Chasing Subgroups

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Abstract

The first chapter of this paper is a review of groups. Chapter two further discusses groups with respect to two ideas: Galois connections and cartesianness.

In chapter three, we revisited Mac Lane's technique of "chasing subgroups" and developed it to a simpler technique which, in the first place, we used to prove diagram lemmas for pointed sets. We then applied this new technique to groups in two ways to prove more complicated diagram lemmas such as the 3×3 Lemma, the Dragon Lemma and the Spider Lemma.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

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1. Introduction

In this paper, we started by discussing the well-known basic ideas of groups and morphisms between them. The basic ideas included the Subgroup criterion, the fact that the image $f_*(S)$ of a subgroup S of G under a group morphism $f : G \longrightarrow G'$ is a subgroup of G' and that the kernel $f^*(0)$ of f is a subgroup of G (a normal subgroup for that matter).

In chapter two we discussed groups in the light of two 'new' ideas: Galois connections and cartesianness. Given two groups G and G' we proved that a group morphism $f : G \longrightarrow G'$ induces a Galois connection between the ordered sets $\mathrm{Sub}(G)$ (of subgroups of G) and $\mathrm{Sub}(G')$ (of subgroups of G'). With this induced Galois connection

$$\mathrm{Sub}(G) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{Sub}(G')\ ,$$

we identified the closed elements of $\mathrm{Sub}(G)$ to be those subgroups of G which contain the kernel $f^*(0)$ of f; while the closed elements of $\mathrm{Sub}(G')$ are those subgroups of G' contained in the image $f_*(G)$ of G under f. In the language of cartesianness, the closed elements of G and G' are the ones said to satisfy the cartesianness condition.

In chapter three we revisited Mac Lane's technique of "chasing subgroups" and developed it to a simpler technique which made it easy to prove a variety of diagram lemmas such as the Five Lemma, the 3×3 Lemma, the Dragon Lemma and the Spider Lemma. Apart from the new technique, the notion of an exact sequence of pointed sets also played an important role in proving the lemmas mentioned above. This notion was introduced when we changed from having say, groups as objects in our diagrams to having ordered sets of subgroups as objects. This transformation, from groups to ordered sets of subgroups, was possible simply because given any group G the set $\mathrm{Sub}(G)$ of all subgroups of G automatically forms an ordered set in which the order relation is subgroup inclusion "$\leq$". This is nice.

Duality is another tool that we have used quite often especially where things have failed. That is to mean that we have always found ourselves in a situation where if we find it hard to show exactness at a certain object, we try and do it at its dual. A typical commutative diagram would look like this:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_2 & \xrightarrow{f} & A_3 & \xrightarrow{g} & A_4 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \\
0 & \longrightarrow & B_2 & \xrightarrow[f']{} & B_3 & \xrightarrow[g']{} & B_4 & \longrightarrow & 0
\end{array}$$

Commutativity means that an element, say, of A_2 is mapped to the same element in B_3 whether it goes through xf or through $f'z$. We defined dual objects to be those objects which swap positions when a given diagram is rotated through a 180° angle. In this diagram, A_2 is dual to B_4; B_2 is dual to A_4 and A_3 is dual to B_3.

2. Review of Groups

2.1 Definition of a Group

2.1.1 Definition. A group G is a set together with a binary operation "$*$" in which the following axioms hold:

(i) **Associativity:** for any $g, h, k \in G$, we have that $(g * h) * k = g * (h * k)$.

(ii) **Existence of the identity element:** G should have an element denoted by e and called the identity element such that $g * e = e * g = g$ for every element g of G.

(iii) **Existence of the inverse element:** for every $g \in G$ there should exist an element g^{-1} in G called the inverse of g such that $g * g^{-1} = e = g^{-1} * g$.

If in addition, $g * h = h * g$ for all g and h in G, then the group G is called an abelian group.

2.1.2 Example (Examples of Groups). (i) $(\mathbb{Z}, +)$ the set of integers with addition.

(ii) $(\mathbb{Z}_7, +)$ the set $\{0, 1, 2, 3, 4, 5, 6\}$ with addition modulo 7 $(+_7)$.

(iii) $(S_3, \circ)$ all the possible bijections on a three-element set with composition of functions.

(iv) $GL_2(\mathbb{R}, \times)$, the set of all 2×2 invertible matrices with real numbers as entries under matrix multiplication.

The first two are abelian while the last two are not abelian.

2.2 Subgroups

2.2.1 Definition. A subset S of G is said to be a subgroup of a group $(G, *)$ if it is a group in its own right with the binary operation "$*$" of G. In this essay, the statement that 'S is a subgroup of G' will be denoted by $S \leqslant G$.

Instead of checking all the four axioms of a group to find out whether a given subset S of a group G is a subgroup or not, there is a faster or better way of doing so; the Subgroup Criterion and hence the theorem below.

2.2.2 Theorem (Subgroup Criterion). *A subset S of a group G is a subgroup of G if and only if the following conditions are satisfied:*

(a) S is non empty.

(b) $xy^{-1} \in S$ whenever $x, y \in S$.

Proof. $(\Leftarrow)$ Suppose (a) and (b) hold. Then since $S \neq \varnothing$ there is an element $x \in S$ and by (b), $xx^{-1} = e \in S$, and so S contains the identity. Let $x \in S$ now that $e \in S$ it follows that $ex^{-1} = x^{-1} \in S$, and so every element in S has an inverse. Since S is closed for inverses, if $x, y \in S$ then $y^{-1} \in S$. Also $x(y^{-1})^{-1} = xy \in S$. Associativity property is inherited from the parent group G and so S is a subgroup of G.

($\Rightarrow$) Suppose S is a subgroup of G. Clearly S is non-empty since $e \in S$. If $x, y \in S$ then $x^{-1}, y^{-1} \in S$. Also $xy^{-1} \in S$ by the closure axiom of groups.

□

2.2.3 Definition. A subgroup N of a group G is called a normal subgroup of G if it is invariant under conjugation; i.e. for each $n \in N$ and each $g \in G$, the element gng^{-1} is still in N. "N is a normal subgroup of G" is usually denoted by $N \lhd G$.

2.2.4 Example (Examples of subgroups). (i) $(\mathbb{Z}, +) \leqslant (\mathbb{Q}, +)$.

(ii) $(\mathbb{Q}, +) \leqslant (\mathbb{R}, +)$.

(iii) $(\{(1), (1, 2)\}, \circ) \leq (S_3, \circ)$.

(iv) $SL_2(\mathbb{R}) \leq GL_2(\mathbb{R})$ the special linear group (the group of all 2×2 matrices with determinant 1) is a subgroup of the general linear group (the group of all invertible 2×2 matrices).

2.3 Group morphisms

In this essay, we will use f_* and f^* to denote direct image map and inverse image map respectively for any given morphism $f : X \longrightarrow Y$. As we can read in Lerdermann (1996) in many fields of mathematics, morphism refers to a structure preserving mapping from one mathematical structure to another. In linear algebra, morphisms are linear transformations, in set theory morphisms are ordinary functions.

2.3.1 Definition. A group morphism from a group $(G, *_1)$ to a group $(G', *_2)$ is a function $f : G \longrightarrow G'$ such that $f(a *_1 b) = f(a) *_2 f(b)$ where the group operation on the left hand side of the equation is that of G and on the right hand side that of G'.

From this property, one can deduce that f maps the identity element e_G of G to the identity element $e_{G'}$ of G', and it also maps inverses to inverses in the sense that$f(g^{-1}) = f(g)^{-1}$.

Intuitively, the purpose of defining a group morphism the way we have is to create functions that preserve the algebraic structure. Hence, an equivalent definition of a group morphism is: the function $f : G \longrightarrow G'$ is a group morphism if whenever $a *_1 b = c$ we have $f(a) *_2 f(b) = f(c)$. In other words, the group G' in some sense has a similar algebraic structure as G and the morphism f preserves that. Given a group G we shall denote the set of all subgroups of G by $\mathrm{Sub}(G)$.

2.3.2 Theorem. *Let $f : G \longrightarrow G'$ be a morphism of groups, then for each subgroup $S \subseteq G$ the set*

$$f_*S = \{g' \in G' | g' = f(s) \quad \text{for some } s \in S\}$$

is a subgroup of G' called the image of S under f.

Proof. We use the Subgroup criterion. Since S is a subgroup of G it contains the identity e_G of G. Since the group morphism f takes the identity e_G of G to the identity $e_{G'}$ of G' , we have that $e_{G'} \in f_*S$. The inverse of the element $f(s)$ is the element $f(s^{-1})$ so that if $f(s)$ and $f(s')$ belong to f_*S, then $f(s')f(s)^{-1} = f(s')f(s^{-1}) = f(s's^{-1}) = f(m) \in f_*S$ since $s's^{-1} = m \in S$ as S is a subgroup. □

2.3.3 Definition. The kernel $f^*(0)$ of a group morphism $f : G \longrightarrow G'$ is defined to be the subset

$$\{g \in G | f(g) = e_{G'}\},$$

where $e_{G'}$ is the identity element of G'.

It is not difficult to show that the kernel $f^*(0)$ of a group morphism $f : G \longrightarrow G'$ is a normal subgroup of G. The assignment $S \mapsto f_*S$, as put by (MacLane and Birkhoff, 1999) , defines a function f_* on the set $\mathrm{Sub}(G)$ of subgroups of G to the set $\mathrm{Sub}(G')$ of subgroups of G'. When $S_1 \subset S_2$ in G, then $f_*S_1 \subset f_*S_2$ in G'; in other words, f_* is a morphism of inclusion. There is a reverse process. For each subgroup $T \subset G'$, the set $f^*(T) = \{g | g \in G, f(g) \in T\}$ is a subgroup of G, called the counterimage of T under f; it is sometimes called the inverse image of f. For example the inverse image of the trivial subgroup 0 of G' is just the kernel of f. The inverse image of any subgroup S' of a group G' always contains the kernel of f. Again f^* is a morphism of inclusion.

