Invariance of differential equations

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Abstract

A brief study of some preliminary concepts from differential geometry are studied (eg, manifolds and Lie group transformations). Invariance of differential equation (as a consequence of the underlying Lie point symmetries) and their applications in reduction and solutions are pursued. An application of these concepts is made to variational differential equation (Lagrangians) with special attention to Noether symmetries and conservation laws. Finally,we apply the results to geodesic equations that arise in the study and analysis of space time manifolds.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Jean Juste Harrisson Bashingwa, 22nd May 2014

Contents

1. Introduction

In this chapter, we recall some basic definitions and theorems(without proofs) of the modern Lie group analysis. We recall concepts of manifolds, one parameter groups, symmetries of differential equations,Lie algebra and invariants. Bluman and Kumei (1989),Guggenheimer (1963).

1.1 Manifolds

1.1.1 Definition. An m-dimensional manifold M is a topological space which is covered by a collection of subsets $W_\alpha \subset M$ called *coordinate charts* and one-to-one maps $\chi_\alpha : W_\alpha \to V_\alpha$ onto connected open subsets $V_\alpha \subset \mathbb{R}^m$ which serve to define local coordinates on M. The manifold is smooth(respectively *analytic*) if the composite " overlap maps", $\chi_{\beta\alpha} = \chi_\beta \circ \chi_\alpha^{-1}$ are smooth (respectively analytic) where defined, i.e. from $\chi_\alpha[W_\alpha \cap W_\beta]$ to $\chi_\beta[W_\alpha \cap W_\beta]$.

We will assume that our manifolds are *separable*, meaning that there is a countable dense subset, satisfying the Hausdorff topological separation axiom.

The manifold will be *connected* if it cannot be written as the disjoint union of two non-empty open subsets, it is *path connected* if for every pair of points exist path in the manifold that connects the points, lastly the manifold is simply connected if every closed curve is reducible to a point.

1.1.2 Definition. A group is a set G that has an associative (but not necessarily commutative) multiplication operation, denoted $g.h$ for group elements $g, h \in G$. The group must also contain a (necessary unique) identity element, denoted e, and each group element g has an inverse g^{-1} satisfying $g.g^{-1} = g^{-1}.g = e$.

1.1.3 Definition. A Lie group is a smooth manifold which is also a group ,such that the group multiplication $(g, h) \mapsto g.h$ and inversion $g \mapsto^{-1} g$ define smooth maps.

1.1.4 Example. The general linear group GL$(n, \mathbb{R})$ consisting of all invertible $n \times n$ real matrices is a n^2-dimension manifold,or n^2 parameter Lie group.

1.2 One parameter groups

1.2.1 Definition. A *transformation group* acting on smooth manifold M is determined by a Lie group G and smooth map $\Phi : G \times M \to M$ denoted by $\Phi(g, x) = g.x$, which satisfies

$e.x = x, \qquad g.(h.x) = (g.h).x, \qquad$ for all $x \in M, \quad g, h \in G.$

We consider invertible transformations, that is a one-to-one and onto map:

$$\bar{x} = \varphi(x, u, \epsilon), \quad \bar{u} = \psi(x, u, \epsilon) \tag{1.2.1}$$

where ϵ is a real parameter. (1.2.1) is called a point transformation.

1.2.2 Definition. The collection G of transformation T_ϵ is called a local one-parameter group if,

(i) $T_0 = I \in G$,

(ii) $T_\epsilon T_\theta = T_{\epsilon+\theta} \in G$,

(iii) $T_{\epsilon^{-1}} = T_{-\epsilon} \in G$.

The following are examples of one-parameter group of transformations:

(a) $\bar{x} = x + \epsilon, \qquad \bar{u} = u$ (translation)

(b) $\bar{x} = e^{\epsilon}x, \qquad \bar{u} = u^{2\epsilon}u$ (dilatation or stretching)

(c) $\bar{x} = x\cos\epsilon - u\sin\epsilon, \qquad \bar{u} = x\sin\epsilon + u\cos\epsilon$ (rotation)

(1.2.1) is the finite form.

The Taylor expansion of the function φ, ψ in the parameter ϵ in a neighbourhood of ϵ is given as

$$\bar{x} \approx x + \xi(x,u)\epsilon, \qquad \bar{u} \approx y + \eta(x,u)\epsilon. \tag{1.2.2}$$

(1.2.2) is called an infinitesimal transformation where

$$\xi(x,u) = \frac{\partial\varphi(x,u,\epsilon)}{\partial\epsilon}\bigg|_{\epsilon=0} \qquad \eta(x,u) = \frac{\partial\psi(x,u,\epsilon)}{\partial\epsilon}\bigg|_{\epsilon=0}.$$

1.2.3 Definition. The vector field (first-order differential operator)

$$X = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}$$

is known as the generator of the group G.

1.3 Prolongation of point transformation

A general system of ordinary differential equations involves p-independent variables $x = (x^1, \cdots, x^p)$ which can be viewed as local coordinates on the Euclidean space $X \simeq \mathbb{R}^p$ and , q-dependent variables $u = (u^1, \cdots, u^q)$, coordinates on $U \simeq \mathbb{R}^q$. Then the *total space* will be the Euclidean space

$E = X \times U \simeq \mathbb{R}^{p+q}$

Let G be a one parameter Lie group acting on E. The generator of group G is written in the form:

$$X = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{1.3.1}$$

The derivatives of u with respect to x are

$$u_i^\alpha = D_i(u^\alpha), \qquad u_{ij}^\alpha = D_j D_i(u^\alpha),$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha\frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha} + \cdots \qquad i = 1, \cdots, p$$

is the operator of total differentiation.

1.3.1 Definition. The *characteristic* of the group generator X(vector field) is the q-tuple of functions $Q(x, u^{(1)})$, depending on x, u and first-order derivatives of u, defined by

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha(x, u) - \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial u^\alpha}{\partial x^i}, \qquad \alpha = 1, \cdots, q. \tag{1.3.2}$$

1.3.2 Theorem. *Let X be a vector field given by (1.3.1) , and let $Q = (Q^1, \cdots, Q^q)$ be its characteristic, the n^{th} prolongation of X is given explicitly by*

$$X^{[n]} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J=j=0}^{n} \varphi_J^\alpha(x, u^{(j)}) \frac{\partial}{\partial u_J^\alpha}, \tag{1.3.3}$$

with the coefficients

$$\varphi_J^\alpha = D_J Q^\alpha + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha$$

Note that these coefficients are related by the relation

$$\varphi_i^\alpha = D_i \varphi^\alpha - \sum_{j=1}^{p} (D_i \xi^j) u_j^\alpha. \tag{1.3.4}$$

2. Invariance of differential equation

2.1 Invariants

2.1.1 Definition. A function $F(x, u)$ is invariant under the group of transformations (1.2.1) if, $F(\bar{x}, \bar{u}) = F(x, u)$, for all $(x, u) \in \mathbb{R}^2$ and $\epsilon \in \mathbb{R}$

2.1.2 Theorem. *A function $F(x, u)$ is invariant under the group of transformation (1.2.1) if, and only if,*

$$XF \equiv \xi(x, u)\frac{\partial F}{\partial x} + \eta(x, u)\frac{\partial F}{\partial u} = 0,$$

where X is the generator of the group.

This concept can be generalized for the multi-dimensional case. We consider group transformations

$$\bar{x}^i = f^i(x, \epsilon), \qquad i = 1, \cdots, n \tag{2.1.1}$$

in the n-dimensional space of points $x = (x^1, \cdots, x^n)$.We consider a system of equations:

$$F_1(x) = 0, \cdots, F_s(x) = 0, \qquad s < n. \tag{2.1.2}$$

2.1.3 Theorem. *The system of equations (2.1.2) is invariant with respect to the group of transformations G (2.1.1) if, and only if,*

$XF_k|_{F_k} = 0 \quad , k = 1, \cdots, s.$

Proof see Ibragimov (1999)

where $X = \xi^i(x)\frac{\partial}{\partial x^i}, \qquad \xi^i = \frac{\partial f^i}{\partial \epsilon}\Big|_{\epsilon=0}$.

