Game Options

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Abstract

In this essay, we will consider Kifer's (2000) work where the pricing and hedging of the game options are obtained in two well-known complete financial market models: the Cox-Ross-Rubinstein's model in the discrete time framework, and the Black-Scholes model in continuous time. The fair price of a game option is the initial value of the value process of a Dynkin game obtained by computing its saddle point which is given by optimal stopping time for the seller and the buyer. Moreover, there exists a self-financing replicating portfolio for the value process. Game options reduce the high level of risk in the market for the investors.

Keywords: Dynkin game, optimal stopping time, Snell envelope, option pricing, game options.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Shimaa Yasser Ahmed Hanafy Elesaely, 22 May 2014

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1. Introduction

One of the most general cases of contingent claims is the Game Option, introduced in Kifer (2000). The Game Option is a contract between a seller (S) and a buyer (B) in which the buyer has the right, not obligation, to either buy (call option) or sell (put option) a specific security at any time until the expiration date T with determined strike price K and the seller has the right to terminate the contract at any time until the expiration date and in this case S has to pay a penalty to the buyer. Following Kifer (2000), Kunita and Seko (2004) studied the value process of a game put and call options and the exercise regions for S and B in complete market, when the penalty is fixed. Kallsen and Kühn (2004) applied the neutral pricing approach to game options in incomplete market. In complete market, Kifer showed that the fair price problem of a game option leads to the zero-sum Dynkin game, and in that case, Kifer proved the existence of the saddle point. In incomplete market, Kühn (2004) showed that there is a connection between the pricing of the game option and the non-zero-sum Dynkin game. In that case Kühn proved the existence of the Nash equilibrium point and he found that this point does not exist except when B and S have exponential utility. Kyprianou (2004) found an explicit formula for the price of perpetual game option in two examples: Russian option and American game put option. Eliasson (2012) found an algorithm to compute the price of the game option. Kifer (2013) studied the game option when the model includes transaction costs.

Unlike an American option, where only the buyer has the right to choose the exercise time, an American game option gives the seller the right to force it. However, in order to avoid immediate exercise/cancellation, it is required that the payoff be higher if the seller wants to cancel. That is why we will assume that the payoff when the seller terminates the contract is greater than the payoff when the buyer exercises the contract. If the seller does not force the exercise time, the game option becomes a standard American option. This shows that the price of an American Contingent Claim (ACC) is cheaper than the Game Contingent Claim (GCC). This is because S can terminate the contract at any time until the expiration date and this reduces the risk for S. Therefore, game options are safer for S from collapsing and they have some elements of game of chance which attract B. Moreover, game options bring diversity in the financial market.

When S or B terminates the contract, S must pay B the payoff of the contract. So B should pay money at the beginning to have such a contract. The problem is to find the fair price that B should pay to S for getting such a contract. The fair price must allow S to safeguard himself. The hedging for such option depends on the choice of the cancellation time and the hedging investment policy. So the fair price should be the lowest price for hedging strategy such that S would be able to choose the cancellation time and to manage a self-financing portfolio. The price of this portfolio at the end of the contract should cover the payment for B whenever B exercises the option until the cancellation time. The pricing of this option is based on the optimal stopping game (Dynkin Game). The first famous example of optimal stopping problem in finance, which involves only one stopping time, is the pricing of American option. The problem was first considered by Samuelson (1965) and solved by McKean (1965). Dynkin Game becomes more interesting since Cvitanić and Karatzas (1996) discovered that the value of the Dynkin Game can be represented as a solution of a reflected Backward Stochastic Differential Equation under some technical conditions.

Consider a game option between S and B. S wants to choose the cancellation time which minimises the payoff. On the other hand, B wants to choose exercise time which maximises the payoff. This lead to a minmax problem involving stopping times. The price of the game option is the solution of the minmax problem. To find the fair price of the game option, we will use the no arbitrage argument to construct a portfolio which replicates the value process of the Dynkin game under the risk neutral, then we will show that the hedging strategy exists and there are optimal stopping times for the seller and the buyer.

The main purpose of this essay is to expose the pricing of the game option. The work is organized as follows. We begin, in the second chapter, with some terminology needed in pricing and hedging game option and then show Kifer's Theorem about the existence and the computation of the fair price in the discrete time, where the dynamic of the stock is described by the binomial CRR model. In the third chapter, after a brief summary of Black-Scholes model, we present the principle results in pricing the game option in continuous time where the dynamic of the stock follows geometric Brownian motion. We end in the last chapter with the investigation of the relation between the value of the zero-sum Dynkin game and the price of the game option.

2. Game Option in Discrete Time

Pricing Game Option in discrete time involves the construction of a *self-financing portfolio strategy* and the choice of *optimal stopping time* for the seller to hedge the Game Option . The price will be given by the value process of the *Dynkin Game*. In this chapter, we will present the pricing and hedging of a game option in the *Cox-Ross-Rubinstein (CRR)* model.

We introduce first some materials that are needed throughout this chapter.

2.1 Portfolio Strategy

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_n\}_{n\in\mathbb{Z}^+}, \mathbb{P})$ be a filtered probability space. The stochastic process $X = (X_n)_{n\in\mathbb{Z}^+}$ is predictable if X_n is $\mathcal{F}_{n-1}$-measurable for all $n \in \mathbb{Z}^+$.

A portfolio strategy is a sequence $\pi = ((x_n, y_n))_{n=1}^N$ of portfolios due to modifications of an initial portfolio at different times, x_n and y_n are predictable stochastic processes. x_n represents the number of units on the savings account or bonds with price $B_n = (1+r)^n$ and y_n represents the number of units on the stocks with stock price S_n at time n. The price of the portfolio at time n is given by

$$\nu_n^\pi = x_n B_n + y_n S_n.$$

The portfolio at time 0 is (x_1, y_1) and the initial value of the portfolio is

$$\nu_0^{(x_1,y_1)} = x_1 B_0 + y_1 S_0.$$

Definition 2.1.1. *A portfolio strategy π is self-financing if*

$$x_n B_n + y_n S_n = x_{n+1} B_n + y_{n+1} S_n,$$

for $n = 1, 2, \ldots, N-1$.

This definition means that the change in the wealth of the strategy depends only on the market movement but not as a result of a withdrawal or infusion of funds.

Remark 2.1.2. Every portfolio strategy that replicates a European Option with payoff M_N and expiration date N is a self-financing portfolio strategy π whose value at time N is $\nu_N^\pi = M_N$.

Lemma 2.1.3. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider a CRR model. There is a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that the discounted value process $(\overline{\nu}_n^\pi)_{n=0}^N = \left(\frac{\nu_n^\pi}{B_n}\right)_{n=0}^N$ is a martingale.

2.2 Stopping Time

Definition 2.2.1 (Stopping time). *Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a filtered probability space. We say that $\tau : \Omega \to \mathbb{N}$ is a stopping time for the filtration $\mathbb{F}$ if for all $n \in \mathbb{N}$ the event $\{\omega : \tau(\omega) \le n\}$ is $\mathcal{F}_n$-measurable.*

$\{\tau \le n\} \in \mathcal{F}_n$ means that the decision whether or not to stop at time n is based on the available information up to time n.

Let $\mathcal{T}_{[n,N]}$ be a finite subclass of stopping times τ with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^N$ such that $\tau(\omega) \in \{n, n+1, \ldots, N\}$.

The stopping time $\sigma \in \mathcal{T}_{[0,N]}$ is said to be optimal for $X = (X_n)_{n=0}^N$, if

$$\mathbb{E}(X_\sigma) = \max_{\tau \in \mathcal{T}_{[0,N]}} \mathbb{E}(X_\tau).$$

Example 2.2.2. Consider a two time-step binary model has 4 possible states, i.e. $\Omega = \{uu, ud, du, dd\}$. A stock with price S_n at time $n = 0, 1, 2$ has the following prices:

$n = 0$

$$S_0 = 100,$$

$n = 1$

$$S_1 = \begin{cases} 110 & \text{if } uu \text{ or } ud \text{ prevails} \\ 90 & \text{if } du \text{ or } dd \text{ prevails,} \end{cases}$$

$n = 2$

$$S_2 = \begin{cases} 120 & \text{if } uu \text{ prevails} \\ 103 & \text{if } ud \text{ prevails} \\ 105 & \text{if } du \text{ prevails} \\ 80 & \text{if } dd \text{ prevails,} \end{cases}$$

with filtration

$$\mathcal{F}_0 = \{\phi, \Omega\},$$
$$\mathcal{F}_1 = \{\phi, \{uu, ud\}, \{du, dd\}, \Omega\},$$
$$\mathcal{F}_2 = \mathcal{P}(\Omega).$$

Let τ be the time when the stock price is less than or equal 90. Let σ be defined by

$$\sigma(\omega) = \begin{cases} 1, & \omega = dd \\ 2, & \omega \in \{uu, ud, du\}. \end{cases}$$

It is easy to check that $\{\tau \le 0\} = \emptyset \in \mathcal{F}_0$, $\{\tau \le 1\} = \{du, dd\} \in \mathcal{F}_1$ and $\{\tau \le 2\} = \{dd\} \in \mathcal{F}_2$. Therefore, τ is a stopping time. For σ, we have $\{\sigma \le 1\} = \{dd\} \notin \mathcal{F}_1$. Hence, σ is not a stopping time.

Proposition 2.2.3. If τ, σ are stopping times, then $\tau \wedge \sigma = \min(\tau, \sigma)$ and $\tau \vee \sigma = \max(\tau, \sigma)$ are stopping times as well.

