Radiative Corrections Associated with the One-loop Vacuum Polarization in QED

Ishmael Takyi (ishmael@aims.ac.za)
African Institute for Mathematical Sciences (AIMS)

Supervised by: Professor Herbert Weigel
from University of Stellenbosch, South Africa

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Abstract

We use the relativistic quantum mechanics equation developed by Dirac and discuss coupling of electromagnetic interaction into this equation. The coupling of any free electron with the electromagnetic field generates a perturbation in the Dirac's Hamiltonian. We explore the concept of perturbation theory and use it with Green's function to discuss the propagator theory. Based on these, we construct the Feynman rules and Feynman integrals. Using the Feynman rules and integrals we compute the radiative correction of the one-loop vacuum polarization diagram and introduce the concept of charge renormalization. Using this concept we compute the Coulomb potential that arises from these radiative corrections.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Ishmael Takyi, 22 May 2014

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1. Introduction

In his hole's theory, Dirac postulated that all the negative-energy states (of the free Dirac operator) are occupied. He described the vacuum as an infinite sea of negative-energy electrons (Dirac, 1934); and that their charges are not measurable. However, if an applied field is introduced (for example a field created by a positive-energy electron) the electrons in the negative-energy sea are repelled and hence polarize the vacuum. Dirac found that the infinite charge density caused by the polarization effects diverges logarithmically, and that the effect cannot be neglected. To find a solution to this, (Oppenheimer, 1930) proposed that a momentum cut-off mass must be introduced to absorb the infinities occurring in the divergent integral; which later became known as charge renormalization.

In their experiment (Lamb Jr and Retherford, 1950) found that the $2S_{\frac{1}{2}} - 2P_{\frac{1}{2}}$ level shift is of size $\Delta E = 4 \times 10^{-6}$eV, which corresponds to a frequency $\Delta\nu = 1057 \pm 0 \cdot 2$ as opposed to Dirac atomic theory which predicts this energy difference to vanish, because they remain degenerate. To relate this finding to Dirac theory, (Lamb Jr and Retherford, 1950) propose that there must be a modification of the Coulomb law of attraction between electrons and protons or a theoretical vacuum polarization to explain this effect. Prior to this Uehling in his paper (Uehling, 1935) explains that the modification of the Coulomb potential is a result of the polarization of the vacuum. Welton in 1948 showed using Dirac hole's theory that the shift is a result of photon cloud "vacuum fluctuations", where the fluctuation δr is determined classical by

$$m\delta\ddot{r} = eE(t), \tag{1.0.1}$$

where $E(t)$ is the electric field strength. Taking the Fourier transform of (1.0.1) Welton found the corresponding energy shift as

$$\Delta E_{2,0} = \frac{mZ^4\alpha^5}{6\pi} \log\left(\frac{1}{\alpha}\right) \approx 2 \cdot 7 \times 10^{-6} Z^4 \mathrm{eV}, \tag{1.0.2}$$

with a corresponding frequency of $650\mathrm{MHz}$ for $Z = 1$.

To construct a relativistic approach in order to find correct results for this shift (Julian, 1949) and (Feynman, 1962) treated the external Coulomb field as a perturbation and suggested that the perturbation contributed to the photon cloud "vacuum fluctuations".

In this work, we compute the supplement to the Coulomb potential that arises from the radiative quantum corrections associated with the one-loop vacuum polarization. For that, we will divide this work in three parts. In chapter 2, we will review Klein-Gordon and Dirac equations. In chapter 3, we will introduce perturbation theory and use it to discuss Feynman rules and Feynman integrals. And in chapter 4, we will develop a systematic way of computing the potential using the Feynman rules and integrals.

1.1 Conventions

The notations and concepts introduced in this work follow the relevant features of special relativity. We will denote the space-time components of a four-vector by Greek indices, and that of a spatial three-vector by Latin letters. In analogy to three-dimensions, we define a four-vector as any set of four components that transforms as

$$x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu}, \tag{1.1.1}$$

where

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mu}_{\ \nu}, \tag{1.1.2}$$

with $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$ and $\beta = \dfrac{v}{c}$; equation (1.1.1) is the notation for the Lorentz boost of x^{μ} along the $x-$ axis. We define a contravariant four-vector as

$$A^{\mu} = \left(A^0, \mathbf{A}\right)^{\mu}, \tag{1.1.3}$$

where A^0 is the time component and $\mathbf{A} = \left(A^1, A^2, A^3\right)$ is the spatial components. Its covariant form is identified by the Minkowski metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}, \tag{1.1.4}$$

as

$$A_{\mu} = g_{\mu\nu} A^{\nu} = \left(A_0, -\mathbf{A}\right)_{\mu}. \tag{1.1.5}$$

We introduce the covariant and contravariant partial derivatives as

$$\frac{\partial}{\partial x^{\mu}} = \partial_{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)_{\mu}, \tag{1.1.6}$$

and

$$\frac{\partial}{\partial x_{\mu}} = \partial^{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)^{\mu}, \tag{1.1.7}$$

respectively, with

$$\partial_{\mu}\partial^{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box^2. \tag{1.1.8}$$

We make use of Einstein's summation convention where this summation is implied whenever a Greek index is repeated in a product; one as a contravariant index and one as a covariant index. Thus, we can write (1.1.1) as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}. \tag{1.1.9}$$

We also make use of the natural units where $\hbar = c = 1$.

2. Review of Klein-Gordon and Dirac Equations

In this chapter, we follow the historical interlude in the quest to search for a relativistic one-particle wave equation by reviewing Klein-Gordon equation for spin 0 particles and Dirac equation for spin $\frac{1}{2}$ particles. In section 2.7 we discuss the relativistic formulation of electromagnetic theory and introduce the concept of gauge invariance.

2.1 Klein-Gordon Equation

We consider an isolated free massive particle. The nonrelativistic energy for this particle is given as

$$E = \frac{\mathbf{p}^2}{2m}, \tag{2.1.1}$$

According to the correspondence principle we construct a quantum theory by introducing operators via

$$E \to \imath \frac{\partial}{\partial t} \quad \text{and} \quad \mathbf{p} \to \frac{1}{\imath} \nabla. \tag{2.1.2}$$

The corresponding free nonrelativistic Schrödinger equation for any state $\psi(\mathbf{r}, t)$ [1] becomes

$$\imath \frac{\partial \psi}{\partial t} = -\frac{\nabla^2}{2m} \psi. \tag{2.1.3}$$

These equations are not Lorentz covariant as their time and space derivative operators appear with different powers. We consider the relativistic case where the energy is given as

$$E = \sqrt{\mathbf{p}^2 + m^2} = \sqrt{-\nabla^2 + m^2}, \tag{2.1.4}$$

and the Schrödinger equation becomes

$$\imath \frac{\partial \psi}{\partial t} = \sqrt{-\nabla^2 + m^2} \psi. \tag{2.1.5}$$

But (2.1.5) has infinitely high spatial derivatives in its Taylor expansion and its coordinates of time and space components occur unsymmetrically. To avoid this we iterate (2.1.5) by squaring (2.1.4) and the relativistic wave-equation becomes

$$-\frac{\partial^2 \psi}{\partial t^2} = \left(-\nabla^2 + m^2\right) \psi,$$

which in compact form becomes

$$\left[\partial_\mu \partial^\mu + m^2\right] \psi = 0. \tag{2.1.6}$$

This is a scalar wave equation and is called the Klein-Gordon equation.

