Multivariable Dilation Theory

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Abstract

We show in this essay the construction of the unitary dilation for a contraction and a pair of commutative contractions on a Hilbert space, and we deduce the one and two variables von Neumann inequality. We expose some examples proving that the inequality fails for more than three contractions. Using d-contraction, a d-tuple of commuting operators $(T_1, T_2 \ldots, T_d)$ acting on the same Hilbert space H which verify

$$\|T_1h_1 + T_2h_2 + \ldots + T_dh_d\|^2 \leq \|h_1\|^2 + \|h_2\|^2 + \ldots + \|h_d\|^2,$$

for all $h_1, h_2, \ldots, h_d \in H$, and a suitable norm on the algebra of polynomials, we give the appropriate norm bound in the von Neumann inequality.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Andriamanankasina Ramanantoanina, 22 May 2014

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1. Introduction

Operator Theory is an area in mathematics which began in the study of integral equations. It takes now some problems from various domains as mathematical physics.

In the Operator Theory on Hilbert spaces, the class of normal operators, including unitary and self-adjoint operators, is well understood. The dilation theory started by Nagy and Foias (1966) considers contractions as part of unitary operator on a bigger space. This leaded to a simplified proof of a famous inequality due to von Neumann. Ando (1963) proved the two variable version of von Neumann inequality using the unitary dilation of a pair of commutative contractions. Parrott (1970) showed that the d-variable version of this inequality fails for $d \geq 3$.

After giving a brief overview of the dilation theory of Nagy and Foias, we will show the short proof of the one variable von Neumann inequality, followed by various examples that show the problem encountered in the attempt to generalise it. In the last chapter, we expose the Drury-Arveson's version of the von Neumann inequality by considering the suitable counterpart of contractions in higher dimension.

2. One variable dilation theory

Notations

Throughout this chapter, H is a complex Hilbert space.

- For a Hilbert space K, $\mathcal{B}(H,K)$ denotes the space of linear bounded operators mapping H to K. If $H = K$ we will simply denote $\mathcal{B}(H) = \mathcal{B}(H,K)$. For $T \in \mathcal{B}(H,K)$, we denote by T^* the adjoint operator of T.
- We will denote by $l^2_n(H)$ the orthogonal sum of n copies of H with its natural norm and inner product.
- $l^2(H)$ is defined as the space of all sequences of elements of H that are norm square summable:

  $$l^2(H) := \left\{ h = (h_n)_{n\in\mathbb{N}}, \sum_{n=0}^{\infty} \|h_n\|^2 < \infty \right\}.$$

  $l^2(H)$ endowed with the inner product

  $$\langle h, k\rangle = \sum_{n=0}^{\infty} \langle h_n, k_n\rangle ,$$

  and the norm

  $$\|h\|^2 = \sum_{n=0}^{\infty} \|h_n\|^2 ,$$

  for $h = (h_n), k = (k_n) \subset H$, is a Hilbert space.

2.1 Contractive operators

2.1.1 Unitary operator. An operator $U \in \mathcal{B}(H,K)$ is called a unitary operator if U is invertible and its inverse is U^*:

$$U^*U = I_H \text{ and } UU^* = I_K.$$

The spectrum $\sigma(U)$ of U is contained in the unit circle $\mathbb{T}$ of the complex plane.

2.1.2 Example (Bilateral Shift)**.** An important example of unitary operator is the *bilateral shift* $S : l^2(H) \oplus l^2(H) \longrightarrow l^2(H) \oplus l^2(H)$ defined by

$$S((\ldots, h_{-3}, h_{-2}, h_{-1}), (h_0, h_1, h_2 \ldots)) = ((\ldots, h_{-3}, h_{-2}), (h_{-1}, h_0, h_1 \ldots)).$$

2.1.3 Contraction. Let $T \in \mathcal{B}(H,K)$. T is called a contraction if $\|T\| \leq 1$ or equivalently:

$$\|Th\| \leq \|h\| \quad (h \in H).$$

If T is a contraction then:

- T^* is a contraction.
- the operator $(I_H - T^*T)^{\frac{1}{2}} : H \mapsto H$ is well defined, it is called the *defect operator* and denoted by D_T.
- for all $h \in H$
$$\|h\|^2 = \|Th\|^2 + \|D_T h\|^2 . \tag{2.1.1}$$

2.1.4 Isometry. $V \in \mathcal{B}(H,K)$ is an isometry if $\|Vh\| = \|h\|$ for all $h \in H$. Using the polarisation identity, it is equivalent to
$$\langle Vh, Vk\rangle = \langle h, k\rangle \quad (h, k \in H).$$
This equation can also be written as
$$\langle (I_H - V^*V)h, k\rangle = 0 \quad (h, k \in H),$$
which shows that V is an isometry if and only if $I_H - V^*V = 0$. That is, the defect operator associated to an isometry is null. The defect operator measures in some sense the failure of a contraction to be an isometry.

An operator T is called coisometric if T^* is isometric. V^* is a contraction. Unlike the case for contraction, the adjoint of an isometry does not need to be an isometry. To see this, let us consider the following example.

2.1.5 Example (Unilateral shift)**.** The *unilateral shift* $S \in \mathcal{B}(l^2(H))$ is defined as
$$Sh = (0, h_0, h_1, h_2, h_3, \ldots) \quad h = (h_0, h_1, h_2, h_3, \ldots) \in l^2(H).$$
S^* acts on $l^2(H)$ as
$$S^*(h_0, h_1, h_2, h_3, \ldots) = (h_1, h_2, h_3, \ldots),$$
so that $S^*S = I$, where I is the identity operator on $l^2(H)$. But $\|S^*h\|$ is not equal to $\|h\|$ unless $h_0 = 0$, which shows that S^* is not an isometry.

2.1.6 Remark. If $V \in \mathcal{B}(H,K)$ is an isometry, then:

- V is one-to-one and has closed range $ImV = VH \subset K$.
- for all $h \in H$ and $k \in K$, we have
$$(I_K - VV^*)(Vh) = 0 \quad \text{and} \ \langle k - VV^*k, Vh\rangle = \langle V^*k - V^*k, h\rangle = 0.$$
Moreover $(I_K - VV^*)^2 = I_K - VV^*$. Thus $I_K - VV^*$ is the orthogonal projection on $K \ominus VH$, the orthogonal complement of the range of V. Also VV^* is the projection on the range of V.

If V is coisometric then $I_K - VV^* = 0$ or equivalently ImV is dense, and since ImV is closed then V is onto. A co-isometric isometry is unitary.

2.1.7 Lemma (Wold decomposition)**.** (Nagy and Foias, 1966, p. 3). Given an isometry $V \in \mathcal{B}(H)$ there exists a decomposition of H into an orthogonal sum
$$H = H_0 \oplus H_1,$$
H_0 and H_1 are reducing subspaces for V, such that $V|_{H_0} : H_0 \longrightarrow H_0$ is unitary and $V|_{H_1} : H_1 \longrightarrow H_1$ is a unilateral shift. H_0 and H_1 are given by
$$H_0 = \bigcap_{n=0}^{\infty} V^n H \quad \text{and} \quad H_1 = \bigoplus_{n=0}^{\infty} V^n L,$$
where $L = H \ominus VH$.

2.2 Isometric and unitary dilation

In the sequel, $T \in \mathcal{B}(H)$ is a contraction acting on H.

For a Hilbert space K containing H as a subspace and an operator B acting on K, the *compression* of B to H is the operator $A \in \mathcal{B}(H)$ defined by

$$A = P_H \, B|_H \, ,$$

where P_H is the projection on H. If U is a unitary operator on K then the compression of U to H is a contraction. Indeed,

$$\|P_H \, U|_H \, h\| \leq \|Uh\| = \|h\| \, .$$

It naturally raises the question whether any contraction on a Hilbert space is a compression of a unitary operator on a bigger space. If T is isometric, consider the decomposition of H given in (2.1.7). Since the unilateral shift $T|_{H_1}$ can be extended to the unitary operator which is the bilateral shift, we expect the answer to be positive for isometries.

In fact, this is true for any contraction T on a Hilbert space H. The condition $T = P_H \, U|_H$ can even be strengthened:

$$T^n = P_H \, U^n|_H \quad (n \in \mathbb{N}).$$

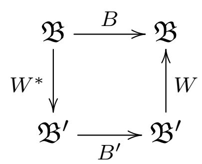
In this case, U is called a (power) *dilation* of T.

