $B \lhd C$. □

Lemma 3.2.3 (Stability of normality under right action)**.** If C and D are clusters of an object X and $C \lhd D$, then $Cf \lhd Df$ for any morphism $f : W \to X$.

Proof.

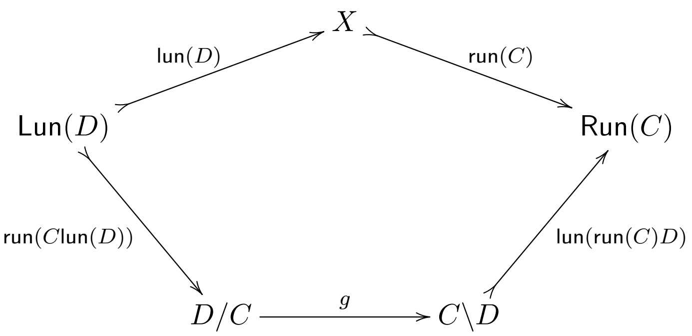


Since $D\text{lun}(D) = 1$ and $Df\text{lun}(Df) = 1$, there exists a unique morphism $f' : \text{Lun}(Df) \to \text{Lun}(D)$ such that $\text{lun}(D)f' = f\text{lun}(Df)$. Now suppose $C \lhd D$, then $C \leqslant D$ and $C\text{lun}(D)$ is normal in $\text{Lun}(D)$. Since the right action preserves normality, we have that $C\text{lun}(D)f'$ is normal in $\text{Lun}(Df)$.The right action of f is order-preserving thus $Cf \leqslant Df$ and $C\text{lun}(D)f' = Cf\text{lun}(Df)$. Hence, $Cf \lhd Df$. ☐

Definition 3.2.4. If C and D are two clusters of the object X the quotient D/C is defined to be the object $\text{Run}(C \cdot \text{lun}(D))$ and the dual notion of the coquotient $C\backslash D$ is defined to be the object $\text{Lun}(\text{run}(C) \cdot D)$.

Theorem 3.2.5 (The Main Theorem on Comparison of Quotients)**.** *If X is an object in a cartesian clustered category $(\mathbb{C}, \varphi)$ and C and D are clusters of X such that C is normal and D is conormal, then $D/C \approx C\backslash D$.*

Proof. Suppose that C is normal and D is conormal. Consider the diagram below.



Notice that $\text{run}(C\text{lun}(D))(C\text{lun}(D)) = 0$ and

$$(\text{run}(C)\text{lun}(D))\,(C\text{lun}(D)) = \text{run}(C)\,(\text{lun}(D)(C\text{lun}(D))) = \text{run}(C)\,(C \wedge D) = 0$$

since $C \wedge D \leqslant C$ and $\text{run}(C)C = 0$. The universal mapping property of $\text{run}(C\text{lun}(D))$ implies that there exists a unique morphism say $y : D/C \to \text{Run}(C)$ such that $y \cdot \text{run}(C\text{lun}(D)) = \text{run}(C)\text{lun}(D)$. We now wish to show that $(\text{run}(C)D)\,y = 1$. Notice that this is the same as showing that

$$\text{run}(C\text{lun}(D))\,((\text{run}(C)D)\text{run}(C)\text{lun}(D)) = 1$$

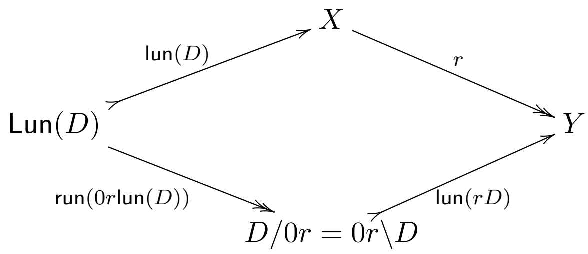
$(\text{run}(C)D)\text{run}(C)\text{lun}(D) = (C \vee D)\text{lun}(D) = 1$, since $D \leqslant C \vee D$ and $D\text{lun}(D) = 1$. Further, $\text{run}(C\text{lun}(D))1 = 1$. Therefore, $(\text{run}(C)D)\,y = 1$. $\text{lun}(\text{run}(C)D)(\text{run}(C)D) = 1$ completes the explanation of the existence of the unique morphism g shown in the diagram above which makes it commute. We wish to prove that g is an isomorphism. To do this we will show that $g1 = 1$ and $0g = 0$. $1\mathsf{run}(C\mathsf{lun}(D)) = 1$ and $\mathsf{lun}(D)1 = D$, since D is conormal and $(\mathsf{run}(C)D)\mathsf{lun}(\mathsf{run}(C)D) = 1$. Thus, $g1 = 1$. $0\mathsf{lun}(\mathsf{run}D) = 0$ and $0\mathsf{run}(C) = C$, since C is normal. Further, $\mathsf{run}(C\mathsf{lun}(D)) \cdot (C\mathsf{lun}(D)) = 0$. Thus, $0g = 0$. Therefore, g is an isomorphism and we have $D/C \approx C\backslash D$. $\square$

Lemma 3.2.6 (Stability of normality under left action of a right universalizer)**.** If C and D are two clusters of the object X such that $C \lhd D$ and D is conormal, then

$$rC \lhd rD$$

for any right universalizer $r : X \to Y$.