2.3.4 Example (An example of a group morphism)**.** The exponential map yields a group morphism from the group of real numbers $\mathbb{R}$ with addition to the group of non-zero real numbers $\mathbb{R}^*$ with multiplication.

$$\phi : (\mathbb{R}, +) \longrightarrow (\mathbb{R}^*, \times),\ x \mapsto e^x.$$

It is easy to see that ϕ is really a group morphism since

$$\phi(x + y) = e^{(x+y)} = e^x \times e^y = \phi(x) \times \phi(y).$$

The kernel is $\{0\}$ and the image consists of positive real numbers.

3. Groups via Cartesianness

3.1 Galois connections

3.1.1 Definition. A pair $(O, \leq)$ of a set O together with an order relation $\leq$ is said to be an ordered set (sometimes called a partially ordered set, Poset in short), if the following three properties hold in it:

(i) **reflexivity property** for each $a \in O$, $a \leq a$.

(ii) **Antisymmetric property** if $a \leq b$ and $b \leq a$ then $a = b$ for all $a, b \in O$.

(iii) **Transitivity property** if $a \leq b$ and $b \leq c$ then $a \leq c$ for all $a, b, c \in O$.

In any ordered set $(O, \leq)$, we will denote the largest element of O by 1 and the smallest element of O by 0.

3.1.2 Definition. A Galois connection f from an ordered set $O_1 = (O_1, \leq_1)$ to an ordered set $O_2 = (O_2, \leq_2)$ usually denoted by (f_*, f^*) consists of two order-preserving maps

$$O_1 \to O_2,\ x \mapsto fx$$

and

$$O_2 \to O_1,\ y \mapsto f^*y$$

such that the following conditions are satisfied:

(i) $x_1 \leq_1 x_2 \Rightarrow fx_1 \leq_2 fx_2$.

(ii) $y_1 \leq_2 y_2 \Rightarrow f^*y_1 \leq_1 f^*y_2$.

(iii) $fx \leq_2 y \Leftrightarrow x \leq_1 f^*y$.

3.1.3 Definition. Given a Galois connection $f : O_1 \longrightarrow O_2$ from an ordered set O_1 to another ordered set O_2, an element $x \in O_1$ is said to be closed under the Galois connection f if it satisfies the following equivalent conditions:

- $\star$ $x = f^*y$ for some $y \in O_2$.
- $\star$ $x = f^*(f_*x)$.

Similarly, an element $y \in O_2$ is closed under f when it satisfies the following equivalent conditions:

- $\star$ $y = fx$ for some $x \in O_1$.
- $\star$ $y = f_*(f^*y)$.

Left and right action preserve closed elements. In fact, it is easy to check that left and right action constitute a bijection between the set of closed elements in O_1 and the set of closed elements in O_2. We can easily show that any function $f : X \longrightarrow Y$ gives rise to a Galois connection from $(P(X), \subseteq)$ to $(P(Y), \subseteq)$, where $P(X)$ and $P(Y)$ denote the power set of X and the power set of Y respectively.

3.1.4 Example. Let $X = \{1, 2, 3\}$, $Y = \{2, 4, 6\}$ and $f : X \longrightarrow Y$ be defined by $f(x) = 2x$. Then

$$P(X) = \{A_1 = \varnothing, A_2 = \{1\}, A_3 = \{2\}, A_4 = \{3\}, A_5 = \{1, 2\}, A_6 = \{1, 3\}, A_7 = \{2, 3\}, A_8 = \{1, 2, 3\}\}$$

and

$P(Y) = B_1 = \{\varnothing, B_2 = \{2\}, B_3 = \{4\}, B_4 = \{6\}, B_5 = \{2,4\}, B_6 = \{2,6\}, B_7 = \{4,6\}, B_8 = \{2,4,6\}\}$

The Galois connection (f_*, f^*) induced is defined by

- ⋆ $f_* : P(X) \longrightarrow P(Y),\ A \mapsto \{2a|a \in A\}$.
- ⋆ $f^* : P(Y) \longrightarrow P(X),\ B \mapsto \{\frac{1}{2}b|b \in B\}$.

It is then easy to check that:

(i) $A_i \subseteq A_j \Rightarrow fA_i \subseteq fA_j$ for all $i, j = 1, 2, \cdots, 8$.

(ii) $B_i \subseteq B_j \Rightarrow f^*B_i \subseteq f^*B_j$ for all $i, j = 1, 2, \cdots, 8$.

(iii) $fA_i \subseteq B_i \Leftrightarrow A_i \subseteq f^*B_i$ for all $i = 1, 2, \cdots, 8$.

3.1.5 Definition. Let $(O, \leq)$ be an ordered set with elements S_1 and S_2. The meet of the elements S_1 and S_2 is the element R such that:

(i) $R \leq S_1$.

(ii) $R \leq S_2$.

(iii) If $Q \leq S_1$ and $Q \leq S_2$ then $Q \leq R$.

The meet of S_1 and S_2 is denoted by $S_1 \wedge S_2$. In other words, it is the largest element contained in both S_1 and S_2.

3.1.6 Definition. Let $(O, \leq)$ be an ordered set with elements S_1 and S_2. The join of the elements S_1 and S_2 is the element Q such that:

(i) $S_1 \leq Q$.

(ii) $S_2 \leq Q$.

(iii) If $S_1 \leq R$ and $S_2 \leq R$ then $Q \leq R$.

In other words, the join of S_1 and S_2 denoted by $S_1 \vee S_2$ is the smallest element containing both S_1 and S_2.

3.1.7 Theorem (Inverse images preserve meets)**.** *That is* $f^*(S' \wedge T') = (f^*S') \wedge (f^*T')$.

Proof. $f^*(S' \wedge T') \leq f^*S'$ since $S' \wedge T' \leq S'$. Equally, $f^*(S' \wedge T') \leq f^*T'$ since $S' \wedge T' \leq T'$. Hence, $f^*(S' \wedge T') \leq (f^*S') \wedge (f^*T')$. Conversely, to show that $(f^*S') \wedge (f^*T') \leq f^*(S' \wedge T')$ it is equivalent to show that $f_*((f^*S') \wedge (f^*T')) \leq S' \wedge T'$, which is equivalent to showing that $f_*((f^*S') \wedge (f^*T')) \leq S'$ and $f_*((f^*S') \wedge (f^*T')) \leq T'$. But $f_*((f^*S') \wedge (f^*T')) \leq S'$ holds since

$$f_*((f^*S') \wedge (f^*T')) \leq f_*(f^*S') \leq S'.$$

Similarly, $f_*((f^*S') \wedge (f^*T')) \leq T'$ holds since

$$f_*((f^*S') \wedge (f^*T')) \leq f_*(f^*T') \leq T'.$$

□

3.1.8 Theorem (Direct images preserve joins)**.** *That is* $f_*(S \vee T) = (f_*S) \vee (f_*T)$.
3.1.9 Definition. The dual of an ordered set $O = (O, \leq)$ denoted by $O^{OP} = (O, \geq)$ is still an ordered set only that the order relation is reversed.

3.1.10 Remark. Notice that Theorem 3.1.8 is "dual" to Theorem 3.1.7, and so having proved one of these Theorems, one could prove the other by duality.

3.2 Cartesianness

Given a morphism of groups $f : G \longrightarrow G'$, the structure which assigns to every group G an ordered set $\mathrm{Sub}(G)$ of subgroups of G and to every morphism $f : G \longrightarrow G'$ a Galois connection (f_*, f^*) is called the subgroup bifibration. The order relation in this ordered set $\mathrm{Sub}(G)$ of subgroups of G is subgroup inclusion "$\subseteq$". We wish to use cartesianness to explore more concepts of subgroups.

3.3 Notation

We shall use the following notation for any group morphism $f : G \longrightarrow G'$ from a group G to a group G':

- ⋆ 0_G, for the identity element of the group G.
- ⋆ $0_{G'}$, for the identity element of the group G'.
- ⋆ 0_{TG}, for the trivial subgroup $\{0_G\}$ of the group G.
- ⋆ $0_{TG'}$, for the trivial subgroup $\{0_{G'}\}$ of the group G'.
- ⋆ f_*1 and f_*G will be used interchangeably for the image of G under the morphism f.
- ⋆ $f^*(0)$ or $f^*(0_G)$ or $f^*(0_{TG})$ for the kernel $\ker(f)$ of f.
- ⋆ $\wedge$ will be used for 'meet' or intersection.
- ⋆ $\vee$ will be used for join.
- ⋆ $\leq$ and $\subseteq$ will be used interchangeably for containment.

There are two conditions that the subgroup bifibration satisfies:

3.4 Condition SB1

By **SB1**, we will refer to the conjunction of several conditions discussed below:

(i) The first thing that **SB1** demands is that for any group G, the ordered set $\mathrm{Sub}(G)$ of subgroups of G is actually a bounded lattice, $i.e.$ it should have a smallest element and a largest element. It is easy to see that the ordered set $\mathrm{Sub}(G)$ of subgroups of G satisfies this condition since the trivial subgroup $\{0\}$ is the smallest subgroup of G while G itself is the largest subgroup of the group G.

(ii) Secondly, this condition also demands that given any two elements S_1 and S_2 of $\mathrm{Sub}(G)$, we should be able to 'meet' them and get another element of $\mathrm{Sub}(G)$ which is easy to check if S_1 and S_2 are two subgroups of G:

3.4.1 Theorem. *If S_1 and S_2 are subgroups of a group G, then the intersection $S_1 \cap S_2$ is also a subgroup of G.*

Proof. (a) Suppose S_1 and S_2 are subgroups of a group G, then $0_G \in S_1 \cap S_2$ since $0_G \in S_1$ and $0_G \in S_2$. Hence, $S_1 \cap S_2$ contains the identity element.