2.2.1 Definition. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two Hilbert spaces.

- $B \in \mathcal{B}(\mathfrak{B})$ is a dilation of $A \in \mathcal{B}(\mathfrak{A})$, or B dilates A, if $\mathfrak{A} \subset \mathfrak{B}$ and

$$A^n = P_{\mathfrak{A}} \, B^n|_{\mathfrak{A}} \, , \quad (n \in \mathbb{N}).$$

- Two dilations of $A \in \mathcal{B}(\mathfrak{A})$, say $B \in \mathcal{B}(\mathfrak{B})$ and $B' \in \mathcal{B}(\mathfrak{B}')$, are said to be isomorphic if there exits a unitary operator $W : \mathfrak{B}' \mapsto \mathfrak{B}$ which fixes $\mathfrak{A}$, $Wa = a$ for all $a \in \mathfrak{A}$, and $B = W^* B' W$:



2.2.2 Proposition. If $B \in \mathcal{B}(\mathfrak{B})$ is a dilation of $A \in \mathcal{B}(\mathfrak{A})$, and $C \in \mathcal{B}(\mathfrak{C})$ is a dilation of B then C dilates A.

2.2.3 Proposition. (Nagy and Foias, 1966, p. 10). For every contraction T on a Hilbert space H there exists an isometric dilation V on some Hilbert space K containing H, which is moreover minimal in the sense that

$$K = \bigvee_{n=0}^{\infty} V^n H.$$

This minimal isometric dilation of T is determined up to isomorphism.

Proof. Since T is a contraction, $D_T = (I - T^*T)^{1/2}$ is a well defined operator on H. Let $K = l^2(H)$ and define $V \in \mathcal{B}(K)$ by

$$V(k) = (Th_0, D_T h_0, h_1, h_2, \dots)$$

for $k = (h_n)_{n\in\mathbb{N}} \in K$.

Since $\|h_0\|^2 = \|Th_0\|^2 + \|D_T h_0\|^2$, V is isometric. Indeed, if $k = (h_0, h_1, h_2, \dots) \in K$ then

$$\|Vk\|^2 = \|Th_0\|^2 + \|D_T h_0\|^2 + \sum_{n=1}^{\infty} \|h_n\|^2 = \sum_{n=0}^{\infty} \|h_n\|^2 = \|k\|^2$$

Moreover, H can be isometrically identified to the subspace of K which consists of the vectors of the form $(h, 0, 0, \dots), (h \in H)$, and the projection of K onto H is defined by:

$$P_H(h_0, h_1, h_2, \dots) = h_0.$$

Then for $h \in H$, h being identified to $(h, 0, 0, \dots) \in K$

$$V^n h = (T^n h, D_T T^{n-1} h, \dots, D_T T h, D_T h, 0, 0, \dots),$$

which hows that $P_H V^n h = T^n h$.

If $V \in \mathcal{B}(\mathfrak{M})$ and $W \in \mathcal{B}(\mathfrak{M}')$,

$$\mathfrak{M} = \bigvee_{i=0}^{\infty} V^i H \quad \text{and} \quad \mathfrak{M}' = \bigvee_{i=0}^{\infty} W^i H,$$

are two minimal isometric dilations of T, then the application $U : \mathfrak{M} \longrightarrow \mathfrak{M}'$ defined on the dense subspace $\{V^j h, j \in \mathbb{N}, h \in H\}$ of $\mathfrak{M}$ by

$$U(V^j h) = W^j h$$

is isometric and onto. So it is unitary and it verifies $U^*WU(V^j h) = U^*WW^j h = V^{j+1} h = V(V^j h)$ for all $h \in H$ and $j \in \mathbb{N}$. As $\{V^j h, j \in \mathbb{N}, h \in H\}$ is dense in $\mathfrak{M}$, $U^*WU = V$. This proves that the isometric dilation of a contraction is unique up to isomorphism. □

2.2.4 Proposition. (Gohberg et al., 1993, chapter XXVII).

$$V = \begin{pmatrix} T & 0 & 0 & 0 & \dots \\ D_T & 0 & 0 & 0 & \dots \\ 0 & I & 0 & 0 & \dots \\ 0 & 0 & I & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} : H \oplus l^2(D_T H) \longrightarrow H \oplus l^2(D_T H)$$

is the matrix representation of a minimal dilation of T.

Proof. Let us note $\mathfrak{M} = H \oplus l^2(D_T H)$. Obviously

$$\bigvee_{n=0}^{\infty} V^n H \subset \mathfrak{M}.$$

For $(h, x_0, x_1, x_2, \ldots) \in \mathfrak{M}$

$$V(h, x_0, x_1, x_2, \ldots) = \begin{pmatrix} T & 0 & 0 & 0 & \ldots \\ D_T & 0 & 0 & 0 & \ldots \\ 0 & I & 0 & 0 & \ldots \\ 0 & 0 & I & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} h \\ x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} = (Th, D_T h, x_0, x_1, \ldots)$$

$V \in \mathcal{B}(\mathfrak{M})$ is isometric and V dilates T.

Let

$$m = (m_0, m_1, m_2, \ldots) \in \mathfrak{M} \ominus \bigvee_{n=0}^{\infty} V^n H.$$

$(m_0, m_1, m_2, \ldots) \perp V^0 H = H$ implies $m_0 = 0$. $(0, m_1, m_2, \ldots) \perp V^1 H$ implies $m_1 \perp D_T H$, but $m_1 \in D_T H$ then $m_1 = 0$. Similarly $(0, 0, \ldots, m_k, \ldots) \perp V^k H$ implies $m_k = 0$ for any $k \in \mathbb{N}$. This proves that

$$\mathfrak{M} \ominus \bigvee_{n=0}^{\infty} V^n H = \{0\},$$

and then

$$\bigvee_{n=0}^{\infty} V^n H$$

is dense in $\mathfrak{M}$. □

2.2.5 Proposition. For an operator T acting on a common Hilbert space H, $\|T\| < 1$, let S be the unilateral shift operator acting on $l^2(D_{T^*}H)$. Therefore, there exists a subspace K of $l^2(D_{T^*}H)$, covariant for S, i.e. invariant for S^*, such that

$$T = P_K S P_K,$$

where D_T is the defect operator associated to T. S is then an isometric dilation for T.

Proof. Define the operator

$$\begin{aligned} L : H &\longrightarrow l^2(D_{T^*}H) \\ h &\longrightarrow (D_{T^*}h, D_{T^*}T^*h, D_{T^*}T^{*2}h, D_{T^*}T^{*3}h, \ldots). \end{aligned}$$

Since

$$\|D_{T^*}T^{*n}h\|^2 + \left\|D_{T^*}T^{*n+1}h\right\|^2 = \|T^{*n}h\|^2 - \left\|T^{*n+1}h\right\|^2 + \left\|T^{*n+1}h\right\|^2 - \left\|T^{*n+2}h\right\|^2 = \|T^{*n}h\|^2 - \left\|T^{*n+2}h\right\|^2,$$

and $\|T\| < 1$, we have

$$\|T^{*n}h\| \leq \|T^{*n}\| \, \|h\| \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

and

$$\|Lh\|^2 = \|h\|^2 - \lim_{n\to\infty} \|T^{*n}h\|^2 = \|h\|^2.$$

So L is isometric and we have $LT^* = S^*L$. Therefore $T = L^*SL$ and since $S^*(LH) = LT^*H \subset LH$ we can identify H to its isometric image $K = LH \subset l^2(D_{T^*}H)$ and write

$$T = P_K S P_K.$$

Since K is invariant by S^*
$$T^n = P_K S^n P_K \quad \text{for all } n \geq 0,$$
and then S is a dilation of T. □

2.2.6 Proposition. (Paulsen, 2002). Let V be an isometry on H and pose $K = H \oplus H$. The operator U defined on K by
$$U = \begin{pmatrix} V & I_H - VV^* \\ 0 & V^* \end{pmatrix} : K \to K$$
is unitary. H can be isometrically identified to the subspace $H \oplus 0$ of K and then U is a unitary dilation of V.

Proof. U is obviously a dilation of V.

The adjoint of U is given by
$$U^* = \begin{pmatrix} V^* & 0 \\ I_H - VV^* & V \end{pmatrix}.$$
Then
$$UU^* = \begin{pmatrix} VV^* + (I_H - VV^*)^2 & V - VV^*V \\ V^* - V^*VV^* & V^*V \end{pmatrix} = \begin{pmatrix} I_H & 0 \\ 0 & I_H \end{pmatrix},$$
and
$$U^*U = \begin{pmatrix} V^*V & V + VV^*V \\ V - VV^*V & (I_H - VV^*)^2 + VV^* \end{pmatrix} = \begin{pmatrix} I_H & 0 \\ 0 & I_H \end{pmatrix}.$$
□

The unitary dilation of an isometry can be chosen to be minimal in the sense of Proposition 2.2.3. We have the following theorem.

2.2.7 Theorem (Sz. Nagy's Dilation Theorem)**.** *(Nagy and Foias, 1966, p .13). For any contraction T on a Hilbert space H there exists a unitary dilation U on a space K containing H which is minimal, that is, such that*
$$K = \bigvee_{n=-\infty}^{+\infty} U^n H.$$
This minimal unitary dilation of T is determined up to isomorphism.