Proof. Since r is a right universalizer, it is a right universalizer of $0r$, which is a normal cluster. Since D is conormal, by the main theorem on comparison of quotients we have the commutative diagram below.



If $C \lhd D$, then $C \leqslant D$ and $C \cdot \mathsf{lun}(D)$ is normal in $\mathsf{Lun}(D)$. Since $C \leqslant D$ we have $rC \leqslant rD$ because the right action of r is order-preserving. Since $C \cdot \mathsf{lun}(D)$ is normal in $\mathsf{Lun}(D)$, we have that $\mathsf{run}(0r\mathsf{lun}(D))\,(C\mathsf{lun}(D))$ is normal in $\mathsf{Run}(0r\mathsf{lun}(D)) = D/0r$ because the left action by a right universalizers preserves normal clusters. Since the diagram commutes,

$$\mathsf{run}(0r\mathsf{lun}(D))\,(C\mathsf{lun}(D)) = rC \cdot \mathsf{lun}(rD).$$

Thus, $rC \cdot \mathsf{lun}(rD)$ is normal and the proof is complete. $\square$

Theorem 3.2.7 (The First Diamond Isomorphism Theorem)**.** *Let G be an object in a cartesian clustered category $(\mathbb{C}, \varphi)$. If N and S are clusters of G such that N is a normal and S is a conormal, then*

$$(N \wedge S) \lhd S \quad \textit{and} \quad N \subset (N \vee S)$$

and

$$S/(N \wedge S) \approx N\backslash(N \vee S).$$

Proof. Consider the diagram of the left universalizer of S below.

$$\mathsf{Lun}(S) \rightarrowtail G$$

Since N is normal, $N\mathsf{lun}(S)$ is normal in $\mathsf{Lun}(S)$ because the right action preserves normal clusters. But

$$N\mathsf{lun}(S) = N\mathsf{lun}(S) \wedge 1 = N\mathsf{lun}(S) \wedge S\mathsf{lun}(S) = (N \wedge S)\mathsf{lun}(S)$$

because the right action by any morphism preserves meets. Since $N \wedge S \leqslant S$, we obtain $N \wedge S \lhd S$. Dually, $N \subset (N \vee S)$. Since $N\mathsf{lun}(S) = (N \wedge S)\mathsf{lun}(S)$ we have $\mathsf{Run}(N\mathsf{lun}(S)) = \mathsf{Run}((N \wedge S)\mathsf{lun}(S))$.

Thus, $S/N = S/(N \wedge S)$. Dually, we obtain $N \backslash S = N \backslash (N \vee S)$. Since N is normal and S is conormal, by the Main Theorem on Comparison of Quotients, we have $S/N \approx N \backslash S$. Therefore,

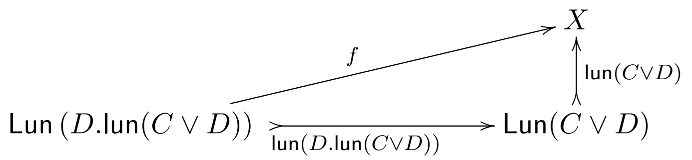
$$S/(N \wedge S) = S/N \approx N \backslash S = N \backslash (N \vee S)$$

. □

Theorem 3.2.8 (The Second Diamond Isomorphism Theorem)**.** *Let X be an object in a cartesian clustered category $(\mathbb{C}, \varphi)$ and let C and D be clusters of X such that D and $C \vee D$ are conormal, and $C \lhd (C \vee D)$. Then $(C \wedge D) \lhd D$ and*

$$D/(C \wedge D) \approx (C \vee D)/C.$$

Proof. Consider the diagram below.