Proof. Let τ, σ be stopping times. Then for all $n \ge 0$, we have $\{\tau \le n\}, \{\sigma \le n\} \in \mathcal{F}_n$. Therefore, $\{\tau \wedge \sigma \le n\} = \{\tau \le n\} \cap \{\sigma \le n\} \in \mathcal{F}_n$ and $\{\tau \vee \sigma \le n\} = \{\tau \le n\} \cup \{\sigma \le n\} \in \mathcal{F}_n$ since $\mathcal{F}_n$ is a $\sigma-$ algebra. □

Definition 2.2.4. *If $\tau : \Omega \to \mathbb{N}$ is a stopping time and X is an $\mathbb{F}$-adapted process, the stopped process $(X_n^\tau)_{n\in\mathbb{N}}$ is defined by the formula*

$$X_n^\tau(\omega) = X(n \wedge \tau(\omega), \omega).$$

2.2.5 Doob's Stopping-Time Principle (STP).

Theorem 2.2.6. *Let τ be a stopping time and $X = (X_n)_{n=0}^N$ be a martingale. Then $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$.*

Proof. Let $\tau \in \mathcal{T}_{[0,N]}$ be a stopping time. We have

$$X_\tau = \sum_{n=0}^N X_n \mathbf{1}_{\{\tau=n\}}.$$

Therefore

$$\begin{aligned}
\mathbb{E}\left(X_\tau\right) &= \mathbb{E}\left(\sum_{n=0}^N X_n \mathbf{1}_{\{\tau=n\}}\right)\\
&= \sum_{n=0}^N \mathbb{E}\left(X_n \mathbf{1}_{\{\tau=n\}}\right) \quad \text{since the expectation is a linear operator}\\
&= \sum_{n=0}^N \mathbb{E}\left(\mathbb{E}(X_N|\mathcal{F}_n)\mathbf{1}_{\{\tau=n\}}|\mathcal{F}_0\right) \quad \text{since } X \text{ is a martingale}\\
&= \sum_{n=0}^N \mathbb{E}\left(X_N \mathbf{1}_{\{\tau=n\}}|\mathcal{F}_0\right) \qquad \text{tower property of conditional expectation}\\
&= \mathbb{E}\left(X_N \sum_{n=0}^N \mathbf{1}_{\{\tau=n\}}|\mathcal{F}_0\right)\\
&= \mathbb{E}(X_N|\mathcal{F}_0) = \mathbb{E}(X_0).
\end{aligned}$$

□

Theorem 2.2.7. *(Doob Decomposition). (Capiński et al., 2012). If $(X_n)_n$ is a supermartingale with respect to the filtration $\mathbb{F}$, then there exists for the same filtration, a martingale M_n and a predictable decreasing process A_n, i.e. $A_{n+1} \le A_n$ for all n with $M_0 = A_0 = 0$, such that*

$$X_n = X_0 + M_n + A_n.$$

This decomposition is unique: this is called the Doob decomposition of X.

2.3 The Snell Envelope

The Snell Envelope is the smallest supermartingale dominating a stochastic process. It is used in pricing American options, as well as game options.

Definition 2.3.1. *Let $X = (X_n)_{n=0}^N$ be an $\mathbb{F}$-adapted stochastic process. The process $Z = (Z_n)_{n=0}^N$ is the Snell envelope of X, if the following conditions are satisfied:*

1. *Z is a supermartingale with respect to the filtration $\mathbb{F}$,*
2. *Z dominates X, i.e. $Z_n \geq X_n$ for all $n \in \{0, 1, \dots, N\}$,*
3. *if $\widehat{Z}$ is a supermartingle dominating X, then we have $\widehat{Z}_n \geq Z_n$ for all $n \in \{0, 1, \dots, N\}$.*

Lemma 2.3.2. If $X = (X_n)_{n=0}^N$ is an $\mathbb{F}$-adapted stochastic process. Then the process $Z = (Z_n)_{n=0}^N$ defined by

$$\begin{cases} Z_N = X_N, \\ Z_n = \max(X_n, \mathbb{E}(Z_{n+1}|\mathcal{F}_n)), \quad n \in \{1, 2, \dots, N-1\} \end{cases} \tag{2.3.1}$$

is the Snell envelope of the payoff process X.

Proof. From the formula (2.3.1), we have

$$Z_n \geq \mathbb{E}(Z_{n+1}|\mathcal{F}_n) \text{ and } Z_n \geq X_n$$

for all n, i.e. Z is an $\mathbb{F}$-supermartingale and Z dominates X. The next step is to show that the condition (3) in Definition (2.3.1) is satisfied. Let $\widehat{Z}$ be another supermartingale dominating X. We now proceed by backward induction. at time N, we have $Z_N = X_N$ and $\widehat{Z}_N \geq X_N$. So the condition (3) holds and we have $\widehat{Z}_n \geq Z_n = X_n$. Suppose that $\widehat{Z}_n \geq Z_n$ for some $n \in \{1, 2, \dots, N\}$. Since the conditional expectation is a monotonic operator and $\widehat{Z}$ is a supermartingale dominating X, we have:

$$\widehat{Z}_{n-1} \geq \mathbb{E}(\widehat{Z}_n|\mathcal{F}_{n-1}) \geq \mathbb{E}(Z_n|\mathcal{F}_{n-1}) \text{ and } \widehat{Z}_{n-1} \geq X_{n-1}.$$

Therefore

$$\widehat{Z}_{n-1} \geq \max(X_{n-1}, \mathbb{E}(Z_n|\mathcal{F}_{n-1})) = Z_{n-1}.$$

Hence Z is the smallest supermartingale dominating X, i.e. Z is the Snell envelope of X. □

Let $Z = (Z_n)_{n=0}^N$ be the Snell envelope of the process $X = (X_n)_{n=0}^N$. The stopping time $\tau^* = \min\{n = 0, \dots, N : Z_n = X_n\}$ is optimal for X, this means that the first time the Snell envelop Z hits the process X is τ^*, we have

$$X_0 = \mathbb{E}(X_{\tau^*}) = \max_{\sigma \in \mathcal{T}_{[0,N]}} \mathbb{E}(X_\sigma).$$

2.4 Game Contingent Claim

Definition 2.4.1. *A Game Contingent Claim (GCC) in a discrete time is a derivative contract between two investors, the seller S and the buyer B, such that*

- *the expiration date N is finite.*
- *S chooses a cancellation time $\tau_s \in \mathcal{T}_{[0,N]}$ and B chooses an exercise time $\tau_b \in \mathcal{T}_{[0,N]}$.*
- *S pledges to pay to B the payoff at some time $n = 0, 1, \dots, N$ when the derivative is ended. The payoff in the event $\{\tau_b = n\} \cup \{\tau_s = n\}$ is given by*

$$R(\tau_s, \tau_b) = U_n \mathbf{1}_{\{\tau_s = n < \tau_b\}} + L_n \mathbf{1}_{\{\tau_b = n \leq \tau_s\}} = \begin{cases} U_n, & \tau_s < \tau_b \\ L_n, & \tau_s \geq \tau_b \end{cases},$$

where U_n (upper payoff) and L_n (lower payoff) are $\mathcal{F}_n$-adapted, finite and non-negative processes and verify $L_n \leq U_n$.

Remark 2.4.2. The process U_n is called the cancellation payoff while the process L_n is called the exercise payoff. Moreover, at time $n = 0$, B should pay S certain amount of money which is *the price of the contract* to motivate S to do the contract. At time $\tau_s \wedge \tau_b$ (*the end of the contract*), S is obligated to pay B the payoff $R(\tau_s, \tau_b)$.

Remark 2.4.3. Since $U_n \geq L_n$, the process $(\delta_n)_{n=0}^N$ which is defined by $\delta_n = U_n - L_n \geq 0$, is the penalty that S pledges to pay to B for terminating the contract. So we can write

$$R(\tau_s, \tau_b) = L_{\tau_s \wedge \tau_b} + \delta_{\tau_s} \mathbf{1}_{\{\tau_s < \tau_b\}}.$$

If terminating the contract at time $n = 0, 1, \ldots, N-1$ is not optimal for S, then we obtain ACC from the GCC in the discrete time framework. This is the case when δ is big enough, i.e. when

$$\delta > \max_{\tau_b \in \{0,1,\ldots,N\}} \mathbb{E}(L_{\tau_b}),$$

and the price V^* of GCC is

$$\max_{\tau_b \in \{0,1,\ldots,N\}} \mathbb{E}\left(\frac{L_{\tau_b}}{B_{\tau_b}}\right),$$

where $B_{\tau_b} = (1+r)^{\tau_b}$.

On the other hand, we can obtain ECC from the GCC in the discrete time framework when it is not suitable for B to exercise before the expiration date N. This occurs when

$$L_n = \begin{cases} 0 & \text{if } n \in \{0, 1, \ldots, N-1\}, \\ L_N > 0 & \text{if } n = N. \end{cases}$$

If $\delta_0 = 0$, then it is optimal for either S or B to terminate the contract at time $n = 0$ and the price $V^* = L_0$. V^* is an increasing function of the penalty δ, with

$$L_0 \leq V^* \leq \max_{\tau_b \in \{0,1,\ldots,N\}} \mathbb{E}\left(\frac{L_{\tau_b}}{B_{\tau_b}}\right).$$

This shows that the contract price for the game option is cheaper than the price of the usual American option since the GCC is safer for the seller. The European option is the cheapest with respect to GCC and ACC.

Definition 2.4.4. *A pair (τ_s, π) which consists of a stopping time $\tau_s \in \mathcal{T}_{[0,N]}$ and a self-financing portfolio strategy π is called a hedge against a GCC with expiration date N if*

$$0 \leq \nu^{\pi}_{\tau_s \wedge n} - R(\tau_s, n),$$

for all $n = 0, 1, \ldots, N$. This means that a hedge against a GCC is a portfolio and a stopping time that give the seller a positive profit.

Definition 2.4.5. *The fair price V^* of the GCC is given by the infimum of $V \geq 0$ such that there exists a hedge (τ_s, π) against this GCC with the initial capital $\nu_0^{\pi} = V$.*

We give an example how to compute the stopping times and the payoffs for the buyer in a game put option and how to hedge a game put option.

Example 2.4.6. Consider a game put option in CRR model described in (2.2.2), with strike price K=115 and $B_0 = 100$, with interest rate $r = 0.1$, where $B_n = (1 + r)^n B_0$. If the seller terminates the contract at time $n = 0$ or $n = 1$, then the seller will pay a penalty to the buyer $\delta_0 = 5$ or $\delta_1 = 8$. Let $\pi = ((x_n, y_n))_{n=1}^2$ be a portfolio strategy with

$$\begin{aligned} (x_1, y_1) &= (90, 30), \\ (x_2(\omega), y_2(\omega)) &= (80, 40), \qquad \text{for } \omega \in \{uu, ud\}, \\ (x_2(\omega), y_2(\omega)) &= \left(\frac{900}{11}, 40\right), \quad \text{for } \omega \in \{du, dd\}. \end{aligned}$$

First we want to show that π is self-financing by checking (2.1.1). Since

$$\left.\begin{aligned} x_1B_1 + y_1S_1(\omega) &= 90(110) + 30(110) = 13200 \\ x_2(\omega)B_1 + y_2(\omega)S_1(\omega) &= 80(110) + 40(110) = 13200, \end{aligned}\right. \quad \text{for } \omega \in \{uu, ud\}$$

and

$$\left.\begin{aligned} x_1B_1 + y_1S_1(\omega) &= 90(110) + 30(90) = 12600 \\ x_2(\omega)B_1 + y_2(\omega)S_1(\omega) &= \tfrac{900}{11}(110) + 40(90) = 12600, \end{aligned}\right. \quad \text{for } \omega \in \{du, dd\}$$

π is self-financing portfolio strategy.

