Cosmology and the Boltzmann Equation

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Abstract

While cosmology has not yet answered all questions, it has already grown into a science with a huge well understood body of knowledge, both theoretical and observational. Nevertheless, the present essay is only an overview of the relativistic cosmology and the Boltzmann equation and its applications to the early universe homogeneous cosmology. Although, the Newtonian approach to studying the universe does yield a good description, general relativity offers a more comprehensive tool needed to do cosmology. We therefore use general relativity to study the dynamics of the expanding universe. Then, we try to understand a number of phenomena that the early, hot and dense universe experiences in the course of its expansion-driven cooling. The essay will cover two phenomena that account for the primordial production of the light elements, and these are the big bang nucleosynthesis and recombination. It will cover also the freeze-out of dark matter. These phenomena all have an equilibrium start which we will describe using the Saha equation. The equilibrium is however broken when the expansion rate exceeds the rate of particle interaction. Therefore, we will continue studying our phenomena in their subsequent out-of-equilibrium stages using the Boltzmann equation in its full form.

Résumé

Alors que la cosmologie n'a pas encore répondu à toutes les questions, elle constitue déjà une discipline suffisamment riche en connaissance tant observationnelle que théorique. Cependant, la présente dissertation porte seulement sur une vue d'ensemble de la cosmologie relativiste et de l'équation de Boltzmann et son application à la cosmologie homogène de l'univers primordial. Même si l'approche Neutonienne fournit aussi une bonne description dynamique de l'univers dans son expansion, le relativité générale d' Einstein s'avère plus efficace dans l'étude de la dynamique de notre univers. Ainsi, nous allons faire usage de la relativité générale pour comprendre l'évolution dynamique de l'univers. Ensuite, nous abordons l'univers très chaud dans son jeune âge où certains phénomènes ont lieu au long du refroidissement occasionné par l'expansion. Cependant, on se limite aux deux processus intervenant dans la synthèse des éléments légers à savoir la nucléosynthèse primordiale et recombinaison, et la formation de la matière noire. On se sert de l'équation de Saha pour décrire les états d'équilibre dans les phénomènes ci-haut mentionnés. L'équilibre thermique est pourtant rompu quand le taux d'expansion de l'univers dépasse le taux d'interaction des particules et ainsi on poursuit l'étude des dits-phénomènes hors équilibre au moyen de l'équation de Boltzmann dans sa forme complète.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Sosthène IRAMBONA, 22 May 2014

Contents

1. Introduction

1.1 The Universe and Cosmology

The universe is larger than everything else we can think of just because everything is a subset of it. The universe or the cosmos is the totality of existence. The next of this this introduction may not make sense unless we get a rough picture of the dimensions of the universe building blocks. In this quest for understanding the universe basic structure, we will introduce different physical quantities that are relevant to astronomical and cosmological studies. Nuclear fusion of hydrogen into helium that takes place in stars is the main source of visible light, a primary source of information about the universe. The sun is a typical star with a mass of about 2^{30} kg. The solar mass $M_\odot$ is used as an astronomical unit of masses. The rate at which the sun radiates its energy, its luminosity $L_\odot$, is 3.8×10^{26} watts and it is used also as a unit of luminosity in astronomy.

The light-year (ly) is the distance that light can travel in a year in vacuum. The light-year is related with the metre by $1ly = 946,073,0472,580,800\,\text{m} \simeq 10^{16}\,\text{m}$. Some other distance units used on different scales. The astronomical unit (AU) is the mean distance between the earth and the sun, $1\text{AU} = 1.5 \times 10^{11}\,\text{m}$. The astronomical unit is useful for distances within the solar system. For interstellar distances, the parsec (pc) is the common unit in cosmology. It is defined through $1\,\text{pc} = 3.261\,\text{ly} = 3.086 \times 10^{16}\,\text{m}$. For example, we are at $1.3\,\text{pc}$ from the sun's nearest neighbour, Proxima Centauri. In cosmology the smallest structural unit considered is is a conglomeration of thousands of millions of stars called the galaxy. It is nonetheless a minuscule object compared to the universe which contains a very large number of galaxies. Our solar system is located about $8\,\text{kpc}$ from the centre of the Milk Way galaxy. The Milk Way galaxy has a mass of $\sim 10^{12}\,M_\odot$ and a luminosity of $L = 3.6 \times 10^{10}\,L_\odot$ (Ryden, 2003). The nearest galaxy, though small and irregular, is the Large Magellanic Cloud (LMC) at $50\,\text{pc}$ from the sun. The Andromeda galaxy at $770\,\text{kpc}$ is the nearest galaxy of similar size to the Milk Way. The average separation between neighbouring galaxies is roughly the megaparsec ($1\,\text{Mpc} = 10^6\,\text{pc} = 3.086 \times 10^{22}\,\text{m}$) which makes it the cosmologist's favourite unit of distances (Liddle, 2003).

Surveying the universe on a scale of the order of $100\,\text{Mpc}$ reveals larger scale structures. In fact, at large scale the universe is made of voids and filament-like structures (White et al., 2012). The filaments can be broken down into superclusters, clusters, galaxy groups and eventually galaxies (Universe Today). Though some galaxies are found lonely in the universe, most of them are found in groups and clusters. A group accommodates up to 50 galaxies whereas a clusters counts 50 to 100 galaxies. The volume of a typical galaxy group is a few cubic megaparsecs. The local group is a small concentrated group of a little bit over 40 galaxies to which our own galaxy, the Milk Way, belongs (Liddle, 2003). A supercluster contains a large number of groups, clusters and individual galaxies. The voids are underdense regions of space that contains few or no galaxies. These vast empty space make 95% of the universe and are typically 11 to $150\,\text{Mpc}$ in diameter (Wiki).

Time, energy and temperature are also important quantities in cosmology. The year (yr) or the time the earth takes to orbit once the sun is, in a cosmological context, a short time and so often the gigayear (Gyr) is used. Mathematically, $1\,\text{Gyr} = 10^9\,\text{yr} = 3.2 \times 10^{16}\,\text{sec}$. As an example, the age of our earth is estimated to $4.6\,\text{Gyr}$. For energy, the megaelectronvolt (MeV) seems more convenient in cosmology. We know that the electronvolt and joule are related by $1\text{eV} = 1.6 \times 10^{-16}\,\text{J}$. For example, the rest mass of the electron is given by $m_e c^2 = 511,000\,\text{eV} = 0.511\,\text{MeV}$ and those of the proton and neutron are $938.3\,\text{MeV}$ and $939.6\,\text{MeV}$ respectively (Ryden, 2003). We can use the Boltzmann constant, $k_B = 8.617 \times 10^{-5}\,\mathrm{eV\,K^{-1}}$, to express temperature in units of energy as it is customary in high energy physics, and indeed in the fourth chapter of the present essay we will be expressing temperature in MeV . For example, we can write the temperature today as $2.34 \times 10^{-4}\,\mathrm{eV}$ but we mean it is $2.34 \times 10^{-4}\,\mathrm{eV}/k_B = (23.4/8.617)\,\mathrm{K} = 2.725\,\mathrm{K}$. Moreover, we will be writing our equations in Plank units whereby the Boltzmann constant k_B, the universal gravitational constant G, the vacuum speed of light c and the h-bar constant (modified Plank's constant) are all set equal to one. Any way we may like to keep G in our equations for the sake of clarity. It is also important to note as early as now that the values of different cosmological quantities and parameters change in time, so let us note that throughout this essay today's values will be indicated by subscript zero (0) unless the context is clear about it otherwise.

We are now in position to talk about cosmology. In what we have said above, it seems that large structures are more relevant than small irregularities such planets and individual stars. This is true because Cosmology is indeed defined as a physical science that seeks to understand the dynamical structure of the universe as a whole. More specifically, cosmology is a study of the origin, evolution and ultimate fate of the universe (Ryden, 2003). Physical cosmology happens to be interdisciplinary in the sense that it involves some other parts of physics including particle physics, astrophysics, statistical physics and most importantly general relativity. However, it turns out to be relatively simple because of two reasons. First, studying large structures such galaxy clusters and superclusters, the only interaction that matters is gravity. Second, on a large enough scale, the universe is homogeneous and isotropic to a good approximation. In fact, what is known as the cosmological principle is the statement that **the universe looks the same everywhere** (Liddle, 2003).

1.2 The Standard Model: The Hot Big Bang Model

The hot big bang theory or standard model is the most accepted cosmological model. In this model, our universe popped up into existence as a "singularity" around 13.824 billion years ago (NASA; Ade et al., 2013). Prior to the singularity nothing existed. And, indeed, the standard model has convincing evidences. First, the observation-supported discovery by Erwin Hubble in 1929 that the universe is expanding suggests that the universe was once compacted. Second, radioastonomers Penzias and Robert Wilson discovered, in 1965, a $2.725\,\mathrm{K}$ cosmic background microwave (CBM) that pervades the observable universe which is thought to be the remnant of heat that existed in the initial hot and dense universe. Last, the abundance of the light elements (hydrogen, helium, lithium) found in the universe today can be traced back to the phenomena known as big bang nucleosynthesis and recombination that took place in the early universe a short time later after the big bang.

So far observations have established the following facts (Mukhanov, 2005; NASA).

- The universe expands following the Hubble law according to which galaxies are moving away from us at velocities proportional to their distances.
- With no significant amount of antimatter in the universe, the baryonic matter contributes a small percentage, 4.9%, to the total density of the universe. The rest is $\sim 26.8\%$ cold dark matter with negligible pressure and $\sim 68.3\%$ dark energy with negative pressure. Additionally, the chemical composition of baryonic matter is $\sim 75\%$ of hydrogen and $\sim 25\%$ of helium plus traces of heavier elements.
- The universe is filled with a $2.73\,\mathrm{K}$ cosmic microwave background (CMB) with an energy density of $4.17 \times 10^{-14}\,\mathrm{J\,m^{-3}}$. There occurred small energy fluctuations in the CMB when the universe

was a thousand times smaller than it is now. The ratio of baryons to photons is one baryon per $\sim 10^{10}$ photons.

Although the standard model does fit the observational data, there are problems with it (Ryden, 2003; Liddle, 2003; NASA). The three problems we are just stating here will be treated, though less rigorously, in the last section of chapter two. First, it doesn't explain why the universe is nearly flat today and was even flatter in the past (flatness problem). Second, it does not answer the question of missing magnetic monopoles. Third and last, the big bang theory does not account for the isotropy and homogeneity of the universe at large scales. These problems call for another theory not to replace the hot big bang theory but to complete it. This is inflation theory which was proposed by the American astrophysicist Alan Guth in 1981 (Liddle, 2003) according to which the early universe did not only expand but also experienced a brief time of exponential expansion. Indeed, this theory of inflation does not only solve the three problem but also accounts better for structure formation in the universe (Dodelson, 2003). Today, research in cosmology is undeniably dominated by search for dark matter particle where various candidates such as the WIMP (weakly interacting particle) are under study, inflation, inhomogeneous cosmology and structure formation. The apparent discovery of B-mode gravitational wave by the BICEP 2 (Background Imaging of Cosmic Extragalactic Polarization) collaboration on 14 March 2014 provides an evidence for Guth's theory (Denis Overbye).

1.3 The Focus of the Essay

The present essay is based on the standard model of the big bang theory. In chapter two, we will use general relativity and the cosmological principle to study the cosmology of the expanding universe. Specifically, we will use the cosmological principle and geometrical considerations to obtain the Robertson-Walker metric which describes spacetime in a spatially homogeneous and isotropic universe. Then, after we have tried to understand the Einstein's field equation, we will use it to study the dynamics of the universe expansion whereby we will obtain the Friedmann equations and the fluid equation, needed to completely describe the universe. We will discuss the inflation theory in relation to the aforementioned imperfections of the standard model. In chapter three, we will turn to a thermodynamical study of the early universe where we will specifically study the production of light elements and dark matter in the early universe using the Boltzmann equation.

2. Fundamentals of Relativistic Cosmology

We review the basics of general relativity, mainly the Einstein field equation, which we then use to get an idea about the dynamics of the universe expansion.