2.1.4 Theorem. *Let the system of equations (2.1.2) admit a group $G, J_1(x), \cdots, J_{n-1}(x)$ a basis of invariants(collection of all functionally independent invariants) of the group G then, (2.1.1) can be written as*

$$\Phi_k(J_1(x), \cdots, J_{n-1}(x)) = 0, \qquad k = 1, \cdots, s.$$

Proof see Ibragimov (1999)

2.2 Lie Algebra

2.2.1 Definition. A *Lie algebra* **g** is a vector space over a field F equipped with a bracket operation [.,.]:**g** $\times$ **g** $\to$ **g** such that the following axiom holds :

(i) Bilinearity: for any $X, Y, Z \in$ **g** and $a, b \in F$

$$[aX + bY, Z] = a[X, Z] + b[Y, Z]$$
$$[X, aY + bZ] = a[X, Y] + b[X, Z]$$

(ii) Anti-symmetric: for $X, Y \in \mathbf{g}$

$$[X, Y] = -[Y, X]$$

(iii) Jacobi identity: for any $X, Y, Z \in \mathbf{g}$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

The Lie bracket is given by

$$[X, Y] = XY - YX.$$

2.2.2 Definition. A sub-algebra of a Lie algebra $\mathbf{g}$ is a subspace of $\mathbf{g}$ which contains the commutator of any two of its elements.

2.3 Symmetries of Differential Equations

Consider a general n^{th} order system of differential equations

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, m \tag{2.3.1}$$

in p-independent variables $x = (x^1, \cdots, x^p)$ and q-dependent variables $u = (u^1, \cdots, u^q)$ with $u^{(n)}$ denoting the derivatives of the u's with respect to x's up to n. For simplicity, we shall restrict our attention to systems defined by smooth functions. The system (2.3.1) can therefore be viewed as defining a variety

$$S_\Delta = \left\{ \left(x, u^{(n)}\right) \middle| \Delta_\nu \left(x, u^{(n)}\right) = 0, \quad \nu = 1, \cdots, m \right\} \tag{2.3.2}$$

contained in the n^{th} order jet space J^n, i.e. consisting of all points $(x, u^{(n)}) \in J^n$ satisfying the system.

A (smooth) function $u = f(x)$ will define a solution to the system of differential equations (2.3.1) if, and only if, its prolongation $f^{(n)}(x)$ satisfies the system: i.e $\Delta_\nu(x, f^{(n)}(x)) = 0, \nu = 1, \cdots, m$. This is equivalent to the requirement that the graph $\Gamma_f^{(n)} = \{(x, f^{(n)}(x)\} = 0$ of the n^{th} prolongation of f is entirely contained in variety S_Δ

2.3.1 Definition. A point transformation $g : E \to E$ acting on the total space $E \simeq X \times U$ of independent and dependent variables is called a *symmetry* of the system (2.3.1) if, whenever $u = f(x)$ is a solution to (2.3.1), and the transformed function $\bar{f} = g.f$ is well defined, then $\bar{u} = \bar{f}(\bar{x})$ is also solution of (2.3.1).

2.3.2 Remark. We assume we are dealing with a connected group of point transformations G, then the infinitesimal generators of from a Lie algebra $\mathbf{g}$ is given by (1.3.1).

2.3.3 Theorem. *A connected group of transformation G is a symmetry group of a system of differential equations $\Delta = 0$ if, and only if, the classical infinitesimal symmetry conditions*

$$X^{[n]}(\Delta_\nu) = 0, \quad \nu = 1, \cdots r \quad \textit{whenever} \quad \Delta = 0 \tag{2.3.3}$$

hold for every infinitesimal generator $X \in \mathbf{g}$.

2.3.4 Example. We determine all symmetries of the differential equation: $y'' - y^{-3} + y = 0$.

The prolongation of the vector field using (1.3.3) is

$$X^{[2]} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta^x \frac{\partial}{\partial y'} + \eta^{xx} \frac{\partial}{\partial y''},$$

where these coefficients are given

$$\begin{aligned} \eta^x &= D_x \eta - (D_x \xi)\, y' \\ &= \eta_x + (\eta_y - \xi_x)\, y' - \xi_y y'^2, \\ \eta^{xx} &= D_x \left(\eta_x + (\eta_y - \xi_x)\, y' - \xi_y y'^2 \right) - (D_x \xi)\, y'' \\ &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})\, y' + (\eta_{yy} - 2\xi_{xy})\, y'^2 + (\eta_y - 2\xi_x)\, y'' - \xi_{yy} y'^3 - 3\xi_y y' y''. \end{aligned}$$

Since $X^{[2]}$ is symmetry of the differential equation $y'' - y^{-3} + y = 0$, we have

$$X^{[2]} \left(y'' - y^{-3} + y \right) = 0 \qquad \text{with} \qquad y'' - y^{-3} + y = 0 = 0.$$

That is, $$\left[\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta^x \frac{\partial}{\partial y'} + \eta^{xx} \frac{\partial}{\partial y''} \right] \left(y'' - y^{-3} + y \right) = 0,$$

which leads to the infinitesimal symmetry criterion,

$$\eta \left(1 + 3y^{-4} \right) + \eta^{xx} = 0,$$

or $$\eta \left(1 + 3y^{-4} \right) + \eta_{xx} + (2\eta_{xy} - \xi_{xx})\, y' + (\eta_{yy} - 2\xi_{xy})\, y'^2 + (\eta_y - 2\xi_x)\, y'' - \xi_{yy} y'^3 - 3\xi_y y' y'' = 0.$$

Replace y'' by $(y^{-3} - y)$ so that

$$\eta \left(1 + 3y^{-4} \right) + \eta_{xx} + \left(2\eta_{xy} - \xi_{xx} - 3\xi_y (y^{-3} - y) \right) y' + (\eta_{yy} - 2\xi_{xy})\, y'^2 + (\eta_y - 2\xi_x)\, (y^{-3} - y) - \xi_{yy} y'^3 = 0.$$

By separating by the power of y', we get

(1) $\eta \left(1 + 3y^{-4} \right) + \eta_{xx} + (\eta_y - 2\xi_x)\, (y^{-3} - y) = 0,$

(2) $2\eta_{xy} - \xi_{xx} - 3\xi_y (y^{-3} - y) = 0,$

(3) $\eta_{yy} - 2\xi_{xy} = 0,$

(4) $-\xi_{yy} = 0.$

From (4) we have

$$\xi = a(x) y + b(x), \tag{2.3.4}$$

(2.3.4) in (3) we get

$$\eta = \dot{a} y^2 + c(x) y + d(x), \tag{2.3.5}$$

where the dot refers to the total derivative with respect to x.

From (2.3.4) and (2.3.5) in (2), we get

$$3y\ddot{a} + 2\dot{c} - 3ay - \ddot{b} - 3y^{-3}a = 0.$$

Separating by the power of y, we get

$$\begin{aligned} &\ddot{a} + a = 0, \\ i.e, \quad &-3a = 0 \Longrightarrow a = 0 \qquad (5), \\ &2\dot{c} - \ddot{b} = 0. \end{aligned}$$

From (2.3.4), (2.3.5) and (5) in (1), we get

$$\ddot{c}y + \ddot{d} + (y^{-3} - y)(c - 2\dot{b}) + (cy + d)(1 + 3y^{-4}).$$

Separatint again by y, we get

$$\begin{aligned} &3d = 0 \Longrightarrow d = 0 \\ &2c - \dot{b} = 0 \qquad (6) \\ &\ddot{c} = -2\dot{b} \qquad (7). \end{aligned}$$

From (6) in (7) we have

$$\ddot{c} + 4\dot{c} = 0 \Longrightarrow c = C_1 \cos 2x + C_2 \sin 2x.$$

Using (6) or (7) we get $b = C_1 \sin 2x - C_2 \cos 2x + C_3$.