(b) Given any two elements $a, b \in S_1 \cap S_2$, we have that $a, b \in S_1$ and $a, b \in S_2$ so that $ab \in S_1$ and $ab \in S_2$, since S_1 and S_2 are groups themselves. Thus, $ab \in S_1 \cap S_2$ and so the closure axiom holds.

(c) Finally, given that $x \in S_1 \cap S_2$, we have that $x \in S_1$ and $x \in S_2$ so that $x^{-1} \in S_1$ and $x^{-1} \in S_2$. Therefore, $x^{-1} \in S_1 \cap S_2$ and so each element in $S_1 \cap S_2$ has an inverse.

□

(iii) The condition (**SB1**) finally demands that given any two elements S_1 and S_2 of $\mathrm{Sub}(G)$, we should be able to 'join' them and get another element of $\mathrm{Sub}(G)$. Again this holds in our case (the case of groups), since to any two subgroups S_1 and S_2 of a group G, there is a least subgroup containing them both; that is, a subgroup L of G such that:

(a) $S_1 \subset L$.

(b) $S_2 \subset L$.

(c) $S_1 \subset M$ and $S_2 \subset M \Rightarrow L \subset M$.

This subgroup L is called the join of S_1 and S_2, written $L = S_1 \vee S_2$. It is usually much larger than the union of the sets S_1 and S_2.

3.4.2 Example. Consider the group G to be $\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, we have that $4\mathbb{Z}_{12} \vee 6\mathbb{Z}_{12} = \{0, 4, 8\} \vee \{0, 6\} = \{0, 2, 4, 6, 8, 10\} = 2\mathbb{Z}_{12}$, while $4\mathbb{Z}_{12} \cup 6\mathbb{Z}_{12} = \{0, 4, 8\} \cup \{0, 6\} = \{0, 4, 6, 8\}$ so that $4\mathbb{Z}_{12} \cup 6\mathbb{Z}_{12} \subset 4\mathbb{Z}_{12} \vee 6\mathbb{Z}_{12}$.

3.5 Condition SB2

We shall call this condition 'the cartesianness condition'. This condition demands that for any given morphism $f : G \longrightarrow G'$ of groups:

(1) If $\ker(f) \subseteq S$ then $f^*(f_*S) = S$

(2) If $T \subseteq f_*G$, then $f_*(f^*T) = T$

There is an equivalent way of stating part (1) of the cartesianness condition without demanding that the subgroup S of the group G should contain the kernel $\ker(f)$ of f in advance.

In the same manner, we shall also restate part (2) of the cartesianness condition without demanding that the subgroup S' of G' should be contained in the image f_*G of the group G under the morphism f in advance. So below are the equivalent ways of restating (1) and (2) respectively:

(I) $f^*(f_*S) = S \vee f^*(0)$.

(II) $f_*(f^*T) = T \cap f_*G$.

Since for any given morphism $f : G \longrightarrow G'$ of groups, we have that the kernel $\ker(f)$ is always contained in the group G, then by part (1) we have that $f^*(f_*G) = G$. Similarly, since the image f_*G of a group G under a group morphism $f : G \longrightarrow G'$ is always a subgroup of the group G' we have that $0_{G'}$ is always contained in f_*G so that the trivial subgroup $0_{TG'}$ is always contained in the image f_*G of G under the morphism f. Hence, by part (2) of cartesianness we have that $f_*(f^*0_{TG'}) = 0_{TG'}$.

By now it should be clear that the smallest elements of $\mathrm{Sub}(G)$ and $\mathrm{Sub}(G')$ denoted by 0_{TG} and $0_{TG'}$ respectively are simply the trivial subgroups of G and G'. The context will always tell which trivial subgroup we are referring to; for example if we take inverse image $f^*(0)$ then we know that this trivial subgroup is actually the trivial subgroup of G'. Similarly, taking direct image f_*0 would imply that the trivial subgroup in question is the trivial subgroup of G. Thus, from now onwards we drop out the notation of 0_{TG} and $0_{TG'}$ and simply use 0 for both of them. Below are the immediate consequences of the cartesianness condition :

- ⋆ $f^*(f_*G) = G$ since $f^*(0) \subseteq G$ always.
- ⋆ $f_*(f^*0) = 0$ since $0 \subseteq f_*G$ always.

3.6 The induced Galois connection

We stated above that in this structure of subgroup bifibration, every morphism $f : G \longrightarrow G'$ of groups induces a Galois connection (f_*, f^*). That is, given S_1 , S_2 and S subgroups of G and T_1, T_2 and T subgroups of G', the following conditions are satisfied:

(i) $S_1 \subset S_2 \Rightarrow f_*S_1 \subset f_*S_2$.

(ii) $T_1 \subset T_2 \Rightarrow f^*T_1 \subset f^*T_2$.

(iii) $f_*S \subset T \Leftrightarrow S \subset f^*T$.

3.6.1 Theorem. *A group morphism $f : G \longrightarrow G'$ between two groups induces a Galois connection.*

$$\mathrm{Sub}(G) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{Sub}(G')$$

Proof. We check all the three axioms that a Galois connection should satisfy as below:

(i) Suppose $S \subseteq T$. We need to show that $f_*S \subseteq f_*T$. Let $m \in f_*S$. Then there exist $s' \in S$ such that $fs' = m$. But $s' \in S \Rightarrow s' \in T$ since $S \subseteq T$ by hypothesis. Hence $fs' = m \in f_*T$. Since we picked the m arbitrarily from f_*S we have that $f_*S \subseteq f_*T$.

(ii) Let S' and T' be two subgroups of G' and suppose that $S' \subseteq T'$. We need to show that $f^*S' \subseteq f^*T'$. Let $m \in f^*S'$, then $fm \in S' \subseteq T'$. Thus, $fm \in T' \Rightarrow m \in f_*T'$ and since m was picked arbitrarily in f^*S' we have that $f^*S' \subseteq f^*T'$ as required.

(iii) First suppose that $f_*S \subseteq T$, where S is a subgroup of G and T is a subgroup of G'. With this supposition we should show that $S \subseteq f^*T$. Let $a \in S$. Then $fa \in f_*S \subseteq T$. Hence, $a \in f^*T$ and since a was arbitrarily chosen from S we have that $S \subseteq f^*T$. Conversely, suppose $S \subseteq f^*T$.

We should show that $f_*S \subseteq T$. Let $n \in f_*S$ then there exists some s in S such that $fs = n$. But s being an element of S implies that $s \in f^*T$ since $S \subseteq f^*T$. And thus, $fs = n \in T$ so that $f_*S \subseteq T$.

□

Hence, by (i) and (ii) of Theorem 3.6.1 we see that both f_* and f^* are morphisms of inclusion "$\subseteq$".

3.6.2 Theorem. *If a morphism of groups $f : G \longrightarrow G'$ satisfies the following four conditions :*

(a) $f_*(A \vee B) = f_*A \vee f_*B$.

(b) $f^*(A' \cap B') = f^*A' \cap f^*B'$.

(c) $f_*(f^*B') = B' \wedge f_*G$.

(d) $f^*(f_*A) = A \vee f^*(0)$.

then (f_*, f^*) *forms a Galois connection.*

Proof. So we check that all the three axioms of a Galois connection hold:

(i) Suppose $A_1 \leq A_2$ then

$$A_2 = A_1 \vee A_2 \Rightarrow f_*A_1 \leq (f_*A_1) \vee (f_*A_2) = f_*(A_1 \vee A_2) = f_*A_2$$

(ii) Suppose $B_1 \leq B_2$ then $B_1 = B_1 \wedge B_2$. Hence

$$f^*B_1 = f^*(B_1 \wedge B_2) = (f^*B_1) \wedge (f^*B_2) \leq f^*B_2.$$

(iii) Here we need to show that $f_*A \leq B \Leftrightarrow A \leq f^*B$. First suppose that $f_*A \leq B$. Then

$$A \leq A \vee f^*(0) = f^*(f_*A) \leq f^*B.$$

The last inequality holds by applying axiom (ii) of a Galois connection to our supposition that $f_*A \leq B$. Conversely, suppose that $A \leq f^*B$. Then

$$f_*A \leq f_*(f^*B) = B \wedge f_*G \leq B.$$

Similarly, the first inequality holds by applying axiom (i) of a Galois connection to the supposition that $A \leq f^*B$; and the equality holds by part (c) of the hypothesis.

□

3.7 Isomorphism theorems for groups

In this section we wish to revisit the isomorphism theorems for groups. Cartesianness allows us to prove the theorem below:

3.7.1 Theorem. *If two subgroups S_1, S_2 both contain the kernel $f^*(0)$ of a group morphism $f : G \longrightarrow G'$, then*

$$f_*(S_1 \cap S_2) = (f_*S_1) \cap (f_*S_2).$$

Proof. Since $f_*(0) \leq S_1$ and $f_*(0) \leq S_2$ we have that $f^*(0) \leq S_1 \cap S_2$. Since there is a bijection between subgroups of G containing $f^*(0)$ and subgroups of G' contained in the image f_*G ($i.e.$ between closed elements of $\text{Sub}(G)$ and closed elements of $\text{Sub}(G')$), we have that $f_*(S_1 \cap S_2) \leq f_*G$ and $(f_*S_1) \cap (f_*S_2) \leq f_*G$. Hence,

$$f_*f^*(f_*(S_1 \cap S_2)) = f_*(S_1 \cap S_2)$$

and

$$f_*f^*((f_*S_1) \cap (f_*S_2)) = (f_*S_1) \cap (f_*S_2).$$

by cartesianness. Therefore,

$$f_*(S_1 \cap S_2) = f_*(f^*f_*S_1 \cap f^*f_*S_2) = f_*f^*(f_*S_1 \cap f_*S_2) = f_*S_1 \cap f_*S_2.$$

The first equality holds since $S_1 = f^*(f_*S_1)$ and $S_2 = f^*(f_*S_2)$ by cartesianness. □

3.7.2 Theorem. *Any morphism $f : G \longrightarrow G'$ of groups yields a bijection*

$$\{S | f^*(0) \leq S \leq G\} \cong \{T | 0 \leq T \leq f_*G\}$$

which assigns to each subgroup S containing the kernel $f^*(0)$ of f its image f_*S, and whose inverse assigns to each subgroup T contained in the image f_*G of f its counterimage f^*T. Each of these sets $\text{Sub}(G)$ and $\text{Sub}(G')$ of subgroups is closed under intersection and join, and this bijection is an isomorphism of these operations.