These are the prices of the portfolio in all different scenarios according to the end of the contract $\tau_s \wedge \tau_b$:

$$\tau_s \wedge \tau_b = 0, \quad \nu_0^\pi = x_1B_0 + y_1S_0 = 90(100) + 30(100) = 12000.$$

$$\begin{aligned} \tau_s \wedge \tau_b = 1, \quad & \nu_1^\pi(\omega) = x_1B_1 + y_1S_1(\omega) = 90(110) + 30(110) = 13200, \quad \text{for } \omega \in \{uu, ud\}. \\ & \nu_1^\pi(\omega) = x_1B_1 + y_1S_1(\omega) = 90(110) + 30(90) = 12600, \quad \text{for } \omega \in \{du, dd\}. \end{aligned}$$

$$\begin{aligned} \tau_s \wedge \tau_b = 2, \quad & \nu_2^\pi(uu) = x_2(uu)B_2 + y_2(uu)S_2(uu) = 80(121) + 40(120) = 14480. \\ & \nu_2^\pi(ud) = x_2(ud)B_2 + y_2(ud)S_2(ud) = 80(121) + 40(103) = 13800. \\ & \nu_2^\pi(du) = x_2(du)B_2 + y_2(du)S_2(du) = \tfrac{900}{11}(121) + 40(105) = 14100. \\ & \nu_2^\pi(dd) = x_2(dd)B_2 + y_2(dd)S_2(dd) = \tfrac{900}{11}(121) + 40(80) = 13100. \end{aligned}$$

We will calculate the payoffs in all scenarios for all possible stopping times. There are five stopping times in this model. Three of them are constant stopping times $\gamma_1(\omega) = 0, \gamma_2(\omega) = 1, \gamma_3(\omega) = 2$ for all $\omega \in \Omega$ and two non-constant stopping times γ_4 and γ_5 given by

$$\gamma_4(\omega) = \begin{cases} 1, & \omega \in \{uu, ud\} \\ 2, & \omega \in \{du, dd\}, \end{cases}$$

$$\gamma_5(\omega) = \begin{cases} 2, & \omega \in \{uu, ud\} \\ 1, & \omega \in \{du, dd\}. \end{cases}$$

If the seller chooses $\tau_s = \gamma_1$ as stopping time

$$R(\gamma_1, \gamma_1) = L_{\gamma_1} = (K - S_0)^+ = (115 - 100)^+ = 15,$$

and we have

$$R(\gamma_1, \gamma_i) = U_{\gamma_1} = (K - S_0)^+ + \delta_0 = (115 - 100)^+ + 5 = 20, \quad \text{for } i = 2, 3, 4, 5.$$

If the seller chooses $\tau_s = \gamma_2$ as stopping time. Then according to the buyer choice of stopping time τ_b, we have:

- $\tau_b = \gamma_2$

$$R(\gamma_2,\gamma_2)(\omega) = (K - S_1(\omega))^+ = (115-110)^+ = 5, \quad \text{for } \omega \in \{uu, ud\}.$$
$$R(\gamma_2,\gamma_2)(\omega) = (K - S_1(\omega))^+ = (115-90)^+ = 25, \quad \text{for } \omega \in \{du, dd\}.$$

- $\tau_b = \gamma_3$

$$R(\gamma_2,\gamma_3)(\omega) = (K - S_1(\omega))^+ + \delta_1 = (115-110)^+ + 8 = 13, \quad \text{for } \omega \in \{uu, ud\}.$$
$$R(\gamma_2,\gamma_3)(\omega) = (K - S_1(\omega))^+ + \delta_1 = (115-90)^+ + 8 = 33, \quad \text{for } \omega \in \{du, dd\}.$$

- $\tau_b = \gamma_4$

$$R(\gamma_2,\gamma_4)(\omega) = (K - S_1(\omega))^+ = (115-110)^+ = 5, \quad \text{for } \omega \in \{uu, ud\}.$$
$$R(\gamma_2,\gamma_4)(\omega) = (K - S_1(\omega))^+ + \delta_1 = (115-90)^+ + 8 = 33, \quad \text{for } \omega \in \{du, dd\}.$$

- $\tau_b = \gamma_5$

$$R(\gamma_2,\gamma_5)(\omega) = (K - S_1(\omega))^+ + \delta_1 = (115-110)^+ + 8 = 13, \quad \text{for } \omega \in \{uu, ud\}.$$
$$R(\gamma_2,\gamma_5)(\omega) = (K - S_1(\omega))^+ = (115-90)^+ = 25, \quad \text{for } \omega \in \{du, dd\}.$$

If the seller chooses $\tau_s = \gamma_3$ as stopping time, then if the buyer chooses

- $\tau_b = \gamma_3$

$$R(\gamma_3,\gamma_3)(uu) = (K - S_2(uu))^+ = (115-120)^+ = 0.$$
$$R(\gamma_3,\gamma_3)(ud) = (K - S_2(ud))^+ = (115-103)^+ = 12.$$
$$R(\gamma_3,\gamma_3)(du) = (K - S_2(du))^+ = (115-105)^+ = 10.$$
$$R(\gamma_3,\gamma_3)(dd) = (K - S_2(dd))^+ = (115-80)^+ = 35.$$

- $\tau_b = \gamma_4$

$$R(\gamma_3,\gamma_4)(\omega) = (K - S_1(\omega))^+ = (115-110)^+ = 5, \quad \text{for } \omega \in \{uu, ud\}.$$
$$R(\gamma_3,\gamma_4)(du) = (K - S_2(du))^+ = (115-105)^+ = 10.$$
$$R(\gamma_3,\gamma_4)(dd) = (K - S_2(dd))^+ = (115-80)^+ = 35.$$

- $\tau_b = \gamma_5$

$$R(\gamma_3,\gamma_5)(uu) = (K - S_2(uu))^+ = (115-120)^+ = 0.$$
$$R(\gamma_3,\gamma_5)(ud) = (K - S_2(ud))^+ = (115-103)^+ = 12.$$
$$R(\gamma_3,\gamma_5)(\omega) = (K - S_1(\omega))^+ = (115-90)^+ = 25, \quad \text{for } \omega \in \{du, dd\}.$$

If the seller chooses $\tau_s = \gamma_4$ as stopping time and

- if the buyer chooses $\tau_b = \gamma_4$, then

$$R(\gamma_4,\gamma_4)(\omega) = (K - S_1(\omega))^+ = (115-110)^+ = 5, \quad \text{for } \omega \in \{uu, ud\}.$$
$$R(\gamma_4,\gamma_4)(du) = (K - S_2(du))^+ = (115-105)^+ = 10.$$
$$R(\gamma_4,\gamma_4)(dd) = (K - S_2(dd))^+ = (115-80)^+ = 35.$$

- if the buyer chooses $\tau_b = \gamma_5$, then

$$R(\gamma_4,\gamma_5)(\omega) = (K - S_1(\omega))^+ + \delta_1 = (115-110)^+ + 8 = 13, \quad \text{for } \omega \in \{uu, ud\}.$$
$$R(\gamma_4,\gamma_5)(\omega) = (K - S_1(\omega))^+ = (115-90)^+ = 25, \quad \text{for } \omega \in \{du, dd\}.$$

If the seller chooses $\tau_s = \gamma_5$ as stopping time and the buyer chooses $\tau_b = \gamma_5$. We have

$$R(\gamma_5,\gamma_5)(uu) = (K - S_2(uu))^+ = (115-120)^+ = 0.$$
$$R(\gamma_5,\gamma_5)(ud) = (K - S_2(ud))^+ = (115-103)^+ = 12.$$
$$R(\gamma_5,\gamma_5)(\omega) = (K - S_1(\omega))^+ = (115-90)^+ = 25, \quad \text{for } \omega \in \{du, dd\}.$$

Obviously, the value process of the portfolio $\nu^{\pi}_{\tau_s \wedge \tau_b}$ is greater than the payoff $R(\tau_s, \tau_b)$ in all scenarios. Therefore the pairs (γ_i, π) for all $i = 1, 2, 3, 4, 5$ represent a hedge against a game put option.

2.5 Valuation and Hedging of The GCC in a CRR Model

In this section, we consider the CRR model introduced by Cox, Ross, and Rubinstein (1979). Since the CRR model is arbitrage-free and complete finite market, there exists a unique risk neutral measure $\mathbb{Q}$. Let us observe the market at time $n = 0, 1, \ldots, N$, where N is the expiration date of the contract. The CRR model consists of a risk-less asset B_n with an interest rate r and positive initial price B_0, such that

$$B_n = (1+r)^n B_0,$$

and a risky asset S_n whose price at time n is

$$S_n = S_0 \prod_{j=1}^{n} (1 + g_j),$$

where g_j, $j = 1, \ldots, N$ is a random growth rate which form a sequence of i.i.d. random variables on the probability space $(\Omega, \mathbb{P})$ such that

$$g_j = \begin{cases} u & \text{with probability } p, \\ d & \text{with probability } 1-p, \end{cases} \qquad \text{where } -1 < d < r < u \text{ and } 0 < p < 1.$$

Let suppose that S and B are two investors, S must assume that B has found the stopping time which gives him the maximum payoff and S must find the stopping time which gives him the minimum payout. Conversely B must assume that S has found the stopping time which gives him the minimum payout and B must find the stopping time which gives him the maximum payoff. These strategies lead to the concept of upper and lower value processes.