2.1 A Bit of General Relativity

Newton's mechanics does account for most of the common particle motions. However, when particles move at speeds close to that of light, it breaks down and then the theory of special relativity (SR), as proposed by Einstein in 1905, takes over. The theory of special relativity relies essentially on its postulates that the form of equations of the fundamental physics laws stays invariant under transitions between inertial reference frames and that the speed of light is the maximum speed there is and is invariant or constant in all inertial frames. The special relativity rejects thus the common sense constancy of space, time and mass measurement. It associates space and time into a 4-dimensional object called spacetime (d'Inverno, 1992). In this spacetime, any event is specified by four numbers, three spatial coordinates and time. The path followed by a particle in spacetime is called the worldline x^μ, with $\mu = 0, 1, 2, 3$. The most important result of special relativity is the famous Einstein mass-energy equation $E = mc^2$ (Duncan, 1985). The theory of special relativity is nevertheless valid in a flat spacetime (Minkowski space), a space of zero curvature or equivalently a space where gravity is absent.

Indeed, in ten years' time, Einstein extended his theory of special relativity to a general theory of gravity known as General Relativity (GR) applicable to nonflat spacetime and which, by the correspondence principle, yields the Newtonian theory and the special theory under the right conditions (Carrol, 2004). In this section, we want to explore the very basic ideas underlying the general theory of gravity of Einstein. The first idea is that spacetime is modelled as a curved four-dimensional mathematical object called pseudo-Riemannian manifold or Lorentzian manifold (d'Inverno, 1992). In this spacetime, the laws of physics must be independent of the choice of coordinate system used to label points. This explains why general relativity is done using tensor formalism. The second idea is known as the strong equivalence principle of Eintein. According to this principle, at any spacetime point in an arbitrary gravitational field there is a "locally inertial" coordinate system in which the effects of gravitation are absent in a sufficiently small spacetime neighborhood of that point (Weinberg, 2008). This principle makes general relativity an extension of special relativity to curved spacetime. Lastly, general relativity regards gravity as an inherent property of curved spacetime or a manifestation of curvature in the geometry of spacetime where matter and momentum flux curves spacetime as described by the Einstein field equation (Carrol, 2004).

At this level it should be no surprise that the famous Einstein field equation is central in general relativity. Our next effort is to understand what it says. This equation is written as

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \qquad (2.1.1)$$

where $G_{\mu\nu}$ is the Einstein tensor, G is the Newton's gravitational constant, c is the speed of light and $T_{\mu\nu}$ is the energy-momentum tensor. We can write the same equation as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \qquad (2.1.2)$$

where the Einstein tensor has been replaced by its value in terms of the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R and the metric $g_{\mu\nu}$.

We need however to try to understand the mathematics of equation (2.1.2) for us to fully appreciate what it actually tells us. In what follows we want to understand the mathematical objects that appear in (2.1.2).

2.1.1 The Metric. Let us consider that a clock ticks once every time interval ds when it is a rest in the absence of a gravitational field. Then, if the clock is moving in absence of a gravitational field, the spacetime separation between successive ticks is

$$d\tau^2 = -ds^2 = -cdt^2 + dx^2 + dy^2 + dz^2. \tag{2.1.3}$$

Introducing the summation convention according to which repeated indices are summed over, we can write

$$d\tau^2 = -ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu. \tag{2.1.4}$$

In (2.1.4) $\eta_{\mu\nu}$ is the Minkowskian metric given by

$$\eta_{\mu\nu} = diag \begin{pmatrix} -c & 1 & 1 & 1 \end{pmatrix}. \tag{2.1.5}$$

We notice that (2.1.3) defines the interval between events in special relativity and components $\eta_{\mu\nu}$ of the Minkowski metric do not depend on coordinates. If now we consider the clock moving in a general gravitational field, then the interval between ticks is given by

$$d\tau^2 = -ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu, \tag{2.1.6}$$

where our new metric g not only depends on coordinates but also has two more properties (Weinberg, 2008). First, the transition of coordinates x^μ to $x^{\mu'}$ transfoms the metric into

$$g_{\rho'\sigma'} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x^{\rho'}} \frac{\partial x^\nu}{\partial x^{\sigma'}}, \tag{2.1.7}$$

where x^μ and $x^{\mu'}$ are the coordinates of the same physical points in different coordinate systems. Second, in inertial local coordinates, $g_{\mu\nu}(x)$ reduces to $\eta_{\mu\nu}$ and its first derivatives vanish. Since x^μ transfoms as

$$dx^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^\mu} dx^\mu, \tag{2.1.8}$$

equation (2.1.7) is covariant.

The tensor $g_{\mu\nu}$ has an inverse $g^{\nu\alpha}$ such that

$$g_{\mu\nu} g^{\nu\alpha} = \delta^\alpha_\mu. \tag{2.1.9}$$

Quantities that transform like (2.1.8) and (2.1.7) are called contravariant vector and covariant tensor respectively. A scalar $\phi(x)$ is a quantity that stays the same under coordinate transformation. The derivative of such a quantity is a covariant vector v_μ

$$v_\mu = \frac{\partial \phi(x)}{\partial x^\mu}, \tag{2.1.10}$$

and it transforms as

$$v_{\nu'} \equiv \frac{\partial \phi}{x^{\nu'}} = \frac{\partial \phi}{x^\mu} \frac{x^\mu}{x^{\nu'}} = v_\mu \frac{x^\mu}{x^{\rho'}}. \tag{2.1.11}$$

Contravariant tensors have raised indices whereas covariant tensors have lowered indices. In general, we can use the metric to raise and lower indices. For example, $S^\beta = g^{\mu\beta} S_\mu$.

2.1.2 Geodesic Equation and Affine Connection. In general relativity, the worldline of a particle that is moving in absence of any external nongravitational force is the geodesic, which is the straightest line there is in curved spacetime. Consider a curve $x^{\nu'}$ parametrised by λ. A tangent vector on this curve is given by

$$\frac{dx^{\nu'}}{d\lambda} = \frac{\partial x^{\nu'}}{\partial x^{\mu}} \frac{dx^{\mu}}{d\lambda}. \tag{2.1.12}$$

By definition, a straight line is a curve that parallel-transports its tangent (the tangent can be moved around on the curve without changing its size and direction). Therefore, if our curve $x^{\nu'}$ is a geodesic or the "straight" line on the curved spacetime, the second derivative of the vector (2.1.12) as computed below must vanish,

$$\frac{d^2 x^{\nu'}}{d\lambda^2} = \frac{d}{d\lambda}\left(\frac{\partial x^{\nu'}}{\partial x^{\mu}} \frac{dx^{\mu}}{d\lambda}\right) = \frac{\partial x^{\nu'}}{\partial x^{\mu}} \frac{d^2 x^{\mu}}{d\lambda^2} + \frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\sigma}} \frac{dx^{\mu}}{d\lambda}\frac{dx^{\sigma}}{d\lambda} = 0. \tag{2.1.13}$$

But there is a problem! What we have got is not a vector because of the the second term. Moreover, (2.1.13) is not covariant. We need a correction to make it covariant. We introduce a nontensor quantity $\Gamma^{\sigma}{}_{\mu\nu}$ which will cancel the second term. It is called the affine connection, whose general transformation law is

$$\Gamma^{\alpha'}{}_{\sigma'\rho'} = \frac{\partial x^{\alpha'}}{\partial x^{\tau}} \frac{\partial x^{\mu}}{\partial x^{\sigma'}} \frac{\partial x^{\nu}}{\partial x^{\rho'}} \Gamma^{\tau}{}_{\mu\nu} - \frac{\partial^2 x^{\alpha'}}{\partial x^{\mu}\partial x^{\nu}} \frac{\partial x^{\mu}}{\partial x^{\sigma'}} \frac{\partial x^{\nu}}{\partial x^{\rho'}}. \tag{2.1.14}$$

The corrected equation of motion for the particle along its geodesic which is covariant becomes

$$\frac{d^2 x^{\tau}}{d\lambda^2} + \Gamma^{\tau}{}_{\mu\nu} \frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0. \tag{2.1.15}$$

This is the geodesic equation. In a coordinate system which is locally inertial and cartesian at x, the affine connection $\Gamma^{\tau}{}_{\mu\nu}$ also known as Christofell symbol vanishes, which leads to the correct equation of motion in absence of gravitational field as $d^2x^a/d\lambda^2 = 0$. We therefore define the covariant derivative of a tensor as

$$\nabla_{\alpha} T^{\mu}{}_{\nu} = \partial_{\alpha} T^{\mu}{}_{\nu} + \Gamma^{\mu}{}_{\alpha\beta} T^{\beta}{}_{\nu} - \Gamma^{\beta}{}_{\nu\alpha} T^{\mu}{}_{\beta}. \tag{2.1.16}$$

The affine connection can be expressed in terms of the metric through

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left(\frac{\partial g_{\beta\nu}}{\partial x^{\mu}} + \frac{\partial g_{\beta\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}}\right). \tag{2.1.17}$$

2.1.3 Riemann Tensor. The curvature tensor or Riemann-Christoffel tensor or Riemann tensor is defined by (Schutz, 1985; d'Inverno, 1992)

$$R^{\beta}{}_{\mu\nu\alpha} = \partial_{\nu} \Gamma^{\beta}{}_{\mu\alpha} - \partial_{\alpha} \Gamma^{\beta}{}_{\mu\nu} + \Gamma^{\sigma}{}_{\mu\alpha} \Gamma^{\beta}{}_{\sigma\nu} - \Gamma^{\sigma}{}_{\mu\nu} \Gamma^{\beta}{}_{\sigma\alpha}. \tag{2.1.18}$$

Looking at (2.1.17) and (2.1.18), we notice that the curvature tensor depends on the metric and its first and second derivatives. The contraction of the Riemann tensor leads to Einstein tensor $G_{\mu\nu}$. Let us first obtain the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R respectively as

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\beta\mu\alpha\nu} \tag{2.1.19}$$

and

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.1.20}$$

The Einstein tensor is then written as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{2.1.21}$$

2.1.4 Energy-Momentum Tensor and Einstien Field Equation. The energy-momentum tensor describes the density and flux of energy and momentum in spacetime. It is the source of gravitational field in general relativity. Let us first consider a matter field of non-interacting incoherent matter or dust. To characterize this matter, we need only the 4-velocity $u^\mu = dx^\mu/d\tau$ of flow and the proper density $\rho_0 = \rho_0(x^\mu)$ as measured by a co-moving observer. The energy-momentum tensor for such a system is written as

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu. \tag{2.1.22}$$

In special relativity, the zero-zero component gives the density ρ as measured by a fixed observer,

$$T^{00} = \rho_0 \frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = \rho_0 \frac{dt^2}{d\tau^2} = \gamma^2 \rho_0 = \rho. \tag{2.1.23}$$

We can show the components of $T^{\mu\nu}$ as

$$T^{\mu\nu} = \rho \begin{pmatrix} 1 & u_x & u_y & u_z \\ u_x & u_x^2 & u_x u_y & u_x u_z \\ u_y & u_x u_y & u_y^2 & u_y u_z \\ u_z & u_x u_x & u_y u_z & u_z^2 \end{pmatrix}. \tag{2.1.24}$$

The equation of motion of a matter field of dust in flat space is

$$\partial_\nu T^{\mu\nu} = 0. \tag{2.1.25}$$

Let's notice that with $\mu = 0$, it reduces to the classical equation of continuity. In general relativity where spacetime is nonflat the law of conservation looks like $\nabla_\nu T^{\mu\nu} = 0$. Unlike incoherent matter, characterization of a perfect fluid will require one more quantity, the scalar pressure $P = P(x^\mu)$ and the energy-momentum tensor becomes

$$T^{\mu\nu} = (\rho_0 + P)\, u^\mu u^\nu - P g^{\mu\nu}. \tag{2.1.26}$$

We can now paraphrase the information contained in the Einstein field equation (EFE). The left hand side of (2.1.1) or (2.1.2) describes the curvature of spacetime as determined by the metric whereas the right hand side of the same equation describes the energy/momentum content of the spacetime. Our tensorial equation is in fact a set of ten second-order partial differential equations for the ten independent components of the metric tensor with the components of the energy-momentum tensor as the source terms. The solutions of these equations are called the metrics of spacetime, ds^2. In fact, given a distribution of matter and energy as the energy-momentum tensor, we can determine the geometry of spacetime by providing the definition of distance.