Then,

$$\begin{aligned} &\xi = C_1 \sin 2x - C_2 \cos 2x + C_3 \qquad C_1, C_2, C_3 \qquad \text{constants} \\ &\eta = (C_1 \cos 2x + C_2 \sin 2x)y \end{aligned}$$

The symmetry algebra is spanned by

$$X = (C_1 \sin 2x - C_2 \cos 2x + C_3)\frac{\partial}{\partial x} + [(C_1 \cos 2x + C_2 \sin 2x)y]\frac{\partial}{\partial y},$$

of which a basis is

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \sin 2x\frac{\partial}{\partial x} + y\cos 2x\frac{\partial}{\partial y}, \\ X_3 &= -\cos 2x\frac{\partial}{\partial y} + y\sin 2x\frac{\partial}{\partial y}. \end{aligned}$$

The commutators are

$$\begin{aligned} &[X_1, X_2] = -2X_3, \\ &[X_1, X_3] = 2X_2, \qquad [X_2, X_3] = 2X_1. \end{aligned}$$

2.3.5 Example. We determine all symmetries of the differential equation: $y''+y=0$. The infinitesimal symmetry criterion is

$$\eta+\eta^{xx}=0,$$
$$\eta+\eta_{xx}+(2\eta_{xy}-\xi_{xx})\,y'+(\eta_{yy}-2\xi_{xy})\,y'^2+(\eta_y-2\xi_x)\,y''-\xi_{yy}y'^3-3\xi_y y' y''=0.$$

Replace y'' by $-y$, we get

$$\eta+\eta_{xx}+(2\eta_{xy}-\xi_{xx}+3\xi_y y)\,y'+(\eta_{yy}-2\xi_{xy})\,y'^2-(\eta_y-2\xi_x)\,y-\xi_{yy}y'^3=0$$

Separate by the power of y' we get

$$\begin{aligned}
&(1)\quad \eta+\eta_{xx}-(\eta_y-2\xi_x)\,y=0,\\
&(2)\quad 2\eta_{xy}-\xi_{xx}+3\xi_y y=0,\\
&(3)\quad \eta_{yy}-2\xi_{xy}=0,\\
&(4)\quad \xi_{yy}=0.
\end{aligned}$$

From (4), we have

$$\xi=a(x)y+b(x). \tag{2.3.6}$$

From (2.3.6) in (3), we get

$$\eta=\dot{a}y^2+cy+d. \tag{2.3.7}$$

From (2.3.6) and (2.3.7) in (2) get

$$2(2y\ddot{a}+\dot{c})-\ddot{a}y-\ddot{b}+3ya=0.$$

Separating by y, we get

$$\begin{aligned}
&\ddot{a}+a=0\Longrightarrow a=c_1\cos x+c_2\sin x, \qquad (5)\\
&2\dot{c}-\ddot{b}=0. \qquad (6)
\end{aligned}$$

From (2.3.6) and (2.3.7) in (1), we get

$$\dot{a}y^2+cy+d+\dddot{a}y^2+\ddot{c}y+\ddot{d}-(2y\dot{a}+c-2\dot{a}y-2\dot{b})y=0.$$

Separating again by y, we get

$$\begin{aligned}
&\dddot{a}+\dot{a}=0,\\
&\ddot{c}+2\dot{b}=0, \qquad (7)\\
&\ddot{d}+d=0\Longrightarrow d=c_3\sin x+c_4\cos x.
\end{aligned}$$

(6) in (7) we have:

$$\dddot{b} + 4\dot{b} = 0 \Longrightarrow b = c_6 \sin 2x + c_7 \cos 2x + c_5.$$

using (6) or (7) we get $c = -c_7 \sin 2x + c_6 \cos 2x + c_8$

Then the symmetry algebra is spanned by

$$\begin{aligned} X =& \left[\left(c_1 \cos x + c_2 \sin x \right) y + \left(c_6 \sin 2x + c_7 \cos 2x + c_5 \right) \right] \frac{\partial}{\partial x} \\ &+ \left[\left(-c_1 \sin x + c_2 \cos x \right) y^2 + \left(c_6 \cos 2x - c_7 \sin 2x + c_8 \right) y + c_3 \sin x + c_4 \cos x \right] \frac{\partial}{\partial y}. \end{aligned}$$

the basis is

$$\begin{aligned} X_1 &= y \cos x \frac{\partial}{\partial x} - y^2 \sin x \frac{\partial}{\partial y}, \\ X_2 &= y \sin x \frac{\partial}{\partial x} + y^2 \cos x \frac{\partial}{\partial y}, \\ X_3 &= \cos x \frac{\partial}{\partial y}, \\ X_4 &= \sin x \frac{\partial}{\partial y}, \\ X_5 &= \frac{\partial}{\partial x}, \\ X_6 &= \sin 2x \frac{\partial}{\partial x} + y \cos 2x \frac{\partial}{\partial y}, \\ X_7 &= \cos 2x \frac{\partial}{\partial x} - y \sin 2x \frac{\partial}{\partial y}, \\ X_8 &= y \frac{\partial}{\partial y}. \end{aligned}$$

2.4 Integration of ordinary differential equation

First order differential equations

Let consider the first-order ordinary differential equation(DE) $\frac{dy}{dx} = \omega(x, y)$. Recall that a transformation $(x, y) \to (\hat{x}, \hat{y})$ is symmetry of the DE if, $\frac{d\hat{y}}{d\hat{x}} = \omega(\hat{x}, \hat{y})$. We get the symmetry condition

$$\begin{aligned} \frac{d\hat{y}}{d\hat{x}} =& \frac{D_x \hat{y}}{D_x \hat{x}} \\ =& \frac{\hat{y}_x + \omega(x, y)\hat{y}_y}{\hat{x}_x + \omega(x, y)\hat{x}_y} = \omega(\hat{x}, \hat{y}) \end{aligned}$$

We can find coordinates in which differential equations become simplest (separables).These coordinates are called *canonical coordinates*.

Canonical coordinates Lets consider the canonical coordinates $(r(x,y), s(x,y))$. A given differential equation becomes separable i.e. $ds/dr = f(r)$ or $ds/dr = g(s)$. The simplest way to choose this transformation is the translation either on $s - axis$ or $r - axis$.

Then $(\hat{r}, \hat{s}) = (r, s + \epsilon)$. Then the tangent vector at (r, s) will be

$$\left.\frac{d\hat{r}}{d\epsilon}\right|_{\epsilon=0} = 0, \quad \left.\frac{d\hat{s}}{d\epsilon}\right|_{\epsilon=0} = 1.$$

Using the chain rule, we get

$$\xi(x,y)r_x + \eta(x,y)r_y = 0,$$
$$\xi(x,y)s_x + \eta(x,y)s_y = 1.$$

We can find r by solving the differential equation $\frac{dy}{dx} = \frac{\eta(x,y)}{\xi(x,y)} = c = r$ where c is a constant and $\xi(x,y)$ is not zero. We use r to find s then,

$$s(r,x) = \left.\left(\int \frac{dx}{\xi(x, y(r,x))}\right)\right|_{r=r(x,y)}.$$

If $\xi(x,y) = 0$, then set $r = x$ and

$$s = \left.\left(\int \frac{dy}{\eta(r,y)}\right)\right|_{r=x}.$$

2.4.1 Example. We want to solve the differential equation

$$\frac{dy}{dx} = xy^2 - \frac{2y}{x} - \frac{1}{x^3}, x \neq .0$$

This equation has the non-trivial Lie symmetries $(\hat{x}, \hat{y}) = (e^{\epsilon}x, e^{-2\epsilon})$

Then $(\xi(x,y), \eta(x,y)) = (x, -2y)$, we have

$$\frac{dy}{dx} = -\frac{2y}{x} = c$$

which gives us $r = x^2 y$. Then $y = rx^{-2}$ and

$$s = \int \frac{dx}{x}.$$

The canonical coordinates are $(r, s) = (x^2 y, ln|x|)$. Rewrite the symmetry condition in terms of r and s

$$\begin{aligned}\frac{ds}{dr} &= \frac{s_x + \omega(x,y)s_y}{r_x + \omega(x,y)r_y} \\ &= \frac{\frac{1}{x} + \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)(0)}{2xy + \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)x^3} \\ &= \frac{1}{x^4 y^2 - 1} \\ &= \frac{1}{r^2 - 1}.\end{aligned}$$

Using separation of variables ,we obtain

$$s = \frac{1}{2} ln \left(\frac{r-1}{r+1} \right) + c$$

for some constant c. Replace x and y we end up with

$$y = \frac{x^2 + k}{x^2(k - x^2)}, \qquad \text{k=constant}$$

High order differential equation

2.4.2 Theorem. *Let $\Delta(x, u^{(n)}) = 0$ be an n^{th} order differential equation admitting a one-parameter group G. Then, all non-tangential solutions can be found by quadrature, from the solutions to an ordinary differential equation $(\Delta G)(x, u^{(n-1)}) = 0$ of order $n - 1$ called symmetry reduced equation.*

2.4.3 Corollary. If u and v are the 0^{th} and 1^{st} order invariants of X, then, a second order invariant is given by $\frac{dv}{du}$.