Proof. In the subsection of Galois connections we defined a closed element $x \in O_1$ under the Galois connection $f : O_1 \longrightarrow O_2$ to be an element x such that $x = f^*(f_*x)$. We easily observe that under the Galois connection

$$\text{Sub}(G) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \text{Sub}(G') \ ,$$

the subgroups of G satisfying this condition are those containing the kernel $f^*(0)$ of f by the cartesinness condition.

Similarly, we observe that the subgroups of G' satisfying the condition that $S' = f_*(f^*S')$ are those subgroups of G' contained in the image f_*G of G under f. Therefore, the set of closed elements in $\text{Sub}(G)$ is $\{S | f^*(0) \leq S \leq G\}$. Similarly, the set of closed elements in $\text{Sub}(G')$ is $\{T | 0 \leq T \leq f_*G\}$. If $f^*(0) \leq S$, then $f^*(f_*S) = S$. Also, if $T \leq f_*(1)$, then $f_*(f^*T) = T$. The assignment $S \mapsto f_*S$ is therefore a bijection as claimed. □

3.7.3 Definition. If a subgroup N of G is normal, then the set of all left cosets xN of N denoted by G/N forms a group called the quotient group.

The name of the Theorem below and other ideas of quotient groups came from MacLane and Birkhoff (1999) in their Algebra book.

3.7.4 Theorem (The main theorem of quotients)**.** *The essential property of the quotient group is that every morphism $f : G \longrightarrow G'$ with domain G and kernel N has its image isomorphic to G/N. That is $G/N \cong f_*G$.*

Proof. Define the group morphism $\theta : G/N \longrightarrow f_*G$ by $\theta(xN) = fx$. We need to show that:

- ⋆ θ is well defined.
- ⋆ θ is really a morphism of groups.
- ⋆ θ is a surjection.
- ⋆ and that it is an injection.

First we show that θ is well defined. Suppose $xN = yN$. Then

$$x^{-1}y \in N \Rightarrow f(x^{-1}y) = (fx^{-1})(fy) = (fx)^{-1}fy = e_{G'} \Rightarrow fx = fy \Rightarrow \theta(xN) = \theta(yN).$$

Next we show that θ is really a morphism of groups:

$$\theta(xNyN) = \theta(xyN) = f(xy) = fxfy = \theta(xN)\theta(yN).$$

We now show that θ is a surjection. This is easy to see since

$$\theta_*(G/N) = \{\theta(xN) : x \in G\} = \{fx : x \in G\} = f_*(G).$$

We finally see that θ is an injection since the kernel of θ is the trivial subgroup N of G/N. And so θ is an isomorphism between G/N and f_*G. □

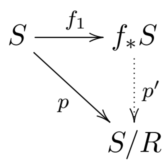
This property has a more general formulation: the projection $p : G \longrightarrow G/N$ defined by $g \mapsto gN$ has kernel $f^*(0) = N$, and every morphism f on G with kernel $f^*(0) = N$ factors uniquely through the projection p. This property is the basic result from which all properties of quotient groups have been developed.

3.7.5 Definition. Let $f : G \longrightarrow G'$ be a morphism of groups. In case the image f_*G is normal in G', the quotient group G'/f_*G is called the cokernel of f. The projection $q : G' \longrightarrow G'/f_*G$ onto the cokernel has the property that the composite qf is trivial or null morphism (maps all elements of G to 0) and q is universal with this property-just like the insertion $i : N \longrightarrow G$ of the kernel of f is universal among morphisms g to G with fg a null morphism.

3.7.6 Theorem. *Let $f : G \longrightarrow G'$ be a morphism of groups with kernel $N \lhd G$, while R and S are subgroups of G with $N \subset R \subset S \subset G$. Then $R \lhd S$ if and only if $f_*R \lhd f_*S$.*

Proof. ($\Leftarrow$) First define the epimorphism $f_1 : S \longrightarrow f_*S$ given by restricting f to S. Now suppose $f_*R \lhd f_*S$; construct the quotient group f_*S/f_*R and the corresponding projection $q : f_*S \longrightarrow f_*S/f_*R$. The composite $q \circ f_1 : S \longrightarrow f_*S/f_*R$ has kernel R. Hence; as the kernel of a morphism, R must be normal in S. Thus $R \lhd S$.

($\Rightarrow$) Conversely, suppose $R \lhd S$. Since the epimorphism $f_1 : S \longrightarrow f_*S$ has kernel N, it is universal for this kernel. Hence, the projection $p : S \longrightarrow S/R$ factors through f_1 as:



$p = p' \circ f_1$, where the morphism $p' : f_*S \longrightarrow S/R$ has kernel f_*R. Thus, $f_*R \lhd f_*S$.

□

3.7.7 Corollary. Let $f : G \longrightarrow G'$ be a morphism of groups with kernel $N \lhd G$. If R and S are subgroups of G with $N \subset R \subset S \subset G$ and $R \lhd S$, then $S/R \cong f_*S/f_*R$, $Rs \longmapsto (f_*R)(fs)$.

Proof. Since $R \lhd S$ we have that $f_*R \lhd f_*S$ by Theorem 3.7.6. Now we need to find the isomorphism between S/R and f_*S/f_*R. Construct the epimorphism $f_1 : S \longrightarrow f_*S$, the quotient group f_*S/f_*R, the corresponding projection $q : f_*S \longrightarrow f_*S/f_*R$ as we did in Theorem 3.7.6. Since the composite $q \circ f_1 : S \longrightarrow f_*S/f_*R$ has kernel R and by the main Theorem of quotients again one then has $S/R \cong f_*S/f_*R$. □

3.7.8 Theorem (The third isomorphism theorem for groups)**.** *If R and S are normal subgroups of a group G and $R \subset S$ then, $G/S = \frac{G/R}{S/R}$.*

Proof. In Corollary 3.7.7, we proved that if S and R are normal subgroups of G with $R \subset S$ and $f : G \longrightarrow G'$ is a morphism of groups then $G/S \cong f_*G/f_*S$. Also by defining projections:

$$p_1 : G \longrightarrow G/R, \text{with kernel} \quad R.$$

and

$$p_2 : S \longrightarrow S/R, \text{with kernel} \quad R.$$

we then have $f_*G \cong G/R$ and $f_*S \cong S/R$ by the main Theorem of quotients. Substituting into $G/S \cong f_*G/f_*S$, we then have that $G/S \cong \frac{G/R}{S/R}$ as claimed. □

4. Diagram lemmas

4.1 Exact Sequences

In the definitions and theorems below, we are working in the context of groups as we will work with groups in our diagrams; and so the term ‘morphism’ will imply a morphism of groups.
4.1.1 Definition. We say that a morphism $f : X \longrightarrow Y$ is a null morphism when the image of f is zero. That is when $f_*(1) = 0$.

Consider a composable pair of morphisms:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

Note that the composite $g \circ f$ is a null morphism if and only if the image of f is wholly contained in the kernel of g i.e. $f_*(1) \leq g^*(0)$. By demanding that this inequality becomes an equality we obtain the concept of an exact sequence.

4.1.2 Definition. Any sequence of morphisms as above such that the image of the first morphism coincides with the kernel of the second is called an exact sequence. A sequence of morphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

is said to be exact at C_n for some n when the sequence

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

is exact. The whole sequence

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

is said to be exact when any two successive morphisms in it form an exact sequence.