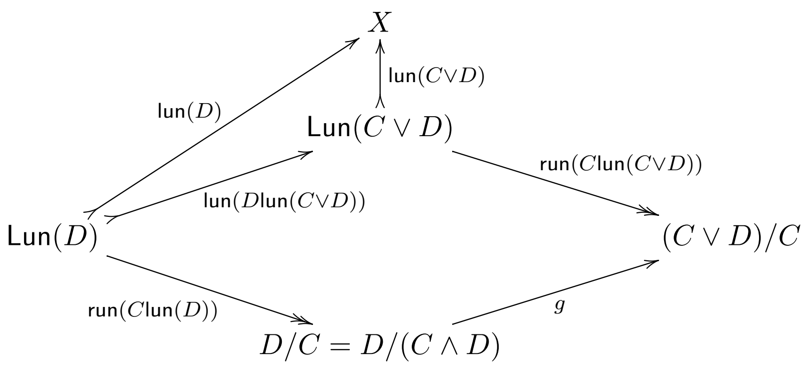


Firstly, we wish to show that $f = \mathsf{lun}(D)$. $0f = 0 \cdot \mathsf{lun}\,(D.\mathsf{lun}(C \vee D))\,\mathsf{lun}(C \vee D) = 0$ implies that f is a left universalizer. Further, since D and $C \vee D$ are conormal, we have

$$\mathsf{lun}(C \vee D)\mathsf{lun}\,(D\mathsf{lun}(C \vee D))\,1 = \mathsf{lun}(C \vee D)\,(D\mathsf{lun}(C \vee D)) = D \wedge (C \vee D) = D.$$

Therefore, f is a left universalizer of D and $\mathsf{Lun}\,(D\mathsf{lun}(C \vee D)) = \mathsf{Lun}(D)$.

Now, consider the diagram

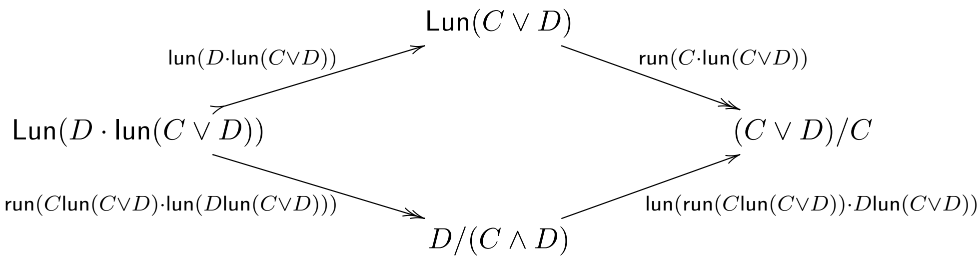


where g is uniquely determined since $\mathsf{run}(C\mathsf{lun}(D))(C\mathsf{lun}(D)) = 0$ and

$$\mathsf{run}(C\mathsf{lun}(C \vee D))\,((\mathsf{lun}(D)(C\mathsf{lun}(D)))\mathsf{lun}(C \vee D)) = \mathsf{run}(C\mathsf{lun}(C \vee D))\,((C \wedge D)\mathsf{lun}(C \vee D)) = 0.$$

If D is conormal, then $D\mathsf{lun}(C \vee D)$ is conormal since the right action by a left universalizers preserves conormal clusters.

$C \lhd (C \vee D)$ implies that $C \leqslant (C \vee D)$ and $C\mathsf{lun}(C \vee D)$ is normal in $\mathsf{Lun}(C \vee D)$. By the main theorem of comparison of quotients, we have the commutative diagram below



Because of the universal mapping property, we obtain

$$g = \mathsf{lun}(\mathsf{run}(C\mathsf{lun}(C \vee D)) \cdot D\mathsf{lun}(C \vee D))$$

$C\mathsf{lun}(C \vee D)$ being normal in $\mathsf{Lun}(C \vee D)$ implies that $C\mathsf{lun}(C \vee D)\mathsf{lun}(D\mathsf{lun}(C \vee D))$ is normal in $\mathsf{Lun}(D\mathsf{lun}(C \vee D)) = \mathsf{Lun}(D)$. But

$$C\mathsf{lun}(C \vee D)\mathsf{lun}(D\mathsf{lun}(C \vee D)) = C\mathsf{lun}(D).$$

Also,

$$C\mathsf{lun}(D) = C\mathsf{lun}(D) \wedge 1 = C\mathsf{lun}(D) \wedge D\mathsf{lun}(D) = (C \wedge D)\mathsf{lun}(D).$$

That is $(C \wedge D)\mathsf{lun}(D)$ is normal. Since $C \wedge D \leqslant D$ we have that $C \wedge D \lhd D$.

To prove that

$$D/(C \wedge D) \approx (C \vee D)/C,$$

we will show that g is an isomorphism. Since g is a left universalizer, $0g = 0$. Therefore, to show that it is an isomorphism it is enough to show that $g1 = 1$. Observe that since $1\mathsf{run}(C\mathsf{lun}(D)) = 1$ and $\mathsf{lun}(D)1 = D$ (since D is cornormal),

$$g1 = \mathsf{run}(C\mathsf{lun}(C \vee D)) \cdot D\mathsf{lun}(C \vee D).$$

We must show that $\mathsf{run}(C\mathsf{lun}(C \vee D)) \cdot D\mathsf{lun}(C \vee D) = 1$. Clearly, $\mathsf{run}(C\mathsf{lun}(C \vee D)) \cdot D\mathsf{lun}(C \vee D) \leqslant 1$. Now,