[1] where $\psi(\boldsymbol{r}, t)$ satisfies the postulates of nonrelativistic quantum equations.

2.1.1 The Continuity Equation. Multiplying (2.1.6) by $\frac{1}{2\imath}\psi^*$ and its complex conjugate by $\frac{1}{2\imath}\psi$, we obtained upon subtracting

$$\frac{1}{2\imath}\psi^*\left[\partial_\mu\partial^\mu + m^2\right]\psi - \frac{1}{2\imath}\psi\left[\partial_\mu\partial^\mu + m^2\right]\psi^* = 0,$$
$$\frac{1}{2\imath}\partial_\mu\left[\psi^*\partial^\mu\psi - \psi\partial^\mu\psi^*\right] = 0.$$

Which can be written as

$$\frac{\partial}{\partial t}\left[\frac{1}{2\imath}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)\right] - \nabla\cdot\left[\frac{1}{2\imath}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)\right] = 0,$$
$$\frac{\partial}{\partial t}\rho + \nabla\cdot\mathbf{j} = 0.$$

Where

$$\rho = \frac{1}{2\imath}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right), \tag{2.1.7}$$

and

$$\mathbf{j} = \frac{\imath}{2}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right), \tag{2.1.8}$$

are identified as the probability and current densities respectively. To interpret the meaning of the probability density we look at free solutions of the Klein-Gordon Equation.

2.1.2 Free Solution of the Klein-Gordon Equation. We consider a free particle solution of the Klein-Gordon equation; that is

$$\psi = Ne^{\imath(\mathbf{p}\cdot\mathbf{r} - Et)}. \tag{2.1.9}$$

Using (2.1.7) and (2.1.8) we have

$$\rho = E|N|^2 \quad \text{and} \quad \mathbf{j} = \mathbf{p}|N|^2 \tag{2.1.10}$$

To find the energy eigenvalues that appear in the probability density we substitute (2.1.9) into the Klein-Gordon equation and we get

$$E = \pm\sqrt{\mathbf{p}^2 + m^2}. \tag{2.1.11}$$

That is we have positive-energy solution

$$E = \sqrt{\mathbf{p}^2 + m^2} > 0,$$

and negative-energy solution

$$E = -\sqrt{\mathbf{p}^2 + m^2} < 0.$$

Consequently we have the probability density as

$$\rho_+ = E|N|^2 \geq 0, \tag{2.1.12}$$

for positive energy solutions and

$$\rho_- = -E|N|^2 \leq 0 \tag{2.1.13}$$

for negative-energy solutions. But we cannot have a negative probability density. To sort for a positive probability density and the interpretation of the negative-energy solution we discuss the Dirac equation in the next section.

2.2 The Dirac Equation

The Dirac equation is a relativistic equation which is linear in its spatial and time coordinates and is given as

$$\imath \frac{\partial \psi}{\partial t} = \left(-\imath \alpha^k \partial_k + \beta m \right) \psi \equiv E \psi. \qquad (2.2.1)$$

Since E is hermitian, α^k and β must also be hermitian and the wave function should also be an $N \times N$ component matrix. That is,

$$\psi = [\psi_1 \cdots \psi_N]^t .$$

Iterating (2.2.1) we have

$$-\frac{\partial^2 \psi}{\partial t^2} = -\sum_{i,j=1}^{3} \frac{1}{2} \left(\alpha^i \alpha^j + \alpha^j \alpha^i \right) \partial_i \partial_j \psi - \imath m \sum_{i=1}^{3} \left(\alpha^i \beta + \beta \alpha^i \right) \partial_i \psi + \beta m^2 \psi. \qquad (2.2.2)$$

Equivalence to the Klein-Gordon equation requires

$$\begin{aligned} \alpha^i \alpha^j + \alpha^j \alpha^i &= 2\delta^{ij}, \\ \alpha^i \beta + \beta \alpha^i &= 0, \\ \alpha^{i\,2} = \beta^2 &= I. \end{aligned} \qquad (2.2.3)$$

2.2.1 Important relations.

- From the second equation of (2.2.3), we have that α^i and β anticommute.
- Since $\alpha^{i\,2} = \beta^2 = I$ we find

$$\begin{aligned} \det \left(\alpha^{i\,2} \right) &= \det \left(\beta^2 \right) = \det(I), \\ \left[\det(\alpha^i) \right]^2 &= \left[\det(\beta) \right]^2 = 1, \\ \implies \det(\alpha^i) &= \det(\beta) = \pm 1. \end{aligned} \qquad (2.2.4)$$

  Using the properties of diagonalizable matrix and (2.2.4) we have that the eigenvalues of α_i and β are ± 1.

- The lowest dimensional matrix satisfying these requirements is that N should be greater than 2. In general, for N dimensions we have N^2 independent Hermitian matrices. Subtracting the identity matrix, we have $N^2 - 1$ Hermitian traceless matrices. For $N = 2$ we have only three, but for $N = 4$ we have fifteen. So we consider $N = 4 \times 4$, in particular the so-called standard representation is

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad (2.2.5)$$

  where σ^i are the 2×2 Dirac-Pauli matrices and I denotes the 2×2 unit matrix, which fulfils (2.2.3).

2.2.2 Covariant form of the Dirac Equation. Multiplying (2.2.1) by β from the left we have

$$-\imath\beta\frac{\partial\psi}{\partial t} - \imath\beta\alpha^i\partial_i\psi + m\psi = 0.$$

This can be written as

$$\left(-\imath\gamma^\mu\partial_\mu + m\right)\psi = 0. \tag{2.2.6}$$

Where γ^μ are the four Dirac matrices which are defined as

$$\gamma^\mu \equiv \left(\gamma^0, \gamma^i\right) = \left(\beta, \beta\alpha^i\right). \tag{2.2.7}$$

In the compact form (2.2.6) can be written as

$$\left(-\imath\partial\!\!\!/ + m\right)\psi = 0,$$

where $\partial\!\!\!/ = \gamma^\mu\partial_\mu$ is the Feynman slash.

2.2.3 Important relations of the Dirac Matrices.

- We have that $\gamma^{0\dagger} = \gamma^0$ and $\left(\gamma^0\right)^2 = I$.
- Also, we have the anticommutative relation from (2.2.3)

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{2.2.8}$$

- We also have that $\gamma^{k\dagger} = -\gamma^k$ and $\left(\gamma^k\right)^2 = -I$.

Combining the hermitian and antihermitian relations we have

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0,$$

and its Dirac adjoint is $\overline{\gamma^\mu} = \gamma^0\gamma^{\mu\dagger}\gamma^0$ which implies that γ^μ is "self barred". We therefore define a new $\gamma-$ matrix

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.2.9}$$

satisfying

$$\overline{\gamma^5} = \gamma^5, \quad \left(\gamma^5\right)^2 = I, \quad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5. \tag{2.2.10}$$

2.3 Dirac Trace Algebra

Using the Dirac matrices relations, we discuss some theorems of the trace algebras of the Dirac matrices that will be needed in later calculations.