2.4.4 Example. Recall that $y'' - y^{-3} + y = 0$ generates the symmetries

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \sin 2x \frac{\partial}{\partial x} + y \cos 2x \frac{\partial}{\partial y}, \\ X_3 &= -\cos 2x \frac{\partial}{\partial y} + y \sin 2x \frac{\partial}{\partial y}. \end{aligned}$$

Consider the the 0^{th} and 1^{st} order invariants given by $X_1^{[1]}$. We have

$$X_1^{[1]} = \frac{\partial}{\partial x} + 0 \frac{\partial}{\partial y} + 0 \frac{\partial}{\partial y'},$$

of which the invariants are given by

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy'}{0}.$$

The 0 order and 1^{st} order invariants are $u = y$ and $v = y'$ respectively.

The second order invariant is

$$\begin{aligned} \frac{dv}{du} &= \frac{\frac{dv}{dx}}{\frac{du}{dx}} \\ &= \frac{y''}{y'} \\ &= \frac{y^{-3} - y}{y'} \\ \frac{dv}{du} &= \frac{u^{-3} - u}{v}. \end{aligned}$$

Which is a reduced form of the second order ODE

3. Symmetries of Euler-Lagrange Equation

3.1 The calculus of variation

As usual we work over an open subset of total space $E = X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$ coordinatized by independent variables $x = (x^1, \cdots, x^p)$ and dependent variables $u = (u^1, \cdots, u^q)$. The associated n^{th} jet space J^n is coordinatized by the derivative $u^{(n)}$. Let $\Omega \subset X$ be a connected open set with smooth boundary $\partial\Omega$. The problem is to find the extremals (maxima and minima) of a functional

$$I = \int_\Omega L(x, u^{(n)})dx \tag{3.1.1}$$

over some space of functions $u = f(x)$, $x \in \Omega$. The integrand $L(x, u^{(n)})$ is the *Lagrangian* of the variation problem and the horizontal p-form $Ldx = Ldx^1 \wedge \cdots \wedge dx^p$ is the *Lagrangian form* .

Let us specialize to $p = q = 1$ (scalar case), then, (3.1.1) can be written as

$$I = \int_a^b L(t, u, u', u'', \cdots, u^{(n)})dt \tag{3.1.2}$$

where the values of function u at the end points are fixed, this mean I satisfy the boundary condition $u(a) = u(b)$, then we need to find function u such that the variation in I be equal to zero, i.e.

$$\delta I = 0$$

The variation of I can be calculated as

$$\begin{aligned}\delta I =& \delta \int_a^b L(t, u, u', u'', \cdots, u^{(n)})dt \\ =& \int_a^b \left(\frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial u'}\delta u' + \cdots + \frac{\partial L}{\partial u^{(n)}}\delta u^{(n)} \right) dt,\end{aligned} \tag{3.1.3}$$

where $\delta u', \cdots, \delta u^{(n)}$ are variation of $u', \cdots, u^{(n)}$.

The second term in (3.1.3) gives

$$\begin{aligned}\int_a^b \frac{\partial L}{\partial u'}\delta u' dt =& \int_a^b \frac{\partial L}{\partial u'}\frac{d}{dt}(\delta u)dt \\ =& \underbrace{\frac{\partial L}{\partial u'}\delta u\Big|_a^b}_{0} - \int_a^b \delta u \frac{d}{dt}\left(\frac{\partial L}{\partial u'}\right) dt \\ =& -\int_a^b \delta u \frac{d}{dt}\left(\frac{\partial L}{\partial u'}\right) dt.\end{aligned} \tag{3.1.4}$$

The third term in (3.1.3) gives

$$
\begin{aligned}
\int_a^b \frac{\partial L}{\partial u''}\delta u'' dt &= \int_a^b \frac{\partial L}{\partial u''}\frac{d^2}{dt^2}(\delta u) dt \\
&= \underbrace{\frac{\partial L}{\partial u''}\frac{d}{dt}(\delta u)\Big|_a^b}_{0} - \int_a^b \frac{d}{dt}(\delta u)\frac{d}{dt}\left(\frac{\partial L}{\partial u''}\right) dt \\
&= -\int_a^b \frac{d}{dt}\left(\frac{\partial L}{\partial u''}\right) d(\delta u) \\
&= -\left[\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial u''}\right)\delta u\Big|_a^b}_{0} - \int_a^b \delta u \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial u''}\right) dt\right] \\
&= \int_a^b \delta u \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial u''}\right) dt. \qquad (3.1.5)
\end{aligned}
$$

Using the same procedure, the $(n+1)^{th}$ term in (3.1.3) gives

$$
\int_a^b \frac{\partial L}{\partial u^{(n)}}\delta u^{(n)} dt = (-1)^n \int_a^b \delta u \frac{d^n}{dt^n}\left(\frac{\partial L}{\partial u^{(n)}}\right) dt. \qquad (3.1.6)
$$

Using the results (3.1.6), (3.1.5), (3.1.4) in (3.1.2), we get

$$
\delta I = \int_a^b \delta u \left[\frac{\partial L}{\partial u} - \frac{d}{dt}\left(\frac{\partial L}{\partial u'}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial u''}\right) + \cdots + (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial L}{\partial u^{(n)}}\right)\right] dt.
$$

Thus $\delta I = 0$ is written as

$$
\int_a^b \delta u \left[\frac{\partial L}{\partial u} - \frac{d}{dt}\left(\frac{\partial L}{\partial u'}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial u''}\right) + \cdots + (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial L}{\partial u^{(n)}}\right)\right] dt = 0. \qquad (3.1.7)
$$

Note that (3.1.7) musts hold for arbitrary δu, the way that can make this possible is

$$
\frac{\partial L}{\partial u} - \frac{d}{dt}\left(\frac{\partial L}{\partial u'}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial u''}\right) + \cdots + (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial L}{\partial u^{(n)}}\right) = 0. \qquad (3.1.8)
$$

The equation (3.1.8) is known as the Euler-Lagrange equation. Its solution give the function that extremizes I .

3.1.1 Example. We can easily prove that the shortest path between two points in a plane is a straight line. In this case the functional should be the length of the curve, in $\mathbb{R}^2$ it is given by $ds = \sqrt{dx^2 + du^2}$ Then, the length is

$$
\begin{aligned}
l &= \int_a^b ds \\
&= \int_a^b \sqrt{dx^2 + du^2} \\
&= \int_a^b \sqrt{\frac{dx^2}{du^2} + 1}dx \\
&= \int_a^b \sqrt{u'^2 + 1}dx.
\end{aligned}
$$

The Lagrangian is $L = \sqrt{u'^2 + 1}$, independent on u.

The Euler-Lagrange equation is

$$
\frac{d}{dx}\left(\frac{\partial}{\partial u'}\sqrt{u'^2 + 1}\right) =0
$$

So that

$$
\frac{C}{\sqrt{1 - C^2}} = u'. \qquad \text{where C is a constant}
$$

Thus, u is a straight line.

3.1.2 Remark. Different Lagrangians may give rise to some Euler-Lagrange equations

3.1.3 Example. Consider two Lagrangian $L_1 = \frac{1}{2}u'^2$ and $L_2 = e^{-u'}$ then the corresponding Euler-lagrange equations are

$$
\begin{aligned}
&(i) \qquad \left[\frac{d}{dx}\left(\frac{\partial}{\partial u'}\right) - \frac{\partial}{\partial u}\right] L_1 = 0 \\
\text{so that} \quad & u'' = 0. \\
&(ii) \qquad \left[\frac{d}{dx}\left(\frac{\partial}{\partial u'}\right) - \frac{\partial}{\partial u}\right] L_2 = 0 \\
\text{which implies} \quad & e^{-u'}u'' = 0 \Longrightarrow u'' = 0 \qquad \text{since} \qquad e^{-u'} \neq 0.
\end{aligned}
$$

3.1.4 Remark. Inverse problem : Given a differential equation is it derivable from a variational principle or, given a differential equation $u'' = E(x, u, u')$, does its solutions optimize some functional $\int L(x, u, u')dx$?