$$\begin{aligned}
&(\mathsf{run}(C\mathsf{lun}(C \vee D)) \cdot D\mathsf{lun}(C \vee D)) \cdot \mathsf{run}(C\mathsf{lun}(C \vee D)) \\
&= D\mathsf{lun}(C \vee D) \vee 0\mathsf{run}(C\mathsf{lun}(C \vee D)) \\
&= D\mathsf{lun}(C \vee D) \vee C\mathsf{lun}(C \vee D) \\
&= (C \vee D) \cdot \mathsf{lun}(C \vee D) \\
&= 1.
\end{aligned}$$

$1 \leqslant (\mathsf{run}(C\mathsf{lun}(C \vee D)) \cdot D\mathsf{lun}(C \vee D)) \cdot \mathsf{run}(C\mathsf{lun}(C \vee D) \Leftrightarrow 1 = \mathsf{run}(C\mathsf{lun}(C \vee D)1 \leqslant \mathsf{run}(C\mathsf{lun}(C \vee D)) \cdot D\mathsf{lun}(C \vee D)$.

Thus, $\mathsf{run}(C\mathsf{lun}(C \vee D)) \cdot D\mathsf{lun}(C \vee D)) = 1$ and hence g is an isomorphism.

□

3.3 Jordan-Hölder Theorem

Theorem 3.3.1 (Zassenhaus Lemma)**.** *Let G be an object in a cartesian clustered category $(\mathbb{C}, \varphi)$ and let $S' \lhd S$ and $T' \lhd T$ be conormal clusters of G. Then*

$$S' \vee (S \wedge T') \lhd S' \vee (S \wedge T), \quad T' \vee (T \wedge S') \lhd T' \vee (T \wedge S), \quad (S \wedge T') \vee (S' \wedge T) \lhd (S \wedge T)$$

and

$$(S' \vee (S \wedge T))/(S' \vee (S \wedge T')) \approx (T' \vee (T \wedge S))/(T' \vee (T \wedge S')).$$

Proof. Observe that if we prove that $S' \vee (S \wedge T') \lhd S' \vee (S \wedge T)$ then it will follow that $T' \vee (T \wedge S') \lhd T' \vee (T \wedge S)$ by changing S' to T' and S to T. Consider the diagram below.

$$G \xleftarrow{\mathsf{lun}(S)} \mathsf{Lun}(S) \xrightarrow{\mathsf{run}(S'\mathsf{lun}(S))} S/S'$$

Since T is conormal, $T\mathsf{lun}(S)$ is conormal in $\mathsf{Lun}(S)$. Since $T' \lhd T$ we have $T'\mathsf{lun}(S) \lhd T\mathsf{lun}(S)$. Using the lemma on stability of normality under the left action of a right universalizer, we have

$$\mathsf{run}(S'\mathsf{lun}(S)) \cdot T'\mathsf{lun}(S) \lhd \mathsf{run}(S'\mathsf{lun}(S)) \cdot T\mathsf{lun}(S)$$

and consequently,

$$(\mathsf{run}(S'\mathsf{lun}(S)) \cdot T'\mathsf{lun}(S)) \cdot \mathsf{run}(S'\mathsf{lun}(S)) \lhd (\mathsf{run}(S'\mathsf{lun}(S)) \cdot T\mathsf{lun}(S)) \cdot \mathsf{run}(S'\mathsf{lun}(S))$$

Now

$$(\mathsf{run}(S'\mathsf{lun}(S)) \cdot T'\mathsf{lun}(S)) \cdot \mathsf{run}(S'\mathsf{lun}(S)) = T'\mathsf{lun}(S) \vee 0 \cdot \mathsf{run}(S'\mathsf{lun}(S)) = T'\mathsf{lun}(S) \vee S'\mathsf{lun}(S)$$

and similarly, $(\mathsf{run}(S'\mathsf{lun}(S)) \cdot T\mathsf{lun}(S)) \cdot \mathsf{run}(S'\mathsf{lun}(S)) = T\mathsf{lun}(S) \vee S'\mathsf{lun}(S)$. Therefore,

$$T'\mathsf{lun}(S) \vee S'\mathsf{lun}(S) \lhd T\mathsf{lun}(S) \vee S'\mathsf{lun}(S).$$

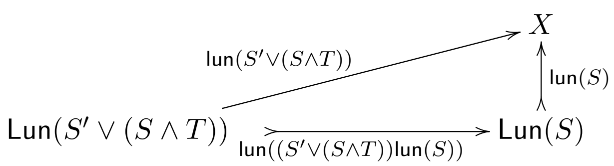
Also, notice that $T'\mathsf{lun}(S) = (S \wedge T')\mathsf{lun}(S)$ and $T\mathsf{lun}(S) = (S \wedge T)\mathsf{lun}(S)$ so that

$$(S \wedge T')\mathsf{lun}(S) \vee S'\mathsf{lun}(S) \lhd (S \wedge T)\mathsf{lun}(S) \vee S'\mathsf{lun}(S).$$