Proof. The minimal unitary dilation U of the minimal isometric dilation of T is a minimal unitary dilation of T. Since K reduces U and H is a subspace of K then the restriction of U on K is still a dilation of T. □

John von Neumann, in 1951, gave an upper bound of the norm of a polynomial in a contraction T on a Hilbert space H: this is the celebrated *von Neumann inequality*. The existence of the unitary dilation for any contraction simplifies the proof of this inequality.

2.2.8 Corollary (von Neumann Inequality)**.** (Paulsen, 2002). Let T be a contraction on a Hilbert space. Then for any polynomial p
$$\|p(T)\| \leq \sup \{|p(z)|, |z| \leq 1\}.$$

Proof. Let U be a unitary dilation of the contraction T and p be any complex polynomial $p \in \mathbb{C}[X]$. U being a dilation of T implies that

$$p(T) = P_H \, p(U)|_H \,,$$

and U being unitary implies that $p(U)$ is normal. Therefore

$$\|p(U)\| = \sup_{\lambda \in p(\sigma(U))} |\lambda| = \sup_{z \in \sigma(U)} |p(z)| \,.$$

Since $\sigma(U) \subset \mathbb{T}$, and $\|p(T)\| = \|P_H \, p(U)|_H\| \leq \|p(U)\|$, we have the desired inequality. □

3. System of Commuting Contractions

The most natural continuation that one would attempt is to generalize the result of the previous chapter to more than one contractions. When dealing several contractions at the same time, one needs to distinguish whether the contractions commute or not. In this chapter we will be focusing on systems of commutative contractions.

First we need to specify what *dilation* means for several commuting operators.

3.0.1 Definition. (Nagy and Foias, 1966, p .19). Let $d \geq 1$ be an integer. Let $\{A_1, A_2, \ldots, A_d\}$ be a set of commuting operators on H. A commutative system $\{B_1, B_2, \ldots, B_d\}$ of bounded operators on a Hilbert space K is called a dilation of $\{A_1, A_2, \ldots, A_d\}$ if

- K contains H as subspace.
- $A_1^{i_1} A_2^{i_2} \ldots A_d^{i_d} = P_H \, B_1^{i_1} B_2^{i_2} \ldots B_d^{i_d}\Big|_H$, for positive integers $i_1, i_2, \ldots, i_d$.

$\{B_1, B_2, \ldots, B_d\}$ will be called isometric, unitary dilation if all B_i's are.

3.1 Pair of commutative contractions

Beginning with two commuting contractions, we have the following result.

3.1.1 Lemma. For every commuting pair of contraction $\{T_1, T_2\}$ on H, there exists an isometric dilation.

Proof. Consider the minimal isometric dilation $V_i \in \mathcal{B}(H \oplus l^2(H))$ of T_i $(i = 1, 2)$, for $(h, x_1, x_2, x_3, \ldots) \in H \oplus l^2(H)$

$$V_i(h, x_1, x_2, x_3, \ldots) = (T_i h, D_{T_i} h, 0, x_1, x_2, x_3, \ldots).$$

Then

$$V_1 V_2 = (T_1 T_2 h, D_{T_1} T_2 h, 0, D_{T_2} h, 0, x_1, x_2, x_3, \ldots)$$

and

$$V_2 V_1 = (T_2 T_1 h, D_{T_2} T_1 h, 0, D_{T_1} h, 0, x_1, x_2, x_3, \ldots).$$

V_1 and V_2 do not generally commute.

Let $U \in \mathcal{B}(l_4^2(H))$ be a unitary operator such that

$$U(D_{T_2} T_1 h, 0, D_{T_2} h, 0) = (D_{T_1} T_2 h, 0, D_{T_1} h, 0), \text{ for all } h \in H.$$

Note that such unitary operator does exist since the application

$$(D_{T_2} T_1 h, 0, D_{T_2} h, 0) \longmapsto (D_{T_1} T_2 h, 0, D_{T_1} h, 0)$$

from the subspace $\{(D_{T_2}T_1h, 0, D_{T_2}h, 0),\ h \in H\} \subset l^2_4(H)$ to the subspace $\{(D_{T_1}T_2h, 0, D_{T_1}h, 0),\ h \in H\} \subset l^2_4(H)$ is isometric as, for all $h \in H$, we have

$$\begin{aligned}
\|(D_{T_2}T_1h, 0, D_{T_2}h, 0)\|^2 &= \|D_{T_2}T_1h\|^2 + \|D_{T_1}h\|^2 \\
&= \|T_1h\|^2 - \|T_2T_1h\|^2 + \|D_{T_1}h\|^2 \\
&= \|h\|^2 - \|T_1T_2h\|^2 \\
&= \|T_2h\|^2 + \|D_{T_2}h\|^2 - \|T_1T_2h\|^2 \\
&= \|D_{T_1}T_2h\|^2 + \|D_{T_2}h\|^2 \\
&= \|(D_{T_1}T_2h, 0, D_{T_1}h, 0)\|^2,
\end{aligned}$$

and since the orthogonal complement of these two subspaces in $l^2_4(H)$ have the same dimension, the isometry above can be extended to the unitary operator U.

Define

$$\mathcal{U} : (h, x_1, x_2, x_3, \ldots) \in H \oplus l^2(H) \longmapsto (h, U(x_1, \ldots, x_4), U(x_5, \ldots, x_8), \ldots).$$

$\mathcal{U}$ is unitary and

$$\mathcal{U}^{-1}(h, x_1, x_2, x_3, \ldots) = (h, U^{-1}(x_1, \ldots, x_4), U^{-1}(x_5, \ldots, x_8), \ldots).$$

Then $W_1 = V_1\mathcal{U}^{-1}$ and $W_2 = \mathcal{U}V_2$ dilate T_1 and T_2 also. We have

$$W_1W_2(h, x_1, x_2, x_3, \ldots) = V_1V_2(h, x_1, x_2, x_3, \ldots) = (T_1T_2h, D_{T_1}T_2h, 0, D_{T_2}h, 0, x_1, x_2, x_3, \ldots)$$

and

$$\begin{aligned}
W_2W_1(h, x_1, x_2, x_3, \ldots) &= \mathcal{U}V_2V_1(h, U^{-1}(x_1, \ldots, x_4), U^{-1}(x_5, \ldots, x_8), \ldots) \\
&= \mathcal{U}(T_2T_1h, D_{T_2}T_1h, 0, D_{T_1}h, 0, U^{-1}(x_1, \ldots, x_4), U^{-1}(x_5, \ldots, x_8), \ldots) \\
&= (T_2T_1h, U(D_{T_2}T_1h, 0, D_{T_1}h, 0), x_1, x_2, x_3, \ldots) \\
&= (T_2T_1h, D_{T_1}T_2h, 0, D_{T_2}h, 0, x_1, x_2, x_3, \ldots) \\
&= W_1W_2, \text{ since } T_1 \text{ and } T_2 \text{ commute.}
\end{aligned}$$

Therefore, $\{W_1, W_2\}$ is an isometric dilation of $\{T_1, T_2\}$. □

3.1.2 Lemma. For every commutative pair of isometric operators $\{V_1, V_2\} \in \mathcal{B}(H)$, there exists a unitary dilation.

Proof. Let $U_1 \in \mathcal{B}(K_1)$ and $U_2 \in \mathcal{B}(K_2)$ be the minimal unitary dilation of V_1 and V_2. Define R_2 on the dense subset of K_1 by

$$R_2\left(\sum_n U_1^n h_n\right) = \sum_n U_1^n V_2 h_n,$$

where $h_n \in H$ and n is taken in a finite subset of $\mathbb{Z}$.

Then

$$\begin{aligned}
\left\| R_2\left(\sum_n U_1^n h_n\right)\right\|^2 &= \sum_{m,n} \langle U_1^n V_2 h_n, U_1^m V_2 h_m\rangle \\
&= \sum_{m\le n} \left\langle U_1^{n-m} V_2 h_n, V_2 h_m\right\rangle + \sum_{m>n} \left\langle V_2 h_n, U_1^{m-n} V_2 h_m\right\rangle.
\end{aligned}$$

Since $V_2 h_m, V_1 h_n \in H$

$$\begin{aligned}\left\| R_2\left(\sum_n U_1^n h_n\right)\right\|^2 &= \sum_{m\le n}\left\langle V_1^{n-m}V_2 h_n, V_2 h_m\right\rangle + \sum_{m>n}\left\langle V_2 h_n, V_1^{m-n}V_2 h_m\right\rangle\\ &= \sum_{m\le n}\left\langle V_2 V_1^{n-m} h_n, V_2 h_m\right\rangle + \sum_{m>n}\left\langle V_2 h_n, V_2 V_1^{m-n} h_m\right\rangle\\ &= \sum_{m\le n}\left\langle V_1^{n-m} h_n, h_m\right\rangle + \sum_{m>n}\left\langle h_n, V_1^{m-n} h_m\right\rangle\\ &= \sum_{m\le n}\left\langle U_1^{n-m} h_n, h_m\right\rangle + \sum_{m>n}\left\langle h_n, U_1^{m-n} h_m\right\rangle\\ &= \sum_{m,n}\langle U_1^n h_n, U_1^m h_m\rangle\\ &= \left\|\sum_n U_1^n h_n\right\|^2.\end{aligned}$$