4.2 Injections and Surjections in terms of Exact sequences

Injectivity and surjectivity of morphisms can be expressed in terms of exact sequences as follows:

4.2.1 Theorem. *A morphism g is an injection if and only if it fits in the exact sequence*

$$0 \xrightarrow{f} Y \xrightarrow{g} Z\ .$$

Proof. Suppose g is an injection. We need to show that the sequence is exact. But g an injection implies that $g^*(0) = 0$. We also know that $f_*(1) = 0$. Hence $f_*(1) = 0 = g^*(0)$, and so the sequence is exact. Conversely, suppose the sequence is exact. We should show that g is an injection. The sequence is exact implies $f_*(1) = g^*(0)$. But $f_*(1) = 0$; hence $g^*(0) = f_*(1) = 0$ and thus g is injective. □

4.2.2 Theorem. *A morphism f is a surjection if and only if it fits in the exact sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} 0\,.$$

Proof. Suppose the sequence is exact. Then $f_*(1) = g^*(0)$. But $g^*(0) = Y$ since a null morphism has the whole of its domain as the kernel. Thus $f_*(1) = g^*(0) = Y$. And so f is a surjection. Conversely, suppose f is a surjection. We need to show that the sequence is exact. Now f is a surjection implies that $f_*(1) = Y$. We also know that $g^*(0) = Y$ as g is a null morphism. Hence $f_*(1) = Y = g^*(0)$. Thus, the sequence is exact. □

Theorems 4.2.1 and 4.2.2 are crucial in the concept of diagram chasing. Consider the commutative diagram below:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle t_1} & & \\
0 & \xrightarrow{f_1} & P_1 & \xrightarrow{f_2} & P_2 & \xrightarrow{f_3} & P_3 & \xrightarrow{f_4} & 0 \\
 & & \downarrow{\scriptstyle r_2} & & \downarrow{\scriptstyle s_2} & & \downarrow{\scriptstyle t_2} & & \\
0 & \xrightarrow{g_1} & Q_1 & \xrightarrow{g_2} & Q_2 & \xrightarrow{g_3} & Q_3 & \xrightarrow{g_4} & 0 \\
 & & \downarrow{\scriptstyle r_3} & & \downarrow{\scriptstyle s_3} & & \downarrow{\scriptstyle t_3} & & \\
0 & \xrightarrow{h_1} & R_1 & \xrightarrow{h_2} & R_2 & \xrightarrow{h_3} & R_3 & \xrightarrow{h_4} & 0 \\
 & & \downarrow{\scriptstyle r_4} & & \downarrow{\scriptstyle s_4} & & \downarrow{\scriptstyle t_4} & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

If we assume that all the rows are exact then by Theorem 4.2.1, f_2, g_2 and h_2 are all injections; while f_3, g_3 and h_3 are all surjections by Theorem 4.2.2. Similarly, if we assume that all the columns are exact then r_2, s_2 and t_2 are all injections by Theorem 4.2.1; while r_3, s_3 and t_3 are surjections by Theorem 4.2.2. Commutativity of a diagram as the one above means that an element, say, of P_1 is mapped to the same element in Q_2 whether it goes through $s_2 f_2$ or through $g_2 r_2$.

4.2.3 Definition. A short exact sequence is an exact sequence of the form

$$0 \longrightarrow K \xrightarrow{k} X \xrightarrow{f} Y \longrightarrow 0\,.$$

From Theorem 4.2.1 and Theorem 4.2.2, we note that the sequence is exact precisely when k is an injection, f is a surjection, and the image of k is the coimage of f.

4.3 Diagram chasing

In the concept of diagram chasing, at least at this stage, we shall often make use of the theorem below:

4.3.1 Theorem. *If $f : X \longrightarrow Y$ is a morphism then $f^*(y \wedge f_*(1)) = f^*(y)$.*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \xrightarrow{l} & X' & \xrightarrow{g} & Y' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \\
0 & \longrightarrow & K & \xrightarrow{k} & X & \xrightarrow{f} & Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \\
0 & \longrightarrow & K'' & \xrightarrow{m} & X'' & \xrightarrow{h} & Y'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Given a commutative diagram as above in which the composite of any two successive arrows is null, if all the rows are exact as well as two out of the three columns, then also the third column is exact.

Proof. We shall assume that the first and the second column are exact and prove exactness of the third. We prove that b is an injection as done by (Van der Linden, 2013).

$$
\begin{aligned}
b^*(0) =& g_*g^*b^*(0) \quad \text{since } g \text{ is a surjection.}\\
=& g_*a^*f^*(0) \quad \text{by commutativity.}\\
=& g_*a^*k_*(1) \quad \text{by horizontal exactness at } X.\\
=& g_*a^*(k_*(1) \wedge a_*(1)) \quad \text{by Theorem 4.3.1.}\\
=& g_*a^*(k_*(1) \wedge x^*(0)) \quad \text{by vertical exactness at } X.\\
=& g_*a^*k_*k^*(k_*(1) \wedge x^*(0)) \quad \text{by cartesianness.}\\
=& g_*a^*k_*k^*x^*(0) \quad \text{by Theorem 4.3.1.}\\
=& g_*a^*k_*z^*m^*(0) \quad \text{by commutativity.}\\
=& g_*a^*k_*z^*(0) \quad \text{since } m \text{ is an injection.}\\
=& g_*a^*k_*c_*(1) \quad \text{by vertical exactness at } K.\\
=& g_*a^*a_*l_*(1) \quad \text{by commutativity.}\\
=& g_*l_*(1) \quad \text{since } a \text{ is an injection.}\\
=& g_*g^*(0) \quad \text{by horizontal exactness at } X'.\\
=& 0 \quad \text{by cartesianness.}
\end{aligned}
$$

□

This is not an easy way to do it. The two Theorems below help us to develop a new technique:

4.3.2 Theorem. *The following statements are equivalent:*

(I) The sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ *is exact.*

(II) In the sequence $(\mathrm{Sub}(X), 0) \xrightarrow{f_*} (\mathrm{Sub}(Y), 0) \xrightarrow{g_*} (\mathrm{Sub}(Z), 0)$, $g_*(y) = 0$ *if and only if* $\exists x \in \mathrm{Sub}(X)$ *such that* $f_*(x) = y$.

$(I) \Rightarrow (II)$.

Proof. Assume condition (I) and suppose $g_*(y) = 0$, we should show that $\exists\, x \in \mathrm{Sub}(X)$ such that $f_*(x) = y$. Now,

$$g_*(y) = 0 \Rightarrow g_*(y) \leq 0 \Rightarrow y \leq g^*(0) \Rightarrow y \leq f_*(1).$$

Hence, by cartesianness we have that $f_*(f^*y) = y \Rightarrow f_*(x) = y$ by taking $x = f^*(y)$.

Suppose the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact and that $f_*(x) = y$ for some $x \in \mathrm{Sub}(X)$. We should show that $g_*(y) = 0$. But $x \leq 1$ for any x and so $y = f_*(x) \leq f_*(1)$. This implies that $y = f_*(x) \leq g^*(0)$ since $g^*(0) = f_*(1)$ by (I). Hence, $y \leq g^*(0) \Rightarrow g_*(y) \leq 0$. But $g_*(y) \leq 0 \Rightarrow g_*(y) = 0$ since 0 is the smallest element. Thus, $(I) \Rightarrow (II)$. Conversely, suppose (II) holds. We should show that the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact; i.e. we show that $f_*(1) = g^*(0)$. Let $y = g^*(0)$, then $g_*(y) = 0$. Thus $f_*(x) = y$ for some $x \in \mathrm{Sub}(X)$ by (II). Now since $x \leq 1$ for any $x \in \mathrm{Sub}(X)$, we have that

$$g^*(0) = y = f_*(x) \leq f_*(1).$$

Hence, $g^*(0) \leq f_*(1)$. To show that $f_*(1) \leq g^*(0)$, let $y = f_*(1)$. So $g_*(y) = 0$ by $(II) \Rightarrow g_*(f_*(1)) \leq 0 \Rightarrow f_*(1) \leq g^*(0)$. By mutual containment, we conclude that $f_*(1) = g^*(0)$ as desired. □

4.3.3 Theorem (The dual of Theorem 4.3.2)**.** *The following statements are equivalent.*

(i) The sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ *is exact.*

(ii) In the sequence $(\mathrm{Sub}(X), 1) \xleftarrow{f^*} (\mathrm{Sub}(Y), 1) \xleftarrow{g^*} (\mathrm{Sub}(Z), 1)$, $f^*(y) = 1$ *if and only if* $\exists\, z \in \mathrm{Sub}(Z)$ *such that* $g^*(z) = y$.

$(i) \Rightarrow (ii)$.

Proof. Suppose (i) holds. We need to show that (ii) also holds. First suppose that $f^*(y) = 1$, then

$$1 \leq f^*(y) \Rightarrow f_*(1) \leq y \Rightarrow g^*(0) \leq y.$$

Thus, by cartesinness $g^*(g_*(y)) = y \Rightarrow g^*(z) = y$ where $z = g_*(y)$. Conversely, suppose $\exists\, z \in \mathrm{Sub}(Z)$ such that $g^*(z) = y$. We should show that $f^*(y) = 1$. But

$$0 \leq z \Rightarrow g^*(0) \leq g^*(z) = y \Rightarrow f_*(1) = g^*(0) \leq y \Rightarrow 1 \leq f^*(y) \Rightarrow 1 = f^*(y).$$

□

$(ii) \Rightarrow (i)$.

Proof. To show that the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an exact sequence, we should show that $f_*(1) = g^*(0)$. Let $y = g^*(0)$, then by taking 0 to be the $z \in \mathrm{Sub}(Z)$, we have that

$$1 = f^*(y) \Rightarrow 1 = f^*(g^*(0)) \Rightarrow 1 \leq f^*(g^*(0)) \Rightarrow f_*(1) \leq g^*(0).$$

To show that $g^*(0) \leq f_*(1)$ take $y = f_*(1)$. Then $f^*f_*(1) = 1 \Rightarrow f_*(1) = g^*(z)$ for some z, by condition (ii). Also,

$$0 \leq z \Rightarrow g^*(0) \leq g^*(z) = f_*(1) \Rightarrow g^*(0) \leq f_*(1).$$

Thus, $f_*(1) = g^*(0)$. □

4.4 Partial diagram lemmas for pointed sets

A pointed set is a set $X = (X, p_x)$ with a distinguished element p_x called a base point. A morphism $(X, p_x) \xrightarrow{f} (Y, p_y)$ of pointed sets is a function $f : X \longrightarrow Y$ from a set X to a set Y such that $f(p_x) = p_y$. It has two separate maps:

$$X \longmapsto (\mathrm{Sub}(X), 0), \quad X \longmapsto (\mathrm{Sub}(X), 1).$$