The Einstein field equation has many particular solutions (d'Inverno, 1992). The Minkowskian metric is the simplest solution that describes the flat spacetime in special relativity. The Schwarzschild metric describes the geometry around a spherical nonrotating mass such as a star. The Kerr metric describes a rotating black hole. There is a solution which describes gravitational waves as well. Most importanta-ntly for us, the Friedmann-Robertson-Walker metric describes the spatially homogeneous and isotropic expanding universe. In fact the latter solution is the centrepiece for the next section.

2.2 General Relativistic Cosmology

Relativistic cosmology that is the object of this section has its basis in three assumptions. The first is the cosmological principle as stated in the introductory chapter. The second is the general relativity as we said in the introduction also and this is why the first section of this chapter is dedicated to some general relativity basics. The third assumption is the Weyl's postulate. Weyl introduced the concept of substratum, a fluid pervading space in which galaxies move like fundamental particles (d'Inverno, 1992). The postulate then states that the particles in the substratum lie in spacetime on a congruence of timelike geodesics diverging from a common point (d'Inverno, 1992).

2.2.1 Robertson-walker (RW) Metric. The cosmological principle of invariance under rotation (isotropy) and invariance under translation (homogeneity) that is assumed on large scales turns out to hold only in space, not in time. This implies that spacetime is not maximally symmetric since the universe is expanding in time. Therefore, the metric for an expanding universe will look like

$$ds^2 = -dt^2 + R^2(t)d\sigma^2, \tag{2.2.1}$$

where $R(t)$ is a function called the scale factor, and $d\sigma^2$ is the metric on a maximally symmetric three-manifold. Here, maximal symmetry is just another way of saying spatial homogeneity and isotropy. We can write $d\sigma^2 = \gamma_{ij}(x)dx^i dx^j$. The Riemann tensor for the maximally symmetric 3-manifold is given by

$$R_{ijkl} = k\left(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}\right), \tag{2.2.2}$$

which can be contracted to the Ricci tensor

$$R_{jl} = 2k\gamma_{jl}. \tag{2.2.3}$$

Maximal symmetry implies spherical symmetry, $d\sigma^2$ can therefore be conveniently written using spherical coordinates,

$$d\sigma^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) = dr^2 + r^2 d\Omega^2. \tag{2.2.4}$$

Furthermore, we can multiply each term by an exponential function of the radial coordinate q which will not disturb our spherical symmetry (the form of $d\Omega^2$ is preserved). We can thus write

$$d\sigma^2 = \gamma_{ij}(x)dx^i dx^j = e^{2f(q)}dq^2 + q^2 d\Omega^2, \tag{2.2.5}$$

where the metric tensors for $d\sigma^2$ now look like

$$\begin{aligned} \gamma_{ij} &= diag\left(e^{2f} \quad q^2 \quad q^2\sin^2\theta\right); \\ \gamma^{ij} &= diag\left(e^{-2f} \quad \tfrac{1}{q^2} \quad \tfrac{1}{q^2\sin^2\theta}\right). \end{aligned} \tag{2.2.6}$$

The nonvanishing components of the Ricci tensor for this static , spherically symmetric 3-manifold are

$$R_{qq} = \frac{2}{q}\partial_q f; \quad R_{\theta\theta} = e^{-2f}(q\partial_q f - 1) + 1; \quad R_{\phi\phi} = \left(e^{-2f}(q\partial_q f - 1) + 1\right)\sin^2\theta. \tag{2.2.7}$$

Equating (2.2.7) to (2.2.3) and using (2.2.6) we solve for f and we get

$$f = -\frac{1}{2}\ln(1 - kq^2). \tag{2.2.8}$$

Using (2.2.8) in (2.2.5), we can write the metric for our 3-manifold as

$$d\sigma^2 = \frac{dq^2}{1 - kq^2} + q^2 d\Omega^2, \tag{2.2.9}$$

where the curvature k can take three values 0 for flat surface, -1 for open surface or 1 for closed surface. We finally get the Robertson-Walker metric, a metric on spacetime which describes a maximally symmetric surface evolving in size (expanding), as

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dq^2}{1 - kq^2} + q^2 d\Omega^2\right]. \tag{2.2.10}$$

In (2.2.10) the scale factor has units of distance and the radial coordinate q is dimensionless. We want a dimensionless scale factor $a(t)$, coordinate r with distance dimensions and a curvature parameter κ with dimensions $(length)^{-2}$. We use a quantity R_0 with distance dimensions and we set

$$a(t) = \frac{R(t)}{R_0}, \quad r = R_0 q \quad \text{and} \quad \kappa = \frac{k}{R_0^2}. \tag{2.2.11}$$

Using the new variables the Robertson-Walker metric becomes

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right], \tag{2.2.12}$$

whose metric tensors now look like

$$\begin{aligned} g_{\mu\nu} &= diag\begin{pmatrix}-1 & \frac{a^2}{1-\kappa^2} & a^2r^2 & a^2r^2\sin^2\theta\end{pmatrix}; \\ g^{\mu\nu} &= diag\begin{pmatrix}-1 & \frac{1-\kappa^2}{a^2} & \frac{1}{a^2r^2} & \frac{1}{a^2r^2\sin^2\theta}\end{pmatrix}. \end{aligned} \tag{2.2.13}$$

Similarly to the 3-manifold case above, we can now evaluate the Christofell symbols, the Riemann and Ricci tensors and then find the Ricci scalar. The nonvanishing components of the Ricci tensor are

$$\begin{aligned} &R_{00} = R_{tt} = -3\frac{\ddot{a}}{a}, \quad R_{11} = R_{rr} = \frac{a\ddot{a} + 2\dot{a}^2 + 2\kappa}{1 - \kappa r^2}, \quad R_{22} = R_{\theta\theta} = r^2\left(a\ddot{a} + 2\dot{a}^2 + 2\kappa\right) \\ &\text{and} \quad R_{33} = R_{\phi\phi} = r^2\left(a\ddot{a} + 2\dot{a}^2 + 2\kappa\right)\sin^2\theta; \end{aligned} \tag{2.2.14}$$

and the Ricci scalar is found to be

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right]. \tag{2.2.15}$$

2.2.2 Friedmann Equations. This equation describes the behaviour of the scale factor. We use the Ricci tensor components (2.2.14) and the Ricci scalar (2.2.15) in the Einstein equation (2.1.2). Following Weyl's postulate, we will model matter and energy by an isotropic perfect fluid which is at rest in a comoving frame. For such a fluid, the the 4-velocity is given by

$$u^\mu = (1, 0, 0, 0). \tag{2.2.16}$$

The energy-momentum tensor defined by equation (2.1.26) becomes

$$T^\mu{}_\nu = diag\begin{pmatrix}-\rho & P & P & P\end{pmatrix}, \tag{2.2.17}$$

where P is the pressure of the isotropic perfect fluid. The 00-component of Einstein equation reduces thus to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{c^2\kappa}{a^2}, \tag{2.2.18}$$

and the ii-component gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right). \tag{2.2.19}$$

Equations (2.2.18) and (2.2.19) are known as the Friedmann equations. We can write Friedmann equations in natural units ($c = 1$) as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}, \tag{2.2.20}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P). \tag{2.2.21}$$

Specifically, (2.2.20) is known as Friedmann equation and (2.2.21) is known as second Friedmann equation or acceleration equation.

We cannot yet use Friedmann equations unless we have an equation of time evolution of the density. As we said earlier, the vanishing covariant derivative of $T^\mu{}_\nu$ expresses the conservation of matter and energy in spacetime. Let's write down the $\nu = 0$ component of this derivative using (2.1.16),

$$\frac{\partial T^\mu{}_0}{\partial x^\mu} + \Gamma^\mu{}_{\alpha\mu}T^\alpha{}_0 - \Gamma^\alpha{}_{0\mu}T^\mu{}_\alpha = 0. \tag{2.2.22}$$

In (2.2.22), only $T^i{}_i$ survive and it turns out that $\mu = 0$ in the first and $\alpha = 0$ in the second terms, which gives us

$$-\frac{\partial \rho}{\partial t} - \rho\Gamma^\mu{}_{0\mu} - \Gamma^\alpha{}_{0\mu}T^\mu{}_\alpha = 0. \tag{2.2.23}$$

The Christofell symbols which do not vanish are $\Gamma^1{}_{01} = \Gamma^2{}_{02} = \Gamma^3{}_{03} = \dot{a}/a$, the conservation law in an expanding universe thus reduces to

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0. \tag{2.2.24}$$

We could use the first law of thermodynamics and consider a reversible expansion ($dS = 0$) in which energy is $E = mc^2$ and still get (2.2.24) which is known as the fluid equation (Liddle, 2003). Equation (2.2.24) is completed by the specification of the universe energy-matter content given as the equation of state $P = P(\rho) = w\rho$ which, in the case of flat universe, can lead to three cases

$$\begin{aligned} a(t) &= \left(\frac{t}{t_0}\right)^{2/3}, \quad \rho_m(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0^2}{t^2}; \\ a(t) &= \left(\frac{t}{t_0}\right)^{1/2}, \quad \rho_r(t) = \frac{\rho_0}{a^4} = \frac{\rho_0 t_0^2}{t^2}; \\ \rho_\Lambda &= const. \end{aligned} \tag{2.2.25}$$

for a matter-dominated or pressureless ($w = 0$) universe, a radiation-dominated universe ($w = 1/3$) and a vacuum energy- dominated universe ($w = -1$) universe, respectively. Solutions can also be found for $\kappa = \pm 1$ cases and cases where we have a mixture of matter, radiation and vacuum energy (also cosmological constant) (Liddle, 2003). Here, by matter and radiation we mean nonrelativistic matter and relativistic particles respectively.

2.3 Basic Cosmological Parameters

2.3.1 Redshift. An electromagnetic radiation is said to be redshifted when the observed wavelength is greater than that emitted by the source. As invoked earlier in the introduction, redshifts of galaxies show that they moving away from us. We define the redshift z in terms of wavelength λ as

$$1 + z = \frac{\lambda_{obs}}{\lambda_{emit}}. \tag{2.3.1}$$

Let us consider the Robertson-Walker metric (2.2.12) for light propagating radially from $r = 0$ to $r = r_0$. Since light does not travel any distance in spacetime, meaning that the light travel geodesics with $ds = 0$, we can write

$$c\frac{dt}{a(t)} = \frac{dr}{\sqrt{1 - \kappa r^2}}. \tag{2.3.2}$$

Therefore, to travel from $r = 0$ to $r = r_0$ light takes the time

$$c\int_{t_{emit}}^{t_{obs}} \frac{dt}{a(t)} = \int_0^{r_0} \frac{dr}{\sqrt{1 - \kappa r^2}}. \tag{2.3.3}$$

Then, since in comoving coordinates galaxies remain at fixed positions, considering a light signal emitted later at $t_{emit} + dt_{em}$ and so observed at $t_{obs} + dt_{obs}$ gives the integral

$$c\int_{t_{emit}}^{t_{obs}} \frac{dt}{a(t)} = c\int_{t_{emit}+dt_{emit}}^{t_{obs}+dt_{obs}} \frac{dt}{a(t)}. \tag{2.3.4}$$

This leads to

$$\frac{dt_{obs}}{a(t)} = \frac{dt_{emit}}{a(t)} \tag{2.3.5}$$

which leads further to (Liddle, 2003)

$$1 + z = \frac{\lambda_0}{\lambda} = \frac{a(t_0)}{a_(t)}. \tag{2.3.6}$$

In an expanding universe, we can obviously see that z will be greater for light emitted closer to the big bang. This allows cosmologists to use the redshift to describe various epochs of the universe history.

2.3.2 Hubble, Density and Deceleration Parameters. Hubble law says the recessional velocity $\mathbf{v}$ of a galaxy is proportional to the physical distance $\mathbf{r}$, or mathematically,

$$\mathbf{v} = H(t)\mathbf{r} = \frac{\dot{a}}{a}\mathbf{r}, \tag{2.3.7}$$

where the Hubble parameter $H(t) = \dot{a}/a$ is the rate of expansion of the universe. The Hubble parameter is constant in space but evolves in time. Its present value (Hubble constant) is $H_0 = 100h$km/sec/Mpc with $h \sim 0.673 \pm 0.012$ (Ade et al., 2013). The physical distance is given by $\mathbf{r} = a\mathbf{x}$, where $\mathbf{x}$ is the comoving distance assumed constant.