R_2 is isometric on K_1 and then can be minimally dilated to a unitary operator W_2 on a Hilbert space $K \supset K_1 \supset H$. Moreover R_2 and U_1 commute. Indeed, they commute on the dense subset of K_1

$$U_1 R_2(U_1^n h) = U_1^{n+1}(V_2 h) = R_2 U_1^{n+1}(h) = R_2 U_1(U_1^n h).$$

Define W_1 on the dense subset $\{W_2^n k, n \in \mathbb{Z}, k \in K_1\}$ of K by

$$W_1\left(\sum_n W_2^n k_n\right) = \sum_n W_2^n U_1 k_n,$$

where $k_n \in K_1$ and n is taken in a finite subset of $\mathbb{Z}$. Then applying the same reasoning as above, W_1 and W_2 commute and dilate respectively V_1 and V_2, but this time the isometry W_1 is onto since every element $\sum_n W_2^n k_n$ can be written as

$$\sum_n W_2^n U_1 k_n' = W_1\left(\sum_n W_2^n k_n'\right).$$

Therefore W_1 and W_2 are unitary, commute and dilate $\{V_1, V_2\}$.

This completes the proof of the proposition. □

3.1.3 Remark. This proposition holds for finitely many commuting isometries. That is, every system of commuting isometries $\{V_1, V_2, \ldots, V_n\}$ acting on the same Hilbert space H has a unitary dilation $\{U_1, U_2, \ldots, U_n\} \subset \mathcal{B}(K)$ on some Hilbert space K containing H:

$$V_1^{i_1} V_2^{i_2} \ldots V_n^{i_n} = P_H\, U_1^{i_1} U_2^{i_2} \ldots U_n^{i_n}\Big|_H,$$

for positive integers $i_1, i_2, \ldots, i_n$.

3.1.4 Theorem (Ando's dilation theorem)**.** *(Paulsen, 2002, p. 61) For any pair of commuting contractions $\{T_1, T_2\}$ acting on a Hilbert space H, there exists a pair of commutative unitaries $\{U_1, U_2\}$ on some Hilbert space K, K containing H as a subspace, such that*

$$T_1^n T_2^m = P_H\, U_1^n U_2^m|_H,$$

for all integers $n, m \ge 0$.

Proof. This is a consequence of the two preceding lemma. □

The two variable von Neumann inequality follows from Ando's Theorem.

3.1.5 Corollary (Two variables von Neumann Inequality)**.** Let T_1, T_2 commuting contractions on a Hilbert space H, and let p be a polynomial in two variables. Then

$$\|p(T_1, T_2)\| \leq \sup_{\|(z_1,z_2)\|\leq 1} |p(z_1, z_2)| \,.$$

3.2 Three and more contractions

Here we give some examples which show that the unitary dilation, as defined in 3.0.1, for a commutative system which consists of more than two contractions sometimes fails to exist.

3.2.1 Example. This example, from (Duen Choi and Davidson, 2012), shows that even in finite dimensional Hilbert space, there are commuting contractions that do not have isometric dilation. However the given contractions still satisfy the von Neumann inequality.

Let $\{e_1, e_2, e_3\}$ be the canonical basis of $H := \mathbb{C}^3$, and $\{f_1, f_2, f_3\}$ its dual basis. Let $a, b, x, y \in \mathbb{R}$ such that

$$a^2 + b^2 = x^2 + y^2 = 1.$$

Define four operators in $l(\mathbb{C}^3)$:

$$\begin{aligned} A_1 &= e_1 f_3, \\ A_2 &= e_2 f_3, \\ A_3 &= (ae_1 + be_2) f_3, \\ A_4 &= (xe_1 + ye_2) f_3. \end{aligned}$$

$\{A_1, A_2, A_3, A_4\}$ is a system of contractions. Since $f_3(e_1) = f_3(e_2) = 0$, we have

$$A_i A_j = A_j A_i = 0 \quad \text{for all } i, j = 1, 2, 3, 4. \tag{3.2.1}$$

Suppose that $\{S_1, \ldots, S_4\}$ is an isometric dilation of $\{A_1, A_2, A_3, A_4\}$ on

$$K = \text{span}\,\{p(S_1, \ldots, S_4)H, p \in \mathbb{C}[z_1, \ldots, z_4]\}\,.$$

$S_i e_3 = P_H S_i e_3 = A_i e_3$ for $i = 1, 2, 3, 4$ since $\|P_H S_i e_3\| = \|A_i e_3\| = \|e_3\| = \|S_i e_3\|$.

Consequently, $(xS_1 + yS_2 - S_4)e_3 = 0$ and then

$$(xS_1 + yS_2 - S_4)p(S_1, \ldots, S_4)e_3 = p(S_1, \ldots, S_4)(xS_1 + yS_2 - S_4)e_3 = 0,$$

for every four variables complex polynomial p. In particular,

$(xS_1 + yS_2 - S_4)e_1 = (xS_1 + yS_2 - S_4)S_1 e_3 = 0$ and $(xS_1 + yS_2 - S_4)e_2 = (xS_1 + yS_2 - S_4)S_2 e_3 = 0.$

And since $H = \text{span}\,\{e_1, e_2, e_3\}$,

$$(xS_1 + yS_2 - S_4)p(S_1, \ldots, S_4)h = 0 \quad \text{for all } h \in H.$$

$xS_1 + yS_2 - S_4 = 0$ and similarly $aS_1 + bS_2 - S_3 = 0$.

Using the fact that the S_i's are isometric,

$$0 = I - S_3^*S_3 = -ab(S_2^*S_1 + S_1^*S_2)$$
$$0 = I - S_4^*S_4 = -i(S_2^*S_1 - S_1^*S_2),$$

and we have $S_2^*S_1 = 0$ which contradicts the commutation relation. So the system $\{A_1, A_2, A_3, A_4\}$ do not have isometric dilation.

To prove that these contractions satisfy the von Neumann inequality, we only need to consider the degree one polynomials because the A_i's are nilpotent of order 2 and the product of any two of them vanishes as in equation (3.2.1).

Let $p(z) = c + p_1 z_1 + \ldots + p_4 z_4$, $p_i \neq 0$. Let $r_i = p_i / |p_i|$ and

$$B = \frac{1}{|p_1| + \cdots + |p_4|}(p_1 A_1 + \ldots + p_4 A_4).$$

Since the A_i's are all contractive, so is B and we have

$$p(A_1, \ldots, A_4) = p(\overline{r_1}B, \ldots, \overline{r_4}B) =: q(B).$$

By Nagy's dilation Theorem 2.2.7,

$$\|p(A_1, \ldots, A_4)\| = q(B) \leq \|q\|_\infty \leq \|p\|_\infty .$$

3.2.2 Example. The construction in this example follows Nagy and Foias (1966, p. 24). We show a system of commuting contractions which has no unitary dilation. Let H be a Hilbert space, and A_1, A_2, A_3 three unitary operators on H. Define T_1, T_2, T_3 on $l_2^2(H)$ by

$$T_i = \begin{pmatrix} 0 & 0 \\ A_i & 0 \end{pmatrix}, (i = 1, 2, 3).$$

$T_i T_j = 0$ and $\|T_i\| = \|A_i\| = 1$, for any i, j.

Suppose that the unitary dilation $\{U_1, U_2, U_3\}$ exists then

$$P_H U_i(h, 0) = T_i(h, 0) = (0, A_i h), ((h, 0) \in l_2^2(H), i = 1, 2, 3).$$

We have

$$U_j^{-1}U_i(h, 0) = U_j^{-1}(0, A_i h) = U_j^{-1}(0, A_j A_j^{-1} A_i h) = (A_j^{-1} A_i h, 0),$$

and

$$U_k U_j^{-1} U_i(h, 0) = (0, A_k A_j^{-1} A_i h).$$

Then $\{U_1, U_2, U_3\}$ does not commute unless $A_k A_j^{-1} A_i = A_i A_j^{-1} A_k$, $i, j, k = 1, 2, 3$.

Let T_1, T_2, T_3 be the three 4×4 complex matrices:

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \text{ and } T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix},$$

$\{T_1, T_2, T_3\}$ does not have a unitary dilation.

3.2.3 Example. This example from (Paulsen, 2002, p .63), due to Varopoulos, exhibits three commuting contractions that fail the multivariable von Neumann inequality, and thus do not have unitary dilation.