4.5 Notion of an exact sequence of pointed sets

In sequences of pointed sets exactness of the sequence

$$(X, p_x) \xrightarrow{f} (Y, p_y) \xrightarrow{g} (Z, p_z)$$

at (Y, p_y), occurs precisely when $g(y) = p_z \Leftrightarrow \exists\, x \in X \quad \text{such that} \quad f(x) = y$. In this sequence, we note that p_x, p_y and p_z are the distinguished elements of the sets X, Y and Z. Hence, taking the distinguished element to be 0, we have that the sequence

$$(\mathrm{Sub}(X), 0) \xrightarrow{f_*} (\mathrm{Sub}(Y), 0) \xrightarrow{g_*} (\mathrm{Sub}(Z), 0)$$

is exact when $g_*(y) = 0 \Leftrightarrow \exists\, x \in \mathrm{Sub}(X)$ such that $f_*(x) = y$, which is the condition we proved to be equivalent to the condition that the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact in Theorem 4.3.2. Also, in the dual of Theorem 4.3.2 we proved that in the sequence

$$(\mathrm{Sub}(X), 1) \xleftarrow{f^*} (\mathrm{Sub}(Y), 1) \xleftarrow{g^*} (\mathrm{Sub}(Z), 1)$$

if $f^*(y) = 1 \Leftrightarrow \exists\, z \in \mathrm{Sub}(Z)$ such that $g^*(z) = y$ then the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact; only that we did not mention it explicitly that we considered 1 to be the distinguished element for all the ordered sets $\mathrm{Sub}(X), \mathrm{Sub}(Y)$ and $\mathrm{Sub}(Z)$. This coincidence gives rise to the new technique below:

4.6 The new technique

As a consequence of Theorem 4.3.2 and its dual, we shall now do diagram chasing in a slightly different and extremely easy way. Exactness of the sequence

$$a \overset{f}{\longmapsto} b \overset{g}{\longmapsto} c$$

means that g maps b to 0 if and only if b 'comes from' some a through f. Stated explicitly, we are saying that:

- ⋆ if f maps a to b, then g maps b to 0.
- ⋆ if g maps b to 0, then b 'comes from' some a through f.

4.7 Applying diagram lemmas for pointed sets in two ways to get complete diagram lemmas for groups

With our new technique from pointed sets, we no longer look at diagrams , for example, the Nine Lemma

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \xrightarrow{l} & X' & \xrightarrow{g} & Y' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a} & & {\scriptstyle b}\downarrow & & \\
0 & \longrightarrow & K & \xrightarrow{k} & X & \xrightarrow{f} & Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle x} & & {\scriptstyle y}\downarrow & & \\
0 & \longrightarrow & K'' & \xrightarrow{m} & X'' & \xrightarrow{h} & Y'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

with the view that the objects are groups or vector spaces. Instead, we look at them in the context that the objects are pointed sets. So we replace all the objects in the diagram above with the corresponding pointed sets; which happen to be ordered sets of subgroups in our case. We also saw from above that when the base point in question is the bottom element (the smallest element), then the direction of the direct map f_* is the same as that of the morphism f. Hence, the diagram below.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\mathrm{Sub}(K'),0) & \xrightarrow{l_*} & (\mathrm{Sub}(X'),0) & \xrightarrow{g_*} & (\mathrm{Sub}(Y'),0) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle a_*} & & {\scriptstyle b_*}\downarrow & & \\
0 & \longrightarrow & (\mathrm{Sub}(K),0) & \xrightarrow{k_*} & (\mathrm{Sub}(X),0) & \xrightarrow{f_*} & (\mathrm{Sub}(Y),0) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle z_*} & & \downarrow{\scriptstyle x_*} & & {\scriptstyle y_*}\downarrow & & \\
0 & \longrightarrow & (\mathrm{Sub}(K''),0) & \xrightarrow{m_*} & (\mathrm{Sub}(X''),0) & \xrightarrow{h_*} & (\mathrm{Sub}(Y''),0) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

On the other hand, if we use the top element (the largest element), before we replace the objects with the induced pointed sets we have to reverse the direction of the arrows as we saw above (in the dual of Theorem 4.3.2) that the inverse map f^* of the map f goes in the opposite direction of the morphism

f. Hence the diagram below:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & (\mathrm{Sub}(K'),1) & \xleftarrow[l^*]{} & (\mathrm{Sub}(X'),1) & \xleftarrow[g^*]{} & (\mathrm{Sub}(Y'),1) & \longleftarrow & 0 \\
 & & c^* \uparrow & & a^* \uparrow & & \uparrow b^* & & \\
0 & \longleftarrow & (\mathrm{Sub}(K),1) & \xleftarrow[k^*]{} & (\mathrm{Sub}(X),1) & \xleftarrow[f^*]{} & (\mathrm{Sub}(Y),1) & \longleftarrow & 0 \\
 & & z^* \uparrow & & x^* \uparrow & & \uparrow y^* & & \\
0 & \longleftarrow & (\mathrm{Sub}(K''),1) & \xleftarrow[m^*]{} & (\mathrm{Sub}(X''),1) & \xleftarrow[h^*]{} & (\mathrm{Sub}(Y''),1) & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

4.8 Duality

Given a sequence of groups $X \xrightarrow{f} Y \xrightarrow{g} Z$, lets take a look at Theorem 4.3.2 and its dual which we proved above. We saw that direct images maintain the direction of the functions from which they originate when we change the objects from mere groups to ordered sets of subgroups:

$$(\mathrm{Sub}(X),0) \xrightarrow{f_*} (\mathrm{Sub}(Y),0) \xrightarrow{g_*} (\mathrm{Sub}(Z),0).$$

On the other hand, we observed that when we change the objects from mere groups to ordered sets of subgroups, inverse images go in the opposite direction to the direction of the original sequence:

$$(\mathrm{Sub}(X),1) \xleftarrow{f^*} (\mathrm{Sub}(Y),1) \xleftarrow{g^*} (\mathrm{Sub}(Z),1).$$

From these two sequences, we observe that it takes a 180° rotation for the second sequence to obtain the direction of the original sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$. This concept leads to the definition below.

4.8.1 Definition. In any commutative diagram, two objects will be said to be dual if and only if when we rotate the original diagram through a 180° angle the two objects swap or exchange positions.

This concept of dual objects will be very useful in diagram chasing (especially in more complicated diagram lemmas) in the sense that when we find it difficult to prove exactness at a certain object, we may try to show exactness at its dual.

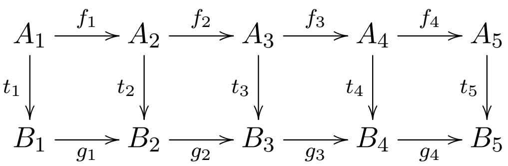
4.9 The Five Lemma

Given a commutative diagram below in which the rows are exact sequences, prove that:

(a) if t_1 is a surjection, while t_2 and t_4 are injections, then t_3 is also an injection.

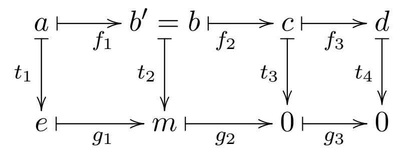
(b) if t_5 is an injection, while t_2 and t_4 are surjections, then t_3 is also a surjection.

(c) if t_1 is a surjection, t_5 is an injection, and t_2 and t_4 are bijections, then t_3 is also a bijection.



Proof in the context of pointed sets:

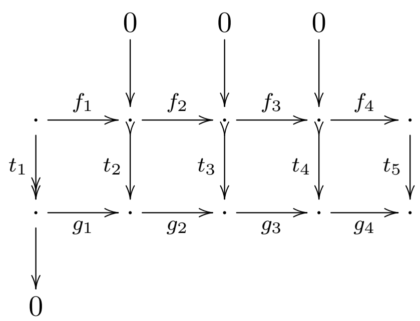
Proof. Proof for part (a):



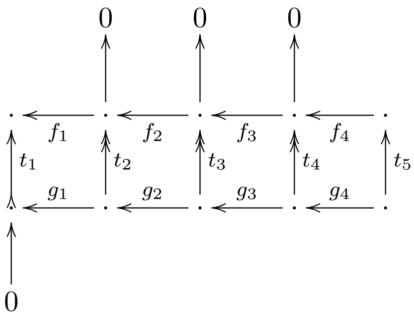
- ⋆ Suppose t_3 maps c to 0; we should show that $c = 0$.
- ⋆ g_3 maps the 0 to 0 horizontally.
- ⋆ By commutativity, $t_4 f_3$ maps c to 0 for some d.
- ⋆ Then $d = 0$ since t_4 is an injection.
- ⋆ Since f_3 maps c to $d = 0$, we have that c comes from some b through f_2.
- ⋆ By commutativity $g_2 t_2$ maps b to 0 i.e. t_2 maps b to m .
- ⋆ Since g_2 maps m to 0, m comes from some e through g_1.
- ⋆ Since t_1 is a surjection, e comes from some a.
- ⋆ By commutativity, $t_2 f_1$ maps a to m i.e. t_2 maps b' to m.
- ⋆ Since t_2 is an injection we have that $b' = b$.
- ⋆ Since b comes from a through f_1, f_2 maps b to 0.
- ⋆ Hence, $c = 0$ as required.

In the context of groups: The proof above proves part (a) of this Theorem in the context of groups directly.

Proof for part (b): The problem we have solved can be represented by the diagram below: If column 1, column 2 and column 4 are all exact, then column 3 is also exact.



If we reverse all the arrows, we have the diagram below:



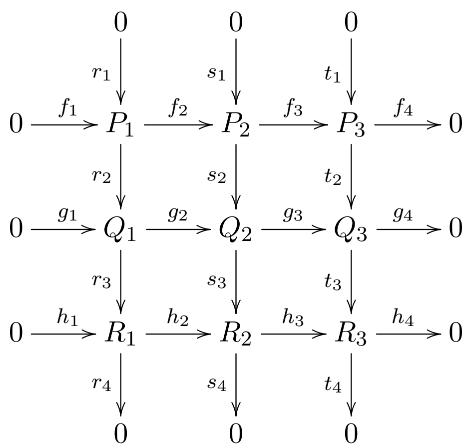
and the statement we proved in part (a) now reads: if t_1 is an injection, while t_2 and t_4 are surjections, then t_3 is also a surjection; which is exactly what we wanted to show in part (b) of this Theorem.