Definition 2.5.1. *Let $\mathbb{E}_{\mathbb{Q}}$ be the expectation with respect to the risk neutral measure $\mathbb{Q}$. The discounted upper value process $\overline{V}^u_n$ of a GCC is an $\mathbb{F}$-adapted process given by the formula*

$$\overline{V}^u_n = \min_{\tau_s \in \mathcal{T}_{[n,N]}} \max_{\tau_b \in \mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{Q}} \left((1+r)^{-\tau_s \wedge \tau_b} R(\tau_s, \tau_b) | \mathcal{F}_n \right).$$

The discounted lower value process $\overline{V}^l_n$ of a GCC is an $\mathbb{F}$-adapted process given by the formula

$$\overline{V}^l_n = \max_{\tau_b \in \mathcal{T}_{[n,N]}} \min_{\tau_s \in \mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{Q}} \left((1+r)^{-\tau_s \wedge \tau_b} R(\tau_s, \tau_b) | \mathcal{F}_n \right).$$

The following theorem, due to Kifer (2000), is the pricing of a GCC in a CRR model.

Theorem 2.5.2. *(Kifer, 2000). Let $\mathbb{Q} = \{\tilde{p}, 1-\tilde{p}\}^N$ be the probability on the space Ω with $\tilde{p} = \frac{r-d}{u-d}$, N is finite and $\mathbb{E}_{\mathbb{Q}}$ is the expectation with respect to the measure $\mathbb{Q}$. Then the fair price $V^* = \overline{V}_0 = V_0$ of the GCC in Definition (2.4.1) can be derived recursively from the relation*

$$\begin{cases} V_N = L_N, \\ V_n = \min\left(U_n, \max\left(L_n, \mathbb{E}_{\mathbb{Q}}\left((1+r)^{-1} V_{n+1} | \mathcal{F}_n\right)\right)\right), \quad \text{for } n = 0, 1, \ldots, N-1. \end{cases}$$

Furthermore, for $n = 0, 1, \ldots, N$, the discounted value process of a GCC is also given by

$$\overline{V}_n = \overline{V}^l_n = \overline{V}^u_n. \tag{2.5.1}$$

Moreover, for $n = 0, 1, \ldots, N$,

$$\begin{aligned} \tau_{sn}^* &= \min \left\{ i \in \{n, n+1, \ldots, N\} \,|U_i = V_i \text{ or } i = N \right\}, \\ \tau_{bn}^* &= \min \left\{ i \in \{n, n+1, \ldots, N\} \,|L_i = V_i \right\}, \end{aligned}$$

where $\tau_{sn}^*, \tau_{bn}^* \in \mathcal{T}_{[n,N]}$ and they satisfy

$$\mathbb{E}_{\mathbb{Q}}\left((1+r)^{-\tau_{sn}^* \wedge \tau_b} R(\tau_{sn}^*, \tau_b) | \mathcal{F}_n \right) \leq \overline{V}_n \leq \mathbb{E}_{\mathbb{Q}}\left((1+r)^{-\tau_s \wedge \tau_{bn}^*} R(\tau_s, \tau_{bn}^*) | \mathcal{F}_n \right), \quad \text{for all } \tau_s, \tau_b \in \mathcal{T}_{[n,N]}.$$

Finally, there exists a self-financing portfolio strategy π^* such that (τ_{s0}^*, π^*) is a hedge against this GCC with $\nu_0^{\pi^*} = \overline{V}_0$ and this strategy is unique up to the time $\tau_{s0} \wedge \tau_{b0}$.

Proof. Let $\pi = ((x_n, y_n))_{n=1}^N$, where (x_n, y_n) is a self-financing portfolio strategy with $\nu_0^\pi = l > 0$. Then the discounted value process is $\overline{\nu}_n^\pi = (1+r)^{-n} \nu_n^\pi$, for all $n = 0, 1, \ldots, N$. By Lemma (2.1.3) $\overline{\nu}_n^\pi$ is a martingale with respect $\mathbb{Q}$. Assume that (τ_s, π) is a hedge of the GCC and $\tau_s \wedge \tau_b$ is a stopping time, then by Doob's Stopping-time Principle Theorem (2.2.6) and the Definition (2.4.4), we have

$$\nu_0^\pi = \mathbb{E}_{\mathbb{Q}}\left((1+r)^{-\tau_s \wedge \tau_b} \nu^\pi_{\tau_s \wedge \tau_b} \right) \geq \mathbb{E}_{\mathbb{Q}}\left((1+r)^{-\tau_s \wedge \tau_b} R(\tau_s, \tau_b) \right).$$

for all $\tau_b \in \mathcal{T}_{[n,N]}$. Therefore

$$\nu_0^\pi \geq \max_{\tau_b \in \mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{Q}}\left((1+r)^{-\tau_s \wedge \tau_b} R(\tau_s, \tau_b) \right).$$

Since by Definition (2.4.5), V^* is the infimum of the ν_0^π, it follows that

$$V^* \geq \min_{\tau_s \in \mathcal{T}_{[n,N]}} \max_{\tau_b \in \mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{Q}}\left((1+r)^{-\tau_s \wedge \tau_b} R(\tau_s, \tau_b) \right). \tag{2.5.2}$$

Conversely, for any $\tau_s \in \mathcal{T}_{[n,N]}$ set

$$\begin{cases} V_N^{\tau_s} = \mathcal{U}_N^{\tau_s}, \\ V_n^{\tau_s} = \max\left(\mathcal{U}_n^{\tau_s}, \mathbb{E}\left(V_{n+1}^{\tau_s}|\mathcal{F}_n\right)\right), \quad n \in \{0,1,2,\ldots,N-1\}, \end{cases}$$

where $\mathcal{U}_n^{\tau_s} = (1+r)^{-\tau_s \wedge n} R(\tau_s, n), n = 0,1,\ldots,N$. Obviously $(V_n^{\tau_s})_{n=0}^N$ is the Snell envelope of $(\mathcal{U}_n^{\tau_s})_{n=0}^N$ by Lemma (2.3.2). Then $V_n^{\tau_s}$ is the smallest supermartingale dominating $\mathcal{U}_n^{\tau_s}$. Hence, by Doob decomposition Theorem (2.2.7), we can write $V_n^{\tau_s}$ as follow

$$V_n^{\tau_s} = M_n - A_n, \qquad \text{for all } n = 0,1,\ldots,N, \text{ with } A_0 = 0, A_n \le A_{n+1}, \tag{2.5.3}$$

where M_n is a martingale and A_n is predictable process. Take $\tau_s \in \mathcal{T}_{[n,N]}$ and construct a self-financing portfolio strategy π^{τ_s} as follows. Let π^{τ_s} be a replicating strategy for a European option, such that its value at the expiration date is equal to $(1+r)^N M_N$, i.e. $\nu_N^{\pi^{\tau_s}} = (1+r)^N M_N$ (Remark (2.1.2)).

$$\overline{\nu}_N^{\pi^{\tau_s}} = \frac{\nu_N^{\pi^{\tau_s}}}{(1+r)^N} = M_N \Rightarrow \mathbb{E}_{\mathbb{Q}}\left(\overline{\nu}_N^{\pi^{\tau_s}}|\mathcal{F}_n\right) = \mathbb{E}_{\mathbb{Q}}\left(M_N|\mathcal{F}_n\right).$$

Since $(M_n)_{n=0}^N$ and $(\overline{\nu}_n^{\pi^{\tau_s}})_{n=0}^N$ are martingale, $\overline{\nu}_n^{\pi^{\tau_s}} = M_n$, for all $n \in \{0,1,2,\ldots,N\}$.

From equation (2.5.3), we have

$$\overline{\nu}_n^{\pi^{\tau_s}} = M_n \ge M_n - A_n = V_n^{\tau_s};$$

i.e.

$$\overline{\nu}_n^{\pi^{\tau_s}} \ge V_n^{\tau_s} \ge \mathcal{U}_n^{\tau_s}.$$

Therefore $\nu_n^{\pi^{\tau_s}} \ge R(\tau_s, n)$ and so (τ_s, π^{τ_s}) is a hedge of a GCC. Thus $\overline{\nu}_0^{\pi^{\tau_s}} = V_0^{\tau_s} = M_0$, for all $\tau_s \in \mathcal{T}_{[0,N]}$.

By Definition (2.4.5), we have $V_0^{\tau_s} = \max_{\tau_b \in \mathcal{T}_{[0,N]}} \mathbb{E}_{\mathbb{Q}}((1+r)^{-\tau_s \wedge \tau_b} R(\tau_s, \tau_b)$. It follows that

$$V^* \le \min_{\tau_s \in \mathcal{T}_{[0,N]}} \max_{\tau_b \in \mathcal{T}_{[0,N]}} \mathbb{E}_{\mathbb{Q}}((1+r)^{-\tau_s \wedge \tau_b} R(\tau_s, \tau_b)) \tag{2.5.4}$$

From Equations (2.5.2) and (2.5.4), the equality in equation (2.5.1) holds for $n = 0$.

The fair price V_N of the GCC in a CRR model is obtained by the relation $V_N = L_N$. The contract is ended at the expiration date so the payoff at this time does not include any penality, which is L_N. The seller must have at least the payoff L_n which is due to the buyer, if the buyer exercises. On the other hand, the portfolio strategy must enable the seller to end up with a positive wealth after paying the other obligation toward the buyer. Then at time $n+1$ the seller must have at least V_{n+1}, this costs

$$\mathbb{E}_{\mathbb{Q}}\left((1+r)^{-1} V_{n+1}|\mathcal{F}_n\right)$$

at time n. So the seller must have

$$\max\left(L_n, \mathbb{E}_{\mathbb{Q}}\left((1+r)^{-1} V_{n+1}|\mathcal{F}_n\right)\right).$$

If

$$U_n \le \max\left(L_n, \mathbb{E}_{\mathbb{Q}}\left((1+r)^{-1} V_{n+1}|\mathcal{F}_n\right)\right),$$

then the seller will terminate the contract. Hence the seller will not need to pay the buyer more than U_n at time n. Actually, the seller needs

$$\min\left(U_n, \max\left(L_n, \mathbb{E}_{\mathbb{Q}}\left((1+r)^{-1}V_{n+1}|\mathcal{F}_n\right)\right)\right),$$

for $n = 0, 1, \ldots, N-1$, which is the infimum value of a hedging portfolio strategy against this GCC.

Finally, we want to find a self-financing portfolio strategy $\pi^* = (x_n^*, y_n^*)$ such that the pair (τ_{s0}^*, π^*) is a hedge for the GCC, such strategy is unique up to the time $\tau_{s0}^* \wedge \tau_{b0}^*$.