2.3.1 Theorem.

$$\begin{aligned}\mathrm{Tr}\left(I\right) &= 4,\\ \mathrm{Tr}\left(\gamma_\mu\gamma_\nu\right) &= 4g_{\mu\nu}.\end{aligned} \tag{2.3.1}$$

Proof. From the properties of trace algebra we have that the trace of the identity matrix is equal to the number of dimensions of the matrix, hence since the dimension of the Dirac matrix is four we have $\mathrm{Tr}\,(I) = 4$. Also, from the cyclic property of the trace algebra we have

$$\mathrm{Tr}\,(\gamma_\mu \gamma_\nu) = \frac{1}{2}\mathrm{Tr}\,\{\gamma_\mu, \gamma_\nu\} = \mathrm{Tr}\,(g_{\mu\nu}) = g_{\mu\nu}\mathrm{Tr}\,(I) = 4g_{\mu\nu}.$$

□

2.3.2 Theorem.

$$\mathrm{Tr}\left(\gamma_{\mu_1} \cdots \gamma_{\mu_{2n+1}}\right) = 0. \tag{2.3.2}$$

Proof. We use the cyclic symmetric property of trace, that is $\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA)$. For n odd we have

$$\begin{aligned}\mathrm{Tr}\left(\not{a}_1 \cdots \not{a}_n\right) &= \mathrm{Tr}\left\{\not{a}_1 \cdots \not{a}_n \overbrace{\gamma^5\gamma^5}^{1}\right\}, \\ &= \mathrm{Tr}\left(\gamma^5 \not{a}_1 \cdots \not{a}_n \gamma^5\right).\end{aligned}$$

From (2.2.10) we have

$$\mathrm{Tr}\left(\not{a}_1 \cdots \not{a}_n\right) = (-1)^n\,\mathrm{Tr}\left(\not{a}_1 \cdots \not{a}_n\right) \tag{2.3.3}$$

for n odd. □

2.3.3 Theorem.

$$\begin{aligned}\mathrm{Tr}\left(\not{a}\not{b}\right) &= 4a\cdot b, \\ \mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) &= 4\left(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right), \\ \mathrm{Tr}\left(\not{a}\not{b}\not{c}\not{d}\right) &= 4\left[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right].\end{aligned} \tag{2.3.4}$$

Proof. Since

$$\begin{aligned}\mathrm{Tr}\,(\gamma_\mu\gamma_\nu) &= 4g_{\mu\nu}, \\ &\implies \mathrm{Tr}\left(\not{a}\not{b}\right) = \frac{1}{2}\mathrm{Tr}\left(\not{a}\not{b} + \not{b}\not{a}\right), \\ &\implies a\cdot b\mathrm{Tr}(I) = 4a\cdot b.\end{aligned}$$

Also

$$\begin{aligned}\mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) &= \mathrm{Tr}\,(\{\gamma_\mu,\gamma_\nu\}\,\gamma_\rho\gamma_\sigma) - \mathrm{Tr}\,(\gamma_\nu\gamma_\mu\gamma_\rho\gamma_\sigma)\,, \\ &= 2g_{\mu\nu}\mathrm{Tr}\,(\gamma_\rho\gamma_\sigma) - \mathrm{Tr}\,(\gamma_\nu\,\{\gamma_\mu,\gamma_\rho\}\,\gamma_\sigma) + \mathrm{Tr}\,(\gamma_\nu\gamma_\rho\gamma_\mu\gamma_\sigma)\,, \\ &= 2g_{\mu\nu}\mathrm{Tr}\,(\gamma_\rho\gamma_\sigma) - 2g_{\mu\rho}\mathrm{Tr}\,(\gamma_\nu\gamma_\sigma) + \mathrm{Tr}\,(\gamma_\nu\gamma_\rho\,\{\gamma_\mu,\gamma_\sigma\}) - \mathrm{Tr}\,(\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu)\,, \\ &= 8g_{\mu\nu}g_{\rho\sigma} - 8g_{\mu\rho}g_{\nu\sigma} + 8g_{\mu\sigma}g_{\nu\rho} - \mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)\,.\end{aligned}$$

Simplifying we have

$$\mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4\left(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right).$$

The proof of the third equation in (2.3.4) is the same as the previous one.

□

2.4 Conserved Current and the Adjoint Equation

We follow the same procedure we used in obtaining the continuity condition for the Klein-Gordon equation; here the hermitian conjugate of ψ becomes

$$\psi^\dagger = [\psi_1^*, \cdots, \psi_N^*] .$$

Taking the hermitian conjugates of (2.2.6) we have

$$\imath\partial_0\psi^\dagger\gamma^0 + \imath\partial_k\psi^\dagger\left(-\gamma^k\right) + m\psi^\dagger = 0. \tag{2.4.1}$$

To have a covariant form of this equation, we remove the minus sign of $-\gamma^k$. Since $\gamma^0\gamma^k = -\gamma^k\gamma^0$, we multiply (2.4.1) from the right by γ^0 and define the adjoint spinor as

$$\overline{\psi} = \psi^\dagger\gamma^0,$$

which in standard representation using (2.2.5) can be written as

$$\overline{\psi} = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*) .$$

We then obtain

$$\begin{aligned} \imath\partial_0\overline{\psi}\gamma^0 + \imath\partial_k\overline{\psi} + m\overline{\psi} &= 0, \\ \imath\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi} &= 0. \end{aligned} \tag{2.4.2}$$

Equation (2.4.2) is called the adjoint equation. We now multiply Dirac equation (2.2.6) from the left by $\overline{\psi}$, and the ajoint equation (2.4.2) from the right by ψ and by subtracting the two

$$\begin{aligned} \left(-\imath\overline{\psi}\gamma^\mu\partial_\mu\psi + \overline{\psi}m\psi\right) - \left(\imath\partial_\mu\overline{\psi}\gamma^\mu\psi + \overline{\psi}m\psi\right) &= 0, \\ \overline{\psi}\gamma^\mu\partial_\mu\psi + \partial_\mu\overline{\psi}\gamma^\mu\psi &= 0, \\ \partial_\mu\left(\overline{\psi}\gamma^\mu\psi\right) &= 0. \end{aligned} \tag{2.4.3}$$

Thus, we have that

$$j^\mu = \overline{\psi}\gamma^\mu\psi \quad \text{with} \quad \partial_\mu j^\mu = 0. \tag{2.4.4}$$

That is, we obtain the continuity equation where the probability density is given as

$$\rho = j^0 = \overline{\psi}\gamma^0\psi = \psi^\dagger\psi = \sum_{i=1}^{4} |\psi_i|^2.$$

Here the probability density is now positive definite. Inserting the charge $-e$ in j^μ we obtained what is called the charge current density given as

$$j^\mu = -e\overline{\psi}\gamma^\mu\psi. \tag{2.4.5}$$

2.5 Free Particle Solution of the Dirac Equation

We consider a free particle eigensolution of the Dirac equation, of the form

$$\psi = u(\mathbf{p})e^{-\imath p\cdot x}, \tag{2.5.1}$$

where we have considered $p_0 = \sqrt{\mathbf{p}^2 + m^2}$ and u is a four-spinor and does not depend on x. To find the energy eigenvalues, we substitute (2.5.1) into (2.2.6) and we have

$$(\gamma^\mu p_\mu - m)\, u(\mathbf{p}) = 0. \tag{2.5.2}$$

The energy eigenvalue equation of (2.5.2) becomes

$$Eu = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\, u. \tag{2.5.3}$$

We will consider two cases for computing the energy eigenvalues.