Consider T_1, T_2, T_3 acting on $\mathbb{C}^5$ defined by

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & 0 \end{pmatrix}, T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 \end{pmatrix}.$$

Then $T_iT_j = T_jT_i$ and $\|T_i\| = 1$, for $i = 1, 2, 3$. Consider the polynomial in three variables

$$p(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3 - 2z_2z_3.$$

$\|p\|_\infty = 5$ but

$$p(T_1, T_2, T_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3\sqrt{3} & 0 & 0 & 0 & 0 \end{pmatrix}$$

has norm $\|p(T_1, T_2, T_3)\| = 3\sqrt{3} > 5 = \|p\|_\infty$. So the given commuting contractions T_1, T_2, T_3 do not verify the von Neumann inequality and then the system do not have unitary dilation.

4. d-contractions and von Neumann Inequality

Notations

In this chapter

- d is a fixed integer ($d = 2, 3, \ldots$).
- we denote $E = \mathbb{C}^d$ and B_d the d-dimensional open unit ball

$$B_d = \left\{(z_1, z_2, \ldots, z_d) \in E; |z_1|^2 + |z_2|^2 + \ldots + |z_d|^2 < 1\right\}.$$

- $\mathcal{P}$ denotes the algebra of analytic polynomial of d complex variables.
- for $n \in \mathbb{N}$, we denote

$$E^{\otimes n} = \underbrace{E \otimes \cdots \otimes E}_{n\times}$$

  and E^n the symmetric subspace of $E^{\otimes n}$. The tensor product of n copies of $z \in E$ is noted $z^n = z \otimes \ldots \otimes z$, for $z \in E$ and we have

$$E^n = \text{span}\left\{z^n, z \in E\right\}.$$

- $$\mathcal{F}_+(E) = E^0 \oplus E^1 \oplus E^2 \oplus \ldots$$

  is the symmetric Fock space where $E^0 = \mathbb{C}$.

4.1 Auxiliary materials

Here we introduce the tools we will be using to develop the appropriate generalisation of the one variable dilation theory of Nagy and Foias (1966).

4.1.1 C^*-algebra. There is an abstract characterization of a C^*-algebra. Since we will be concerned with operators on Hilbert space the following definition is appropriate. A C^*-algebra is a subalgebra of the algebra of operators on a Hilbert space that is self adjoint [1] and closed under the operator norm topology. A *-homomorphism is a homomorphism of algebra which preserves the adjoint.

Let H be an Hilbert space.

- The positive elements of a C^*-algebra $\mathcal{A} \subset \mathcal{B}(H)$ are the elements that can be written as T^*T, where $T \in \mathcal{A}$.
- Let $\{T_i\}_{i\in I} \subset \mathcal{B}(H)$ be a system of operators on H, then the C^*-algebra generated by $\{T_i\}_{i\in I}$ is the smallest C^*-algebra containing $\{T_i\}_{i\in I}$.

[1] closed under the adjoint operation

4.1.2 Completely Positive Maps. Let $\mathcal{A}$ be a C^*-algebra. Let $\mathcal{S}$ be an operator system, that is a self-adjoint subspace of $\mathcal{A}$ which contains the unit element of $\mathcal{A}$, of course we assume that $\mathcal{A}$ is unital. Let $\mathcal{B}$ be an other C^*-algebra. A linear map $\phi : \mathcal{S} \longrightarrow \mathcal{B}$ is said to be a *positive map* if it maps a positive element of $\mathcal{S}$ to a positive element of $\mathcal{B}$.

For a C^*-algebra $\mathcal{A}$ and $n \geq 0$, $M_n(\mathcal{A})$ is the C^*-algebra of the $n \times n$ square matrices with entries in $\mathcal{A}$. A map $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ induces a map

$$\begin{aligned} \phi_n : M_n(\mathcal{A}) &\longrightarrow M_n(\mathcal{B}) \\ (m_{ij}) &\longmapsto (\phi(m_{ij})). \end{aligned}$$

ϕ is called *completely positive* if ϕ_n is positive for every $n \in \mathbb{N}$. In general, the term *completely* is used to denote a property that all ϕ_n share. If the ϕ_n's are all bounded, ϕ is said to be completely bounded. If the ϕ_n's are all contractive, ϕ is said to be completely contractive.

4.1.3 Proposition. If ϕ is a *-homomorphism that maps $1 \in \mathcal{A}$ to $1 \in \mathcal{B}$ then ϕ is completely positive and completely contractive.

4.1.4 Theorem (Stinespring's Dilation Theorem)**.** *(Paulsen, 2002, p. 43) For any completely positive map $\phi : \mathcal{A} \longrightarrow \mathcal{B}(H)$, where $\mathcal{A}$ is a unital C^*-algebra and H an Hilbert space, there exists a Hilbert space K, a bounded operator $V : H \longrightarrow K$ and a unital *-homomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}(K)$ such that*

$$\phi(a) = V^* \pi(a) V,$$

for all $a \in \mathcal{A}$.

If $\phi(1) = I$, where I is the identity operator on H, then V is isometric. Consequently H can be identified to its isometric image VH as a subspace of K. If P_H is the projection on H as subspace of K, then $V^* = P_H$ due to 2.1.6. The equality above thus take the more familiar form

$$\phi(a) = P_H \, \pi(a)|_H \,,$$

for all $a \in \mathcal{A}$.

4.1.5 The H^2_d space. Given an homogeneous polynomial $h \in \mathcal{P}$, the total degree of h being $k \in \mathbb{N}$, we can define a linear functional $\tilde{h}$ on E^k by setting

$$\tilde{h}(z^k) = h(z), z = (z_1, z_2, \ldots, z_d) \in E,$$

and extend on E^k by linearity. The theorem of representation of Riesz guarantees the existence of a unique element ξ of E^k which satisfies $\tilde{h}(e) = \langle e, \xi \rangle$ for all $e \in E^k$. In particular, we have $h(z) = \tilde{h}(z^k) = \langle z^k, \xi \rangle_{E^k}$, for all $z = (z_1, z_2, \ldots, z_d) \in E$.

Any $p \in \mathcal{P}$ can be written as a finite sum of homogeneous polynomials

$$p(z) = p_0(z) + p_1(z) + \cdots + p_n(z), \quad z = (z_1, z_2, \ldots, z_d)$$

where the total degree of p_k is $k \geq 0$. Let ξ_k be the unique element of E^k satisfying

$$p_k(z) = \left\langle z^k, \xi_k \right\rangle_{E^k}, \quad (k = 0, 1, 2, \ldots n).$$

Then

$$\|p\|^2 = \|\xi_0\|^2 + \|\xi_1\|^2 + \ldots + \|\xi_n\|^2$$

obviously defines a norm on $\mathcal{P}$. We define the H_d^2 *space* as the completion of $\mathcal{P}$ in this norm. We have

$$\left\| z_1^{i_1} z_2^{i_2} \dots z_d^{i_d} \right\|^2 = \frac{i_1! i_2! \dots i_d!}{(i_1 + i_2 + \dots + i_d)!},$$

and H_d^2 is spanned by the polynomials $\left\{ z_1^{i_1} z_2^{i_2} \dots z_d^{i_d};\ i_1, i_2, \dots, i_d \geq 0 \right\}$(Arveson, 1998).

If f is an element of H_d^2 then

$$f(z) = \sum_{k=0}^{\infty} \left\langle z^k, \phi_k \right\rangle_{E^k}, \tag{4.1.1}$$

where the sequence $(\phi_k)_{k\in\mathbb{N}}$, $\phi_k \in E^k$, is uniquely determined and satisfies

$$\sum_{k=0}^{\infty} \|\phi_k\|^2 < \infty.$$

The right-hand side of (4.1.1) will be referred to as the *Taylor expansion* of f.

We can see that the sequence (ϕ_k) which determines f is an element of the symmetric Fock space $\mathcal{F}_+(E)$. Hence, the application $J : f \longmapsto (\phi_0, \phi_1, \phi_2, \dots) \in \mathcal{F}_+(E)$ is an isomorphism. Since, for two elements f, g of H_d^2,

$$\langle f, g \rangle_{H_d^2} = \langle Jg, Jf \rangle_{\mathcal{F}_+(E)},$$

J is anti-unitary and we identify an element f of H_d^2 with $Jf \in \mathcal{F}_+(E)$.

4.1.6 Proposition. For $x \in B_d$ and $f \in H_d^2$:

(i) the function $u_x : z \in B_d \longmapsto \dfrac{1}{1 - \langle z, x \rangle}$ is an element of H_d^2.

(ii) $\langle u_x, u_y \rangle = \dfrac{1}{1 - \langle y, x \rangle} = u_y(x)$ and also $f(x) = \langle f, u_x \rangle$.

(iii) H_d^2 is spanned by $\{u_x, x \in B_d\}$.

Proof. Let $x \in b_d$.

(i) For $z \in B_d$

$$u_x(z) = \frac{1}{1 - \langle z, x \rangle} = \sum_{k=0}^{\infty} \left\langle z^k, x^k \right\rangle_{E^k}.$$

Since

$$\sum_{k=0}^{\infty} \left\| x^k \right\|_{E^k}^2 = \sum_{k=0}^{\infty} \|x\|^{2k} < \infty,$$

$u_x \in H_d^2$ with $Ju_x = (1, x, x^2, x^3, \dots) \in \mathcal{F}_+(E)$.