Take $\tau_{s0}^* \in \mathcal{T}_{[0,N]}$ and construct the corresponding portfolio strategy π^* such that the pair (τ_{s0}^*, π^*) is a hedge as above with initial capital $V_0^{\tau_s^*}$

$$V_0^{\tau_s^*} = \nu_0^{\pi^*} = \mathbb{E}_{\mathbb{Q}}(\nu_0^{\pi^*}) = \mathbb{E}_{\mathbb{Q}}\left((1+r)^{-\tau_{s0}^*\wedge\tau_{b0}^*}\nu^{\pi^*}_{\tau_{s0}^*\wedge\tau_{b0}^*}\right) \geq \mathbb{E}_{\mathbb{Q}}\left((1+r)^{-\tau_{s0}^*\wedge\tau_{b0}^*}R(\tau_{s0}^*, \tau_{b0}^*)\right) = V_0^{\tau_s^*},$$

since $(1+r)^{-\tau_{s0}^*\wedge\tau_{b0}^*}\nu^{\pi^*}_{\tau_{s0}^*\wedge\tau_{b0}^*}$ is a martingale. Hence, $\nu^{\pi^*}_{\tau_{s0}^*\wedge\tau_{b0}^*} = R(\tau_{s0}^*, \tau_{b0}^*)$. Therefore, there exists a portfolio strategy π^* such that (τ_{s0}^*, π^*) is a hedge against the GCC with initial capital $V_0^{\tau_s^*}$.

Now we want to prove the uniqueness of such portfolio up to time $\tau_{s0}^* \wedge \tau_{b0}^*$. Let $\phi^{**} = ((x_n^{**}, y_n^{**}))_{n=1}^N$ be another self-financing portfolio strategy. Since $(1+r)^{-n}\nu_n^{\phi^{**}}$ is a martingale, $\nu^{\phi^{**}}_{\tau_{s0}^*\wedge\tau_{b0}^*} = R(\tau_{s0}^*, \tau_{b0}^*)$.

So

$$\nu_n^{\phi^{**}} = \nu_n^{\pi^*}, \quad \text{for all } n \leq \tau_{s0}^* \wedge \tau_{b0}^*.$$

Thus $x_n^{**} = x_n^*$ and $y_n^{**} = y_n^*$ for $n \leq \tau_{s0}^* \wedge \tau_{b0}^*$. Therefore the self-financing portfolio strategy π^* is unique.

□

The next example shows us how to compute the fair price of a game call option and also explains the behaviour of the fair price when the penalty δ changes.

Example 2.5.3. Consider a one time step CRR model as shown in the figure (2.1) with strike price $K = 95$, $r = 0.1$, $u = 0.25$, $d = -0.25$ and with penalty $\delta_0 = 6$ at time $n = 0$. we want to compute the fair price of a game call option by using the recursive relation mentioned in the Theorem (2.5.2).

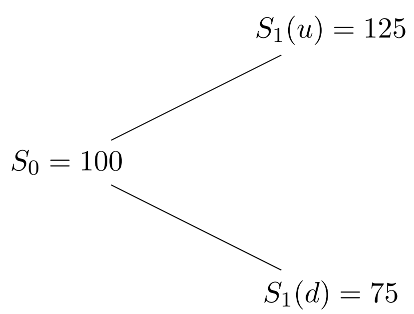

Figure 2.1: A one time step CRR model

The equivalent martingale measure is $(\tilde{p}, 1-\tilde{p}) = (\frac{r-d}{u-d}, \frac{u-r}{u-d}) = (\frac{7}{10}, \frac{3}{10})$.

The fair price of a game call option at $n = N = 1$ is

$V_1(u) = L_1(u) = (S_1(u) - K)^+ = (125 - 95)^+ = 30$

$V_1(d) = L_1(d) = (S_1(d) - K)^+ = (75 - 95)^+ = 0.$

The fair price of a game call option at $n = 0$ is determined by the equation

$$V_0 = \min\left(U_0, \max\left(L_0, \mathbb{E}_{\mathbb{Q}}\left((1+r)^{-1}V_1|\mathcal{F}_0\right)\right)\right)$$

$$\mathbb{E}_{\mathbb{Q}}\left((1+r)^{-1}V_1|\mathcal{F}_0\right) = (1+r)^{-1}(\tilde{p}V_1(u) + (1-\tilde{p})V_1(d)) = (1.1)^{-1}\left(\frac{7}{10}(30)\right) = 19.091$$

$U_0 = (S_0 - K)^+ + \delta_0 = (100 - 95)^+ + 6 = 11$

$L_0 = (S_0 - K)^+ = (100 - 95)^+ = 5.$

Therefore

$$V_0 = \min(11, \max(5, 19.091)) = 11.$$

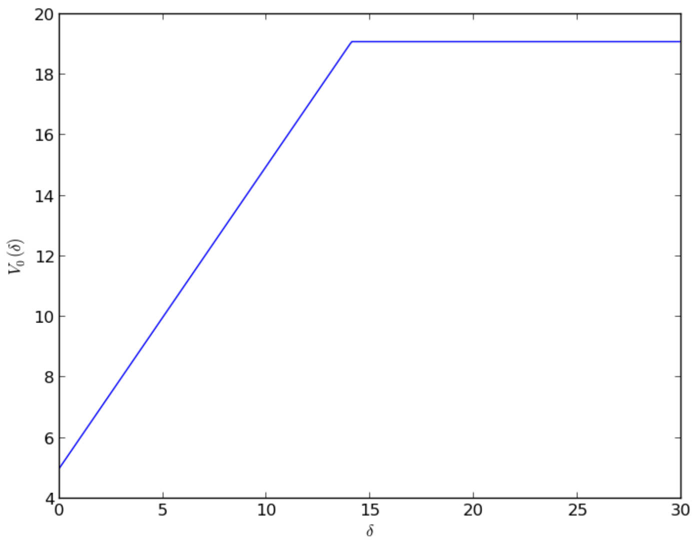

Figure 2.2: The fair price when δ changes

Let us change the value of the penalty and see the behaviour of the fair price when S_0 is fixed. We can see from the figure (2.2) that the $V_0(\delta)$, for all $\delta \geq 0$, is a continuous non-decreasing function. We conclude that for $\delta \in [0, 14.09]$ the fair price $V_0 = U_0$. If $\delta = 0$, it is better either for the seller or the buyer to exercise at time $n = 0$, i.e. $V_0 = U_0 = L_0$ and we can treat it as a ECC. For $\delta \in (14.09, \infty)$, it is better for the seller to wait for the buyer to exercise the contract and it can be considered as an ACC.

3. Game Option in Continuous Time

In this chapter, we will show how the game option in the discrete time can be extended to a continuous time framework. We will study the pricing of the game option in the *Black-Scholes* model. Since the tools needed to price the game option in the continuous time are technical, we will concentrate on the results and the fundamental ideas.

First, we present some concepts that are needed throughout this chapter.

Definition 3.0.1 (Predictable)**.** *. (Etheridge, 2002). Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space. The stochastic process $X = (X_t)_{t\geq 0}$ is predictable or previsible if X_t is $\mathcal{F}_{t-}$-measurable for all $t \geq 0$ where*

$$\mathcal{F}_{t-} = \cup_{k<t}\mathcal{F}_k.$$

Definition 3.0.2 (Self-financing strategy)**.** *(Etheridge, 2002). A self-financing strategy $\pi = ((x_t, y_t))_{0\leq t\leq T}$ consists of pairs (x_t, y_t), where $(x_t)_{0\leq t\leq T}$ and $(y_t)_{0\leq t\leq T}$ are predictable processes representing the quantities of risk-less and risky asset respectively held in the portfolio at time t, satisfying*

$$\int_0^T |x_t| dt + \int_0^T |y_t|^2 dt < \infty \text{ a.s. },$$

and

$$\nu_t^\pi = \nu_0^\pi + \int_0^t x_u dB_u + \int_0^t y_u dS_u \text{ a.s. }, \text{ for all } t \in [0, T].$$

Remark 3.0.3. In the case of self-financing strategy, we put the second condition in the above definition in differential form, then we have

$$d\nu_t^\pi = x_t dB_t + y_t dS_t, \tag{3.0.1}$$

i.e. the change of the portfolio price over an infinitesimal time interval depends only on the changes in the value of the risky and the risk-less asset.

Theorem 3.0.4. *(Ito's Lemma, one-dimensional). (Munk, 2013). The stochastic dynamic of $Y = (Y_t)_{t\geq 0}$ is a real-valued Ito process defined by*

$$dY_t = \mu_t dt + \sigma_t dW_t,$$

where μ and σ are real-valued processes, and W is a one-dimensional standard Brownian motion. Let $f(Y, t)$ be a real valued function which is twice continuously differentiable in Y and continuously differentiable in t. Then the process $X = (X_t)_{t\geq 0}$ given by

$$X_t = f(Y_t, t)$$

is an Ito process with dynamics

$$dX_t = \left(\frac{\partial f}{\partial t}(Y_t, t) + \mu_t \frac{\partial f}{\partial Y_t}(Y_t, t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial Y_t^2}(Y_t, t)\right) dt + \sigma_t \frac{\partial f}{\partial Y_t}(Y_t, t) dW_t.$$

Theorem 3.0.5 (Wald's Martingale)**.** *Let $\sigma \in \mathbb{R}$. Then the process $Q_t = \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)$, $t \geq 0$ is a martingale.*

3.1 Black-Scholes Model

The Black–Scholes model is used in pricing options in the continuous time. It was introduced by Black and Scholes (1973). They derived the Black–Scholes partial differential equation which estimates the option price over the time. Robert C. Merton extended the work in option pricing model. Scholes and Merton received the Nobel price in economics (1997).

We will consider a financial complete market which consists of two securities. The first is the risk-less asset with time evolution

$$B_t = B_0 e^{rt}.$$

This is known as continuous compounding with rate of growth $r \geq 0$ and initial wealth $B_0 > 0$. The second security is a risky asset whose price at time t is S_t which follows a geometric Brownian motion if it satisfies the following stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t) \tag{3.1.1}$$

where μ and σ represent the trend and the volatility of the stock price respectively, and $(W_t)_{t\geq 0}$ is the standard Brownian motion. The term $(\mu dt + \sigma dW_t)$ is accountable for random fluctuation of the stock price in short time interval.

In the Black-Sholes model, we assume that:

1. the drift μ and the volatility σ are constant in time,
2. the underlying security is continuously traded and does not pay a dividend,
3. there are no arbitrage opportunity and no transaction costs.