The right action by $\mathsf{lun}(S)$ preserves joins less than S, thus

$$(S' \vee (S \wedge T'))\mathsf{lun}(S) \lhd (S' \vee (S \wedge T))\mathsf{lun}(S).$$

$S' \vee (S \vee T') \leqslant S' \vee (S \vee T) \leqslant S$ so that in the diagram



$(S' \vee (S \wedge T'))\mathsf{lun}(S) \lhd (S' \vee (S \wedge T))\mathsf{lun}(S)$ implies that $((S' \vee (S \wedge T'))\mathsf{lun}(S))\,\mathsf{lun}\,((S' \vee (S \wedge T))\,\mathsf{lun}(S))$ is normal in $\mathsf{Lun}(S' \vee (S \wedge T))$. But

$$\left((S' \vee (S \wedge T'))\mathsf{lun}(S)\right) \mathsf{lun} \left(\left(S' \vee (S \wedge T)\right) \mathsf{lun}(S)\right) = (S' \vee (S \wedge T))\mathsf{lun}\left(S' \vee (S \wedge T)\right).$$

Hence, $S' \vee (S \wedge T') \lhd S' \vee (S \wedge T)$, our desired result.

Let $C = S' \vee (S \wedge T')$ and $D = S \wedge T$. D is conormal since S and T are conormal.

$$C \vee D = S' \vee (S \wedge T') \vee (S \wedge T) = S' \vee (S \wedge T), \quad \text{since } S \wedge T' \leqslant S \wedge T.$$

Also, $C \lhd C \vee D$ since $S' \vee (S \wedge T') \lhd S' \vee (S \wedge T)$ from above. By the Second Diamond Isomorphism Theorem, $C \wedge D \lhd D$ and $D/(C \wedge D) \approx (C \vee D)/C$. That is

$$\big((S' \vee (S \wedge T')) \wedge (S \wedge T)\big) \lhd S \vee T$$

and

$$(S \wedge T)/\,\big((S' \vee (S \wedge T')) \wedge (S \wedge T)\big) \approx \big(S' \vee (S \wedge T)\big)\,/\,\big(S' \vee (S \wedge T')\big)$$

If we show that $((S' \vee (S \wedge T')) \wedge (S \wedge T)) = (S \wedge T') \vee (S' \wedge T)$, then

$$\big(S' \vee (S \wedge T)\big)\,/\,\big(S' \vee (S \wedge T')\big) \approx (S \wedge T)/(S \wedge T') \vee (S' \wedge T) \approx (T' \vee (T \wedge S))/(T' \vee (T \wedge S'))$$

and the proof would be complete.

Since $((S' \vee (S \wedge T')) \wedge (S \wedge T)) \leqslant S$ and $(S \wedge T') \vee (S' \wedge T) \leqslant S$, showing the above equality is equivalent to showing

$$\big((S' \vee (S \wedge T')) \wedge (S \wedge T)\big)\,\mathsf{lun}(S) = \big((S \wedge T') \vee (S' \wedge T)\big)\,\mathsf{lun}(S)$$

$$\begin{aligned}
\text{Now } \big((S' \vee (S \wedge T')) \wedge (S \wedge T)\big)\,\mathsf{lun}(S) &= (S' \vee (S \wedge T'))\mathsf{lun}(S) \wedge (S \wedge T)\mathsf{lun}(S) \\
&= (S'\mathsf{lun}(S) \vee (S \wedge T')\mathsf{lun}(S)) \wedge (S\mathsf{lun}(S) \wedge T\mathsf{lun}(S)) \\
&= (S'\mathsf{lun}(S) \vee (S\mathsf{lun}(S) \wedge T'\mathsf{lun}(S))) \wedge (S\mathsf{lun}(S) \wedge T\mathsf{lun}(S)) \\
&= (S'\mathsf{lun}(S) \vee (1 \wedge T'\mathsf{lun}(S))) \wedge (1 \wedge T\mathsf{lun}(S)) \\
&= (S'\mathsf{lun}(S) \vee T'\mathsf{lun}(S)) \wedge T\mathsf{lun}(S) \\
&= (T'\mathsf{lun}(S) \vee S'\mathsf{lun}(S)) \wedge T\mathsf{lun}(S),
\end{aligned}$$

$$\begin{aligned}
\text{and } \big((S \wedge T') \vee (S' \wedge T)\big)\,\mathsf{lun}(S) &= (S \wedge T')\mathsf{lun}(S) \vee (S' \wedge T)\mathsf{lun}(S) \\
&= (S\mathsf{lun}(S) \wedge T'\mathsf{lun}(S)) \vee (S'\mathsf{lun}(S) \wedge T\mathsf{lun}(S)) \\
&= (1 \wedge T'\mathsf{lun}(S)) \vee (S'\mathsf{lun}(S) \wedge T\mathsf{lun}(S)) \\
&= T'\mathsf{lun}(S) \vee (S'\mathsf{lun}(S) \wedge T\mathsf{lun}(S)).
\end{aligned}$$