3.2 Conservation Law

Given a differential equation $u'' = E(x, u, u')$, if there exist a function $I = I(x, u, u')$ such that $\frac{dI}{dx} = 0$ on the differential equation, then I is a conservation quantity.

3.3 Symmetries of variational Problem, Noether symmetries

3.3.1 Definition. A point transformation g is called a *variational symmetry* of the functional (1.3.1) if, and only if, the transformed functional agrees with the original one, which means that for every smooth function f define on a domain Ω, with transformed $\bar{f} = g.f$ define on $\bar{\Omega}$, we have

$$\int_{\Omega} L(x, f^{(n)}(x))dx = \int_{\bar{\Omega}} L(\bar{x}, \bar{f}^{(n)}(x))d\bar{x}.$$

Consider the one parameter Lie group of transformation(restriction to one independent and one dependent variable)

$$\begin{aligned} T_a :&\bar{x} = x + \epsilon\xi(x,u), \\ &\bar{u} = u + \epsilon\eta(x,u). \end{aligned}$$

It leaves invariant the functional in the calculus of variations up to a *gauge term* $f(x,u)$ if

$$\int L(\bar{x},\bar{u},\bar{u}')d\bar{x} = \int L(x,u,u')dx + \epsilon f(x,u)$$

$$\text{or } \int L(\bar{x},\bar{u},\bar{u}')d\bar{x} = \int L(x,u,u')dx + \epsilon \int \frac{df(x,u)}{dx}dx$$

differentiate the two sides with respect to $\bar{x}$ we have

$$\begin{aligned}
&\frac{d}{d\bar{x}}\int L(\bar{x},\bar{u},\bar{u}')d\bar{x} = \frac{d}{d\bar{x}}\left[\int L(x,u,u')dx + \epsilon\int\frac{df}{dx}dx\right], \\
&L(\bar{x},\bar{u},\bar{u}') = \frac{dx}{d\bar{x}}\frac{d}{dx}\left[\int L(x,u,u')dx + \epsilon\int\frac{df}{dx}dx\right], \\
&\frac{d\bar{x}}{dx}L(\bar{x},\bar{u},\bar{u}') = L(x,u,u') + \epsilon\frac{df}{dx}, \\
&\left(1+\epsilon\frac{d\xi}{dx}\right)L(x+\epsilon\xi, u+\epsilon\eta, u'+\epsilon\eta^x) = L(x,u,u') + \epsilon\frac{df}{dx}.
\end{aligned}$$

Taylor expension

$$\begin{aligned}
&\left(1+\epsilon\frac{d\xi}{dx}\right)\left[L(x,u,u') + \epsilon\xi\frac{\partial L}{\partial x} + \epsilon\eta\frac{\partial L}{\partial u} + \epsilon\eta^x\frac{\partial L}{\partial u'} + \cdots\right] = L(x,u,u') + \epsilon\frac{df}{dx}, \\
&\xi\frac{\partial L}{\partial x} + \eta\frac{\partial L}{\partial u} + \eta^x\frac{\partial L}{\partial u'} + L\frac{d\xi}{dx} = \frac{df}{dx},
\end{aligned}$$

Or

$$X^{[1]}L + L\frac{d\xi}{dx} = \frac{df}{dx}, \tag{3.3.1}$$

Where $X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}$.

3.3.2 Example. Consider the Lagranian $L = \frac{1}{2}(y'^2 - y^2)$, associated to the differential equation $y'' + y = 0$. Let find the Noether's symmetries by solving

$$X^{[1]}\frac{1}{2}\left(y'^2 - y^2\right) + \frac{1}{2}\left(y'^2 - y^2\right)\frac{d\xi}{dx} = \frac{df}{dx}. \tag{3.3.2}$$

The infinitesimal form is

$$-y\eta + y'\eta^x + \frac{1}{2}\left(y'^2 - y^2\right)\frac{d\xi}{dx} = \frac{df}{dx},$$

$$\text{or } -y\eta + y'\left[\eta_x + (\eta_y - \xi_x) - y'^2(\xi_y)\right] + \frac{1}{2}\left(y'^2 - y^2\right)(\xi_x + y'\xi_y) - f_x - y'f_y = 0.$$

Separating by the power of y', we have

$$\text{for } y'^3 \Rightarrow -\xi_y + \frac{1}{2}\xi_y = 0 \quad \text{or} \quad \xi = a(x) \qquad \text{function of x only.}$$

$$\text{for } y'^2 \Rightarrow \eta_y - \frac{1}{2}\xi_x = 0,$$

$$\eta_y = \frac{1}{2}\dot{a} \quad \text{or} \quad \eta = \frac{1}{2}\dot{a}y + b(x).$$

$$\text{for } y'^1 \Rightarrow \eta_x = f_y,$$

$$f_y = \frac{1}{2}\ddot{a}y + \dot{b} \quad \text{or} \quad f = \frac{1}{y}\ddot{a}y^2 + \dot{b}y + c(x).$$

$$\text{for } y'^0 \Rightarrow -y\eta - \frac{1}{2}y^2\xi_x = f_x,$$

$$-y\left(\frac{1}{2}\dot{a}y + b\right) - \frac{1}{2}y^2\dot{a} = \frac{1}{4}\dddot{a}y^2 + \ddot{b}y + \dot{c}.$$

Now separating by the power of y, we get

$$\text{for } y^2 \Rightarrow \dddot{a} + 4\dot{a} = 0 \qquad \text{or} \qquad a = A + B\cos 2x + C\sin 2x.$$

$$\text{for } y^1 \Rightarrow -b = \ddot{b} \qquad \text{or} \qquad b = D\cos x + E\sin x.$$

$$\text{for } y^0 \Rightarrow \dot{c} = 0 \qquad \text{or} \qquad c = F.$$

where A, B, C, D, E, F are constant, then we have

$$X = (A + B\cos 2x + C\sin 2x)\frac{\partial}{\partial x} + \left[\frac{1}{2}y(2C\cos 2x - 2B\sin 2x) + D\cos x + E\sin x\right]\frac{\partial}{\partial y},$$

$$f = \frac{1}{4}y^2[-4C\sin 2x - 4B\cos 2x] + y[E\cos x - D\sin x] + F.$$

The basis is spanned by

$$X_1 = \frac{\partial}{\partial x}, \qquad f_1 = 0$$

$$X_2 = \cos 2x\frac{\partial}{\partial x} - y\sin 2x\frac{\partial}{\partial y}, \qquad f_2 = -y^2\cos 2x$$

$$X_3 = \sin 2x\frac{\partial}{\partial x} + y\cos 2x\frac{\partial}{\partial y}, \qquad f_3 = -y^2\sin 2x$$

$$X_4 = \cos x\frac{\partial}{\partial y}, \qquad f_4 = -y\sin x$$

$$X_5 = \sin x\frac{\partial}{\partial y}, \qquad f_5 = y\cos x$$

We have the commutators

$$
\begin{aligned}
&[X_1, X_2] = -2X_3, \\
&[X_1, X_3] = 2X_2, \qquad [X_2, X_3] = 2X_1, \\
&[X_1, X_4] = -X_5, \qquad [X_2, X_4] = X_5, \qquad [X_3, X_4] = -X_4, \\
&[X_1, X_5] = X_4, \qquad [X_2, X_5] = X_4, \qquad [X_3, X_5] = X_5, \qquad [X_4, X_5] = 0.
\end{aligned}
$$

3.3.3 Remark. The Noerther symmetries form a Lie algebra sub Lie algebra of Lie symmetries (2.3.5) of the corresponding Euler-Lagrange equation.

3.3.4 Theorem. *If X is a Noerther symmetry, f a corresponding gauge function for a lagrangian $L(x, u^\alpha, u^\alpha_x) \qquad \alpha = 1, \cdots, q$ then*

$$I = L\xi + \sum_{\alpha=1}^{q} (\eta_\alpha - u^\alpha_x \xi) \frac{\partial L}{\partial u^\alpha_x} - f$$

is a conserved quantity of the Euler-Lagrange equation.