(ii) Since $\left|\langle y^k, x^k \rangle_{E^k}\right| = \left|\langle y, x \rangle^k\right| < 1$, for $y \in B_d$, we have

$$\langle u_x, u_y \rangle = \langle Ju_y, Ju_x \rangle = \sum_{k=0}^{\infty} \left\langle y^k, x^k \right\rangle_{E^k} = \sum_{k=0}^{\infty} \langle y, x \rangle^k = \frac{1}{1 - \langle y, x \rangle} = u_y(x).$$

(iii) Let $f \in H_d^2$, $f \perp u_x$. Then $f(x) = \langle f, u_x \rangle = 0$, x being an arbitrary element of B_d, $f = 0$. This proves that $\{u_x, x \in B_d\}$ is dense in H_d^2.

□

The proposition 4.1.6 ii shows that H_d^2 is a *Reproducing Kernel Hilbert Space*. Then it has a multiplier algebra which is defined as follow.

4.1.7 Multiplier Algebra of H_d^2. An application $f : B_d \longmapsto \mathbb{C}$ is called a *multiplier* of H_d^2 if $fg \in H_d^2$ for all $g \in H_d^2$, fg denotes the point-wise multiplication.

$$\mathcal{M} = \left\{ f : B_d \longmapsto \mathbb{C}, f H_d^2 \subset H_d^2 \right\}$$

is called the multiplier algebra of H_d^2. For an $f \in \mathcal{M}$, the multiplication operator

$$M_f : g \in H_d^2 \longmapsto fg \in H_d^2$$

is a bounded operator on H_d^2.

Since the application $f \in \mathcal{M} \longmapsto M_f \in \mathcal{B}(H_d^2)$ is a homomorphism of algebra, $\|f\|_{\mathcal{M}} = \|M_f\|$ defines a norm on $\mathcal{M}$.

4.1.8 Remark.

- H_d^2 contains the function $\mathbf{1} : z \in B_d \longmapsto 1$, and consequently $H_d^2 \ni f\mathbf{1} = f$, for all $f \in \mathcal{M}$. Moreover, if we consider the inequality $\|fg\|_{H_d^2} \leq \|f\|_{\mathcal{M}} \|g\|_{H_d^2}$ for the function $g = \mathbf{1}$, then we have $\|f\|_{H_d^2} \leq \|f\|_{\mathcal{M}}$. Therefore

$$\mathcal{M} \subset H_d^2 \quad \text{and} \quad \|f\|_{H_d^2} \leq \|f\|_{\mathcal{M}} \quad (\text{ for all } f \in H_d^2). \tag{4.1.2}$$

- Since $\mathcal{M}$ is an algebra, for $p \in \mathcal{P}$ and $f_1, f_2, \ldots, f_d \in H_d^2$, $M_{p(f_1,\ldots,f_d)} = p(M_{f_1}, \ldots, M_{f_d})$.

4.1.9 Proposition. Let H^∞ be the algebra of bounded functions on B_d. Then

$$\|f\|_\infty \leq \|f\|_{\mathcal{M}},$$

for every $f \in \mathcal{M}$, which shows that $\mathcal{M} \subset H^\infty$.

Proof. For $x \in B_d$, $|f(x)u_x(x)| = |\langle M_f u_x, u_x \rangle| \leq \|M_f u_x\| \|u_x\| \leq \|M_f\| \|u_x\|^2 = \|M_f\| |u_x(x)|$. Since $u_x(x) \neq 0$ then $|f(x)| \leq \|M_f\| = \|f\|_{\mathcal{M}}$ for all $x \in B_d$. □

One of the important result in the operator theory on H_d^2 is given by the following theorem.

4.1.10 Theorem. *$\mathcal{M}$ is a proper subalgebra of H^∞.*

Proof. Let $(c_n) \subset \mathbb{C}$ be a sequence of complex number satisfying

$$\sum_{n=0}^{\infty} |c_n| < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |c_n|^2 n^{\frac{d-1}{2}} = \infty. \tag{4.1.3}$$

The function f defined by

$$f(z_1, \ldots, z_d) = \sum_{n=0}^{\infty} \sqrt{d^{dn}} c_n (z_1 \ldots z_d)^n$$

is bounded on B_d. Indeed, for $(z_1, z_2, \ldots, z_d) \in B_d$

$$(|z_1|^2 \ldots |z_d|^2)^{\frac{1}{d}} \leq \frac{1}{d}(|z_1|^2 + \ldots + |z_d|^2) \leq \frac{1}{d}.$$

Therefore

$$\sqrt{d^{dn}}\, |c_n|\, |z_1 z_2 \ldots z_d|^n = \sqrt{d^{dn}}\, |c_n|\, |z_1 z_2 \ldots z_d|^{2\frac{n}{2}} \leq \frac{\sqrt{d^{dn}}\, |c_n|}{\sqrt{d^{dn}}};$$

and we conclude using (4.1.3).

But if we note $f_N(z_1, \ldots, z_d) = \sum_{n=0}^{N} \sqrt{d^{dn}} c_n (z_1 \ldots z_d)^n$ then

$$\|f_N\|_{H_d^2}^2 = \left\| \sum_{n=0}^{N} \sqrt{d^{dn}} c_n (z_1 \ldots z_d)^n \right\|_{H_d^2}^2 = \sum_{n=0}^{N} d^{dn} |c_n|^2 \, \|(z_1 \ldots z_d)^n\|_{H_d^2}^2 = \sum_{n=0}^{N} d^{dn} |c_n|^2 \frac{n!^d}{(nd)!}.$$

Using Stirling's formula $n! \sim \sqrt{2\pi} \frac{n^{\frac{n+1}{2}}}{e^n}$, we have

$$d^{dn} |c_n|^2 \frac{(n!)^d}{(nd)!} \sim d^{dn} |c_n|^2 \frac{n^{\frac{nd+d}{2}} \sqrt{2\pi}^d}{e^{nd}} \frac{e^{nd}}{nd^{\frac{nd+1}{2}} \sqrt{2\pi}} \sim \sqrt{\frac{(2\pi)^{d-1}}{d}} |c_n|^2 n^{\frac{d-1}{2}}.$$

Since $\sum_{n=0}^{\infty} |c_n|^2 n^{\frac{d-1}{2}} = \infty$, $\|f_N\|_{H_d^2} \to \infty$ where $(f_N) \subset \mathcal{P} \subset H^\infty$ is the sequence of the partial sum of f, and taking account of the equation (4.1.2), the bounded function f is not a multiplier. □

4.2 d-contraction

One of the key concept which the success of the multivariable dilation theory is due to is the d-*contraction*. These are the correct counterpart for contraction in the higher dimension. The presentation in this section follows Arveson (1998).

4.2.1 Definition (d-contraction)**.** A d-contraction is a d-tuple of commuting operators, acting on the same Hilbert space, which satisfy

$$\|T_1\xi_1 + T_2\xi_2 + \cdots + T_d\xi_d\|^2 \leq \|\xi_1\|^2 + \|\xi_2\|^2 + \ldots + \|\xi_d\|^2,$$

or equivalently

$$T_1T_1^* + T_2T_2^* + \cdots + T_dT_d^* \leq I.$$

4.2.2 Remark. For a d-contraction $\underline{T} = (T_1, \ldots, T_d)$

(i) the application

$$P : \mathcal{B}(H) \longrightarrow \mathcal{B}(H)$$
$$A \longmapsto T_1 A T_1^* + \ldots + T_d A T_d^*$$

is completely positive. The sequence $(Q_n)_{n\geq 0}$, with $Q_n = P^n(I)$, is decreasing:

$$I \geq Q_1 \geq Q_2 \geq \ldots \geq 0$$

so that it converges in the weak-operator topology. We will write

$$P_\infty = \lim_{n\to\infty} P^n(I).$$

$(T_1, \ldots, T_d)$ is called a *null d-contraction* if $P_\infty = 0$.

(ii) the operator $I - P(I) \in \mathcal{B}(H)$ is positive, therefore , we can define a defect-like operator

$$\Delta = (I - P(I))^{\frac{1}{2}} = (I - T_1 T_1^* - \ldots - T_d T_d^*)^{1/2},$$

which is a contraction on H. Δ satisfies

$$\|\Delta h\|^2 = \|h\|^2 - \sum_{k=1}^{d} \|T_k^* h\|^2$$

for all $h \in H$.

4.2.3 Example (d-shift)**.** Given an orthonormal basis $\{e_1, e_2, \ldots, e_d\}$ of E, the *system of coordinate functions* associated to this basis is the system $\{Z_1, Z_2, \ldots, Z_d\} \subset H_d^2$ where

$$Z_k(x) = \langle x, e_k \rangle, \quad (x \in \mathbb{C}^d, k = 1, 2, 3, \ldots, d).$$

The Z_k's belong to $\mathcal{M}$ and the d-tuple of operators $(M_{Z_1}, \ldots, M_{Z_d})$ noted $\underline{S} = (S_1, \ldots, S_d)$, $S_k = M_{Z_k}$, is called the *d-shift*.