The critical density ρ_c given by

$$\rho_c(t) = \frac{3H^2}{8\pi G}, \tag{2.3.8}$$

is obtained by setting $\kappa = 0$ in (2.2.20). It is the total density required to make the geometry of the universe flat. Its value today is $\rho_c(t_0) = 1.88h^2 \times 10^{-26}\,\text{kg}\,\text{m}^{-3} = 2.78h^{-1} \times 10^{11}\,M_\odot/\left(h^{-1}\,\text{Mpc}\right)^3$ (Liddle, 2003). We define a dimensionless quantity $\Omega(t)$, the parameter density, by

$$\Omega(t) \equiv \frac{\rho}{\rho_c}. \tag{2.3.9}$$

When used in the Friedmann equation, it makes it look like

$$\Omega - 1 = \frac{\kappa}{a^2H^2}. \tag{2.3.10}$$

If we set the right hand side of (2.3.10) equal to $-\Omega_\kappa$, where Ω_κ is the density parameter associated to the universe curvature, then we can rewrite the Friedmann equation as

$$\Omega + \Omega_\kappa = 1. \tag{2.3.11}$$

Here we need to understand that the energy density ρ as it appears in the Friedmann equation is a summed contribution from various components (see chapter one). Therefore, we refer to the density parameter defined above as Ω_{tot}, the sum of fractional density parameters, Ω_{rad} for radiation and relativistic matter, Ω_b for baryonic matter, and Ω_{dm} for dark matter and Ω_Λ for dark energy or vacuum energy or cosmological constant Λ. Or, mathematically,

$$\Omega_{tot} = \Omega_{rad} + \Omega_b + \Omega_{dm} + \Omega_\Lambda = \Omega_r + \Omega_m + \Omega_\Lambda, \tag{2.3.12}$$

where Ω_m stands for both baryonic and dark matter. Observations have shown that today, while $\Omega_r = 8.4 \times 10^{-5}$ (negligibly small contribution from photons and neutrinos), $\Omega_m = 0.315 \pm 0.017$, $\Omega_\Lambda = 0.6825$, $\Omega_\kappa = -0.0326$, $\Omega_b h^2 = 0.02205 \pm 0.00028$ and $\Omega_{dm}h^2 = 0.1199 \pm 0.0027$, ($\Omega_m h^2 = 1.4300$) (Ade et al., 2013). More importantly, Ω_{tot} is very close to 1, which suggests that the universe total density is nearly critical. In other words, our universe is nearly flat (Ruiz-Lapuente, 2010).

We now talk about another parameter, deceleration parameter q_0 which we obtain by taking the Taylor expansion of the scale factor $a(t)$ about the present time t_0 and then divide by $a(t_0)$. This is

$$a(t) = a(t_0) + (t - t_0)\dot{a}(t_0) + \frac{1}{2}(t - t_0)^2\ddot{a}(t_0) + \cdots. \tag{2.3.13}$$

We then divide by $a(t_0)$ to get

$$\frac{a(t)}{a(t_0)} = 1 + (t - t_0)H_0 - \frac{1}{2}(t - t_0)^2 H_0^2 q_0, \tag{2.3.14}$$

where our deceleration parameter is

$$q_0 = -\frac{1}{H_0^2}\frac{\ddot{a}(t_0)}{a(t_0)}. \tag{2.3.15}$$

Using equations the second Friedmann equation (acceleration equation) for a pressureless universe and our definition of the critical density, we obtain that

$$q_0 = \Omega_{r,0} + \frac{1}{2}\Omega_{m,0} - \Omega_\Lambda. \tag{2.3.16}$$

Using the omega values above we get a negative q_0 due to the dominant value of Ω_Λ, meaning $\ddot{a} > 0$. Therefore, we conclude that the universe is not only expanding but also the expansion is accelerated.

2.3.3 Time and Distances. In many contexts it is important to define the conformal time η by

$$\eta \equiv \int_0^t \frac{dt'}{a(t')}, \tag{2.3.17}$$

which is the comoving horizon and therefore regions separated by distances greater than η are causally disconnected. The comoving distance or proper distance between us and a distant astronomical object that emits light is

$$d_p(t_0) = \int_{t(a)}^{t_0} \frac{dt'}{a(t')} = \int_a^1 \frac{da'}{a'^2 H(a')}, \tag{2.3.18}$$

where the scale factor today is $a(t_0) = 1$. In a spatially flat matter-dominated universe, the proper distance is

$$d_p(t_0) = \frac{2}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]. \tag{2.3.19}$$

Since the current proper distance to a galaxy is not measurable, some techniques for measuring distances have been elaborated in cosmology and astronomy. Within our solar system the radar technique is quite useful. The distance between the earth and a celestial body is given by $d = (1/2)ct$, where t is the time taken by the radar signal sent from the earth to travel to the body and back to the earth after being reflected by the body. Distances in our galaxy are better determined by the trigonometric parallax method. This method yields

$$d_\theta = 1pc\left(\frac{b}{1\text{AU}}\right)\left(\frac{\theta}{1\,\text{arcsec}}\right)^{-1}. \tag{2.3.20}$$

In fact, when a star is observed from two points separated by a distance b along a baseline, its position is seen to shift by an an angle θ. It is customary to use the earth's orbit round the sun so that $b = 2$AU. The difficult task is then to measure the angle θ with sufficient accuracy, which is not always feasible (Ryden, 2003). Luminosity distance is another alternative, it is defined as

$$d_L = \left(\frac{L}{4\pi f}\right)^{1/2}, \tag{2.3.21}$$

where L is the luminosity in watts of a "standard candle", an object whose luminosity is known and f in watts per square metres is the radiation flux measured. With our well supported assumption that the universe is nearly flat, it can shown that $d_L = (1+z)d(t_0)$ (Ryden, 2003). Finally, we define the angular-diameter distance. Instead of luminosity, the known size l of a cosmological object can be used to determine its its angular-diameter distance

$$d_A = \frac{l}{\theta}, \tag{2.3.22}$$

where θ is the angle it subtends to the eye. Under our usual assumptions, one can show (Liddle, 2003; Ryden, 2003) that

$$d_A(1+z) = d_p(t_0) = d_L/(1+z). \tag{2.3.23}$$

2.4 Inflation

We talked about it in the introduction, here is now a brief account for it. We once again look at the Friedmann equation $|\Omega_{tot}(t) - 1| = |\kappa|/a^2H^2$ which tells us that $|\Omega_{tot}(t) - 1| \propto a^2 \propto t$ during the radiation domination era and $|\Omega_{tot}(t) - 1| \propto a \propto t^{2/3}$ during matter domination epoch. In either case $|\Omega_{tot}(t) - 1|$ is an increasing function of time, which would imply an unstable flat geometry. As observed $\Omega_{tot}(t)$ is close to one today, $|\Omega_{tot}(t) - 1| \leq 0.2$. In the past at the point of matter-radiation equality, we can show that $|\Omega_{mr} - 1| \leq 2 \times 10^{-4}$. This shows that the density gets closer and closer to critical as we go back into the past. However, the pure hot big bang model or standard model of cosmology does not provide any explanation for this evidenced situation. It is what we referred to as the flatness problem in the introduction. It can be solved rather easily if we assume that the universe expanded very rapidly in its very early time and then got a high enough flatness level that has been falling so far with the ordinary expansion up to the still flat situation observed today. It is this extremely rapid expansion $(\ddot{a}(t) > 0)$ which is known as inflation and it is thought to have taken place when the universe was 10^{-34} sec old. Here we can only guess that it has to do with something that has a negative pressure.

In the previous section, the conformal time shows that the universe has a finite age and light has travelled a finite distance by today which gives rise to a region we call the observable universe. Moreover, the standard model is based on the cosmological principle of universe homogeneity and isotropy and indeed, one example is the thermal equilibrium of the 2.724K cosmic microwave background radiation (whose origin is discussed in the fourth chapter). Nevertheless, it is not explained how two distant points can be in thermal equilibrium with no prior contact or interaction. This is the horizon problem as invoked in the introduction also. The solution to the puzzle is rather simple if we think again about inflation. In fact, a patch of the universe so small that it thermalizes can expand during inflation to a size larger than the observable universe, explaining that two points no matter how distant they are, were once in equilibrium.

The inflationary expansion solves also the problem of relic particle abundances such as magnetic monopole. The dramatic expansion dilutes away these particles so much that their abundance is too faint to detect today. Inflation is currently an active research area but researchers have not yet understood what caused it.

3. Overview of Boltzmann Equation

In the present chapter we want to recall the meaning of the phase space distribution function of particles and then obtain the relativistic Boltzmann equation it satisfies. The Boltzmann equation is used to study different out-of-equilibrium phenomena involving particles. By making use of some practical cosmological features, we also derive Saha equation from the Boltzmann one. The Saha equation is useful when studying equilibrium interactions of particles. And, indeed these two equations are the only tools we need in the fourth chapter.

3.1 Distribution Function

The central piece of the third chapter is the Boltzmann equation. It makes sense however to first look at the distribution function which satisfies it. Indeed, the distribution function f_i is a 7-D function of time t, coordinates x^μ and momenta p^μ; it measures the average particle number density in phase space or position-momentum space. That is, $f_i(x^\mu, p^\mu)d^3x d^3p$ is the number of particles of species i that are found in a small but finite volume element d^3x around the point x^μ and have momenta in the vicinity of p^μ (Dodelson, 2003). In our definition, we are talking about proper momentum, not comoving momentum, which falls as a^{-1}.

With the help of the distribution function, we can compute various macroscopic properties of systems of particles. In the case of a spatially homogeneous and isotropic distribution, the distribution function obviously depends on the three-momentum and time. We can therefore obtain the number density, energy density and pressure for species i particles as

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} f(p,t); \quad \rho = g_i \int \frac{d^3p}{(2\pi)^3} f_i(p,t)E(p); \quad P_i = g_i \int \frac{d^3p}{(2\pi)^3 3E(p)} f_i(p,t)p^2 \tag{3.1.1}$$

respectively. where, according to quantum considerations, g_i is the number of degrees of freedom of the particle of species i and $(2\pi\hbar)^3$ rises due to the Heisenberg uncertainty principle setting the unit phase space volume element to $d^3p/(2\pi\hbar)^3$. For particles in thermodynamic equilibrium, the function f is the Bose-Einstein/Fermi-Dirac distribution function given by

$$f_i(E) = \frac{1}{e^{(E_i - \mu_i)/T} \pm 1} \tag{3.1.2}$$

where the positive sign works for fermions and the negative one for bosons. In case of chemical equilibrium, the chemical potentials μ_i of interacting particles are related. For example, when a particle of type a interacts with particles of species b, c and d according to $a + b \longleftrightarrow c + d$, we can write

$$\mu_a + \mu_b = \mu_c + \mu_d. \tag{3.1.3}$$

One can show that for cases where the chemical potential is much smaller than the temperature, as it is in cosmology, the distribution function depends only on E/T and the pressure satisfies

$$\frac{\partial P_i}{\partial T} = \frac{\rho_i + P_i}{T}, \tag{3.1.4}$$

which can be used also to show that the entropy density is given by

$$s = \frac{\rho + P}{T}. \tag{3.1.5}$$

In the radiation-dominated universe, the contribution to energy density and pressure come from the relativistic particles ($m_i \ll T$). We can compute them to obtain

$$\rho_{rad} = \frac{\pi^2}{30} g_* T^4; \quad P_{rad} = \rho_{rad}/3 = \frac{\pi^2}{90} g_* T^4, \tag{3.1.6}$$

where g_* is the effective number of degrees of freedom of the contributing particles, given by (Kolb and Turner, 1990)

$$g_* = \sum_{i=boson} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i=fermion} g_i \left(\frac{T_i}{T}\right)^4. \tag{3.1.7}$$