Proof

We want to show that $\frac{dI}{dx} = 0$.

$$
\begin{aligned}
\frac{dI}{dx} &= \xi \frac{dL}{dx} + L\frac{d\xi}{dx} + \sum_{\alpha=1}^{q} \left(\frac{d\eta_\alpha}{dx} - u^\alpha_x \frac{d\xi}{dx} - u^\alpha_{xx}\xi \right) \frac{\partial L}{\partial u^\alpha_x} + \sum_{\alpha=1}^{q} (\eta_\alpha - u^\alpha_x \xi) \frac{d}{dx} \frac{\partial L}{\partial u^\alpha_x} - \frac{df}{dx} \\
&= \left(\frac{\partial L}{\partial x} + \sum_{\alpha=1}^{q} u^\alpha_x \frac{\partial L}{\partial u^\alpha} \right) \xi + L\frac{d\xi}{dx} + \sum_{\alpha=1} \underbrace{\left(\frac{d\eta_\alpha}{dx} - u^\alpha_x \frac{d\xi}{dx} \right)}_{\eta^x_\alpha} \frac{\partial L}{\partial u^\alpha_x} + \sum_{\alpha=1}^{q} (\eta_\alpha - u^\alpha_x \xi) \frac{d}{dx} \frac{\partial L}{\partial u^\alpha_x} - \frac{df}{dx} \\
&= \underbrace{\xi \frac{\partial L}{\partial x} + \sum_{\alpha=1}^{q} \eta_\alpha \frac{\partial L}{\partial u^\alpha} + \sum_{\alpha=1}^{q} \eta^x_\alpha \frac{\partial L}{\partial u^\alpha_x}}_{X^{[1]}} + L\frac{d\xi}{dx} - \frac{df}{dx} - \sum_{\alpha=1}^{q} \eta_\alpha \frac{\partial L}{\partial u^\alpha} + \sum_{\alpha=1}^{q} (\eta_\alpha - u^\alpha_x \xi) \frac{d}{dx} \frac{\partial L}{\partial u^\alpha_x} + \sum_{\alpha=1}^{q} u^\alpha_x \xi \frac{\partial L}{\partial u^\alpha} \\
&= X^{[1]} + L\frac{d\xi}{dx} - \frac{df}{dx} + \sum_{\alpha=1}^{q} (\eta_\alpha - u^\alpha_x \xi) \left[\frac{d}{dx} \left(\frac{\partial L}{\partial u^\alpha_x} \right) - \frac{\partial L}{\partial u^\alpha} \right] \\
&= 0.
\end{aligned}
$$

3.3.5 Theorem. *If X is a Noerther symmetry and I the corresponding conserved quantity then $X^{[1]} I = 0$.*

3.3.6 Example. Let take the symmetry $X_4 = \cos x \frac{\partial}{\partial y}$ given in (3.3.2), the Lagrangian is

$$L = \frac{1}{2} y'^2 - \frac{1}{2} y^2.$$

Then the invariant is

$$\begin{aligned} I =& \left(\frac{1}{2}y'^2 - \frac{1}{2}y^2\right)(0) + (\cos x - y'.0)y' + y\sin x \\ =& y'\cos x + y\sin x. \end{aligned}$$

$$\begin{aligned} \frac{dI}{dx} =& (y'' + y)\cos x \\ =& 0 \qquad \text{since} \quad y'' + y = 0. \end{aligned}$$

So the differential equation

$$y'\cos x + y\sin x = K \quad \text{is a reduced form of} \quad y'' + y = 0$$

The prolongation of X_4 is

$$X_4^{[1]} = \cos x\frac{\partial}{\partial y} - \sin x\frac{\partial}{\partial y'}.$$

Lets apply the prolonged vector field $X_4^{[1]}$ on the reduced equation, we have

$$\begin{aligned} X_4^{[1]}(y'\cos x + y\sin x - K) =& \cos x(\sin x) + (-\sin x)(\cos x) \\ =& 0 \end{aligned}$$

$X_4^{[1]}$ is the symmetry of the reduced equation ,then the following corollary

3.3.7 Corollary. The Noether symmetries reduce twice the differential equation

3.3.8 Definition. Two Lagrangian $L(x, y, y')$ and $\bar{L}(X, Y, Y')$ are said to be equivalent up to gauge $f = f(x, y)$ if, $L(x, y, y') = \bar{L}(X, Y, Y')\frac{dX}{dx} + \frac{df}{dx}$

4. Invariance of Geodesic Equations

In this chapter we are comparing symmetries of Lagrangian(Noether symmetries) constructed from a Weyl re-scaled metric and Lagrangian from Minkowski metric with Lie symmetries of the Geodesic equation which are given in Feroze et al. (2006)

4.1 Lagrangian from Minkowski metric

Let consider the well know Minkowski metric in polar coordinates

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

The Lagrangian is given by

$$L = \dot{t}^2 - \dot{r}^2 - r^2\dot{\theta}^2 - r^2 \sin^2\theta \dot{\phi}^2,$$

where the dot refers to the derivative with respect to s. The Euler-Lagrange equations(geodesic equations) are

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0,$$

$$\text{or} \quad r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - \ddot{r} = 0, \tag{4.1.1}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0,$$

$$\text{or} \quad 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{4.1.2}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0,$$

$$\text{or} \quad 2\dot{r}\sin\theta\dot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi} = 0, \tag{4.1.3}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right) - \frac{\partial L}{\partial t} = 0,$$

$$\text{or} \quad \ddot{t} = 0. \tag{4.1.4}$$

The first prolongation of the Noether symmetries is

$$X^{[1]} = \sigma\frac{\partial}{\partial s} + \tau\frac{\partial}{\partial t} + \rho\frac{\partial}{\partial r} + J\frac{\partial}{\partial \theta} + F\frac{\partial}{\partial \phi} + \tau^s\frac{\partial}{\partial \dot{t}} + \rho^s\frac{\partial}{\partial \dot{r}} + J^s\frac{\partial}{\partial \dot{\theta}} + F^s\frac{\partial}{\partial \dot{\phi}},$$

where

$$\tau^s = \tau_s + \tau_t\dot{t} + \tau_r\dot{r} + \tau_\theta\dot{\theta} + \tau_\phi\dot{\phi} - \sigma_s\dot{t} - \sigma_t\dot{t}^2 - \sigma_r\dot{r}\dot{t} - \sigma_\theta\dot{\theta}\dot{t} - \sigma_\phi\dot{\phi}\dot{t},$$
$$\rho^s = \rho_s + \rho_t\dot{t} + \rho_r\dot{r} + \rho_\theta\dot{\theta} + \rho_\phi\dot{\phi} - \sigma_s\dot{r} - \sigma_t\dot{t}\dot{r} - \sigma_r\dot{r}^2 - \sigma_\theta\dot{\theta}\dot{r} - \sigma_\phi\dot{\phi}\dot{r},$$
$$J^s = J_s + J_t\dot{t} + J_r\dot{r} + J_\theta\dot{\theta} + J_\phi\dot{\phi} - \sigma_s\dot{\theta} - \sigma_t\dot{t}\dot{\theta} - \sigma_r\dot{r}\dot{\theta} - \sigma_\theta\dot{\theta}^2 - \sigma_\phi\dot{\phi}\dot{\theta},$$
$$F^s = F_s + F_t\dot{t} + F_r\dot{r} + F_\theta\dot{\theta} + F_\phi\dot{\phi} - \sigma_s\dot{\phi} - \sigma_t\dot{t}\dot{\phi} - \sigma_r\dot{r}\dot{\phi} - \sigma_\theta\dot{\theta}\dot{\phi} - \sigma_\phi\dot{\phi}^2.$$