Case I For the case in which the particle is at rest, we have $\mathbf{p} = 0$ and using (2.2.5) we have

$$Eu = \beta m u = \begin{pmatrix} mI & 0 \\ 0 & -mI \end{pmatrix} u. \tag{2.5.4}$$

From the eigenvalue problem, we have the eigenvalues as $E = m, m, -m, -m$, with corresponding eigenvectors as

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{2.5.5}$$

That is we have positive-energy solution $E = m > 0$ and negative-energy solution $E = -m < 0$.

Case II For the case in which the particle is not at rest, we have $\mathbf{p} \neq 0$ and using (2.2.5) we have

$$Eu = \begin{pmatrix} mI & \boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -mI \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = E \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{2.5.6}$$

where we have decompose u into two-component column vectors u_1 and u_2. Which can also be written as

$$\begin{aligned} \boldsymbol{\sigma}\cdot\mathbf{p} u_2 &= (E - m)\, u_1, \\ \boldsymbol{\sigma}\cdot\mathbf{p} u_1 &= (E + m)\, u_2. \end{aligned} \tag{2.5.7}$$

To find the positive-energy solutions following from **Case I**, we take $u_1^{(r)} = \chi^{(r)}$, where

$$\chi^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}, \chi^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}. \tag{2.5.8}$$

Using the second equation of (2.5.7) we have the lower components of u as

$$u_2^{(r)} = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\chi^{(r)}, \tag{2.5.9}$$

hence the positive-energy solutions of the Dirac's equation becomes

$$u = N \begin{pmatrix} 1 \\ 0 \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \\ 0 \end{pmatrix} \quad \text{or} \quad u = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \end{pmatrix} \tag{2.5.10}$$

with N as the normalization constant. Similarly, to find the negative-energy solutions we take $u_2^{(r)} = \chi^{(r)}$, and from (2.5.7) we have the upper components of u as

$$u_1^{(r)} = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E-m}\chi^{(r)} = -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|E|+m}\chi^{(r)}, \tag{2.5.11}$$

hence the negative-energy solution of the Dirac's equation becomes

$$u = N \begin{pmatrix} -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|E|+m} \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{or} \quad u = N \begin{pmatrix} 0 \\ -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|E|+m} \\ 0 \\ 1 \end{pmatrix}. \tag{2.5.12}$$

2.5.1 Coupling to the Electromagnetic Field. The solution of the Dirac's equation for a free electron of four momentum p^μ is given as

$$\psi = u(\mathbf{p})e^{-\imath p\cdot x}.$$

As before u is a four-spinor and does not depend on x. In covariant and contravariant form the momentum operator becomes

$$p^\mu \to \imath\partial^\mu \quad \text{and} \quad p_\mu \to \imath\partial_\mu. \tag{2.5.13}$$

To find the equation for an electron in an electromagnetic field A^μ, where $A^\mu = (\phi, \mathbf{A})$ is the four-potential, we make use of the minimal substitution (of non relativistic quantum mechanics)

$$p^\mu \to p^\mu + eA^\mu, \tag{2.5.14}$$

where $-e$ is the charge of the particle. From the prescription (2.5.13) and (2.5.14) we find

$$\imath\partial^\mu \to \imath\partial^\mu + eA^\mu, \tag{2.5.15}$$

which in components form can be written as

$$\begin{cases} \imath\frac{\partial}{\partial t} \to \imath\frac{\partial}{\partial t} + e\phi \\ \imath\nabla \to \imath\nabla - e\mathbf{A} \end{cases} \tag{2.5.16}$$

Substituting (2.5.15) into the Dirac equation we have

$$\begin{aligned} \left(-\gamma_\mu\left(\imath\partial^\mu + eA^\mu\right) + m\right)\psi = 0, \\ \left(\gamma_\mu p^\mu - m\right)\psi = -e\gamma_\mu A^\mu\psi. \end{aligned} \tag{2.5.17}$$

Equation (2.5.17) is the relativistically covariant form of the Dirac equation when an electromagnetic field is introduced. Here $-e\gamma_\mu A^\mu$ is the perturbation. We introduce the concept of perturbation theory in the next chapter to explain its quantum effects on the electron.

2.6 Interpretation of the Negative-Energy Solutions

We discuss the Dirac as well as Feynman and Stückelberg interpretation of the negative energy solutions and introduce the concept of Feynman diagrams.

2.6.1 Dirac Interpretation of the Negative-Energy Solutions. In the quantum theory of radiation, if such negative-energy states exist; then the interaction of atomic electrons with the radiation field will make transitions into the negative-energy states and since the negative-energy spectrum has no lower bond, its energy falls indefinitely by emitting photons.

In an attempt to avoid this, Dirac invoked the Pauli exclusion principle, and postulated that the negative energy levels are all filled up with electrons and referred the filled up negative-energy levels as the vacuum state. This ensures the stability of the hydrogen atom since by the Pauli exclusion principle the positive-energy electrons are prevented from collapsing into the negative-energy levels.

However, there is a possibility for an electron to be excited from the negative-energy states into the positive-energy states after an absorption of radiation (Bjorken and Drell, 1964, p.65). This excitation is possible provided we create a "hole" in the sea. The absence of an electron of charge $-e$ and energy $-E$ in the "hole" is interpreted in relation to the vacuum as the presence of charge $+e$ and energy $+E$ (Halzen and Martin, 2008, p.76). The consequence of this is the pair production of particles. Similarly, the emission of radiation by an electron in the negative-energy state results in electron-positron annihilation.

The consequence of the Dirac "hole" theory leads to the many-particle theory for the Dirac theory (Bjorken and Drell, 1964, p.65) instead of the one-particle theory. Also, the interpretation of the "hole" theory is not applicable to bosons as one cannot fill up the sea with bosons (Halzen and Martin, 2008, p.76).

2.6.2 Feynman and Stückelberg Interpretation of the Negative-energy Solutions. In order to explain the many-particle theory (pair production and pair annihilation) encountered in the Dirac "hole" theory, Feynman (1948) and Stückelberg (1941) interpreted the solutions of the Dirac equations differently so as to replace the "hole" theory. Using the principle of space-time $(-E)t = E(-t)$ they explained that the negative-energy states can be seen as waves propagating backwards in time, away from an external potential.