Let $\mathcal{L}_t$ be a differential operator with respect to the dynamics of the stock such that

$$\mathcal{L}_t \equiv \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + rs\frac{\partial}{\partial s} - r. \tag{3.1.2}$$

In this model, the price of the option satisfies the Black-Scholes differential equation, which is given by

$$\mathcal{L}_t V(s,t) = 0,$$

where V is the function price of the option.

We want to find the stock price S_t in the explicit form by solving the stochastic differential equation $dS_t = \mu(S_t,t)dt + \sigma(S_t,t)dW_t$, where $\mu(S_t,t) = \mu S_t$ and $\sigma(S_t,t) = \sigma S_t$. Clearly, $(S_t)_{t\geq 0}$ is an Ito process. Let us define a function f by $f(S_t,t) = \log(S_t)$, where f is a smooth function when $S_t > 0$, for all $t \geq 0$. Then we have

$$\frac{\partial f}{\partial t}(S_t,t) = 0, \quad \frac{\partial f}{\partial S_t}(S_t,t) = \frac{1}{S_t} \quad \text{and} \quad \frac{\partial^2 f}{\partial S_t^2}(S_t,t) = \frac{-1}{S_t^2}.$$

By applying the Ito's Theorem 3.0.4, we get

$$\begin{aligned} d(\log S_t) &= \left(0 + \frac{1}{S_t}\mu S_t - \frac{1}{2}\frac{\sigma^2 S_t^2}{S_t^2}\right) dt + \frac{1}{S_t}\sigma S_t dW_t \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \end{aligned}$$

$$\implies \int_0^t d(\log S_m) = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) dm + \int_0^t \sigma dW_m$$
$$\implies \log S_t - \log S_0 = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma(W_t - W_0)$$

and since $W_0 = 0$
$$\log\left(\frac{S_t}{S_0}\right) = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t.$$

Taking exponentials on both sides, we get

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right).$$

Let $(W_t)_{t\geq 0}$ be a stochastic process on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, where Ω is the space of continuous paths $\omega = (\omega_t)_{t\geq 0}$ starting at zero, i.e. $\omega_0 = 0$, $\mathcal{F}$ is the Borel $\sigma-$ algebra generated by the cylinder sets. Then $W_t(\omega) = \omega_t$, $t \geq 0$. Let $\mathcal{F}_t^*$ be a $\sigma-$ algebra generated by $\{W_s, s \leq t\}$, $\mathcal{F}_t$ be the minimal $\sigma-$ algebra containing $\mathcal{F}_t^*$ and $\mathbf{P}$ be the standard measure on C^0, known as Wiener measure and
$$S_t = S_0 e^{\mu t} Q_t, \tag{3.1.3}$$
where $Q_t = e^{\sigma W_t - \left(\frac{\sigma^2}{2}\right)t}$, $t \geq 0$, is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ by Walt's theorem (3.0.5).

Lemma 3.1.1. (Etheridge, 2002). Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, consider a Black-Scholes model. There is a probability measure $\mathbf{P}^*$, equivalent to $\mathbf{P}$, such that the discounted stock price $(\overline{S}_t)_{t\geq 0} = (e^{-rt}S_t)_{t\geq 0}$ is a martingale. Furthermore, the Radon-Nikodym derivative of $\mathbf{P}^*$ with respect to $\mathbf{P}$ is defined by
$$\left.\frac{d\mathbf{P}^*}{d\mathbf{P}}\right|_{\mathcal{F}_t} = \exp\left(-\frac{\mu - r}{\sigma} W_t(\omega) - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right).$$

Set
$$W_t^* = W_t + \frac{\mu - r}{\sigma} t, \tag{3.1.4}$$

then $(W_t^*)_{t\geq 0}$ is the standard Brownian process with respect to $\mathbf{P}^*$.

From (3.1.4), we have
$$dW_t^* = dW_t + \frac{\mu - r}{\sigma} dt. \tag{3.1.5}$$

By substituting (3.1.5) in (3.1.1), we have

$$\begin{aligned} dS_t = S_t\left(\mu dt + \sigma dW_t\right) &= S_t\left(\mu dt + \sigma\left(dW_t^* - \frac{\mu - r}{\sigma} dt\right)\right) \\ &= S_t\left(\mu dt + \sigma dW_t^* - \mu dt + r dt\right) \\ &= S_t\left(r dt + \sigma dW_t^*\right). \end{aligned}$$

Therefore

$$dS_t = S_t(rdt + \sigma dW_t^*). \quad (3.1.6)$$

By substituting (3.1.6) in (3.0.1), we have

$$\begin{aligned} d\nu_t^\pi = x_t dB_t + y_t dS_t &= x_t B_0 r e^{rt} dt + y_t\left(S_t r dt + S_t \sigma dW_t^*\right) \\ &= x_t r B_t dt + y_t S_t r dt + y_t S_t \sigma dW_t^* \\ &= (x_t B_t + y_t S_t) r dt + S_t y_t \sigma dW_t^* \\ &= \nu_t^\pi r dt + S_t y_t \sigma dW_t^*. \end{aligned}$$

Therefore

$$d\nu_t^\pi = r\nu_t^\pi dt + \sigma y_t S_t dW_t^*, \quad (3.1.7)$$

for any self financing portfolio strategy π .

3.2 Game Contingent Claim

Denote by $\mathcal{T}_{[t,T]}$ the subclass of stopping times τ with respect to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ such that $\tau(\omega) \in [t, T]$.

Definition 3.2.1. *A Game Contingent Claim (GCC) in a continuous time is a contract between investors S and B consisting of a finite expiration date T. Assume that S chooses a cancellation time $\tau_s \in \mathcal{T}_{[0,T]}$ and B chooses an exercise time $\tau_b \in \mathcal{T}_{[0,T]}$. U_t and L_t are $\mathcal{F}_t$-adapted payoff Càdlàg processes such that $U_t \geq L_t \geq 0$ a.s. and $\mathbb{E}^*\left(\sup_{0\leq t\leq T} e^{-rt} U_t\right) < \infty$ where $\mathbb{E}^*$ is the expectation with respect to the risk neutral measure. So that S pledges to pay to B at time $\tau_b \wedge \tau_s$ the sum*

$$R(\tau_s, \tau_b) = U_{\tau_s}\mathbf{1}_{\{\tau_s<\tau_b\}} + L_{\tau_b}\mathbf{1}_{\{\tau_b\leq\tau_s\}}.$$

Remark 3.2.2. This shows how to obtain ACC from the GCC in the continuous time framework, when terminating the contract at time $t < T$ is not optimal for S. This is the case when δ is big enough i.e. when

$$\delta > \sup_{0\leq\tau_b\leq T} \mathbb{E}^*(L_{\tau_b})$$

and the price V^* of GCC is

$$\sup_{0\leq\tau_b\leq T} \mathbb{E}^*\left(\frac{L_{\tau_b}}{B_{\tau_b}}\right),$$

where $B_{\tau_b} = e^{r\tau_b}$

On the other hand, we obtain ECC from the GCC in the continuous time framework, if it is not suitable for B to exercise before the expiration date T, which occurs when

$$L_t = \begin{cases} 0, & \text{if } t < T, \\ L_T > 0, & \text{if } t = T. \end{cases}$$

If $\delta_0 = 0$, then it is optimal for either S or B to terminate the contract at time $t = 0$ and the price $V^* = L_0$. V^* is increasing function of penalty, with $L_0 \leq V^* \leq \sup_{0 \leq \tau_b \leq T} \mathbb{E}\left(\frac{L_{\tau_b}}{e^{r\tau_b}}\right)$.

Example 3.2.3. δ-penalty game put option is an American put option with additional possibility that the seller can terminate the contract at any time until the expiration date by paying a fixed penalty, i.e. $\delta_t = \delta > 0$ for all $T \geq t \geq 0$. If the buyer exercises the option before the seller terminates the contract or if both exercise the contract at the same time, then the exercise payoff is the payoff of a normal American put option

$$L_t = (K - S_t)^+.$$

If the seller terminates the contract, then the cancellation payoff will be the American payoff plus a constant (a non-negative fixed penalty) which is

$$U_t = (K - S_t) + \delta, \quad \delta > 0.$$

Example 3.2.4. δ-penalty Russian option is a Russian option that gives the seller the right to terminate the contract at any time until the expiration date, implying a payment exceeding the exercise payoff at that moment.

In this case, The buyer can exercise before the seller terminates, as a normal Russian option

$$L_t = e^{-rt} \max\left(m, \sup_{l \in [0,t]} S_l\right) \quad \text{for } r > 0, m > S_0.$$

On the other hand, the seller will be punished by that amount $e^{-rt}\delta S_t$ to terminate the contract before the exercise time

$$U_t = L_t + e^{-rt}\delta S_t = e^{-rt}\left(\max\left(m, \sup_{l \in [0,t]} S_l\right) + \delta S_t\right) \quad \text{for } \delta > 0.$$

3.3 Pricing of GCC in a Black-Scholes (B-S) Model

Definition 3.3.1. *A pair (τ_s, π) which consists of a stopping time $\tau_s \in \mathcal{T}_{[0,T]}$ and a $\{\mathcal{F}_t\}_{0 \leq t \leq T}$-progressively measurable self-financing portfolio π is called a hedge against a GCC with expiration date T if $\nu^\pi_{\tau_s \wedge t} \geq R(\tau_s, t)$ a.s., for all $t \in [0, T]$. The fair price V^* of the GCC is the infimum of $V \geq 0$ such that there exists a hedge (τ_s, π) against this GCC with initial capital $\nu^\pi_0 = V$.*

Let us suppose that S and B are two investors, S must assume that B has found the stopping time which gives him the maximum payoff and S must find the stopping time which gives him the minimum payout. Conversely B must assume that S has found the stopping time which gives him the minimum payout and B must find the stopping time which gives him the maximum payoff. These strategies rise to the upper and lower value processes.