The Noether's symmetries are given by (3.3.1)

$$X^{[1]}L + LD_s\sigma = D_s f,$$

which give

$$\begin{aligned}
&-\rho 2r(\dot\theta^2+\sin^2\theta\dot\phi^2)+J(-2\sin\theta\cos\theta r^2\dot\phi^2)+\tau^s(2\dot t)+\rho^s(-2\dot r)+J^s(-2r^2\dot\theta)+(\dot t^2-\dot r^2-r^2\dot\theta^2-r^2\sin^2\theta\dot\phi^2)D_s\sigma\\
&=f_s+f_t\dot t+f_r\dot r+f_\theta\dot\theta+f_\phi\dot\phi,
\end{aligned}$$

or

$$\begin{aligned}
&-\rho 2r(\dot\theta^2+\sin^2\theta\dot\phi^2)-2J\sin\theta\cos\theta r^2\dot\phi^2+2\dot t\tau_s+2\tau_t\dot t^2+2\tau_r\dot r\dot t+2\tau_\theta\dot\theta\dot t+2\tau_\phi\dot\phi\dot t\\
&-\sigma_s\dot t^2-\sigma_t\dot t^3-\sigma_r\dot r\dot t^2-\sigma_\theta\dot\theta\dot t^2-\sigma_\phi\dot\phi\dot t^2+-2\dot r\rho_s-2\rho_t\dot t\dot r-2\rho_r\dot r^2-2\rho_\theta\dot\theta\dot r-2\rho_\phi\dot\phi\dot r\\
&+\sigma_s\dot r^2+\sigma_t\dot t\dot r^2+\sigma_r\dot r^3+\sigma_\theta\dot\theta\dot r^2+\sigma_\phi\dot\phi\dot r^2+-2r^2\dot\theta J_s-2r^2J_t\dot t\dot\theta-2r^2J_r\dot r\dot\theta-2r^2J_\theta\dot\theta^2-2r^2J_\phi\dot\phi\dot\theta\\
&r^2\sigma_s\dot\theta^2+r^2\sigma_t\dot t\dot\theta^2+r^2\sigma_r\dot r\dot\theta^2+r^2\sigma_\theta\dot\theta^3+r^2\sigma_\phi\dot\phi\dot\theta^2-2r^2\sin^2\theta\dot\phi F_s-2r^2\sin^2\theta F_t\dot t\dot\phi-2r^2\sin^2\theta F_r\dot r\dot\phi-\\
&2r^2\sin^2\theta F_\theta\dot\theta\dot\phi-2r^2\sin^2\theta F_\phi\dot\phi^2+r^2\sin^2\theta\sigma_s\dot\phi^2+r^2\sin^2\theta\sigma_t\dot t\dot\phi^2+r^2\sin^2\theta\sigma_r\dot r\dot\phi^2+r^2\sin^2\theta\sigma_\theta\dot\theta\dot\phi^2+r^2\sin^2\theta\sigma_\phi\dot\phi^3\\
&=f_s+f_t\dot t+f_r\dot r+f_\theta\dot\theta+f_\phi\dot\phi.
\end{aligned}$$

Separating by the power of the derivative of t, r, θ, ϕ, we get

$$\begin{aligned}
\dot\theta^2:&\quad 2\rho+2rJ_\theta-r\sigma_s=0,\\
\dot\phi^2:&\quad 2rJ\cos\theta+2\rho\sin\theta+2r\sin\theta F_\phi-r\sin\theta\sigma_s=0,\\
\dot t:&\quad 2\tau_s=f_t,\\
\dot t^2:&\quad 2\tau_t-\sigma_s=0,\\
\dot t\dot r:&\quad \tau_r-\rho_t=0,\\
\dot r^2:&\quad 2\rho_r-\sigma_s=0,\\
\dot t\dot\theta:&\quad \tau_\theta-r^2J_t=0,\\
\dot t\dot\phi:&\quad \tau_\phi-r^2\sin^2\theta F_t=0,\\
\dot r:&\quad -2\rho_s=f_r,\\
\dot r\dot\theta:&\quad \rho_\theta+r^2J_r=0,\\
\dot r\dot\phi:&\quad \rho_\phi+r^2\sin^2\theta F_r=0,\\
\dot\theta:&\quad \sigma-2r^2J_s=f_\theta,\\
\dot\theta\dot\phi:&\quad J_\phi+\sin^2\theta F_\theta,\\
\dot\phi:&\quad -r^2\sin^2\theta F_s=f_\phi,\\
1:&\quad 0=f_s,\\
\dot t^3:&\quad \sigma_t=0,\\
\dot t^2\dot r:&\quad \sigma_r=0,\\
\dot t^2\dot\theta:&\quad \sigma_\theta=0,\\
\dot t^2\dot\phi:&\quad \sigma_\phi=0.
\end{aligned}$$

After lengthy calculation we obtain 17 Noether symmetries. The seventeen-dimensional Lie algebra of the Noethersymmetries is spanned by the following Noether symmetries

$$
\begin{aligned}
X_1 &= s\frac{\partial}{\partial s} + \frac{1}{2}t\frac{\partial}{\partial t} + \frac{1}{2}r\frac{\partial}{\partial r}, \qquad f = 0\\
X_2 &= \frac{\partial}{\partial s}, \qquad f = 0\\
X_3 &= \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}, \qquad f = 0\\
X_4 &= \frac{\partial}{\partial\phi}, \qquad f = 0\\
X_5 &= -r\cos\phi\sin\theta\frac{\partial}{\partial t} - t\cos\phi\sin\theta\frac{\partial}{\partial r} - \frac{t}{r}\cos\phi\cos\theta\frac{\partial}{\partial\theta} + \frac{t}{r}\sin\phi\csc\theta\frac{\partial}{\partial\phi}, \qquad f = 0\\
X_6 &= r\sin\phi\sin\theta\frac{\partial}{\partial t} + t\sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{t}{r}\sin\phi\cos\theta\frac{\partial}{\partial\theta} + \frac{t}{r}\cos\phi\csc\theta\frac{\partial}{\partial\phi}, \qquad f = 0\\
X_7 &= s^2\frac{\partial}{\partial s} + ts\frac{\partial}{\partial t} + sr\frac{\partial}{\partial r}, \qquad f = \frac{1}{2}(t^2 - r^2)\\
X_8 &= -\cos\theta\frac{\partial}{\partial r} + \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}, \qquad f = 0\\
X_9 &= s\cos\theta\frac{\partial}{\partial r} - \frac{s}{r}\sin\theta\frac{\partial}{\partial\theta}, \qquad f = -r\cos\theta\\
X_{10} &= -r\cos\theta\frac{\partial}{\partial t} + \frac{t}{r}\sin\theta\frac{\partial}{\partial\theta} - t\cos\theta\frac{\partial}{\partial r}, \qquad f = 0\\
X_{11} &= \frac{\partial}{\partial t}, \qquad f = 0\\
X_{12} &= s\frac{\partial}{\partial t}, \qquad f = t\\
X_{13} &= \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}, \qquad f = 0\\
X_{14} &= s\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{s}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{s}{r}\csc\theta\cos\phi\frac{\partial}{\partial\phi}, \qquad f = r\sin\theta\sin\phi\\
X_{15} &= s\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{s}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{s}{r}\csc\theta\sin\phi\frac{\partial}{\partial\phi}, \qquad f = -r\sin\theta\sin\phi\\
X_{16} &= -\sin\theta\cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\csc\theta\sin\phi\frac{\partial}{\partial\phi}, \qquad f = 0\\
X_{17} &= \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\csc\theta\cos\phi\frac{\partial}{\partial\phi}, \qquad f = 0
\end{aligned}
$$

Each of one provide a conserved quantity by Noether's theorem. In particular, X_1 leads to the conservation quantity

$$I_1 = t\dot{t} - r\dot{r} - sL$$

In fact,

$$
\begin{aligned}
\frac{dI_1}{ds} =& \dot{t}^2 + t\ddot{t} - \dot{r}^2 - r\ddot{r} - L - s\frac{dL}{ds} \\
=& r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 + \underbrace{t\ddot{t}}_{0} - r\ddot{r} - s\frac{dL}{ds} \\
=& \underbrace{[(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)r - \ddot{r}]}_{0} r - s\frac{dL}{ds} \\
=& -s(2\dot{t}\ddot{t} - 2\dot{r}\ddot{r} - 2r\dot{r}\dot{\theta}^2 - 2r^2\dot{\theta}\ddot{\theta} - 2r\dot{r}\sin^2\theta\dot{\phi}^2 - 2r^2\cos\theta\sin\theta\dot{\theta}\dot{\phi}^2 - 2r^2\sin^2\theta\dot{\phi}\ddot{\phi}) \\
=& -s(-4r\dot{r}\theta^2 - 2r^2\dot{\theta}\ddot{\theta} + 2r^2\cos\theta\sin\theta\dot{\theta}\dot{\phi}^2 - 4r\dot{r}\sin^2\theta\dot{\phi}^2 - 4r^2\cos\theta\sin\theta\dot{\theta}\dot{\phi}^2 - 2r^2\sin^2\theta\dot{\phi}\ddot{\phi}) \\
=& -s[-2r\dot{\theta}\underbrace{(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2)}_{0} - 2r\sin\theta\dot{\phi}\underbrace{(2\dot{r}\sin\theta\dot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi})}_{0}] \\
=& 0.
\end{aligned}
$$