They considered the electron as a particle propagating forward in time and the positron as the one propagating backwards in time. They postulated that in the presence of an external potential, the electron may be scattered forward in time (ordinary scattering) or backward in time (pair annihilation). Similarly, the scattered positron forward in time in the presence of an external potential was referred to as pair production and backward in time as positron scattering. Their explanation led to what later became known as the Feynman diagrams.

2.6.3 Feynman Diagrams. Following from the Feynman and Stückelberg interpretation, the photons serve as a force mediator in the interaction of particles. In other words, force is transmitted whenever an electron emits or absorbs a photon. The pictorial representation for the absorption and emission of photons is as shown in Figure 2.1.

This pictorial representation is what is known as the Feynman diagrams. Where

- in the diagram, time flows from left to right.

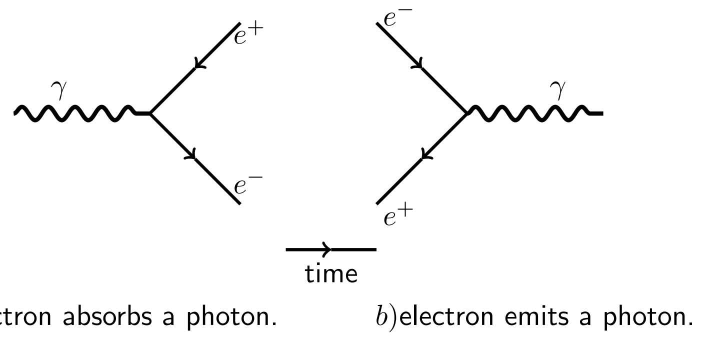


Figure 2.1: Pictorial representation of the emission and absorption of a photon by an electron.

- arrows forward in time denotes electrons (particles) and arrows backward in time denotes positrons (antiparticles).
- the wavy line corresponds to a real photon.

In the interaction process in quantum electrodynamics (QED) the Feynman diagrams are achieved by the combination of these two diagrams.

2.7 Relativistic Formulation of Electromagnetism

Here we introduce Maxwell's equations of electrodynamics in a vacuum and discuss the relativistic covariant form of this equation. We will then introduce the concept of gauge invariance.

2.7.1 Covariant Form of Maxwell Equations. Maxwell's equations of electrodynamics in a vacuum are given as

$$\begin{aligned} \nabla \cdot \mathbf{E} = \rho, \quad \nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \\ \nabla \cdot \mathbf{B} = 0, \quad \nabla \wedge \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}, \end{aligned} \tag{2.7.1}$$

where $\mathbf{j}$ and ρ are components of the four-charge densities with $j^\mu = (\rho, \mathbf{j})$, and $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic fields identified as

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi, \quad \text{and} \quad \mathbf{B} = \nabla \wedge \mathbf{A}, \tag{2.7.2}$$

where $\mathbf{A}$ and ϕ are the components of the four-vector potential. The antisymmetric electromagnetic field tensor is given as

$$\begin{aligned} F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu, \\ &= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}^{\mu\nu} . \end{aligned} \tag{2.7.3}$$

Now, we have that

$$\begin{aligned}
\partial_\mu F^{\mu\nu} &= \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \partial_\mu \partial^\mu A^\nu - \partial^\nu \left(\partial_\mu A^\mu\right), \\
&= \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)(\phi, \mathbf{A}) - \left(\frac{\partial}{\partial t}, -\nabla\right)\left(\frac{\partial \phi}{\partial t} + \nabla \mathbf{A}\right), \\
&= \left(\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi, \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A}\right) - \left(\frac{\partial^2 \phi}{\partial t^2} + \nabla \cdot \frac{\partial \mathbf{A}}{\partial t}, -\frac{\partial \nabla \phi}{\partial t} - \nabla\left(\nabla \cdot \mathbf{A}\right)\right), \\
&= \left(-\nabla^2 \phi - \nabla \cdot \frac{\partial \mathbf{A}}{\partial t}, \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} + \frac{\partial \nabla \phi}{\partial t} + \nabla\left(\nabla \cdot \mathbf{A}\right)\right), \\
&= \left(\nabla \cdot \mathbf{E}, \nabla \wedge \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}\right) = (\rho, \mathbf{j}) = j^\nu.
\end{aligned}$$

Thus, we have

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{2.7.4}$$

or equivalently

$$\Box^2 A^\nu - \partial^\nu \left(\partial_\mu A^\mu\right) = j^\nu. \tag{2.7.5}$$

Equations (2.7.4) and (2.7.5) are covariant forms of Maxwell's equations.

2.7.2 Gauge Invariance. From Maxwell's equations we have seen that the four-potential A^μ determines the fields $\mathbf{E}$ and $\mathbf{B}$. Now, we transform the potential as

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu f, \tag{2.7.6}$$

where f is an arbitrary function of x. Using (2.7.6) the antisymmetric field tensor becomes

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu \left(A_\nu + \partial_\nu f\right) - \partial_\nu \left(A_\mu + \partial_\mu f\right), \\
&= \partial_\mu A_\nu + \Box^2 f - \partial_\nu A_\mu - \Box^2 f = \partial_\mu A_\nu - \partial_\nu A_\mu.
\end{aligned} \tag{2.7.7}$$

Thus, we have that changing the potential corresponds to one and same fields. Also the equation of motion under such transformation becomes

$$\begin{aligned}
\Box^2 \left(A_\mu + \partial_\mu f\right) - \partial_\mu \left[\partial^\nu \left(A_\nu + \partial_\nu f\right)\right] &= \Box^2 A_\mu + \Box^2 \partial_\mu f - \partial_\mu \left(\partial^\nu A_\nu + \Box^2 f\right), \\
&= \Box^2 A_\nu - \partial_\mu \left(\partial^\nu A_\nu\right).
\end{aligned} \tag{2.7.8}$$

Thus, the covariance equation of motion is invariant under this transformation. This invariance is what is known as the gauge invariance. Now, if $\partial^\nu A_\nu = 0$, then we have

$$\Box^2 A_\mu = j_\mu. \tag{2.7.9}$$

This transformation is referred to as the Lorentz gauge with the Lorentz condition $\partial^\nu A_\nu = 0$. For a free photon the wave function A^μ satisfies

$$\Box^2 A^\mu = 0, \tag{2.7.10}$$

whose solutions is given as

$$A^\mu = \epsilon^\mu(\mathbf{q}) e^{-\imath q \cdot x}, \tag{2.7.11}$$

where ϵ^μ is a four-vector polarization of the photon, which describe the spin of the photon. Substituting (2.7.11) into (2.7.10) we have

$$q^\mu q_\mu = 0, \quad \text{or} \quad E = |\mathbf{q}|, \quad \text{that is,} \quad m_\gamma = 0, \tag{2.7.12}$$

and the Lorentz condition gives

$$q_\mu \epsilon^\mu = 0. \tag{2.7.13}$$

This reduces the number of independent components of ϵ^μ to three (Halzen and Martin, 2008, p.134). Now writing the arbitrary function in (2.7.6) as

$$f = \imath a e^{-\imath q \cdot x}, \tag{2.7.14}$$

where a is a normalization factor, satisfies

$$\Box^2 f = 0. \tag{2.7.15}$$

Substituting (2.7.14) and (2.7.11) into (2.7.6) under the transformation

$$\epsilon_\mu \to \epsilon'_\mu = \epsilon_\mu + a q_\mu, \tag{2.7.16}$$

does not change the physics of the system. Choosing $a = \frac{-\epsilon_0}{q_0}$, we have $\epsilon'_0 = 0$ and (2.7.13) reduces to

$$\begin{aligned} \boldsymbol{\epsilon'} \cdot \mathbf{q} &= (\boldsymbol{\epsilon} + a\mathbf{q})\,\mathbf{q} = \boldsymbol{\epsilon} \cdot \mathbf{q} + a\mathbf{q}^2, \\ &= \boldsymbol{\epsilon} \cdot \mathbf{q} = 0, \end{aligned} \tag{2.7.17}$$

since $\mathbf{q}^2 = 0$. This condition, together with the choice of f is referred as the Coulomb gauge.