By the modularity law,

$$\big(T'\mathsf{lun}(S) \vee S'\mathsf{lun}(S)\big) \wedge T\mathsf{lun}(S) = T'\mathsf{lun}(S) \vee \big(S'\mathsf{lun}(S) \wedge T\mathsf{lun}(S)\big)$$

since $S' \lhd S$ ($S'\mathsf{lun}(S)$ is normal) and $T\mathsf{lun}S$ is conormal.

Thus, $((S' \vee (S \wedge T')) \wedge (S \wedge T)) = (S \wedge T') \vee (S' \wedge T)$ and the proof is complete. □

Definition 3.3.2. Let G be an object in a clustered category $(\mathbb{C}, \varphi)$. Then a normal series of G is the sequence

$$0 = S_0 \subset S_1 \subset \cdots \subset S_{n-1} \subset S_n = 1, \quad \text{where the } S_i \text{ are normal clusters of } G.$$

Dually, the sequence

$$0 = S_n \lhd S_{n-1} \lhd \cdots \lhd S_1 \lhd S_0 = 1$$

of conormal clusters is called a conormal series of G.

If $0 = S_0 \subset S_1 \subset \cdots \subset S_{n-1} \subset S_n = 1$ and $0 = S_n \lhd S_{n-1} \lhd \cdots \lhd S_1 \lhd S_0 = 1$ are normal series and conormal series respectively, then sequences

$$S_0 \backslash S_1, S_1 \backslash S_2, ..., S_{n-1} \backslash S_n \quad \text{and} \quad S_{n-1}/S_n, S_{n-2}/S_{n-1}, ..., S_1/S_2, S_0/S_1$$

of coquotients and quotients, respectively, are called the factor sequences of the series.

Two (conormal) normal series are said to be factor-equivalent if they have the same number n of terms in their factor sequences and there is a bijection $p : \{1, ..., n\} \to \{1, ..., n\}$ such that each i-th term of the first factor sequence is isomorphic to the $p(i)$-th term of the second factor sequence.

A refinement of a series is a series obtained by adding new members between the existing members of the series.

Theorem 3.3.3 (The Schreier-Zassenhaus Refinement Theorem)**.** *Any two conormal series have factor-equivalent refinements.*

Proof. Let

$$0 = S_n \lhd S_{n-1} \lhd \cdots \lhd S_1 \lhd S_0 = 1,$$
$$0 = T_m \lhd T_{m-1} \lhd \cdots \lhd T_1 \lhd T_0 = 1.$$

We define

$$\overline{S}_{(i-1)m+j} = S_i \vee (S_{i-1} \wedge T_j), \quad \text{for} \quad i \in \{i, ..., n\}, j \in \{1, ..., m\}$$
$$\overline{S}_0 = 1.$$

The first part of Zassenhaus lemma implies $\overline{S}_{(i-1)m+j} \lhd \overline{S}_{(i-1)m+(j-1)}$, that is

$$S_i \vee (S_{i-1} \wedge T_j) \lhd S_i \vee (S_{i-1} \wedge T_{j-1}).$$

Thus, we have the conormal series

$$0 = \overline{S}_{nm} \lhd \overline{S}_{nm-1} \lhd \overline{S}_{nm-2} \lhd \cdots \lhd \overline{S}_1 \lhd \overline{S}_0 = 1,$$

of length $nm + 1$, which is a refinement of the S conormal series.

Similarly, if we let

$$\overline{T}_{(j-1)n+i} = T_j \vee (T_{j-1} \wedge S_i), \quad \text{for} \quad i \in \{i, ..., n\}, j \in \{1, ..., m\}$$
$$\overline{T}_0 = 1.$$

we obtain

$$0 = \overline{T}_{mn} \lhd \overline{T}_{mn-1} \lhd \overline{T}_{mn-2} \lhd \cdots \lhd \overline{T}_1 \lhd \overline{T}_0 = 1$$

a refinement of the T conormal series, having the same length $nm + 1$.