3.2 Boltzmann Equation

This is the equation that governs the evolution in phase space of the distribution function discussed above and it simply says the number of particles in a phase space volume element is constant unless there are collisions. It can be written as

$$\hat{\mathbf{L}}[f] = \mathbf{C}[f], \tag{3.2.1}$$

where C is collision operator and $\hat{\mathbf{L}}$ is the Liouville operator for the phase space distribution function f. For a species of particles of mass m acted upon by an external force $\mathbf{F} = \mathbf{p}/dt$, the nonrelativistic Liouville's operator is

$$\hat{\mathbf{L}} = \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla + \mathbf{F} \cdot \frac{\partial}{\partial \mathbf{p}}. \tag{3.2.2}$$

We are however interested in a relativistic version of the Liouville operator. The covariant relativistic Liouville operator is written as

$$\hat{\mathbf{L}} = p^\mu \frac{\partial}{x^\mu} - \Gamma^\mu_{\beta\nu} p^\beta p^\nu \frac{\partial}{\partial p^\mu}, \tag{3.2.3}$$

where the connection $\Gamma^\mu_{\beta\nu}$ introduces the curvature of spacetime and therefore, the gravity. For a spatially homogeneous and isotropic universe, the Liouville operator for $f_i = f_i(p,t) = f_i(E,t)$ reduces to the zeroth component of the operator in (3.2.3) which turns out to be

$$\hat{\mathbf{L}} = E\frac{\partial}{\partial t} - \frac{\dot{a}}{a} p^2 \frac{\partial}{\partial E}. \tag{3.2.4}$$

The relativistic Boltzmann equation that governs the evolution of $f(E,t)$ in phase space can then be written as

$$E\frac{\partial f_i}{\partial t} - Hp^2 \frac{\partial f_i}{\partial E} = \mathbf{C}[f_i], \tag{3.2.5}$$

where $H = \dot{a}/a$ is the Hubble parameter. On integrating by parts the Boltzmann equation (3.2.5) over momentum and then use (3.1.1), we get

$$\frac{dn_i}{dt} + 3Hn_i = \frac{1}{(2\pi)^3} \int \mathbf{C}[f_i] \frac{d^3p}{E} \quad \text{or} \quad a^{-3} \frac{d(a^3 n_i)}{dt} = \frac{1}{(2\pi)^3} \int \mathbf{C}[f_i] \frac{d^3p}{E}. \tag{3.2.6}$$

It turns out that in the expanding universe the physical number density of particles falls off as a^{-3}. The evaluation of the collision integral on the right hand side of (3.2.6) depends on the nature of particle interactions. Let us now examine the situation where the number density n_a of species a particles is affected only by annihilation with species b producing two particles c and d, shown schematically as $a + b \longleftrightarrow c + d$. The double arrow shows that inverse process can produce c and d. The Boltzmann equation for this equation is

$$\frac{dn_a}{dt} + 3Hn_a = \int \frac{d^3p_a}{(2\pi)^3 2E_a} \int \frac{d^3p_b}{(2\pi)^3 2E_b} \int \frac{d^3p_c}{(2\pi)^3 2E_c} \int \frac{d^3p_d}{(2\pi)^3 2E_d} \times A \tag{3.2.7}$$

with

$$A = (2\pi)^4 \delta^3(p_a + p_b - p_c - p_d)\delta(E_a + E_b - E_c - E_d)|\mathcal{M}|^2 \times S, \tag{3.2.8}$$

where we have, in addition to the delta functions of energies and momenta enforcing the conservation of energy and momentum, the amplitude $\mathcal{M}$ determined from the fundamental physics of interaction in question and the quantity S defined as

$$S = \{f_c f_d(1 \pm f_a)(1 \pm f_b) - f_a f_b(1 \pm f_c)(1 \pm f_d)\}. \tag{3.2.9}$$

In S the production term proportional to $f_c f_d$ and the loss term proportional to $f_a f_b$ both contain a quantity $1 \pm f_i$ representing Pauli blocking/Bose enhancement effects where the plus sign works for bosons and the minus for fermions.

We now have recourse to simplifying features and assumptions. First, we realize that rapid scattering processes enforce a kinetic equilibrium so that the distributions for the interacting particles retain the Fermi-Dirac (FD)/Bose-Einstein (BE) distribution functions. Therefore, f_i, with $i = a, b, c, d$, in (3.2.9) is the BE/FD distribution function (occupation number) of particles of species i. Second, since we want to study out-of-equilibrium phenomena, the system is not in chemical equilibrium but at least the assumed kinetic equilibrium will reduce the problem to solving a single ordinary differential equation for the chemical potential $\mu_i(T)$ or $\mu_i(t)$ rather than the complicated Boltzmann equation (3.2.7)

Finally, we will be interested in systems at temperatures T smaller than $(E_i - \mu_i)$. This reduces the quantum BE/FD distribution (3.1.2) to the classical Boltzmann distribution

$$f_i(E) = e^{-(E_i - \mu_i)/T}. \tag{3.2.10}$$

But also, $f_i + 1 \approx 1$. Making use of our simplifications, we can reduce expression (3.2.9) for S to

$$S = e^{-(E_a+E_b)/T}\{e^{(\mu_c+\mu_d)/T} - e^{(\mu_a+\mu_b)/T}\}. \tag{3.2.11}$$

Using (3.2.10), we can write (3.1.1) for the time-dependent number density as

$$n_i(t) = \frac{g_i e^{\mu_i/T}}{2\pi^2} \int p^2 e^{-E_i/T} dp. \tag{3.2.12}$$

Furthermore, since $n_i(t)$ is a function of $\mu_i(t)$ we can now reset our problem, instead of solving for μ_i we solve for n_i. To proceed, let us define the species-dependent equilibrium ($\mu = 0$) number density as

$$n_i^0 \equiv \frac{g_i}{2\pi^2} \int p^2 e^{-E_i/T} dp \tag{3.2.13}$$

which, upon integration, yields

$$n_i^0 = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T} \quad \text{and} \quad n_i^0 = g_i \frac{T^3}{\pi^3} \tag{3.2.14}$$

for the nonrelativistic case, $m_i \gg T$, and the relativistic one, $m_i \ll T$, respectively. Using (3.2.12) and (3.2.13), we can write $e^{\mu_i/T} = n_i/n_i^0$. Using this result we can transform (3.2.11) into

$$S = e^{-(E_a+E_b)/T} \left\{ \frac{n_c n_d}{n_c^0 n_d^0} - \frac{n_a n_b}{n_a^0 n_b^0} \right\}. \tag{3.2.15}$$

We now define the thermally averaged cross section as

$$\langle \sigma v \rangle = \frac{1}{n_a^0 n_b^0} \int \frac{d^3 p_a}{(2\pi)^3 2E_a} \int \frac{d^3 p_b}{(2\pi)^3 2E_b} \int \frac{d^3 p_c}{(2\pi)^3 2E_c} \int \frac{d^3 p_d}{(2\pi)^3 2E_d} \times A. \tag{3.2.16}$$

Using (3.2.16) and (3.2.15) into equation (3.2.7), we obtain a simplified Boltzmann equation

$$\frac{dn_a}{dt} + 3Hn_a = n_a^0 n_b^0 \langle \sigma v \rangle \left\{ \frac{n_c n_d}{n_c^0 n_d^0} - \frac{n_a n_b}{n_a^0 n_b^0} \right\}. \tag{3.2.17}$$

We need to do something further to get the final form of Boltzmann equation we want. The cosmological time is H^{-1} and therefore the left hand side of (3.2.17) is of the order n_a/t or Hn_a. The right hand side of the same equation is of the order $n_a n_b \langle \sigma v \rangle$. We therefore realize that if the reaction rate $\Gamma_b = n_b \langle \sigma v \rangle$ is much larger than the expansion rate H as it was in the early universe, then to keep equality in (3.2.17) the content in the braces must vanish which finally lead us to the famous Saha equation,

$$\frac{n_c n_d}{n_c^0 n_d^0} = \frac{n_a n_b}{n_a^0 n_b^0}. \tag{3.2.18}$$

We will use this equation to describe the equilibrium part of our next discussion.

4. Boltzmann Equation in Cosmology

In the very early, dense and hot universe, interactions among particles were so frequent that a thermal equilibrium existed. However, departures from equilibrium did happen and resulted in important phenomena including big bang nucleosynthesis, recombination and production of dark matter, photon and neutrino decoupling (becoming free of interactions), inflation, etc. (Dodelson, 2003). In this chapter we want to use the Boltzmann and Saha equations to study the homogeneous big bang nucleosynthesis, primordial recombination, photon decoupling resulting in the cosmic microwave background and free-out of dark matter. So far, We have established that as the universe expands the radiation density, matter density and temperature fall off as a^{-4}, a^{-3} and a^{-1} respectively. We still need two more pieces of physics to do our job. First, normally a particle can only be formed if its binding energy is greater than the energies of particles it is interacting with. Second, in an expanding universe, interacting particles will remain in equilibrium as long as the interaction rate is greater than the expansion rate.

4.1 Big Bang Nucleosynthesis

While heavier elements such as carbon, silicon, sulphur and iron are formed in the stars as they age in a process known as stellar nucleosynthesis, the fact that young stars seem to start their lives with nonzero abundances of the light elements, helium, hydrogen, deuterium, helium-3 and lithium suggests that these light elements are the components of the primordial gas that was formed in the early universe (Liddle, 2003; NASA).

When the universe has cooled down to a temperature $\sim 1\,\text{MeV}$, the cosmic plasma comprises two parts, relativistic and nonrelativistic particles. The relativistic particles are the electrons, positrons and photons kept in equilibrium by electromagnetic interactions such as $e^+e^- \leftrightarrow \gamma\gamma$ and the decoupled neutrinos. The nonrelativistic particles are the baryons with survived an initial asymmetry that did exist in the number of baryons and antibaryons. The baryon-to-photon ration is (Dodelson, 2003)

$$\eta_b \equiv n_b/n_\gamma = 2.75 \times 10^{-8}\Omega_b h^2. \tag{4.1.1}$$

Solving (3.2.17) reveals the fate of the baryons. We make two assumptions. First, above $\sim 0.1\,\text{MeV}$ there exist only free protons and neutrons, no light elements form. Second, when elements start to form, no elements heavier than helium are formed at a significant rate and this is due to the short life time of the neutron as we will see later in this chapter. To start, we examine the formation of the deuterium nucleus in the equation $\text{n} + \text{p} \longleftrightarrow \text{D} + \gamma$. Realising that $n_\gamma = n_\gamma^0$ for the photon, (3.2.18) leads to

$$\frac{n_D}{n_n n_p} = \frac{n_D^0}{n_n^0 n_p^0} \tag{4.1.2}$$

which, upon using result (3.2.14), yields

$$\frac{n_D}{n_n n_p} = \frac{3}{4}\left(\frac{2\pi m_D}{m_n m_p}\right)^{3/2} e^{B_D/T}, \tag{4.1.3}$$

where the numbers of spin states $g_D = 3$ and $g_n = g_p = 2$ have resulted into the fraction we can see in the prefactor. Furthermore, we have set $m_D = 2\,m_n = 2\,m_p$ in the prefactor but kept $m_n + m_P - m_D =$

B_D (deuterium binding energy) in the exponential. Then, realizing that $n_p \propto n_b$ and $n_n \propto n_b$, we obtain a rough estimate

$$\frac{n_D}{n_b} \sim \eta_b \left(\frac{T}{m_p}\right)^{3/2} e^{B_D/T} \tag{4.1.4}$$

which is dominated by the prefactor as long as B_D/T is not too large. The small value of η_b thus prevents nuclei formation until the temperature falls well below the binding energy B_D.

As it might have been pointed out somewhere above, the abundance of neutrons in the early universe is very crucial to our discussion. We will study it by considering the neutron-proton ratio. In the nonrelativistic limit $E_i = m_i + p_i^2/2m_i$, the proton-neutron equilibrium ratio can be obtained using (3.2.14) as

$$\frac{n_p^0}{n_n^0} = e^{\mathcal{Q}/T}, \tag{4.1.5}$$

where the integrals give the ratio $(\frac{m_p}{m_n})^{3/2} \simeq 1$ and $\mathcal{Q} \equiv m_n - m_p = 1.293$MeV.