3. Feynman Rules and Integrals

Here we discuss old fashioned perturbation theory to describe the interaction of particles and use it with the help of Green functions to describe the propagator theory. In section 3.3 we use the propagator theory to discuss the invariant amplitude that arises from the Feynman rules. We then discuss the divergent integrals that arise in the computation of the invariant amplitude by exploring the Feynman integrals.

3.1 Old Fashioned Perturbation Theory

We consider a time-independent Hamiltonian H_0 of a free particle moving in the presence of an interacting potential $V(\mathbf{x}, t)$ and whose solution to the Schrödinger equation is known, that is

$$H_0\psi_n = E_n\psi_n \quad \text{with} \quad \int_V \psi_m^*\psi_n \mathrm{d}^3x = \delta_{mn}. \tag{3.1.1}$$

The Schrödinger equation corresponding to this interaction is given as

$$\left(H_0 + V(\mathbf{x}, t)\right)\psi = \imath\frac{\partial\psi}{\partial t}, \tag{3.1.2}$$

whose solution can be expressed as

$$\psi = \sum_n a_n(t)\psi_n(\mathbf{x})e^{-\imath E_n t}. \tag{3.1.3}$$

To find the unknown coefficients $a_n(t)$, we substitute (3.1.3) into (3.1.2) and we obtain

$$\left(H_0 + V\left(\mathbf{x}, t\right)\right)\sum_n a_n(t)\psi_n(\mathbf{x})e^{-\imath E_n t} = \imath\sum_n \frac{\mathrm{d}a_n}{\mathrm{d}t}\psi_n(\mathbf{x})e^{-\imath E_n t} + \sum_n E_n a_n(t)\psi_n(\mathbf{x})e^{-\imath E_n t}. \tag{3.1.4}$$

From (3.1.1) we have that $H_0\psi_n = E_n\psi_n$, so this equation reduces to

$$\imath\sum_n \frac{\mathrm{d}a_n}{\mathrm{d}t}\psi_n(\mathbf{x})e^{-\imath E_n t} = \sum_n V\left(\mathbf{x}, t\right)a_n(t)\psi_n(\mathbf{x})e^{-\imath E_n t}. \tag{3.1.5}$$

Multiplying from the left by ψ_f^* and integrating over the volume, using the orthogonality relation in (3.1.1) we have

$$\frac{\mathrm{d}a_f}{\mathrm{d}t} = -\imath\sum_n a_n(t)\int_V \psi_f^*(\mathbf{x})V\left(\mathbf{x}, t\right)\psi_n(\mathbf{x})e^{\imath(E_f - E_n)t}\mathrm{d}^3x. \tag{3.1.6}$$

Now, assume that before the interaction of the potential with the particle, the potential is small and impermanent. Then, using Born approximation we have

$$a_n \approx \delta_{ni} + a_{n,i}^{(1)} + a_{n,i}^{(2)} + \cdots, \tag{3.1.7}$$

where the order of potential of $a_{n,i}^{(1)}$ and $a_{n,i}^{(2)}$ are $\mathcal{O}(V)$ and $\mathcal{O}(V^2)$ respectively. Substituting the value of a_n in (3.1.7) in (3.1.6) we have

$$\begin{aligned}
\frac{\mathrm{d}a_f}{\mathrm{d}t} &\approx -\imath \sum_n \left[\delta_{ni} + a_{n,i}^{(1)} + a_{n,i}^{(2)} + \cdots\right] \int_V \psi_f^*(\mathbf{x}) V\left(\mathbf{x}, t\right) \psi_n(\mathbf{x}) e^{\imath(E_f - E_n)t} \mathrm{d}^3x, \\
&= -\imath \sum_n \delta_{ni} \int_V \psi_f^*(\mathbf{x}) V\left(\mathbf{x}, t\right) \psi_n(\mathbf{x}) e^{\imath(E_f - E_n)t} \mathrm{d}^3x + \cdots, \\
\frac{\mathrm{d}a_f^{(1)}}{\mathrm{d}t} &= -\imath \int_V \psi_f^*(\mathbf{x}) V\left(\mathbf{x}, t\right) \psi_n(\mathbf{x}) e^{\imath(E_f - E_i)t} \mathrm{d}^3x.
\end{aligned} \tag{3.1.8}$$

Solving for $a_f^{(1)}(t)$ it was shown in (Halzen and Martin, 2008, p.80–81) that the first order perturbation theory in the covariant form is

$$T_{fi} \equiv a_{fi}^{(1)} = -\imath \int \mathrm{d}^4x \psi_f^*(x) V(\mathbf{x}) \psi_i(x), \tag{3.1.9}$$

where $|T_{fi}|^2$ is interpreted as the probability of scattering a particle from an initial state i to a final state f and where we have considered $V\left(\mathbf{x}, t\right) = V\left(\mathbf{x}\right)$ to be time independent. By repeated iteration of (3.1.6) the second order perturbation was found to be

$$\begin{aligned}
T_{fi} &= \cdots - \sum_{n \neq i} V_{fn} V_{ni} \int_{-\infty}^{\infty} \mathrm{d}t e^{\imath(E_f - E_n)t} \int_{-\infty}^{t} \mathrm{d}t' e^{\imath(E_f - E_i)t'}, \\
&= \cdots - \sum_{n \neq i} V_{fn} V_{ni} \int_{-\infty}^{\infty} \mathrm{d}t e^{\imath(E_f - E_n)t} \int_{-\infty}^{t} \mathrm{d}t' e^{\imath(E_f - E_i - \imath\epsilon)t'},
\end{aligned} \tag{3.1.10}$$

where we have introduced ϵ (small positive quantity) to make the integral over dt' meaningful, and

$$V_{fn} = \int \psi_f^*(\mathbf{x}) V(\mathbf{x}) \psi_n(\mathbf{x}) \quad \text{and} \quad V_{ni} = \int \psi_n^*(\mathbf{x}) V(\mathbf{x}) \psi_i(\mathbf{x}). \tag{3.1.11}$$