Since $\overline{S}_{(i-1)m+j} \lhd \overline{S}_{(i-1)m+(j-1)}$ and $\overline{T}_{(j-1)n+i} \lhd \overline{T}_{(j-1)n+(i-1)}$, by the second part of Zassenhaus lemma we have

$$\overline{S}_{(i-1)m+(j-1)}/\overline{S}_{(i-1)m+j} \approx \overline{T}_{(j-1)n+(i-1)}/\overline{T}_{(j-1)n+i}$$

or

$$(S_i \vee (S_{i-1} \wedge T_{j-1}))/(S_i \vee (S_{i-1} \wedge T_j)) \approx (T_j \vee (T_{j-1} \wedge S_{i-1}))/(T_j \vee (T_{j-1} \wedge S_i)).$$

This proves that the two refinements are factor-equivalent. □

A series is said to be maximal if it is not possible to add new members in between existing members of the series.

Theorem 3.3.4 (**Jordan-Hölder Theorem**). *Let G be an object of a cartesian clustered category $(\mathbb{C}, \varphi)$.*

1. *If $\varphi(G)$ is finite, then G has a maximal conormal series.*
2. *Any two maximal conormal series of G are factor-equivalent.*

Proof. 1. If 0 and 1 denote the bottom and top elements, respectively, of $\varphi(G)$, then $0 \lhd 1$ is a conormal series of $\varphi(G)$. We look for a cluster normal to 1 other than 0. If we cannot find one then $0 \lhd 1$ is a maximal conormal series. If we can, we fit it into $0 \lhd 1$ to obtain another conormal series of $\varphi(G)$. We continue this process of adding new members between existing members of the series. This process will terminate because $\varphi(G)$ is finite and terminates at a maximal conormal series.

2. The proof is an easy application of the Schreier-Zassenhaus refinement theorem. Suppose that

$$0 = S_n \lhd S_{n-1} \lhd \cdots \lhd S_1 \lhd S_0 = 1,$$
$$0 = T_m \lhd T_{m-1} \lhd \cdots \lhd T_1 \lhd T_0 = 1,$$

are two maximal conormal series. By the Schreier-Zassenhaus refinement theorem these two series have equivalent refinements. By definition of maximal conormal series, they cannot be refined without repeating the existing members. It therefore follows that the two series are equivalent.

□

Theorem 3.3.5. *If G is an object of a cartesian clustered category $(\mathbb{C}, \varphi)$, then any two maximal normal series of G are factor-equivalent.*

Proof. This follows from Theorem 3.3.4 by duality. □

Observe that in the case of groups, a normal series of a group G is also a conormal series of G because conormal clusters in $(\mathbf{Grp}, \varphi)$ are just subgroups.

Example 3.3.6 (The symmetric group of degree 4, S_4)**.** The symmetric group of degree 4, denoted S_4, is the group of permutations on four symbols. The elements of S_4 in cycle notation are:

$$S_4 = \{(1), (1\,2), (1\,3), (1\,4), (2\,3), (2\,4), (3\,4), (1\,2\,3), (1\,3\,2), (1\,2\,4), (1\,4\,2), (1\,3\,4), (1\,4\,3), (2\,3\,4), (2\,4\,3),$$
$$(1\,2\,3\,4), (1\,4\,3\,2), (1\,3\,4\,2), (1\,2\,4\,3), (1\,3\,2\,4), (1\,4\,2\,3), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$$

$4! = 24$ and so S_4 has 24 elements.

To compute all the normal subgroups of S_4, will first compute the conjugacy classes of S_3. We know that elements with the same cycle notation belong to the same conjugacy class. Therefore, the conjugacy classes of S_4 are:

$$\{(1)\}, \quad \{(1\,2),(1\,3),(1\,4),(2\,3),(2\,4),(3\,4)\}, \quad \{(1\,2\,3),(1\,3\,2),(1\,2\,4),(1\,4\,2),(1\,3\,4),(1\,4\,3),(2\,3\,4),(2\,4\,3)\},$$
$$\{(1\,2\,3\,4),(1\,4\,3\,2),(1\,3\,4\,2),(1\,2\,4\,3),(1\,3\,2\,4),(1\,4\,2\,3)\} \quad \text{and} \quad \{(1\,2)(3\,4),(1\,3)(2\,4),(1\,4)(2\,3)\}$$

It follows that if N is a normal subgroup of S_4, then N is a union of conjugacy classes. Since the identity element is always in the subgroup and by Lagrange theorem, which states that the order of a subgroup divides the order of the group we get that

$$V = \{(1),(1\,2)(3\,4),(1\,3)(2\,4),(1\,4)(2\,3)\} \quad \text{and}$$
$$A_4 = \{(1),(1\,2\,3),(1\,3\,2),(1\,2\,4),(1\,4\,2),(1\,3\,4),(1\,4\,3),(2\,3\,4),(2\,4\,3),(1\,2)(3\,4),(1\,3)(2\,4),(1\,4)(2\,3)\}$$

are the only non-trivial subgroups of S_4. Using a similar argument we get that V is the only non-trivial normal subgroup of A_4.