Until $T \sim 1$MeV, protons can be converted into neutrons via equilibrium weak interactions such as $p + e^- \longleftrightarrow n + \nu_e$. Assuming that for the two leptons (electron and its neutrino), $n_{e^-} = n_{e^-}^0$ and $n_{\nu_e} = n_{\nu_e}^0$, the Saha equation for this interaction results in

$$\frac{n_p^0}{n_n^0} = \frac{n_p}{n_n} = e^{\mathcal{Q}/T}, \tag{4.1.6}$$

showing that at high enough temperatures protons and neutrons are equally abundant. But, as the temperature drops below 1 MeV the neutron fraction falls and therefore, if the weak interactions are efficient enough to sustain equilibrium indefinitely, then the neutron fraction would drop to zero (Dodelson, 2003).

However, what happens to the neutron fraction in the real world where weak interactions are not efficient enough to maintain equilibrium indefinitely, is different and that is what we want to deal with. We conveniently define X_n the ratio of neutrons to the total nucleons as

$$X_n \equiv \frac{n_n}{n_n + n_p} \tag{4.1.7}$$

which, in equilibrium, reduces to

$$X_{n,eq} \equiv \frac{n_n^0}{n_n^0 + n_p^0} = \frac{1}{1 + (n_p^0/n_n^0)}. \tag{4.1.8}$$

As a start let us now consider (3.2.17), with a and c being the neutron and proton respectively, and b and d are the leptons in equilibrium. Also, we assume that $n_l = n_l^0$. This leads to

$$\frac{dn_n}{dt} + 3Hn_n = n_l^0 \langle \sigma v \rangle \left\{ \frac{n_p n_n^0}{n_p^0} - n_n \right\}. \tag{4.1.9}$$

Using (4.1.7) and (4.1.8) into (4.1.9) we get

$$\frac{dX_n}{dt} = \lambda_{np} \left\{ (1 - X_n) e^{-\mathcal{Q}/T} - X_n \right\}, \tag{4.1.10}$$

where $\lambda_{np} = n_l^0\langle\sigma v\rangle$ is the rate for neutron-to-proton conversion. Let us change variable by using $x = \mathcal{Q}/T$. Both T and λ_{np} are function of time. Moreover, the temperature in the expanding universe scales as a^{-1}. This can lead us to $dx/dt = -x(dT/T) = Hx$. Since nucleosynthesis occurs in the early radiation-dominated universe, the major part of energy comes from relativistic particles. These are photons $g_\gamma = 2$, six flavours of neutrinos $g_\nu = 6$, and electrons and positrons $g_{e^-} = g_{e^+} = 2$ (Dodelson, 2003; Kolb and Turner, 1990). Considering that all these particles have the same temperature, we can get $g_* = 10.75$ giving us $\rho = 10.75\pi^2T^4/30$. Furthermore, the radiation energy density scales as T^4 (3.1.6) and Friedmann equation tells thus that $H \propto T^2$. This allows us to write $H = H_1/x^2$, where H_1 is the Hubble parameter at $T = \mathcal{Q}$ or $x = 1$. We change (4.1.10) to

$$\frac{dX_n}{dx} = \frac{x\lambda_{np}}{H_1}\left\{e^{-x} - X_n(1+e^{-x})\right\}. \tag{4.1.11}$$

Using (2.2.20) we get $H_1 = \left((4\pi^3 G\mathcal{Q}^4/45)\times 10.75\right)^{1/2} = 1.13\text{sec}^{-1}$. The neutron-proton conversion rate λ_{np} under our assumptions is given by (Bernstein et al., 1989)

$$\lambda_{np} = \frac{255}{\tau_n x^5}\left(12 + 6x + x^2\right), \tag{4.1.12}$$

where $\tau_n = 886.7\,\text{sec}$ is the life time of the neutron (Dodelson, 2003). It turns out that when $x = 1$ or $T = \mathcal{Q}$, the conversion rate is $\lambda_{np} = 5.5\,\text{sec}^{-1}$ which is greater than the expansion rate $H = 1.13\,\text{sec}^{-1}$ implying that the conversion rate is efficient enough to maintain equilibrium. However, beneath $1\,\text{MeV}$, the conversion fades out. Indeed, integration of (4.1.10) shows that neutrons froze out to a fraction of $X_n^f = 0.15$ at $0.5\,\text{MeV}$ (Bernstein et al., 1989). Below $0.1\,\text{MeV}$, two reactions become important and these are deuterium production, $\text{n} + \text{p} \to \text{D} + \gamma$, and neutron decay, $\text{n} \to \text{p} + e^- + \bar{\nu}$.

Let us now call T_{nuc} the instantaneous temperature at which light nuclei are produced and use it in (4.1.4) whose we take the logarithm. This gives us

$$\ln n_D - \ln n_b \sim \ln \eta_b + \frac{3}{2}\ln\left(\frac{T_{nuc}}{m_p}\right) + \frac{B_D}{T_{nuc}}. \tag{4.1.13}$$

If the world remained in equilibrium, all neutrons and protons would form deuterium. Therefore, $n_D \sim n_b$ and

$$\ln \eta_b + \frac{3}{2}\ln\left(\frac{T_{nuc}}{m_p}\right) \sim -\frac{B_D}{T_{nuc}}, \tag{4.1.14}$$

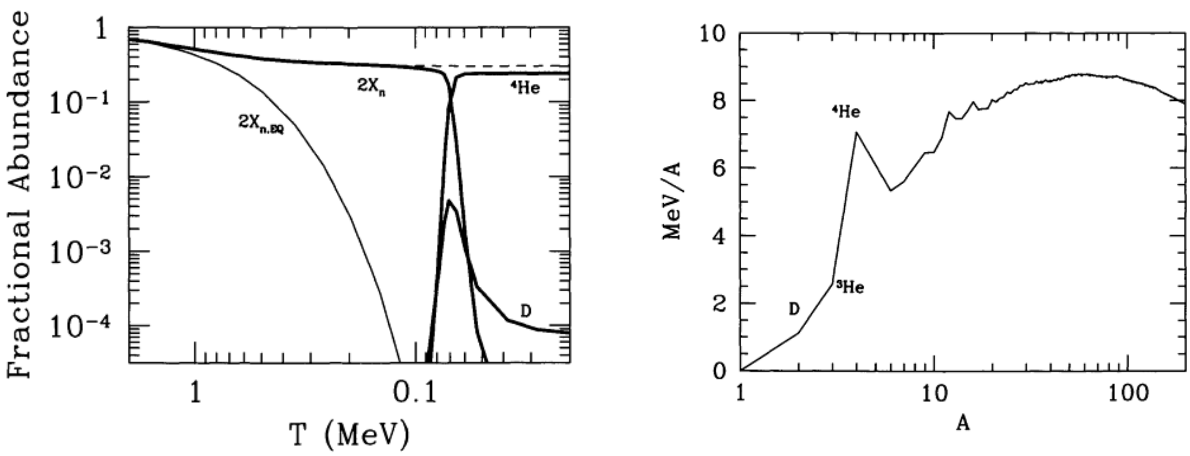
which simply tells us that deuterium forms at $T_{nuc} \sim 0.07\,\text{MeV}$. Using Friedmann equation for a radiation-dominated universe where electrons and positrons have annihilated ($g_* = 3.36$), we can get the time-temperature relation as (Kolb and Turner, 1990; Dodelson, 2003)

$$t = 132\,\text{sec}\left(\frac{0.1\text{MeV}}{T}\right)^2. \tag{4.1.15}$$

The number of neutrons on the onset of nucleosynthesis can be calculated using result (4.1.15) in the decay equation. This gives us

$$X_n(T_{nuc}) = X_n^f e^{-t/\tau_n} = 0.15e^{-[(132/886.7)(0.1/0.07)^2]} = 0.11. \tag{4.1.16}$$

We now consider the production of helium. After freeze-out, there occurred a brief period of neutron decay. Since the binding energy of helium is larger than that of deuterium (see fig. 4.1b) all the neutrons



(a) Light element abundances in the early universe

(b) Binding energy per nucleon

Figure 4.1: On the left (4.1a): Dashed curve results from integration of (4.1.11); light solid line is twice the neutron equilibrium abundance; heavy solid curves emerge from the exact solution code (Wagoner, 1973). Neutron fraction is in equilibrium until $\sim 1\,$MeV. The exact solution and equation (4.1.11) agree until neutron decay. On the right (4.1b): Among the light elements ^{4}He has the highest binding energy per nucleon. Nucleosynthesis stops at ^{4}He because of lack of tightly bound isotopes in the range $A = 5 - 8$ (Dodelson, 2003).

that survived the decay go into helium ^{4}He at $T \sim T_{nuc}$, but via deuterium. In fact, the exponential $e^{B/T}$ favours helium over deuterium (Dodelson, 2003). However, there is not enough energy allowing four-body reactions such as $\mathrm{p+n+p+n} \longrightarrow {}^4\mathrm{He}+\gamma$ to take place. Instead, the reactions of the sort $\mathrm{p+n} \longrightarrow \mathrm{D}+\gamma$; $\mathrm{D+p} \longrightarrow {}^3\mathrm{He}+\gamma$ and $\mathrm{D+D} \longrightarrow {}^4\mathrm{He}+\gamma$ take place first. Moreover, deuterium having the lowest binding energy of all nuclei, energetic photons prevent its formation until the temperature T_{nuc} is of the order $0.1\,$Mev and therefore the general big bang nucleosynthesis process is delayed. This is called deuterium bottleneck (Steigmann, 2006).

Now, the only baryons present are helium (helium nuclei) and hydrogen (protons). Since helium ^{4}He contains two protons and two neutrons, the number density of protons after helium production is $n_H = n_p - n_n$ and therefore the fractional mass of hydrogen (protons) is $X_H = (n_p - n_n)/(n_p + n_n)$. The mass fraction of helium is thus

$$Y = 1 - X_H = 1 - \frac{n_p - n_n}{n_p + n_n} = 2\left(\frac{n_n}{n_n + n_p}\right) = 2X_n = 2 \times 0.11 = 0.22. \qquad (4.1.17)$$

There is agreement between this rough estimate and the exact analytic fit to the primordial mass fraction of ^{4}He (Kolb and Turner, 1990),

$$Y = 0.2262 + 0.0135 \ln\left(10^{10}\eta_b\right). \qquad (4.1.18)$$

Thus both T_{nuc} and Y depend only logarithmically on the baryon fraction. Finally, we have seen the reaction which change deuterium into helium at T_{nuc}. The rate of this reaction is not however strong enough to deplete deuterium which eventually freezes at the fractional mass $\sim 10^{-5}$. The other light nuclei formed are ^{3}He $\sim 10^{-5}$, and ^{7}Li $\sim 10^{-14}$. Our cosmic plasma contains now photons, electrons, neutrinos, protons, helium nuclei and some other light nuclei.

4.2 Recombination

In the process of light element production in the early universe, the big bang nucleosynthesis as already discussed was followed by the process where the nuclei, or in other words the ionised atoms, captured free electrons to become neutral atoms subsequently forming the corresponding elements. This is recombination. The vast majority of the nuclei which capture free electrons are in fact the free protons that survived the formation of helium. This explains why recombination is dominated by hydrogen formation, of course there is also helium recombination which even takes place earlier than that of hydrogen because of its greater ionisation potential.