The second-order correction to T_{fi} is given as

$$T_{fi} = \cdots - 2\pi\imath \sum_{n \neq i} \frac{V_{fn} V_{ni}}{E_i - E_n + \imath\epsilon} \delta(E_f - E_i). \tag{3.1.12}$$

The value of T_{fi} is the amplitude of the perturbation theory and its description ensures that, in the discussion of scattering theory, energy is conserved at the vertex. From (3.1.12) the correction term of the vertex factors V_{fi} for the $i \to f$ transition is given as

$$\upsilon_{fi} \to \upsilon_{fi} + \sum_{n \neq i} \upsilon_{fn} \frac{1}{E_i - E_n + \imath\epsilon} \upsilon_{ni} + \cdots \tag{3.1.13}$$

3.1.1 Definition. The term $\sum_n \frac{\psi_n(x)\psi_n^*(y)}{(E_i - E_n + \imath\epsilon)}$ is identified as the propagator, where

$$\begin{aligned}
G(x, y) &= \sum_n \frac{\psi_n(x)\psi_n^*(y)}{(E_i - E_n + \imath\epsilon)}, \\
\left(E_i - H_0 + \imath\epsilon\right) G(x, y) &= \sum_n \psi_n(x)\psi_n^*(y) = \delta(x - y).
\end{aligned}$$

A more conversant approach of obtaining the propagator is discussed in section 3.2.

3.2 The Green Function of Propagator Theory

Staying away from singularities, the Green function of (2.5.17) is defined as

$$(\imath\gamma_\mu\partial^\mu - m)\, G_F = \delta^4(x - x'), \tag{3.2.1}$$

where δ is the delta function defined as follows.
3.2.1 Definition. The delta function is defined as

$$\delta(x) = \begin{cases} undefined & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{3.2.2}$$

satisfying the following properties

- $\displaystyle\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1 = \int_{-\infty}^{\infty} e^{-\imath kx}\delta(x)\mathrm{d}x.$
- $\displaystyle\delta(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty} e^{-\imath\mathbf{k}\cdot\mathbf{x}}\mathrm{d}^3x.$

Taking the Fourier transform of (3.2.1) to momentum space we have

$$G_F(x - x') = \frac{1}{(2\pi)^4}\int S_F(p)e^{-\imath p\cdot(x-x')}\mathrm{d}^4p. \tag{3.2.3}$$

Substituting this equation into (3.2.1) we have

$$\frac{1}{(2\pi)^4}\int (\not{p} - m)\, S_F(p)e^{-\imath p\cdot(x-x')}\mathrm{d}^4p = \frac{1}{(2\pi)^4}\int e^{-\imath p\cdot(x-x')}\mathrm{d}^4p,$$

where $S_F(p)$ satisfies

$$(\not{p} - m)\, S_F(P) = 1,$$

which then reduces to

$$S_F(p) = \frac{1}{\not{p} - m} = \frac{\not{p} + m}{p^2 - m^2}, \tag{3.2.4}$$

since

$$\begin{aligned} \not{p}\not{p} = p_\mu p_\nu\gamma^\mu\gamma^\nu &= \frac{1}{2}p_\mu p_\nu\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right), \\ &= \frac{1}{2}p_\mu p_\nu\left(2g^{\mu\nu}\right) = g^{\mu\nu}p_\mu p_\nu = p^2. \end{aligned} \tag{3.2.5}$$

We now consider the singularities at

$$p^2 - m^2 = p_0^2 - \left(\mathbf{p}^2 + m^2\right) = (p_0 - E)(p_0 + E) = 0$$

where the poles of integration is at $p_0 = \pm E = \pm\sqrt{\mathbf{p}^2 + m^2}$. From (3.2.3) and (3.2.4) we have

$$\begin{aligned} G_F(x - x') &= \frac{1}{(2\pi)^4}\int \frac{\not{p} + m}{(p_0 - E)(p_0 + E)}e^{-\imath p\cdot(x-x')}\mathrm{d}^4p, \\ &= \frac{1}{(2\pi)^4}\int \mathrm{d}^3pe^{\imath p\cdot(x-x')}\int_{-\infty}^{\infty}\mathrm{d}p_0\frac{(\gamma_0p_0 - \boldsymbol{\gamma}\cdot\mathbf{p} + m)}{(p_0 - E)(p_0 + E)}e^{-\imath p\cdot(x-x')}. \end{aligned} \tag{3.2.6}$$

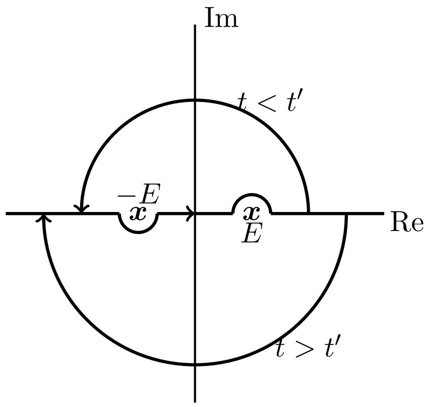


Figure 3.1: The contours in the complex p_0 plane used to evaluate the $\mathrm{d}p_0$ integral in (3.2.6).

The value of S_F for which we seek in this case corresponds to the propagation of positive-energy electrons forward in time $(t > t')$ and to negative-energy electrons backward in time $(t < t')$. It was shown in (Halzen and Martin, 2008, p.148–150) and (Peskin and Schroeder, 1995, p.83) that if the required boundary conditions were imposed by displacing the contour around the poles at $p_0 = \pm E$, the electron propagator becomes

$$\imath S_F = \imath \frac{\not{p} + m}{p^2 - m^2 + \imath\epsilon} \tag{3.2.7}$$

where $\imath\epsilon$ is infinitesimal and positive. From this, the respective propagator for:

1. a spinless particle is

$$\imath S_F = \frac{\imath}{p^2 - m^2 + \imath\epsilon}. \tag{3.2.8}$$

2. a massless spin-1 photon by making use of the Lorentz condition is

$$\imath S_F = -\frac{\imath g_{\mu\nu}}{p^2 + \imath\epsilon}. \tag{3.2.9}$$

3.3 Feynman Rules

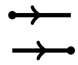
Here we discuss the Feynman rules (Griffiths, 2008, p.242–243) for calculating the invariant amplitude $\mathfrak{M}$.

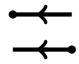
3.3.1 Definition. The invariant amplitude is obtained by identifying the covariant replacement for the vertex factors V_{fi} and the propagators $\dfrac{1}{E_i - E_f + \imath\epsilon}$ of the amplitude T_{fi}.

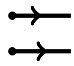
The rules for computing the invariant amplitude are as stated below:

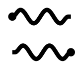
- We draw all the Feynman diagrams for the interaction processes.
- At each vertices of the diagram, associate a vertex factor containing the electromagnetic coupling $-e$ and a four-vector index connecting with the photon index. The vertex factor for a spin 0 photon is given as $\imath e(p + p')^\mu$ and that of spin $\dfrac{1}{2}$ is given as $\imath e\gamma^\mu$.
- The coupling constant is given as $\alpha = \dfrac{e^2}{4\pi} \equiv \dfrac{1}{137}$.