We can also show that Noether symmetries reduce twice the geodesic equation. This is due to the invariance of the first integral under the associated Noether symmetry. In the above case, we have

$$
\begin{aligned}
X_1^{[1]}I_1 =& \left(s\frac{\partial}{\partial s} + \frac{1}{2}t\frac{\partial}{\partial t} + \frac{1}{2}r\frac{\partial}{\partial r} - \frac{1}{2}\dot{t}\frac{\partial}{\partial \dot{t}} - \frac{1}{2}\dot{r}\frac{\partial}{\partial \dot{r}} - \dot{\theta}\frac{\partial}{\partial \dot{\theta}} - \dot{\phi}\frac{\partial}{\partial \dot{\phi}}\right)[-s(\dot{t}^2 - \dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2) + t\dot{t} - r\dot{r}] \\
=& \frac{1}{2}t\dot{t} - s(\dot{t}^2 - \dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2) + \frac{1}{2}r[2sr(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - \dot{r}] - \frac{1}{2}\dot{t}[-2s\dot{t} + t] \\
&- \frac{1}{2}\dot{r}[2s\dot{r} - r] - \dot{\theta}(2sr^2\dot{\theta}) - \dot{\phi}(2sr^2\sin^2\theta\dot{\phi}) \\
=& 0,
\end{aligned}
$$

then X_1 is symmetry of I_1 .

4.2 Lagrangian from a Weyl re-scaled metric

Let now consider a Lagrangian constructed from a Weyl re-scaled metric

$$ds^2 = \frac{1}{r^2}dt^2 + \frac{1}{r^2}dr^2 + d\theta^2 + \sin^2\theta d\phi^2.$$

The Lagrangian for this metric is

$$L = \frac{1}{r^2}\dot{t}^2 + \frac{1}{r^2}\dot{r}^2 + \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2.$$

The corresponding Euler-Lagrange equations are given by

$$
\begin{aligned}
& r\ddot{t} - 2\dot{r}\dot{t} = 0, \\
& r\ddot{r} + \dot{t}^2 - \dot{r}^2 = 0, \\
& \ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \\
& 2\cos\theta\dot{\theta}\dot{\phi} + \sin\theta\ddot{\phi} = 0.
\end{aligned}
$$

The Noether symmetries are of the form

$$X = \sigma\frac{\partial}{\partial s} + \tau\frac{\partial}{\partial t} + \rho\frac{\partial}{\partial r} + J\frac{\partial}{\partial \theta} + F\frac{\partial}{\partial \phi}.$$

Substitute in (3.3.1) leads , after separation with respect to the power of variables t, r, θ, ϕ to the following linear system of pdes

$$\begin{aligned}
\dot{t}^2 :&\quad -2\rho + 2r\tau_t - r\sigma_s = 0,\\
\dot{r}^2 :&\quad -2\rho + 2r\rho_r - r\sigma_s = 0,\\
\dot{\theta}^2 :&\quad 2J_\theta - \sigma_s = 0,\\
\dot{\phi}^2 :&\quad 2\cos\theta J + 2\sin\theta F_\phi - \sin\theta\sigma_s = 0,\\
\dot{t}\dot{r} :&\quad \tau_r + \rho_t = 0,\\
\dot{t}\dot{\theta} :&\quad \tau_\theta + r^2 J_t = 0,\\
\dot{t}\dot{\phi} :&\quad \tau_\phi + r^2\sin^2\theta F_t = 0,\\
\dot{\theta}\dot{r} :&\quad r^2 J_r + \rho_\theta = 0,\\
\dot{\phi}\dot{r} :&\quad r^2\sin^2\theta F_r + \rho_\phi = 0,\\
\dot{\phi}\dot{\theta} :&\quad \sin^2\theta F_\theta + J_\phi = 0,\\
\dot{t} :&\quad 2\tau_s = r^2 f_t,\\
\dot{r} :&\quad 2\rho_s = r^2 f_r,\\
\dot{\theta} :&\quad 2J_s = f_\theta,\\
\dot{\phi} :&\quad 2\sin^2\theta F_s = f_\phi,\\
1 :&\quad 0 = f_s,\\
\dot{t}^3 :&\quad \sigma_t = 0,\\
\dot{r}^3 :&\quad \sigma_r = 0,\\
\dot{\theta}^3 :&\quad \sigma_\theta = 0,\\
\dot{\phi}^3 :&\quad \sigma_\phi = 0.
\end{aligned}$$

After some tedious manipulation,we obtain the seven-dimension Lie algebra of Noether point symmetries with basis

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\\
X_2 &= (r^2 - t^2)\frac{\partial}{\partial t} - 2rt\frac{\partial}{\partial r},\\
X_3 &= t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r},\\
X_4 &= \sin\phi\frac{\partial}{\partial \phi} + \cos\theta\cos\phi\frac{\partial}{\partial \phi},\\
X_5 &= \cos\phi\frac{\partial}{\partial \phi} - \cos\theta\sin\phi\frac{\partial}{\partial \phi},\\
X_6 &= \frac{\partial}{\partial s},\\
X_7 &= \frac{\partial}{\partial \phi}.
\end{aligned}$$

The nonzero commutators are

$$\begin{aligned}
&[X_1, X_2] = -2X_3,\\
&[X_1, X_3] = X_1, \qquad [X_2, X_3] = -X_2,\\
&[X_4, X_5] = X_7, \qquad [X_5, X_7] = -X_5.
\end{aligned}$$

Each of these leads to a conservation law. For instance the conserved quantity corresponding to X_6 is

$$I_6 = -\left(\frac{1}{r^2}(\dot{t}^2 + \dot{r}^2 + \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right)$$

In fact,

$$\begin{aligned}
\frac{dI_6}{ds} &= -\left[-\frac{2\dot{r}}{r^3}(\dot{t}^2 + \dot{r}^2) + \frac{1}{r^2}(2\dot{t}\ddot{t} + 2\dot{r}\ddot{r}) + 2\dot{\theta}\ddot{\theta} + 2\sin\theta\cos\theta\dot{\theta}\dot{\phi}^2 + 2\sin^2\theta\dot{\phi}\ddot{\phi}\right]\\
&= -\left[-\frac{1}{r^3}(-2\dot{r}\dot{t}^2 - 2\dot{r}(r\ddot{r} + \dot{t}^2) + 2r\dot{t}\ddot{t} + 2r\dot{r}\ddot{r}) + 2\dot{\theta}\ddot{\theta} + \sin^2\theta\dot{\phi}\ddot{\phi} + \sin\theta\dot{\phi}\underbrace{(2\cos\theta\dot{\phi}\dot{\theta} + \sin\theta\ddot{\phi})}_{0}\right]\\
&= -\left[-\frac{1}{r^3}(4\dot{r}\dot{t}^2 + 2r\dot{t}\ddot{t}) + 2\dot{\theta}\sin\theta\cos\theta\dot{\phi}^2 + \sin^2\theta\dot{\phi}\ddot{\phi}\right]\\
&= -\left[\frac{1}{-r^3}2\dot{t}\underbrace{(-2\dot{r}\dot{t} + r\ddot{t})}_{0} + \sin\theta\dot{\phi}\underbrace{(2\dot{\theta}\dot{\phi}\cos\theta + \sin\theta\ddot{\phi})}_{0}\right]\\
&= 0.
\end{aligned}$$

Also X_6 is symmetry of I_6, implying invariance in fact,

$$X_6^{[1]}I_6 = \frac{\partial}{\partial s}I_6 = 0.$$

4.3 Conclusion

We have shown that the Noether's symmetries arising from the usual Lagrangian form a subalgebra of Lie point symmetries of a differential equation. We were also able to show that Noether's symmetries play an important role in solving of differential equation(DE) as they reduce twice twice the DEs associated to the given Lagrangian. Finally, the Noether's theorem allow us to find conservation quantity from each Noether's symmetry.

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