Definition 3.3.2. *Let $\mathbf{P}^*$ be the probability on the space Ω and $\mathbb{E}^*$ denote the expectation with respect to the measure $\mathbf{P}^*$. The upper value process V^u_t of a GCC is an $\mathbb{F}$-adapted Càdlàg process given by the formula*

$$V^u_t = \operatorname{essinf}_{\tau_s \in \mathcal{T}_{[t,T]}} \operatorname{esssup}_{\tau_b \in \mathcal{T}_{[t,T]}} \mathbb{E}^*\left(e^{-r(\tau_s \wedge \tau_b - t)} R(\tau_s, \tau_b) | \mathcal{F}_t\right). \tag{3.3.1}$$

The lower value process V_t^l of a GCC is an $\mathbb{F}$-adapted càdlàg process given by the formula

$$V_t^l = \text{esssup}_{\tau_b \in \mathcal{T}_{[t,T]}} \text{essinf}_{\tau_s \in \mathcal{T}_{[t,T]}} \mathbb{E}^* \left(e^{-r(\tau_s \wedge \tau_b - t)} R(\tau_s, \tau_b) | \mathcal{F}_t \right).$$

Definition 3.3.3. *Let $\varepsilon > 0$, $V = V^u = V^l$ be the value process of a GCC. We say that τ_b^ε is an ε-optimal stopping time strategy for buyer B if:*

$$R(\tau_s, \tau_b^\varepsilon) + \varepsilon \geq V \quad \forall \tau_s \in \mathcal{T}_{[0,T]}.$$

Similarly, we say that τ_s^ε is an ε- optimal stopping time strategy for seller S if:

$$R(\tau_s^\varepsilon, \tau_b) - \varepsilon \leq V \quad \forall \tau_b \in \mathcal{T}_{[0,T]}.$$

The following Theorem is the Kifer's result about pricing the GCC in a B-S model.

Theorem 3.3.4. *(Kifer, 2000). The fair price V^* of the GCC in Definition (3.2.1) is V_0 where V_t is a Càdlàg process such that $V_t = V_t^u = V_t^l$ a.s. . Moreover, for each $t \in [0,T]$ and $\varepsilon > 0$, the stopping times*

$$\tau_{st}^{\varepsilon} = \inf \{ u \geq t : U_u \leq V_u + \varepsilon \text{ or } u = T \} \tag{3.3.2}$$

and

$$\tau_{bt}^{\varepsilon} = \inf \{ u \geq t : L_u \geq V_u - \varepsilon \} \tag{3.3.3}$$

satisfy

$$\mathbb{E}^* \left(e^{-r(\tau_{st}^\varepsilon \wedge \tau_b - t)} R(\tau_{st}^\varepsilon, \tau_b) | \mathcal{F}_t \right) - \varepsilon \leq V_t \leq \mathbb{E}^* \left(e^{-r(\tau_s \wedge \tau_{bt}^\varepsilon - t)} R(\tau_s, \tau_{bt}^\varepsilon) | \mathcal{F}_t \right) + \varepsilon \text{ a.s.} \tag{3.3.4}$$

for any τ_b, $\tau_s \in \mathcal{T}_{[t,T]}$.

Moreover, there exists a self-financing portfolio strategy π^* such that a pair (τ_{s0}^0, π^*) is a hedge against this GCC with the initial capital $\nu_0^{\pi^*} = V_0$ and with P^*-probability one such strategy is unique this case up to the time $\tau_{s0}^0 \wedge \tau_{b0}^0$.

Let $U, L, V : \mathbb{R}^+ \times [0,T] \longrightarrow \mathbb{R}$ be three functions which satisfy some regularity conditions and such that $U_t = U(S_t, t)$, $L_t = L(S_t, t)$, $V_t = V(S_t, t)$ and $U(s,t) \geq L(s,t)$. We will assume the value process of the game option exists, which is given by

$$V_t = V(S_t, t) = \inf_{\tau_s \in \mathcal{T}_t} \sup_{\tau_b \in \mathcal{T}_t} \mathbb{E}^* \left(e^{-r(\tau_s \wedge \tau_b - t)} R(\tau_s, \tau_b) | S_t = s \right) = \sup_{\tau_b \in \mathcal{T}_t} \inf_{\tau_s \in \mathcal{T}_t} \mathbb{E}^* \left(e^{-r(\tau_s \wedge \tau_b - t)} R(\tau_s, \tau_b) | S_t = s \right),$$

where $R(\tau_s, \tau_b) = U(S_{\tau_s}, \tau_s) \mathbf{1}_{\{\tau_s < \tau_b\}} + L(S_{\tau_s}, \tau_s) \mathbf{1}_{\{\tau_b \leq \tau_s\}}$.

For each $t \in [0,T]$, we define the stopping times as

$$\tau_{st}^* = \inf \{ u \geq t : (S_u, u) \in D_U \text{ or } u = T \}$$

and

$$\tau_{bt}^* = \inf \{ u \geq t : (S_u, u) \in D_L \text{ or } u = T \},$$

where

$$D_U = \left\{ (s,t) \in \mathbb{R}^+ \times [0,T] : V(s,t) = U(s,t) \right\}$$

and

$$D_L = \left\{(s,t) \in \mathbb{R}^+ \times [0,T] : V(s,t) = L(s,t)\right\}.$$

Furthermore, ${\tau_s}_t^*$ and ${\tau_b}_t^*$ satisfy the inequality

$$\mathbb{E}^*\left(e^{-r({\tau_s}_t^* \wedge \tau_b - t)} R({\tau_s}_t^*, \tau_b)|S_t = s\right) \le V_t \le \mathbb{E}^*\left(e^{-r(\tau_s \wedge {\tau_b}_t^* - t)} R(\tau_s, {\tau_b}_t^*)|S_t = s\right), \text{ for all } \tau_s, \tau_b \in \mathcal{T}_{[t,T]}. \quad (3.3.5)$$

Theorem 3.3.5. *Musiela and Rutkowski (2006). The value process $V_t = V(S_t, t)$ of the GCC satisfies at least one of the following conditions:*

1. *If the seller and the buyer do not exercise the option before the expiration date, i.e. $L(S_t, t) \le V(S_t, t) \le U(S_t, t)$, then the payoff $R(T,T) = L_T$ is a martingale and the payoff satisfies Equation (3.1.2).*
2. *If the buyer exercises before the seller i.e. $V(s,t) = L(s,t)$, then the payoff is a supermartingale and we have $\mathcal{L}_t V(s,t) \ge 0$.*
3. *If the seller exercises before the buyer i.e. $V(s,t) = U(s,t)$, then the payoff is a submartingale and we have $\mathcal{L}_t V(s,t) \le 0$.*

If the seller chooses a cancellation time ${\tau_s}_t^*$ and the buyer chooses an exercise time ${\tau_b}_t^*$, then the inequality (3.3.5) holds. By Theorem (3.3.5), there exist a martingale portfolio π with value ν_t^π at time t such that $V(S_t, t) = \nu^p i_t$, for all $t \in [0,T]$, and the discounted gains is a martingale under the risk neutral measure $\mathbf{P}^*$. So the fair price of the GCC is given by $\mathbb{E}^*\left(e^{-r{\tau_s}_0^* \wedge {\tau_b}_0^*} R({\tau_s}_0^*, {\tau_b}_0^*)\right)$.

4. Dynkin Game

The concept of the Dynkin Game was introduced first by Dynkin (1969). It is a game between two players B_1 and B_2 that can each end the game at any time. At the end of the game, B_2 receives a certain amount of money, that is referred to as *payoff*, from B_1.

4.1 Dynkin Game in Discrete Time

This section is based on the work of Musiela and Rutkowski (2006). Throughout the sequel, $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in\{0,\dots,N\}}, \mathbb{P})$ is a filtered probability space, N is an integer which denotes the end of the game and $\mathcal{T}_{[n,N]}$ is the set of all stopping times τ with respect to the filtration $\mathbb{F}$ such that $\tau(\omega) \in \{n, n+1, \dots, N\}$.

Three stochastic processes govern the payoff of the Dynkin Game:

- the process that gives the amount B_2 will receive if the player B_1 ends first the game.
- the process giving the amount that B_2 will receive if B_2 ends the game first.
- the process which governs the amount B_2 will receive if both of them end the game at the same time.

In such a game, it is obvious that B_2 wants to maximise the payoff and B_1 wants to minimise his payout.

Definition 4.1.1. *Let $n = 0, 1, \dots, N$ be any fixed date and $(\Omega, \mathcal{F}, \{\mathcal{F}_k\}_{k\in[0,N]}, \mathbb{P})$ be a filtered probability space. The Dynkin Game starting at time n is defined as the game between B_1 and B_2, where B_1 chooses a stopping time σ and B_2 a stopping time τ, with $\tau, \sigma \in \mathcal{T}_{[n,N]}$. At the time $\tau \wedge \sigma$, B_2 receives the payoff*

$$H(\sigma, \tau) = X_\sigma \mathbf{1}_{\{\sigma<\tau\}} + Y_\tau \mathbf{1}_{\{\tau<\sigma\}} + Z_\tau \mathbf{1}_{\{\sigma=\tau\}},$$

where $\mathbf{1}_A$ is the characteristic function and $Y \le Z \le X$ are Borel functions adapted to the filtration $\mathcal{F}$. The expected payoff is given by

$$\mathbb{E}_{\mathbb{P}}(H(\sigma, \tau)|\mathcal{F}_n).$$

We are interested in finding the amount of money that B_2 must pay B_1 at the beginning of the game to entice him to play. The price of the game is the amount that B_2 have to pay to play the game. We are also interested in finding the corresponding optimal stopping times.

Let suppose that B_1 and B_2 are playing the game in the optimal way. Each of the players try to optimize the profit knowing that the other one does so for himself. B_1, assuming that B_2 has found the stopping time maximizing the payoff, try to find the stopping time to minimize it. Conversely B_2 must assume that B_1 has found the stopping time which yields the minimum payoff and B_2 must find the stopping time which gives him the maximum payoff. The upper and lower value processes of Dynkin Game arises from this problem.

Definition 4.1.2. *The upper value process V_n^u of the Dynkin game is an $\mathbb{F}$-adapted process*

$$V_n^u = \min_{\sigma\in\mathcal{T}_{[n,N]}} \max_{\tau\in\mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{P}}(H(\sigma, \tau)|\mathcal{F}_n).$$

The lower value process V_t^l of the Dynkin game is an $\mathbb{F}$-adapted process. It is given by the formula

$$V_n^l = \max_{\tau\in\mathcal{T}_{[n,N]}} \min_{\sigma\in\mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{P}}(H(\sigma,\tau)|\mathcal{F}_n).$$

Definition 4.1.3. *If the equality $V_n^u = V_n^l$ is satisfied, we say that the Stackelberg equilibrium holds for a Dynkin game. Then the process $V^u = V^l = V$ is called the value process of the Dynkin game.*

Remark 4.1.4. Stackelberg equilibrium is a game between two players, the leader and the follower, such that the leader put his strategy knowing the optimal response of the follower.