Proof for part (c): The proof for the last part is easy since t_2 and t_4 bijections and t_1 a surjection gives us the conditions we needed in part (a) and so t_3 is an injection. Also, t_2 and t_4 bijections and t_5 an injection gives us the conditions we needed in part (b); and so t_3 is a surjection. Therefore, t_3 is indeed a bijection.

□

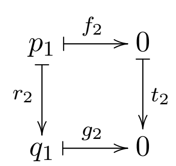
4.10 The 3×3 Lemma

Given a commutative diagram below :



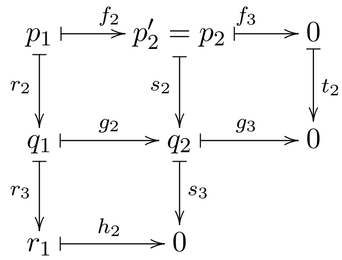
suppose that all the columns are exact and that the second and third rows are exact. We need to prove that the first row is also exact.

Proof. (i) Exactness at P_1:

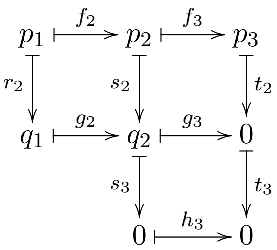


- ⋆ To show exactness at P_1 it is enough to show that f_2 is an injection.
- ⋆ Suppose f_2 maps p_1 to 0; we should show that $p_1 = 0$.
- ⋆ s_2 maps the 0 to 0 vertically.
- ⋆ By commutativity, we have that $g_2 r_2$ maps p_1 to 0 for some q_1.
- ⋆ Since g_2 is an injection, we have that $q_1 = 0$.
- ⋆ Since r_2 is an injection, we have that $p_1 = 0$.
- ⋆ And so, f_2 is an injection as required.

(ii) Exactness at P_2:



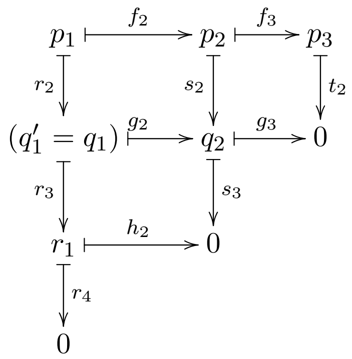
- ⋆ Suppose f_3 maps p_2 to 0, we should show that p_2 comes from some p_1 through f_2
- ⋆ t_2 maps the 0 to 0, vertically.
- ⋆ By commutativity, $g_3 s_2$ maps p_2 to 0 for some q_2.
- ⋆ since q_2 comes from p_2 through s_2 , s_3 maps q_2 to 0.
- ⋆ Since g_3 maps q_2 to 0, we have that q_2 comes from some q_1 through g_2.
- ⋆ By commutativity $h_2 r_3$ maps q_1 to 0 i.e. h_2 maps r_1 to 0.
- ⋆ since h_2 is an injection we have that $r_1 = 0$.
- ⋆ Since r_3 maps q_1 to $r_1 = 0$ by q_1 comes from some p_1 through r_2.
- ⋆ By commutativity, $s_2 f_2$ maps p_1 to q_2 for some p_2' i.e. s_2 maps p_2' to q_2.
- ⋆ Now we have that s_2 maps both p_2' and p_2 to q_2.
- ⋆ Since s_2 is an injection, we have that $p_2' = p_2$.
- ⋆ Hence, p_2 comes from p_1 through f_2 as required.
- ⋆ Conversely, suppose p_2 comes from some p_1 through f_2. We need to show that f_3 maps p_2 to 0.



- ⋆ s_2 maps p_2 to q_2 for some q_2.

- ⋆ By commutativity, $g_2 r_2$ maps p_1 to q_2 for some q_1.
- ⋆ Since q_2 comes from p_2 through s_2, s_3 maps q_2 to 0 by vertical exactness.
- ⋆ Also since q_2 comes from q_1 through g_2, g_3 maps q_2 to 0 by horizontal exactness.
- ⋆ By commutativity, we have h_3 and t_3.
- ⋆ Also since $g_3 s_2$ maps p_2 to 0, we have f_3 and t_2 by commutativity for some p_3.
- ⋆ i.e. f_3 maps p_2 to p_3 and t_2 maps p_3 to 0.
- ⋆ Since t_2 is an injection, we have that $p_3 = 0$.

(iii) Exactness at P_3:

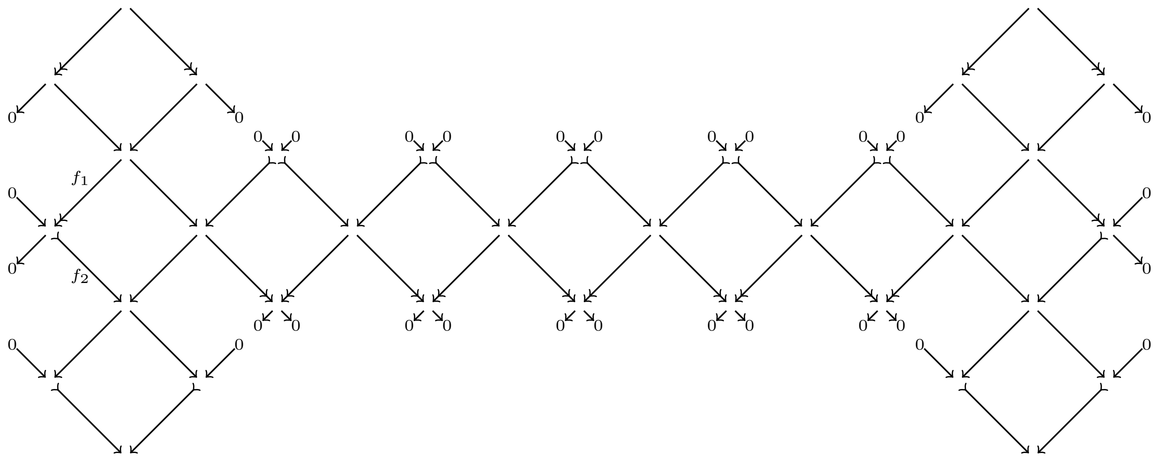


- ⋆ It is enough to show that f_3 is a surjection.
- ⋆ Not easy to do.
- ⋆ We show exactness at the dual of P_3 which is R_1.
- ⋆ Enough to show that h_2 is an injection.
- ⋆ Suppose h_2 maps r_1 to 0; we should show that $r_1 = 0$.
- ⋆ r_4 maps r_1 to 0 vertically.
- ⋆ So by exactness, r_1 comes from q_1 through r_3.
- ⋆ By commutativity, we have that $s_3 g_2$ maps q_1 to 0.
- ⋆ i.e. g_2 maps q_1 to q_2.
- ⋆ By horizontal exactness, g_3 maps q_2 to 0.
- ⋆ By vertical exactness, q_2 comes from some p_2 through s_2.
- ⋆ By commutativity, $t_2 f_3$ maps p_2 to 0 for some p_3.
- ⋆ Since t_2 is an injection, we have that $p_3 = 0$.
- ⋆ By horizontal exactness, p_2 comes from some p_1 through f_2.
- ⋆ By commutativity, $g_2 r_2$ maps p_1 to q_2 for some q_1', i.e. g_2 maps q_1' to q_2.
- ⋆ Since g_2 is an injection, we have that $q_1' = q_1$.

- ⋆ So q_1 comes from p_1 through r_2.
- ⋆ Hence, by vertical exactness, r_3 maps q_1 to 0.
- ⋆ Thus, $r_1 = 0$ and so h_2 is indeed an injection.

□

4.11 The Dragon Lemma

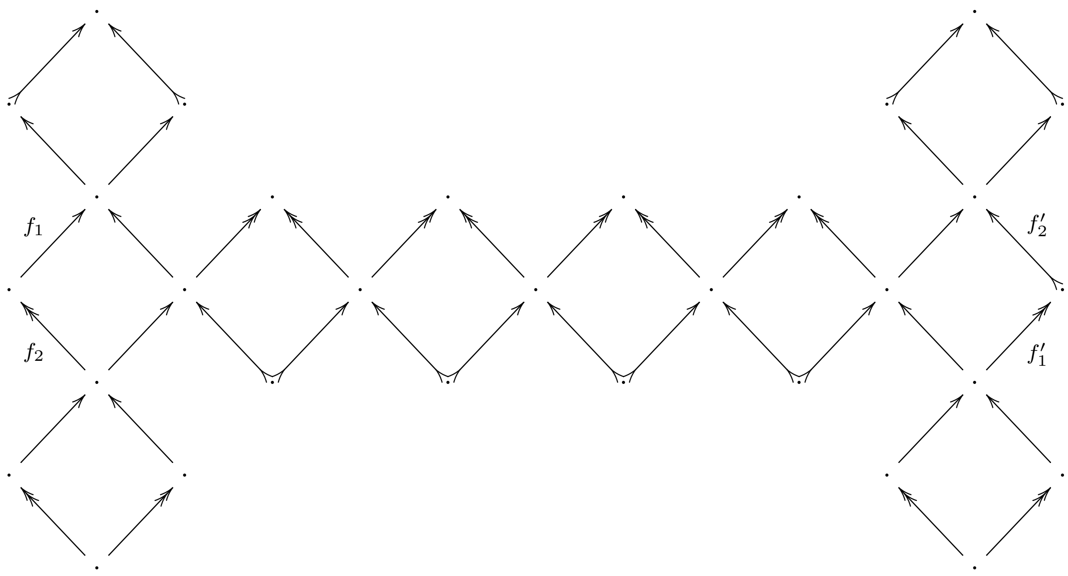


4.11.1 Theorem (The Dragon Lemma)**.** *In the commutative diagram above in which all arrows " $\twoheadrightarrow$ " (except f_1) are surjections and in which all arrows " $\rightarrowtail$ " (except f_2) are injections, f_2 is an injection if and only if f_1 is a surjection. We split this Theorem into two problems:*