Therefore,

$$(1) \subset V \subset A_4 \subset S_4 \tag{3.3.1}$$

is a maximal normal series of S_4 with factors

$$\begin{aligned} &(1)\backslash\ V \approx V/(1) \approx V, \\ &V\backslash A_4 \approx A_4/V \approx \mathbb{Z}_3 \\ &\text{and} \quad A_4\backslash S_4 \approx S_4/A_4 \approx \mathbb{Z}_2. \end{aligned}$$

The fact that the coquotients are isomorphic to the quotients follows from Theorem 3.2.5. By Lemma 3.2.2 and since conormal clusters in the cartesian clustered category $(\mathbf{Grp}, \varphi)$ are subgroups, a normal series of a group is also a conormal series of that group. Thus, (3.3.1) is a conormal series of S_4 . However, it is not maximal because we can fit in a new member. Observe that the elements of V other than (1) have order 2. Since V is abelian, $\{(1),(1\,2)(3\,4)\},\{(1),(1\,3)(2\,4)\}$ and $\{(1),(1\,4)(2\,3)\}$ are the normal subgroups of V. Thus, the conormal series

$$(1) \lhd \{(1),(1\,2)\} \lhd V \lhd A_4 \lhd S_4 \tag{3.3.2}$$

is maximal with factors

$$\begin{aligned} &\{(1),(1\,2)\}/(1) \approx \mathbb{Z}_2, \\ &V/\{(1),(1\,2)\} \approx \mathbb{Z}_2, \\ &A_4/V \approx \mathbb{Z}_3, \\ &\text{and} \quad S_4/A_4 \approx \mathbb{Z}_2. \end{aligned}$$

4. Some Applications

4.1 The Fundamental Theorem of Arithmetic

Theorem 4.1.1. *Every integer $n > 1$ can be decomposed as a product of prime numbers and this decomposition is unique up to rearrangement.*

Proof. Let n be an integer greater than one, then the group $\mathbb{Z}_n$ is finite and has order n. Since it is finite it has a maximal conormal series. Let $X_1, X_2, ..., X_t$ be a factor sequence of a maximal conormal series of $\mathbb{Z}_n$. The groups in the factor sequence are simple since our series is maximal. Now, we know that a finite abelian group is simple if and only if its order is prime. Therefore, the groups in the factor sequence have prime order. We also know that if N is a normal subgroup of G, then $|G| = |G/N||N|$. Therefore, $|\mathbb{Z}_n| = |X_1||X_2|...|X_t|$ which proves that n is a product of primes. The Jordan-Hölder theorem ensures that such a decomposition is unique up to isomorphism and rearrangement. Hence the proof. □

4.2 Jordan-Hölder Theorem and classification of finite groups

Let 0 denote the trivial group. A sequence $0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} G_3 \xrightarrow{f_4} 0$ of groups G_1, G_2, G_3 and group homomorphisms f_1, f_2, f_3, f_4 is called a short exact sequence if im(f_i) = ker(f_{i+1}), $i = 1, 2, 3$ and we say that the group G_2 is an extension of G_1 by G_3. Observe that when this is the case, G_1 is a normal subgroup of G_2 and the quotient G_2/G_1 is isomorphic to the group G_3.

Now suppose that we are given groups G_1 and G_3, is it possible to classify all groups G_2 so that the sequence $0 \to G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \to 0$ is short exact? This problem is unsolved and has come to be known as the extension problem.

Let G be a finite group and let $0 = G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 \lhd G_0 = G$ be a maximal conormal series. Then we have short exact sequences

$$\begin{gathered} 0 \to G_1 \to G_0 \to G_0/G_1 \to 0 \\ 0 \to G_2 \to G_1 \to G_1/G_2 \to 0 \\ \cdots \\ 0 \to G_{n-1} \to G_{n-2} \to G_{n-2}/G_{n-1} \to 0 \\ 0 \to G_n \to G_{n-1} \to G_{n-1}/G_n \to 0 \end{gathered}$$

where $G_0/G_1, G_1/G_2, ..., G_{n-2}/G_{n-1}, G_{n-1}/G_n$ are simple groups. Jordan-Hölder theorem ensures that this sequence of short exact sequences is unique.

Now, all simple finite groups have been classified (Gorenstein et al. (2002)). So suppose G_{n-1} is a simple group. Then we have the short exact sequence $0 \to 0 \to G_{n-1} \to G_{n-1}/0 \to 0$. If we knew the solution to the extension problem, we could then find a group say G_{n-2} such that the sequence $0 \to G_{n-1} \to G_{n-2} \to G_{n-2}/G_{n-1} \to 0$ is exact. Continuing this way, we could build up to any group and the classification problem would have been solved.

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