Let $f \in H_d^2$ such that $Jf = (f_0, f_1, f_2, \ldots)$,

$$Z_n(x) f(x) = \sum_{k-0}^{\infty} \langle x, e_n \rangle \left\langle x^k, f_k \right\rangle = \sum_{k-0}^{\infty} \left\langle x^{k+1}, e_n f_k \right\rangle.$$

Then $J(Z_n f) = (0, e_n f_0, e_n f_1, \ldots) = e_n Jf$. If we denote $A_k : \mathcal{F}_+(E) \longmapsto \mathcal{F}_+(E)$ the application defined by $A_n(\xi) = e_n \xi$ then $JM_{Z_n} = A_n J$.

4.2.4 Example (Spherical Operator)**.** A d-tuple $(Z_1, \ldots, Z_d)$, acting on a Hilbert space H, is called a *spherical operator* if the Z_k's are commuting normal operators and

$$Z_1^* Z_1 + \ldots + Z_d^* Z_d = I_H.$$

Therefore $(Z_1^*, \ldots, Z_d^*)$ is a d-contraction.

Every non-degenerate representation of $C(\partial B_d)$ on $\mathcal{B}(H)$ gives a spherical operator and conversely, any spherical operator arises from a non-degenerate representation of $C(\partial B_d)$.

4.2.5 Proposition. Let us note $K = \overline{\Delta H}$, where $\overline{\Delta H}$ is the closure of the range of Δ in K. If $(e_1, \ldots, e_d)$ is an orthonormal basis of E then the application $W : H \longmapsto \mathcal{F}_+(E) \otimes K$ defined by

$$Wh = (x_0, x_1, x_2, \ldots),$$

where $x_0 = 1 \otimes \Delta h$ and

$$x_k = \sum_{i_1,i_2,\ldots,i_n=1}^{d} e_{i_1} \otimes \ldots \otimes e_{i_n} \otimes \Delta T^*{}_{i_1} \ldots T^*{}_{i_n} h \; (n \geq 1)$$

is a contraction. Moreover, if $(T_1, \ldots, T_d)$ is a null d-contraction then W is isometric.

Proof. Notice first that since the operators $(T_1, \ldots, T_d)$ commute, so do their adjoints and then x_k is an element of $\mathcal{F}_+(E) \otimes K$; W is well defined. Since $(e_1, \ldots, e_d)$ is an orthonormal basis of E

$$\begin{aligned}
\|x_k\|^2 &= \left\langle \sum_{i_1,i_2,\ldots,i_n=1}^{d} e_{i_1} \otimes \ldots \otimes e_{i_n} \otimes \Delta T^*{}_{i_1} \ldots T^*{}_{i_n} h, \sum_{i_1,i_2,\ldots,i_n=1}^{d} e_{i_1} \otimes \ldots \otimes e_{i_n} \otimes \Delta T^*{}_{i_1} \ldots T^*{}_{i_n} h \right\rangle \\
&= \sum_{i_1,i_2,\ldots,i_n=1}^{d} \|\Delta T^*{}_{i_1} \ldots T^*{}_{i_n} h\|^2 \\
&= \sum_{i_1,i_2,\ldots,i_n=1}^{d} \langle \Delta T^*{}_{i_1} \ldots T^*{}_{i_n} h, \Delta T^*{}_{i_1} \ldots T^*{}_{i_n} h \rangle \\
&= \sum_{i_1,i_2,\ldots,i_n=1}^{d} \langle \Delta T^*{}_{i_1} \ldots T^*{}_{i_n} h, \Delta T^*{}_{i_1} \ldots T^*{}_{i_n} h \rangle \\
&= \sum_{i_1,i_2,\ldots,i_n=1}^{d} \langle T_{i_1} \ldots T_{i_n} \Delta\Delta T^*{}_{i_1} \ldots T^*{}_{i_n} h, h \rangle \\
&= \langle P^n(I - P(I))h, h \rangle \\
&= \langle P^n(I)h, h \rangle - \left\langle P^{n+1}(I)h, h \right\rangle .
\end{aligned}$$

Then

$$\sum_{k=0}^{\infty} \|x_k\|^2 = \left\langle P^0(I)h, h \right\rangle - \langle P_\infty h, h \rangle = \langle h, h \rangle - \langle P_\infty h, h \rangle \leq \|h\|^2 .$$

Therefore $\|Wh\|^2 \leq \|h\|^2$ for all $h \in H$ with equality if $P_\infty = 0$ □

4.3 $\mathcal{A}$-morphism and von Neumann inequality

4.3.1 Lemma. Let us note $K = \overline{\Delta H}$. If $(e_1, \ldots, x_d)$ is an orthonormal basis of E then, for any d-contraction $(T_1, \ldots, T_d)$, there exists a bounded operator $L : \mathcal{F}_+(E) \otimes K \longmapsto H$ which sends $1 \otimes \xi$ to $\Delta\xi$ and $e_{i_1} \otimes \ldots \otimes e_{i_n} \otimes \xi$ to $T_{i_1} \ldots T_{i_n} \Delta h$.

Proof. Let W be the operator defined in the previous theorem. We have

$$\langle 1 \otimes \xi, Wh \rangle = \langle 1 \otimes \xi, 1 \otimes \Delta h \rangle = \langle \xi, \Delta h \rangle .$$

Given $0 \leq j_1, \ldots, j_n \leq d$,

$$\begin{aligned}\langle e_{j_1} \ldots e_{j_n} \otimes \xi, Wh \rangle &= \sum_{i_1,i_2,\ldots,i_n=1}^{d} \langle e_{j_1} \ldots e_{j_n} \otimes \xi, x_n \rangle \\ &= \sum_{i_1,i_2,\ldots,i_n=1}^{d} \langle e_{j_1} \ldots e_{j_n}, e_{i_1} \ldots e_{i_n} \rangle \langle \xi, \Delta T^*{}_{i_1} \ldots T^*{}_{i_n} h \rangle \\ &= \langle \xi, \Delta T^*{}_{j_1} \ldots T^*{}_{j_n} h \rangle \\ &= \langle \Delta T_{j_1} \ldots T_{j_n} \xi, h \rangle .\end{aligned}$$

Therefore the application $L = W^*$ has the required property. □

4.3.2 Remark. Since the adjoint of a contraction is a contraction, L is contractive. And if the d-contraction $(T_1, \ldots, T_d)$ is a null d-contraction then the application L is co-isometric.

4.3.3 Definition ($\mathcal{A}$-morphism)**.** Let $\mathcal{A}$ be a subalgebra of a unital C^*-algebra $\mathcal{B}$. An $\mathcal{A}$-morphism is a completely positive map ϕ which satisfy:

1. $\phi(\mathbf{1}) = \mathbf{1}$.
2. $\phi(AX) = \phi(A)\phi(X)$, when $A \in \mathcal{A}$ and $X \in \mathcal{B}$.

4.3.4 Definition (Toeplitz C^*-algebra)**.** The Toeplitz C^*-algebra $\mathcal{T}_d$ is the C^*-algebra generated by $\mathbf{1}, S_1, S_2, \ldots, S_d$.

4.3.5 Remark. We will denote by $\mathcal{A}$ the subalgebra of $\mathcal{T}_d$ which consists of all polynomial in the d-shift:

$$\mathcal{A} = \{p(S_1, \ldots, S_d), p \text{ is a polynomial of } d \text{ complex variables } \}.$$

- $\mathcal{A}\mathcal{A}^*$ is dense in $\mathcal{T}_d$.
- $\mathcal{T}_d$ contains the compact operators on $\mathcal{F}_+(E)$. Indeed, for any $f, g \in \mathcal{F}_+(E)$ the application $\xi \in \mathcal{F}_+(E) \longmapsto \langle \xi, g \rangle f$ is in $\mathcal{T}_d$. Then $\mathcal{T}_d$ contains all rank-one operators on $\mathcal{F}_+(E)$ and consequently contains the compact operators.