Since the ionisation potential of hydrogen is $13.6\,\text{eV}$ we would expect the formation of neutral hydrogen at the temperature $13.6\,\text{eV}$ but until $\sim 1\,\text{eV}$ photons remain coupled to electrons via Compton scattering and electrons to protons via Coulomb scattering (Dodelson, 2003). Recombination is thus delayed by the high photon-to-baryon ratio and becomes only significant at $T \sim 0.3\,\text{eV}$. Before we discuss the real recombination, let us show that at $T = 13.6\,\text{eV}$ all hydrogen is still ionised. We consider hydrogen formation in $e^- + \text{p} \longrightarrow \text{H} + \gamma$ for which (3.2.18) leads to

$$\frac{n_e n_p}{n_H} = \frac{n_e^0 n_p^0}{n_H^0} \tag{4.2.1}$$

which, by using results (3.2.13), leads to

$$\frac{n_e n_p}{n_H} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T}, \tag{4.2.2}$$

where we have assumed $m_p = m_H$ in the prefactor and $\epsilon_0 = m_e + m_p - m_H = 13.6\,\text{eV}$ in the exponential is the hydrogen ionisation energy. As we did for the proton-neutron case, we define the free electron fraction as

$$X_e \equiv \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H}, \tag{4.2.3}$$

where we have used the neutrality of the universe condition, $n_e = n_p$. We can use (4.2.2) and (3.2.17) to obtain

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \left\{ \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T} \right\}. \tag{4.2.4}$$

We can neglect the small number of helium atoms and assume $n_p + n_H = n_b = \eta_b\, n_\gamma \sim 10^{-9}\, T^3$ as it follows from (4.1.1). Here we have used the fact that the number density of massless particles $\simeq T^3$ from (3.2.14). At $T = \epsilon_0$ as we said earlier the recombination is not yet effective, we have instead

$$\frac{X_e^2}{1 - X_e} \sim 10^9 \left(\frac{m_e}{T}\right)^{3/2} \simeq 10^{15}, \tag{4.2.5}$$

which can only be satisfied if X_e is close to 1, meaning that all the electrons are still free or all hydrogen is ionised, what we wanted to show. The Saha equation tells us more than this! If we define recombination as the point where 90% of free electrons have found their nuclei to form neutral atoms or $X_e = 0.1$, we can solve (4.2.4) for T_{rec} which we then use in the relation $1 + z = T/T_0$ to estimate the redshift of recombination z_{rec}. This gives $T_{rec} = 3575\,\text{K} = 0.308\,\text{eV}$ and $z_{rec} = 1300$.

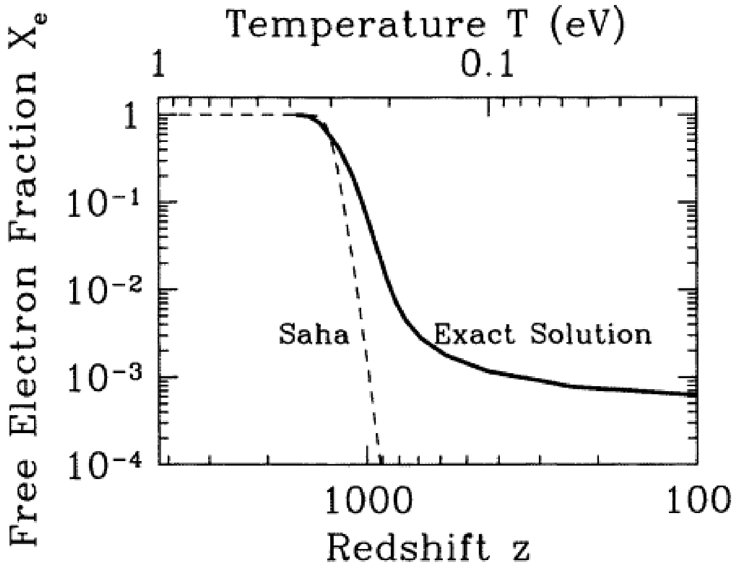


Figure 4.2: Free electron as a function of redshift (Dodelson, 2003). Recombination occurs at $z \sim 1000$ corresponding to $T \sim 0.25\,\mathrm{eV}$. The equilibrium Saha equation (4.2.4) identifies only the redshift of recombination. Here $\Omega_b = 0.06$, $\Omega_m = 1$ and $h = 1$.

In spite of this, we have to reject the equilibrium Saha equation for a while if we are to examine the out-of-equilibrium situation or what happens as X_e falls and instead solve (3.2.17) just as we did when we were calculating the neutron-proton ratio. The equation becomes

$$\frac{dn_e}{dt} + 3Hn_e = n_e^0 n_p^0 \langle \sigma v \rangle \left\{ \frac{n_H}{n_H^0} - \frac{n_e^2}{n_e^0 n_p^0} \right\} \tag{4.2.6}$$

which, on using equation (4.2.4), gives

$$n_b \frac{dX_e}{dt} + X_e \frac{dn_b}{dt} + 3Hn_b X_e = n_b \langle \sigma v \rangle \left\{ (1 - X_e) \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-\epsilon_0/T} - X_e^2 n_b \right\}. \tag{4.2.7}$$

Using that $n_b \propto a^{-3}$ and defining $x \equiv \epsilon_0/T$ as we did in the previous section, we can simplify our expression to

$$\frac{dX_e}{dx} = \frac{\langle \sigma v \rangle}{xH} \left\{ (1 - X_e) \left(\frac{m_e \epsilon_0}{2\pi x} \right)^{3/2} e^{-x} - X_e^2 n_b \right\}. \tag{4.2.8}$$

Now let us call H_1 the Hubble rate at $\epsilon_0 = T$. Taking the universe to be matter-dominated during recombination, we have $\rho \propto a^{-3} \propto T^3$, and therefore we can obtain $H = H_1 x^{-3/2}$. Moreover, the recombination rate and photoionisation rate are given by $\langle \sigma v \rangle$ and $\beta \equiv \langle \sigma v \rangle \left(m_e \epsilon_0 / 2\pi x \right)^{3/2} e^{-x}$ respectively. All these allow us to simplify further our equation (4.2.8) to

$$\frac{dX_e}{dx} = \frac{\sqrt{x}}{H_1} \left[(1 - X_e)\beta - n_b X_e^2 \langle \sigma v \rangle \right]. \tag{4.2.9}$$

The recombination rate producing hydrogen in any state n is a function of $E_n = \epsilon_0/n^2$ (Kolb and Turner, 1990). However, only the recombination to an excited state ($n > 1$) is what is relevant to producing neutral hydrogen. In fact, recombination to ground state ($n = 1$) produces ionising photons which destroy then neutral atoms, producing no net recombination. A good approximation for recombination rate to excited states ($n = 2, 3, \cdots$) gives (Dodelson, 2003)

$$\langle \sigma v \rangle = 9.78 \frac{\alpha^2}{m_e^2} \sqrt{x} \ln x. \tag{4.2.10}$$

Solving equation (4.2.9) though not analytically easy can be made by realizing that at low temperatures or at large x we can neglect the photoionisation term and then try to get the residual electron fraction which remains in the universe at $x = \infty$ after recombination by solving

$$\frac{dX_e}{dx} = -\lambda x \ln x X_e^2, \tag{4.2.11}$$

where $\lambda = 9.78 n_b \alpha^2 / m_e^2 H_1$. Once again we change the variable, we introduce the redshift $z = T/T_0 - 1$, which leads to

$$\frac{dX_e}{dz} = -\frac{\lambda \epsilon_0^2}{T_0^2} (z+1)^{-3} \ln \left(\frac{\epsilon_0}{T_0(z+1)} \right). \tag{4.2.12}$$

Evaluating this integral from $z = z_f$ to $z = 0$ and then considering recombination continues for a so long time after freeze-out that X_e^f being very much larger than X_e^∞ we can neglect $1/X_e^f$, we get the residual electron fraction X_e^∞ as

$$X_e^\infty = \frac{2}{\lambda} \frac{\epsilon_0^2 / T_0^2}{\ln(z_f + 1)}. \tag{4.2.13}$$

Before computing the numerical value of the residual ionisation, let us first analyse what happens at electron freeze-out. At temperatures higher than T_{rec}, the rate of electron-photon interaction was greater than the expansion rate, photons were coupled to matter and were so energetic that they ionized any hydrogen atom that formed. The mean free path of photon was very short $\sim 1/n_e \sigma_T$ and the universe was opaque. As the universe expanded further, the photons lost energy because of redshift and therefore, became gradually unable to ionize neutral atoms being formed. As a result, after electrons were now part of neutral atoms, photons stopped interacting with matter, they became free and the universe was now transparent. This is photon decoupling which occurred effectively when the scattering rate equalled the expansion rate. These photons which decoupled at that time are what form the cosmic microwave background seen today at the temperature of 2.75 K.

We can show that decoupling took place during recombination. Using the Thomson cross section $\sigma_T = 0.665 \times 10^{-24}$ cm^2, and using $m_p n_b / \rho_c = \Omega_b a^{-3}$, we can compute the rate of photon-electron scattering $n_e \sigma_T$ as

$$\Gamma(z) = n_e \sigma_T = 7.477 \times 10^{-29} \text{m}^{-1} X_e \Omega_b h^2 a^{-3} = 7.477 \times 10^{-29} \text{m}^{-1} X_e \Omega_b h^2 (1+z)^3 \tag{4.2.14}$$

and its ratio to the expansion rate as

$$\frac{n_e \sigma_T}{H} = 0.0692 (1+z)^3 X_e \Omega_b h \frac{H_0}{H}. \tag{4.2.15}$$

The ratio H_0/H in a matter-dominated flat universe is $\left(\Omega_m / a^3\right)^{1/2} = \left[\Omega_m (1+z)^3\right]^{-1/2}$. Let us now normalize $1 + z$ by 1000, the baryon density by 0.02 and the matter density by 0.15, with this we can write

$$\frac{n_e \sigma_T}{H} = 113 X_e \left(\frac{\Omega_b h^2}{0.02} \right) \left(\frac{0.15}{\Omega_m h^2} \right)^{0.5} \left(\frac{1+z}{1000} \right)^{1.5}. \tag{4.2.16}$$

We can see that when X_e goes beneath $\sim 10^{-2}$, H goes larger than $n_e\sigma_T$, photons decouple, implying that decoupling takes place during recombination since $X_e^\infty < 10^{-2}$. When released at decoupling with a temperature $T_{dec} \sim 3000\,\mathrm{K}$, the photons had more energy than today where they have $T_0 = 2.75\,\mathrm{K}$ and therefore were not in the microwave range. The background has preserved its equilibrium energy distribution, the black body spectrum

$$\epsilon(f)df = \frac{8\pi h}{c^3}\frac{f^3 df}{e^{h\nu/k_\mathrm{B}T} - 1}, \tag{4.2.17}$$

because of two reasons. First, the redshift effect in the frequency f is cancelled by that in the temperature T in the denominator since both scale as a^{-1} and appear only in f/T. Second, in the numerator f^3 scales as a^{-3} and on the left hand side, the energy density $\epsilon(f)$ scales as a^{-3}, resulting in no net effect. This is why the CMB radiation as seen today is still a black body spectrum with a peak in the microwave region.

We can now carry on with computing X_e^∞. We can use the ratio H^2/H_0^2 in the case of a matter-dominated flat universe to compute H_1. In fact, we can show that $H = \Omega_m^{1/2}a^{-3/2}H_0 = \Omega_m^{1/2}(1+z)^{3/2}H_0 = \Omega_m^{1/2}\left(T/T_0\right)^{3/2}H_0$. Therefore, to compute H_1 we just set T equal to ϵ_0 and get $H_1 = \Omega_m^{1/2}\left(\epsilon_0/T_0\right)^{3/2}H_0$. We use today's values of parameters $\Omega_m = 0.315$, $\Omega_b h^2 = 0.0221$, $h = 0.67$, the fine structure constant $\alpha = 7.3\times 10^{-3}$, $\epsilon_0 = 13.6\times 1.6\times 10^{-19}\,\mathrm{J}$, $T_0 = 2.43\times 1.6\times 10^{-19}\,\mathrm{J}$, $\rho_c = 1.88h^2\times 10^{-26}\,\mathrm{kg\,m^{-3}}$ and $H_0 = 100{,}000\,h\,\mathrm{m}\times\left(3.08\times 10^{22}\,\mathrm{m\,sec}\right)^{-1}$.

The analysis above has shown us that decoupling takes place when $X_e \sim 10^{-2}$ during recombination. We can use this to compute the scattering rate at decoupling and then we solve $H(z_f) = \Gamma(z_f)$ for z_f. Using (4.2.14) and the expression for $H(Z)$ above, we get

$$H_0(1+z_f)^{3/2}\,\Omega_m^{1/2} = 7.477\times 10^{-28}\mathrm{m}^{-1}X_e\Omega_b h^2\,(1+z_f)^3\,. \tag{4.2.18}$$

To give sense to this equality, we use speed of light c to change the unit on its right hand side of to sec because the unit of H on the left hand side turns out to be sec. This leads to

$$1.235\times 10^{-18}(1+z_f)^{3/2} = 4.4\times 10^{-21}X_e(z_f)(1+z_f)^3 \tag{4.2.19}$$

or

$$1+z_f = \frac{43}{\left(X_e(z_f)\right)^{2/3}}. \tag{4.2.20}$$

For $X_e(z_f) = 1/113 = 8.85\times 10^{-3}$, we get $1+z_f = 1005$. A better analysis using the visibility function shows however that photons decoupled at $1+z_f \sim 1100$. Finally, we use this value, together with the values for different constants written above, in (4.2.13) to compute roughly the residual ionisation, which gives $X_e^\infty \simeq 10^{-3}$.