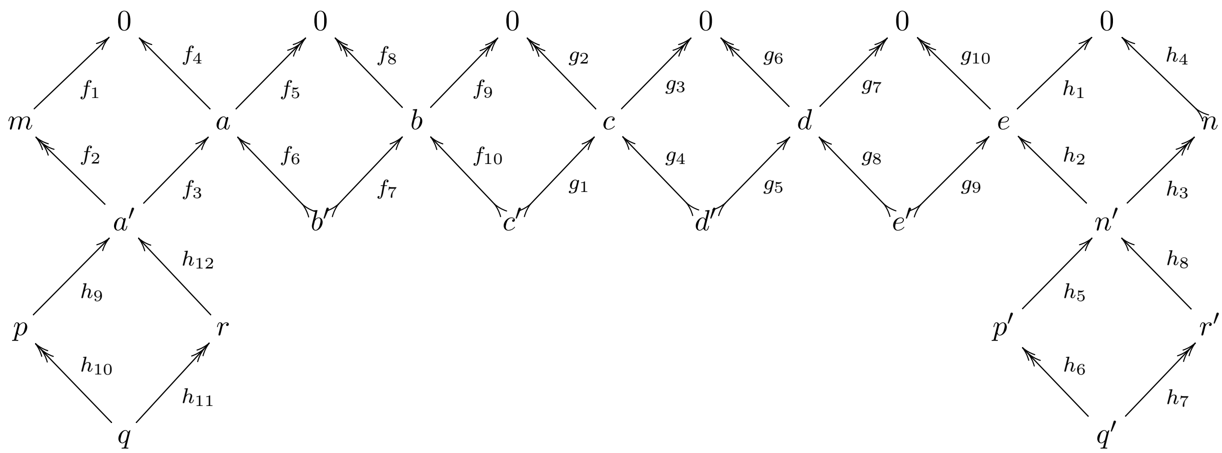
(i) Show that if f_2 is an injection then f_1 is a surjection.

(ii) Show that if f_1 is a surjection then f_2 is an injection.

Proof. We prove part (i) of the Theorem as follows: Since we do not find it easy to show that if f_2 is an injection then f_1 is a surjection directly, we transform the problem by reversing the direction of all the arrows and obtain the diagram below. Note that all the arrows which were injections in our original problem are now surjections and those which were surjections are now injections. Our new problem now is "In the new diagram , show that if f_2 is an surjection then f_1 is an injection." Below is The Translated Dragon Lemma.



We chase 0 around the diagram :



- ⋆ suppose f_1 maps m to 0.
- ⋆ m comes from some a' since f_2 is a surjection.
- ⋆ by commutativity, we have f_3 and f_4 for some a.
- ⋆ f_5 maps a to 0 since a comes from a', also a comes from some b' through f_6 since f_4 maps a to 0.
- ⋆ by commutativity, we have f_7 and f_8 for some b.
- ⋆ f_9 maps b to 0 since b comes from b', also b comes from some c' through f_{10} since f_8 maps b to 0.
- ⋆ by commutativity, we have g_1 and g_2 for some c.
- ⋆ g_3 maps c to 0 since c comes from c', also c comes from some d' through g_4 since g_2 maps c to 0.
- ⋆ by commutativity, we have g_5 and g_6 for some d.

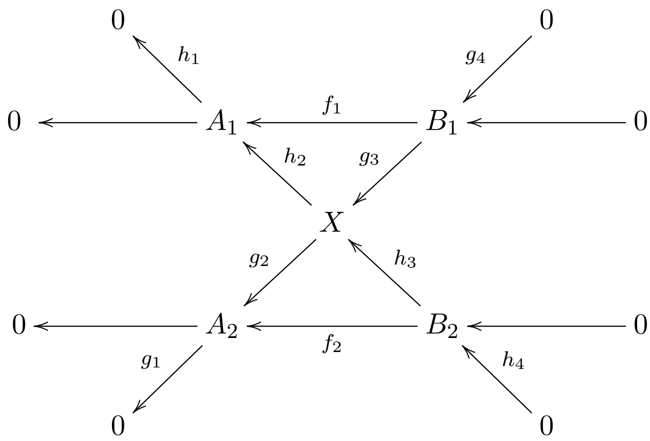
- ⋆ g_7 maps d to 0 since d comes from d', also d comes from some e' through g_8 since g_6 maps d to 0.
- ⋆ by commutativity, we have g_9 and g_{10} for some n.
- ⋆ h_1 maps e to 0 since e comes from e', also e comes from n' through h_2 since g_{10} maps e to 0
- ⋆ by commutativity, we have h_3 and h_4
- ⋆ since h_4 is an injection, we have that $n = 0$
- ⋆ since h_3 maps n' to $n = 0$, n' comes from some p' through h_5
- ⋆ aslo p' comes from some q' since h_6 is a surjection.
- ⋆ by commutativity, we have h_7 and h_8 for some r'.
- ⋆ since n' comes from r' through h_8, h_2 maps n' to 0; that is $e = 0$
- ⋆ since $g_9, g_8, g_5, g_4, g_1, f_{10}, f_7$and f_6 are all injections, we have that $e' = d = d' = c = c' = b = b' = a = 0$.
- ⋆ since f_3 maps a' to $a = 0$, a' comes from some p.
- ⋆ Also p comes from some q since h_{10} is a surjection.
- ⋆ by commutativity, we have h_{11} and h_{12} for some r.
- ⋆ since a' comes from r through h_{12}, f_2 maps a' to 0 and so $m = 0$.
- ⋆ Thus, f_1 is indeed an injection.

□

Now that we have solved part (i) of our problem, to see that we have actually solved part (ii) equally, we only need to flip the translated Dragon diagram (i.e. rotate it through $180°$ angle) and read the 'new problem' in reverse as: show that if f_1 is a surjection then f_2 is an injection. Note that in such a rotation, the dual of f_1 is f'_1 while the dual of f_2 is f'_2 as shown in the translated Dragon diagram.

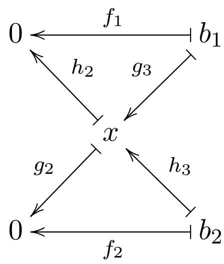
4.12 The Spider Lemma

The Spider Lemma diagram:



4.12.1 Theorem. *In the commutative diagram above if the two diagonals and the first row are exact, then the second row is also exact. This is called the Spider Lemma sometimes referred to as the Cross Lemma.*

Proof.



- ⋆ we need to show that f_2 is a bijection.
- ⋆ First we show that it is an injection.
- ⋆ Suppose f_2 maps b_2 to 0; we show that $b_2 = 0$.
- ⋆ By commutativity, we have h_3 and g_2 for some x.
- ⋆ Since g_2 maps x to 0, x comes from some b_1 through g_3 by exactness.
- ⋆ Since h_3 maps b_2 to x, h_2 maps x to 0 by exactness.
- ⋆ By commutativity, we have that f_1 maps b_1 to 0.
- ⋆ Since f_1 is a bijection, it is an injection and so $b_1 = 0$.
- ⋆ Since g_3 is a group morphism, we have that $x = 0$.
- ⋆ Since h_3 is an injection, we have that $b_2 = 0$.
- ⋆ And hence f_2 is an injection.

□

By showing that f_2 is an injection, we have shown exactness at B_2; but if we reverse the arrows then we have dually shown exactness at A_2 also.

5. Conclusion

It is so interesting to study groups and other mathematical structures from the category Theory point of view. This is so because the area of study simplifies the abstraction in these structures. Drawing of morphisms also makes it easy to understand the very abstract ideas of Algebra such as exact sequences.

Acknowledgements

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Thank you so much to Martha, my Tutor. To my supervisor I appreciate your commitment.

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References

S. Awodey. *Category Theory*. Oxford University Press, 2nd edition, 2010.

O. Bogopolski. *Introduction to Group Theory*. Institute of Computer Science (Moscow-Izhevsk), 2nd edition, 2002.

M. Grandis. *Homological Algebra, in strongly non-abelian settings*. World Scientific Publishing, 2013.

D. Holgate and A. Razafindrakoto. *An Introduction to Cartegory Theory*. University of Stellenbosch, June 2011.

Z. Janelidze. *The pointed subobject functor, 3 by 3 lemmas, and subtractivity of spans*. University of Stellenbosch, 2010.

Z. Janelidze. An axiomatic survey of diagram lemmas for non-abelian group-like structures. *Journal of Algebra 370 (2012) 387-401*, 2012.

Z. Janelidze. On the form of subobjects in semi-abelian and regular protomodular categories. *Applied Categorical Structures*, 2013.

Z. Janelidze. *Theory of Forms*. University of Stellenbosch, september 2013.

S. M. Lane. *Categories for the Working Mathematician*. Springer - Verlag, 2nd edition, 1971.

W. Lerdermann. *Introduction to Group Theory*. Addison wesley longman Inc, 2nd edition, 1996.

S. MacLane and G. Birkhoff. *Algebra*. AMS Chelsea Publishing, 3rd edition, 1999.

H. Simmons. *An Introduction to Cartegory Theory*. Cambridge University Press, 2011.

T. Van der Linden. Ordered sets in homological algebra. Lecture notes for the 1st Workshop on Mathematical Sciences in Muizenberg, South Africa, 2013.