Definition 4.1.5. *If for any n there exists stopping times $\sigma_n^*, \tau_n^* \in \mathcal{T}_{[n,N]}$ such that*

$$\mathbb{E}_{\mathbb{P}}(H(\sigma_n^*,\tau)|\mathcal{F}_n) \le \mathbb{E}_{\mathbb{P}}(H(\sigma_n^*,\tau_n^*)|\mathcal{F}_n) \le \mathbb{E}_{\mathbb{P}}(H(\sigma,\tau_n^*)|\mathcal{F}_n)$$

for all stopping times $\tau,\sigma \in \mathcal{T}_{[n,N]}$, then Nash equilibrium holds.

The point (σ_n^*, τ_n^*) is a saddle point of the Dynkin game. The next result shows that the Stackelberg equilibrium is a special case from Nash equilibrium.

Lemma 4.1.6. Assume that Nash equilibrium holds. Then the Stackelberg equilibrium holds and

$$V_n = \mathbb{E}_{\mathbb{P}}(H(\sigma_n^*,\tau_n^*)|\mathcal{F}_n),$$

so that σ_n^* and τ_n^* are optimal stopping times as of time n.

Proof. Suppose that Nash equilibrium holds, then

$$\max_{\tau\in\mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{P}}(H(\sigma_n^*,\tau)|\mathcal{F}_n) \le \mathbb{E}_{\mathbb{P}}(H(\sigma_n^*,\tau_n^*)|\mathcal{F}_n) \le \min_{\sigma\in\mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{P}}(H(\sigma,\tau_n^*)|\mathcal{F}_n).$$

Therefore,

$$\begin{aligned} V_n^u &= \min_{\sigma\in\mathcal{T}_{[n,N]}} \max_{\tau\in\mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{P}}(H(\sigma,\tau)|\mathcal{F}_n) \\ &\le \mathbb{E}_{\mathbb{P}}(H(\sigma_n^*,\tau_n^*)|\mathcal{F}_n) \\ &\le \max_{\tau\in\mathcal{T}_{[n,N]}} \min_{\sigma\in\mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{P}}(H(\sigma,\tau)|\mathcal{F}_n) = V_n^l. \end{aligned} \tag{4.1.1}$$

For any $\tau_0, \sigma_0 \in \mathcal{T}_{[n,N]}$, we have

$$\min_{\sigma\in\mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{P}}(H(\sigma,\tau_0)|\mathcal{F}_n) \le \mathbb{E}_{\mathbb{P}}(H(\sigma_0,\tau_0)|\mathcal{F}_n) \le \max_{\tau\in\mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{P}}(H(\sigma_0,\tau)|\mathcal{F}_n),$$

$$\max_{\tau_0\in\mathcal{T}} \min_{\sigma\in\mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{P}}(H(\sigma,\tau_0)|\mathcal{F}_b) \le \min_{\sigma_0\in\mathcal{T}} \max_{\tau\in\mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{P}}(H(\sigma_0,\tau)|\mathcal{F}_n).$$

Therefore,

$$V_n^l = \max_{\tau\in\mathcal{T}_{[n,N]}} \min_{\sigma\in\mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{P}}(H(\sigma,\tau)|\mathcal{F}_n) \le \min_{\sigma\in\mathcal{T}_{[n,N]}} \max_{\tau\in\mathcal{T}_{[n,N]}} \mathbb{E}_{\mathbb{P}}(H(\sigma,\tau)|\mathcal{F}_n) = V_n^u. \tag{4.1.2}$$

From equations (4.1.1) and (4.1.2), we have $V_n^u = V_n^l$ and the value process is well defined and satisfies $V_n = \mathbb{E}_{\mathbb{P}}(H(\sigma_n^*,\tau_n^*)|\mathcal{F}_n)$, for any $n = 0, 1, \ldots, N$. □

4.2 The Continuous Time Nonzero-sum Dynkin Game

This section is based on the work of Hamadène and Zhang (2010). Throughout the sequel, T is the horizon time of the system, $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t<T}, \mathbb{P})$ is a filtered probability space and $\mathcal{T}_{[t,T]}$ is a set of all stopping times τ with respect to the filtration $\mathbb{F}$ such that $\tau(\omega) \in [t, T]$.

Dynkin Game in continuous time is a game between two players B_1 and B_2 such that each player chooses his decision independently. B_1 and B_2 control on the system, when one of the player decides to stop at a stopping time $\tau_1 \in \mathcal{T}_{[0,T]}$ for B_1 and $\tau_2 \in \mathcal{T}_{[0,T]}$ for B_2. At time $\tau_1 \wedge \tau_2$, the system is dissolved where the expected cost for B_1 is $\mathbb{E}(H_1(\tau_1, \tau_2))$ and for B_2 is $\mathbb{E}(H_2(\tau_1, \tau_2))$ such that for $i = 1, 2$, we have

$$H_i(\tau_1, \tau_2) = \begin{cases} X^i_{\tau_i} & \text{if } \tau_i < \tau_j, \\ Y^i_{\tau_j} & \text{if } \tau_j < \tau_i, \\ Y^i_{\tau_i}\delta_{2i} + X^i_{\tau_i}\delta_{1i} & \text{if } \tau_1 = \tau_2, \end{cases}$$

where δ_{ij} is Kronecker delta, X^i and Y^i are $\mathbb{F}$-adapted càldàg processes.

Definition 4.2.1. *If there exists two stopping times $\tau_1^*, \tau_2^* \in \mathcal{T}_{[0,T]}$ such that*

$$\mathbb{E}(H_1(\tau_1, \tau_2^*)) \le \mathbb{E}(H_1(\tau_1^*, \tau_2^*))$$

and

$$\mathbb{E}(H_2(\tau_1^*, \tau_2)) \le \mathbb{E}(H_2(\tau_1^*, \tau_2^*)),$$

for all stopping times $\tau_1, \tau_2 \in \mathcal{T}_{[0,T]}$, then we say that the non-zero-sum Dynkin game has a Nash equilibrium point.

This definition means that if B_2 chooses his stopping time strategy, say τ_2^*, then B_1 can not improve his reward by changing his stopping time strategy since $\mathbb{E}(H_1(\tau_1, \tau_2^*))$ is always bounded above by $\mathbb{E}(H_1(\tau_1^*, \tau_2^*))$ for all the changes in the stopping time strategy τ_1 that B_1 can do and vice versa.

Remark 4.2.2. In the non-zero-sum Dynkin Game, we want to prove the existence of the Nash equilibrium.

When the sum of lose and gain of the players B_1 and B_2 equals zero, i.e. $(H_1 = -H_2)$ as it is the case in Convertible Bonds and Game Option, the game is called the *zero-sum Dynkin Game*. In this case, it is reasonable to only consider the reward for one player, we will consider H_2. We say that the zero-sum Dynkin Game has a Nash equilibrium, if there exits a pair of stopping times (τ_1^*, τ_2^*) satisfying the inequality

$$\mathbb{E}(H_2(\tau_1^*, \tau_2)) \le \mathbb{E}(H_2(\tau_1^*, \tau_2^*)) \le \mathbb{E}(H_2(\tau_1, \tau_2^*)), \tag{4.2.1}$$

where B_2 wants to maximise his payoff and B_1 wants to minimise the B_2 payoff's. If B_2 changes the stopping time strategy to have a better payoff $\mathbb{E}(H_2(\tau_1, \tau_2^*))$, then B_1 will response by adjusting his stopping time strategy and B_2 will have less payoff $\mathbb{E}(H_2(\tau_1^*, \tau_2))$ than they were at the saddle point. So the strategies of B_1 and B_2 tend to stabilize at the saddle point.

Moreover, the game has a value process which is

$$\inf_{\tau_1} \sup_{\tau_2} \mathbb{E}(H_2(\tau_1, \tau_2)) = \sup_{\tau_2} \inf_{\tau_1} \mathbb{E}(H_2(\tau_1, \tau_2)). \tag{4.2.2}$$

Remark 4.2.3. In the zero-sum Dynkin Game, the Nash equilibrium for the game is just the saddle point for the game and it can be solved using minmax method. In such a game, we want to prove the existence of the saddle point.

This kind of game becomes interesting since it has application in financial markets like the pricing of game option. Here we give some example of zero-sum Dynkin Game.

Example 4.2.4.

- American Game Option is an American Option with additional property that the buyer can terminate the contract.
- Unlike the regular bond, a convertible bond (see, Brennan and Schwartz (1979)) is a type of bond that the buyer can convert into a share and the issuer has the right to force the buyer to convert.

  The fair price of the game option and the convertible bond are given by the solution of the optimal stopping problem

$$\inf_{\tau_1} \sup_{\tau_2} \mathbb{E}(e^{-\tau_1 \wedge \tau_2} H_2(\tau_1, \tau_2)) = \sup_{\tau_2} \inf_{\tau_1} \mathbb{E}(e^{-\tau_1 \wedge \tau_2} H_2(\tau_1, \tau_2)). \tag{4.2.3}$$

Many works have been done to analyse the relation between Game Option and the zero-sum Dynkin Game, namely Hamadène and Hassani (2005), Bahlali, Hamadène, and Mezerdi (2005), Hamadène and Hdhiri (2006), Hamadène (2006), Crépey and Matoussi (2008), Hamadène and Wang (2009) followed by Bielecki, Jeanblanc, and Rutkowski (2009). They have shown that under the Mokobodski condition there exists a solution to the Backward Stochastic Differential Equations (BSDE) with doubly reflection which describes the price of a convertible bond and the price of a game option.

5. Conclusion

We have discussed the problem of valuation and hedging Game option. These depend mainly on an optimal stopping time problem and the existence of a replicating portfolio for the value process of the game option. Stackelberg equilibrium, which is a consequence of Nash equilibrium, guarantees the existence of the optimal stopping time for this problem.

In the CRR model, finding the fair price is just evaluating the saddle point of the zero-sum Dynkin game and such a point does exist and it is unique up to the end of the contract, with optimal stopping time for the seller and the buyer. The numerical example we have constructed shows how to use the recursive formula to compute the fair price in a one step CRR model. In the case where it has the large number of steps, Eliasson (2012) suggested an algorithm using dynamic programming (recursive relation).

Actually, in discrete time, there is no explicit formula for the fair price but we could find the way to compute this recursively by calculating the value of the process going backwards from the expiration of the contract. Some conditions in the real market can be exploited to give an explicit formula for the fair price. Finding these additional conditions can make the object of further work.

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