4.3 Freeze-out of Dark Matter

As we said earlier, the most likely candidate in the search for dark matter particle is a weakly interacting particle (WIMP). We want to use the Boltzmann equation to show that the relic abundance of the WIMP is $\Omega_{dm} = 0.3$. We consider that before freeze-out, the equilibrium interaction consists of two

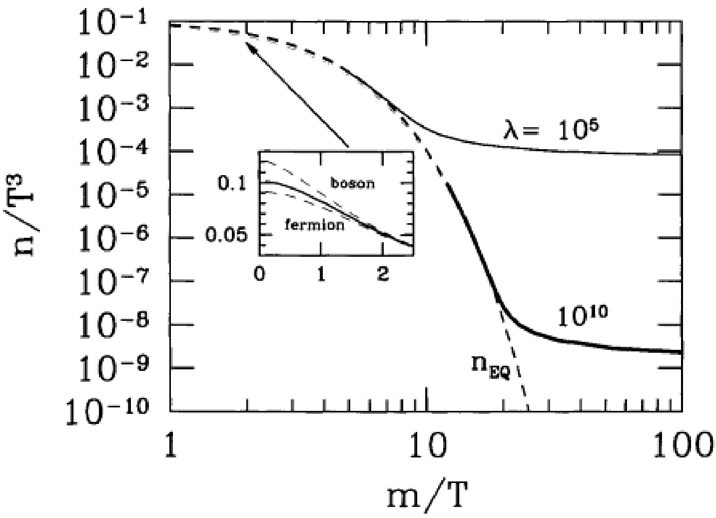


Figure 4.3: Abundance of the WIMP as T drops beneath its mass (Dodelson, 2003). Dashed curve is equilibrium abundance. Two solid lines indicate abundance for two different values of λ. Difference between Botzmann statistcs and quantum statistic is important at T larger than m.

heavy particles (two WIMPs) that annihilate to produce two massless leptons l. Let m and n_w be the mass and the number density of the WIMP. We can, as usual, write

$$\frac{dn_w}{dt} + 3Hn_w = \langle\sigma v\rangle \left\{(n_w^0)^2 - (n_w)^2\right\}. \tag{4.3.1}$$

The fact that temperature T scales as a^{-1} serves us again. We define $Y \equiv n_w/T^3$ and $Y_{eq} = n_w^0/T^3$ which we use to change equation (4.3.1) to

$$\frac{dY}{dt} = T^3\langle\sigma v\rangle \left\{(Y_{eq}^2 - Y^2\right\}. \tag{4.3.2}$$

We apply the same trick of changing the time variable, we define $x \equiv m/T$. Equilibrium requires high temperatures, $x \ll 1$ and the particles behave relativistically and can therefore be taken as massless, $m \ll T$. This implies $Y \simeq 1 \simeq Y_{eq}$. When the temperature falls beneath m or, in other words, for high x, the interactions are not efficient enough to maintain equilibrium; the particles will then freeze out. In radiation-dominated universe we have $H = H_1/x^2$, with $H_1 = H_{m=T}$. Working with the variable x rather than t allows us to write

$$\frac{dY}{dx} = -\frac{m^3}{Hx^4}\langle\sigma v\rangle \left\{(Y^2 - Y_{eq}^2\right\}. \tag{4.3.3}$$

Defining the ratio of the annihilation rate to the expansion rate when $m = T$ by $\lambda \equiv m^3\langle\sigma v\rangle/H_1$, we simplify further our equation to

$$\frac{dY}{dx} = -\frac{\lambda}{x^2} \left\{(Y^2 - Y_{eq}^2\right\}. \tag{4.3.4}$$

Although (4.3.4) has no analytical solution, we can still use it to determine the the value of Y after freeze-out. We start by realizing that when $m = T$ or $x = 1$, the left and the right hand sides of, (4.3.4) are of the orders Y and λY^2 respectively. Since λ is typically large, the right hand side must vanish as long as Y is not too small. This just means $Y = Y_{eq}$ at $x \leq 1$ in accordance to what we said earlier. However, as the universe expands, the temperature drops and Y_{eq} fall exponentially. Therefore, at a certain temperature, our WIMP will not be able to annihilate fast enough to maintain equilibrium, it will freeze out. At late times or when $x \gg 1$, we can write

$$\frac{dY}{dx} = -\frac{\lambda}{x^2} Y^2 \tag{4.3.5}$$

which, on integrating from the freeze-out time x_f to a very late time $x = \infty$, yields

$$\frac{1}{Y_\infty} - \frac{1}{Y_f} = \frac{\lambda}{x_f}. \tag{4.3.6}$$

Assuming that the annihilation of the WIMPs into leptons continues for a so long time after freeze-out that Y_f being much larger than Y_∞, we can neglect $1/Y_f$. This leads us to

$$Y_\infty = \frac{x_f}{\lambda} = \frac{H_1}{\langle \sigma v \rangle m^2 T_f}. \tag{4.3.7}$$

When Y has reached its constant value Y_∞, the number density at this sufficiently late time is $Y_\infty T_1^3$. since the particle number density scales as a^{-3}, the energy density today can be written as

$$\rho_w = m Y_\infty T_0^3 \left(\frac{a_1 T_1^3}{a_0 T_0^3} \right) = \frac{H_1}{\langle \sigma v \rangle m T_f} T_0^3 \left(\frac{a_1 T_1^3}{a_0 T_0^3} \right). \tag{4.3.8}$$

From our definition of entropy density as $s \equiv (\rho + P)/T$ in (3.1.5), we can infer that the product $(sa)^3$ remains constant, allowing us to write $(s_1 a_1)^3 = (s_0 a_0)^3$ or equivalently,

$$\left[g_* (aT)^3 \right]_{T=T_1} = \left[g_* (aT)^3 \right]_{T=T_0}. \tag{4.3.9}$$

At high temperatures, the entropy in the cosmic plasma comes from fermions: quarks, antiquarks, leptons and antileptons, and bosons: photons and gluons. Realizing the top quark does not contribute because it is too heavy, the effective number of degrees of freedom can be computed as $g_* = 91.5$ (Dodelson, 2003). Today, the entropy comes only from neutrinos and photons and therefore, $g_* = 3.36$. Using this assumption together with (3.1.5) and (3.1.6), we can show that $(a_1 T_1)^3/(a_0 T_0)^3 \simeq 1/30$. Using this result, we write

$$\rho_w = \frac{H_1 T_0^3}{30 \langle \sigma v \rangle m T_f}. \tag{4.3.10}$$

We recall that H_1 is the expansion rate when the temperature equaled the WIMP mass m, therefore $H_1 = \left(4\pi^3 G g_* / 45 \right)^{1/2}$. Then, the density parameter for the WIMP is

$$\Omega_w = \left(\frac{4\pi^3 G g_*}{45} \right)^{1/2} \frac{x_f T_0^3}{30 \langle \sigma v \rangle \rho_{cr}}. \tag{4.3.11}$$

Furthermore, the production of the WIMP takes place at a high temperature where all the particles of the standard model are relativistic so that g_* scales as 100. Using the values of the constants and conveniently normalizing x_f by 10, we get

$$\Omega_w = \frac{0.3}{h^2} \frac{x_f}{10} \left(\frac{g_*}{100} \right)^{1/2} \frac{10^{-39} \text{cm}^2}{\langle \sigma v \rangle}, \tag{4.3.12}$$

which is promising since the WIMP, being weakly interacting, should have a cross section of the order $10^{-39}\,\text{cm}^2$.

5. Summary

Cosmology is made relatively easy by the cosmological principle that there is no special place in the universe or more usefully, the universe is homogeneous and isotropic on large enough scales; the evidence is the isotropy of the $2.75\,\mathrm{K}$ cosmic microwave background. The universe being in expansion as observed or in other words, being homogeneous and isotropic only spatially (not temporally), the distance in spacetime is given by the Robertson-Walker metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right].$$

Let us notice that the equation above has nothing yet to do with general relativity! General relativity then comes in because we need to know the energy content of the universe so that we can understand the evolution of the function of time $a(t)$, the scale factor that appears in the Robertson-Walker metric. Indeed, the Einstein's field equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{5.0.1}$$

tell us that energy, represented by the energy-momentum tensor $T_{\mu\nu}$, present in the universe curves spacetime or in other words, makes the spatial components of the metric tensor $g_{\mu\nu}$ depend on time. Furthermore, modelling the universe energy/matter content as a fluid (Weyl's postulate), the form of the energy-momentum tensor becomes concrete and the Einstein equation together with the $g_{\mu\nu}$ in the Robertson-Walker metric yield the Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} \quad \text{and} \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

which, when the state equation $P = P(\rho)$ is specified, can be solved for $a(t)$. With $a(t)$ known, the dynamics of the universe is in one's hand, for example the Hubble's law of expansion says that galaxies are receding away from us with velocities v proportional to their distance r, i.e. $v = Hr$, where H is the Hubble's parameter or the universe expansion rate given by $H = \dot{a}/a$; the red shift z in the light caused by expansion is given by $1 + z = a_0/a$ and the temperature is related with the scale factor by $T = T_0/a$.

Applying the Boltzmann equation to the early universe has allowed us to understand the observed high abundances of light elements (hydrogen, helium, lithium) in the universe today. In fact, unlike the heavy elements which are formed in ageing stars, the light elements were formed in the early universe as it cooled with expansion. There occurred a process called the big bang nucleosynthesis in which nucleons fused into nuclei when the universe cooled to $\sim 0.1\,\mathrm{Mev}$, well below the typical binding energy of a light nucleus. It explains the observed high abundance of helium, $\sim 25\%$ of baryonic matter and that of hydrogen, $\sim 75\%$ of baryonic matter. By using the Boltzmann equation, we got the mass fraction of helium as $Y = 0.22$, in good agreement with observations $Y = 0.247710$ (Ade et al., 2013). It was followed by the recombination process where mainly free electrons joined free protons forming neutral hydrogen when the universe cooled to $\sim 0.3\,\mathrm{eV}$, well below the hydrogen ionisation energy $13.6\,\mathrm{eV}$. Consequently, the photons which missed electrons to interact with decoupled subsequently forming the cosmic microwave background radiation that pervades the universe. However, not every free electron found a nucleus to join, there is a residual fraction that remained, $\sim 10^{-3}$.

Hypothetically, a different type of matter, a nonbaryonic matter known as dark matter, froze out in early universe at the temperature $\sim 100\,\mathrm{GeV}$. Although the nature of this matter is not yet known, its presence seems today undeniable. Applying the Boltzmann equation to the most potential candidate, the weakly interacting massive particle (WIMP), reveals that the WIMP must have a cross section of the order 10^{-39}. According to particle physics, cross sections of this order emerge naturally, so there is hope that the dark matter puzzle will be solved.

The standard model does an excellent job accounting for most of the observations. However, another theory such as inflation is needed to solve the problems that are in the standard model of cosmology. In its essence, inflation is the exponential expansion that the universe experienced as early as 10^{-34} sec. This theory solves the flatness, horizon and missing relic problems that the pure standard model cannot solve. It also account for the formation of structures in the universe.

In a nutshell, it is true that the quest for a general understanding or even a basic understanding of cosmology is challenging partly if there has been no one's prior exposure to the subject matter of cosmology and partly if it has to be done within a certain fixed time but what I have learnt is that one can learn cosmology at least little by little. A bigger overview should tackle more than this present essay and possibly include among others a "bigger bit" of general relativity, neutrino cosmology, more on inflation, anisotropies in the CMB and structure formation and a better account for dark matter and dark energy.

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