4.3.6 Lemma. For every null d-contraction $(T_1, \ldots, T_d)$ acting on a Hilbert space H, there exists a unique $\mathcal{A}$-morphism

$$\phi : \mathcal{T} \longmapsto \mathcal{B}(H),$$

which satisfies

$$\phi(S_k) = T_k, \quad K = 1, 2, 3, \ldots, d.$$

Proof. Consider the application L of the Lemma 4.3.1. As stated in the Remark 4.3.2, for a null d-contraction $(T_1, \ldots, T_d)$ the application L is co-isometric. Let p be a polynomial of d complex variables. Using the realisation of the d-shift as creation operator on $\mathcal{F}_+(E)$, $S_k : e \longmapsto e_k e$, we have

$$L(p(S_1, \ldots, S_d) \otimes I_K)(e \otimes \xi) = L(p(e_1 e, \ldots, e_n e) \otimes \xi) = p(T_1, \ldots, T_d)L(e \otimes \xi).$$

Then for any X in $\mathcal{T}_d$

$$L(p(S_1, \ldots, S_d)X \otimes I_K)(e \otimes \xi) = L(p(S_1, \ldots, S_d) \otimes I_K)(Xe \otimes \xi) = p(T_1, \ldots, T_d)L(Xe \otimes \xi).$$

Therefore

$$L(p(S_1, \dots, S_d) \otimes I_K) = p(T_1, \dots, T_d)L \tag{4.3.1}$$

Moreover $L(I \otimes I_K) = L$, therefore the application ϕ defined on $\mathcal{T}_d$ by

$$\phi(X) = L(X \otimes I_K)L^*$$

satisfies

- $\phi(I) = LL^* = I_H$
- $\phi(p(S_1, \dots, S_d)X) = p(T_1, \dots, T_d)L(X \otimes I_K)L^* = p(T_1, \dots, T_d)\phi(X)$

for $A \in \mathcal{A}$ and $X \in \mathcal{T}_d$. □

4.3.7 Theorem. *(Arveson, 1998) Given a d-contraction $(T_1, \dots, T_d)$ acting on a Hilbert space H, there exists a unique $\mathcal{A}$-morphism $\phi : \mathcal{T} \longmapsto \mathcal{B}(H)$ such that*

$$\phi(S_k) = T_k, \quad k = 1, 2, 3, \dots, d.$$

Proof. Let $0 < r < 1$ and $\underline{T}_r = (rT_1, \dots, rT_d)$. There exists ϕ_r which satisfies

- $\phi_r(I) = LL^* = I_H$,
- $\phi_r(p(S_1, \dots, S_d)X) = p(rT_1, \dots, rT_d)\phi_r(X)$,

for any polynomial in d variables p. As $\|\phi_r\| \leq 1$ for $0 < r < 1$,

$$\lim_{r \to 1} \phi_r = \phi \in \mathcal{B}(\mathcal{T}_d, \mathcal{B}(H)),$$

in the operator norm. Moreover,

$$\phi(p(S_1, \dots, S_d)X) = \lim_{r \to 1} \phi_r(p(S_1, \dots, S_d)X) = \lim_{r \to 1} p(rT_1, \dots, rT_d)\phi_r(X) = p(T_1, \dots, T_d)\phi(X),$$

for $X \in \mathcal{T}$. □

The existence of the contractive $\mathcal{A}$-morphism proved in the Theorem above gives the appropriate multivariable version of the *von Neumann Inequality*.

4.3.8 Corollary. (Arveson, 1998, p. 44) If $(T_1, \dots, T_d)$ is a d-contraction acting on a Hilbert space H, then for any polynomial $p \in \mathbb{C}[z_1, z_2, \dots, z_d]$, we have

$$\|p(T_1, \dots, T_d)\| \leq \|p\|_{\mathcal{M}}.$$

Proof. By 4.3.7, there is a contractive $\mathcal{A}$-morphism ϕ such that $p(T_1, \dots, T_d) = \phi(p(S_1, \dots, S_d))$. Then $\|p(T_1, \dots, T_d)\| \leq \|p(S_1, \dots, S_d)\|$. But $p(S_1, \dots, S_d) = p(M_{z_1}, \dots, M_{z_d}) = M_p$ by 4.1.8, so that $\|p(S_1, \dots, S_d)\| = \|p\|_{\mathcal{M}}$. □

The Theorem 4.3.7 also gives rise to a model theory for null d-contractions.

4.3.9 Corollary. For any null d-contraction $(T_1, \ldots, T_d)$ acting on a common Hilbert space H, there exist a subspace K of $H^2_d \otimes \Delta H$ such that

$$T_i = P_K(S_i \otimes I_{\Delta H})P_K,$$

Δ being the defect operator associated to $(T_1, \ldots, T_d)$.

Proof. Consider the coisometry L of the Lemma 4.3.6 and the induced $\mathcal{A}$-morphism ϕ such that $\phi(S_i) = T_i$. Let $K = \Delta H$. From (4.3.1), we have

$$L^*T_i^* = (S_i^* \otimes I_{\Delta H})L^*.$$

Then $(S_i^* \otimes I_{\Delta H})L^*H = L^*T_i^*H \subset L^*H$, so L^*H is invariant by $(S_i^* \otimes I_{\Delta H})$. Since ϕ is the $\mathcal{A}$-morphism associated to $(T_1, \ldots, T_d)$ we have

$$T_i = L(S_i \otimes I_{\Delta H})L^*,$$

and if one identifies H with its isometric image $K = L^*H$ one has

$$T_i = P_K(S_i \otimes I_{\Delta H})P_K.$$

□

The operator $S_i \otimes I_{\Delta H}$ in the previous corollary can be considered as the direct sum of n copies of S_i, that we will note $n.S_i$, where $n = dim\Delta H$. Then the null d-contraction considered is the compression of $n.S = (n.S_1, \ldots, n.S_d)$ on the space L^*H which is invariant by $(S^* \otimes I_{\Delta H}) = n.S^*$.

One can prove a similar result corresponding to d-contractions in general. Let $(T_1, \ldots, T_d)$ be a d-contraction acting on H. According to the Stinespring Theorem 4.1.4, the $\mathcal{A}$-morphism ϕ in Theorem 4.3.7, which is a completely positive unital map, is the *compression* of a representation π, π being a representaion of the Toeplitz algebra $\mathcal{T}_d$ on some Hilbert space K. That is, there exists a Hilbert space K and an isometry V on K such that

$$\phi(X) = V^*\pi(X)V, \quad X \in \mathcal{T}_d.$$

Since a representation of $\mathcal{T}_d$ is the direct sum of a representation of $\mathcal{K}$, the ideal of $\mathcal{B}(H)$ which consists of the compact operators, on H^2_d and a representation of $\mathcal{T}_d/_{\mathcal{K}}$, and knowing that every representation of the ideal $\mathcal{K}$ is a multiple of the identity representation, there exists n [2] and a representation π_0 of $\mathcal{T}_d/_{\mathcal{K}}$ such that

$$T_k = \phi(S_k) = V^*\pi(S_k)V = V^*(n.S_k \oplus \pi_0(S_k))V, \quad (k = 1, 2, \ldots, d).$$

Thus, identifying H with its image under the isometry V, we have

$$T_k = P_{VH}(n.S_k \oplus \pi_0(S_k))P_{VH}.$$

One can also prove that the quotient $\mathcal{T}_d/_{\mathcal{K}}$ is isometric to the C^*-algebra $C(\partial B_d)$ of the continuous complex-valued functions on the complex d-sphere. From the Example 4.2.4, it follows that there exists a spherical operator $Z = (Z_1, Z_2, \ldots, Z_d)$ acting on a Hilbert space H_Z such that $\pi_0(S_k) = Z_k$ (Arveson, 1998).

Therefore we have the following corollary which gives the general form of the d-contractions.

[2] n can be any integer or ∞.

4.3.10 Corollary. For any d-contraction $(T_1, \ldots, T_d)$ acting on a common Hilbert space H, there exists n that can be any integer or ∞, a spherical operator $(Z_1, Z_2, \ldots, Z_d)$ acting on a Hilbert space H_Z such that

$$T_i = P_H(n.S_i \oplus Z_i)P_H,$$

for $i = 1, 2, \ldots, d$, the $n.S_i \oplus Z_i$'s acting on $H_d^2 \oplus H_Z$.

5. Conclusion

In this project, we study the space H_d^2 as the function space completion of the polynomials under a suitable norm. This space can also be represented as the symmetric Fock space. H_d^2 as a function space is moreover a reproducing kernel Hilbert space. These properties all coming together make H_d^2 an appropriate space to study operator theory.

We have seen in chapter 3 that the intuitive Definition 3.0.1 of the dilation of commuting contractions $\{T_1, T_2, \ldots, T_d\}$ acting on a common Hilbert space and the norm conditions $\|T_k\| \leq 1$ for $k = 1, 2, \ldots, d$, are not flexible enough do get a polynomial norm bound for a system of commutative contraction. Following the work of Arveson (1998), we let $(T_1, T_2, \ldots, T_d)$ act together as a d-contraction. Then, every d-contraction can be represented as part of a universal d-contraction which is the d-shift acting on H_d^2 and get thereby the appropriate bound in the von Neumann inequality.

The representation of the d-contraction as part of the d-shift uses a sophisticated language more than a simple compression. It is formulated in term of contractive completely positive maps in the Theorem 4.3.7 and the corresponding *-representation. These kind of maps are of central interest in today's dilation theory.

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"Raha ho avy ilay fotoana handaozako ny tany,

dia izao no mba faniriako sy masaka ato antsaiko:

Ianao Jesoa sy Maria reninao mba ho eo anilako hatrany,

ho eo anilako eo mandrapahatapitry ny aiko."

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