- Each internal line corresponds to a propagator as described in section 3.2.
- The contributing factors to the external lines are shown below:

spin $\frac{1}{2}$ fermion(in,out) : $\begin{cases} u \\ \overline{u} \end{cases}$ 

spin $\frac{1}{2}$ antifermion(in,out) : $\begin{cases} v \\ \overline{v} \end{cases}$ 

spin 0 boson(in,out) : $\begin{cases} 1 \\ 1 \end{cases}$ 

spin 1 photon(in,out) : $\begin{cases} \epsilon_\mu \\ \epsilon^*_\mu \end{cases}$ 

- Conservation of energy and momentum at each vertex is reflected by a delta function

$$(2\pi)^4\,\delta^4\,(p_1+p_2+p_3)$$

where the p's are the three four-momenta coming into the vertex.

- For each internal momentum, p has an integration factor of the form

$$\frac{\mathrm{d}^4 p}{(2\pi)^4}$$

- The overall energy-momentum conservation

$$(2\pi)^4\,\delta^4\,(p_1+p_2+p_3+\cdots+p_n)$$

is cancel out; the resulting term is multiplied by $\imath$ and this multiplied term is what is referred to as the invariant amplitude.

- For each fermion loop, a factor of (-1) should be included in the invariant amplitude.

3.4 Feynman Integrals

We consider a general one-loop Feynman integral

$$I=\int\frac{d^dk}{(2\pi)^d}\frac{N(k)}{a_1a_2\cdots a_n}, \tag{3.4.1}$$

where the a_i are second degree polynomials in the four vectors k_μ, $N(k)$ is an n-degree polynomial in the loop momentum k_μ and d denote the number of space-time dimensions. The advantage of using this integral is that

- It is covariant.
- It reduced the number of poles in the k_0-plane to two.
- It is applied to all integrals which occur in the computing of the invariant amplitude (Jauch and Rohrlich, 1976, p.455).

The evaluation of these integrals proceed in two steps

- We combine the different denominators into a single denominator, which is reduced to a standard form by shifting the internal loop momentum.
- We then compute the integral by using an integral identity (Gross, 2008, p.606).

We introduce Feynman parameters x_i, so that the denominators become n^{th} power of a second degree polynomial in k_μ:

$$\begin{aligned}\frac{1}{a_1 a_2 \cdots a_n} &= (n-1)! \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{n-2}} \mathrm{d}x_{n-1} \\ &\times \frac{1}{\left[a_1 x_{n-1} + a_2\left(x_{n-2} - x_{n-1}\right) + \cdots + a_n(1-x_1)\right]^n}.\end{aligned} \tag{3.4.2}$$

For $n = 2$ we have

$$\begin{aligned}\frac{1}{a_1 a_2} &= \int_0^1 \mathrm{d}x \frac{1}{\left[a_1 x + a_2(1-x)\right]^2}, \\ &= \int_0^1 \mathrm{d}x \frac{1}{\left[D(x)\right]^2},\end{aligned} \tag{3.4.3}$$

where D is the combined denominator and is formally written as

$$D = A^2 + 2k.Q - k^2, \tag{3.4.4}$$

where k represents the internal loop momentum and Q is a vector function of the external momentum and the Feynman parameters. To complete the square in D we shift the origin by introducing $k = k' + Q$ so that

$$D \to D' = A^2 + Q^2 - k'^{\,2}. \tag{3.4.5}$$

We apply this shift also to $N(k)$ and we have

$$N(k) = N_0 + k'_\mu N_1^\mu + k'_\mu k'_\nu N_2^{\mu\nu} + k'_\mu k'_\nu k'_\sigma N_3^{\mu\nu\sigma} + \cdots \tag{3.4.6}$$

where N^i are tensors which do not depend on k'. We have that D is even in k', so all the odd terms reduce to zero and we simplify the even terms using the following identities

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{D(k^2)} = \frac{g^{\mu\nu}}{d} \int \frac{d^d k}{(2\pi)^d} \frac{k^2}{D(k^2)}, \tag{3.4.7}$$

and

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu k^\rho k^\sigma}{D(k^2)} = \frac{\left[g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}\right]}{d(d+2)} \int \frac{d^d k}{(2\pi)^d} \frac{k^4}{D(k^2)}. \tag{3.4.8}$$

This reduce the integral I to the standard form

$$I' = \int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^m}{\left[c^2 - k^2 - \imath\epsilon\right]^n}. \tag{3.4.9}$$

Where

$$I' \begin{cases} \text{converges} & \text{if} \quad d + 2m - 2n \leq -1 \\ \text{diverges} & \text{if} \quad d + 2m - 2n > -1 \end{cases} \tag{3.4.10}$$

It was shown in (Gross, 2008, p.346) that in the region of convergence

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{[c^2 - k^2 - \imath\epsilon]^n} = \frac{\imath}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{c^2}\right)^{n-\frac{d}{2}}, \tag{3.4.11}$$

and its covariant part is given as

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{[c^2 - k^2 - \imath\epsilon]^n} = \frac{-\imath d g^{\mu\nu}}{2\,(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(n - 1 - \frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{c^2}\right)^{n-1-\frac{d}{2}}, \tag{3.4.12}$$

where $\Gamma(n)$ denotes the gamma function which is defined as

$$\Gamma(n) = \int_0^\infty \mathrm{d}t\, t^{n-1} e^{-t}. \tag{3.4.13}$$

Outside the region of convergence the integral is defined by analytic continuation. Poles are removed by renormalization. A particular example is discussed in the next chapter. The gamma function satisfies the following properties

- If n is non-negative integer we have

$$\Gamma(n+1) = n\Gamma(n). \tag{3.4.14}$$

- If n is an integer then

$$\Gamma(n) = (n-1)!. \tag{3.4.15}$$

- For $\epsilon \ll 1$ we have

$$\begin{aligned} \Gamma\left(1 + \frac{\epsilon}{2}\right) &= \frac{\epsilon}{2}\Gamma\left(\frac{\epsilon}{2}\right) = \frac{\epsilon}{2}\left[\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon^2)\right] \\ &= 1 - \frac{\epsilon}{2}\gamma + \mathcal{O}(\epsilon^2), \end{aligned} \tag{3.4.16}$$

where $\gamma = 0.5772..$ is Euler's constant.

4. Vacuum Polarization and its Applications

In this chapter we will show that if a virtual cloud of photons surrounds a free-electron, it generates a radiative correction to the free-electron. If this effect takes place in a bound-electron in an atom (Hydrogen atom), it modifies the static interaction between the electron and the nucleus of the atom contributing to the $2S_{\frac{1}{2}} - 2P_{\frac{1}{2}}$ energy level shift, also known as the Lamb shift. We will use Feynman rules and integrals to derive these corrections. In section 4.2 we will use this derive correction to compute the charge renormalization and based on this we will show that the Lamb shift affected the correction in the Coulomb potential.

4.1 Vacuum Polarization

As stated, if a virtual cloud of photons surrounds a free-electron it generates radiative corrections to the free electrons. This effect is called the self-energy of the photon. The corresponding Feynman diagram for this effect is shown below.

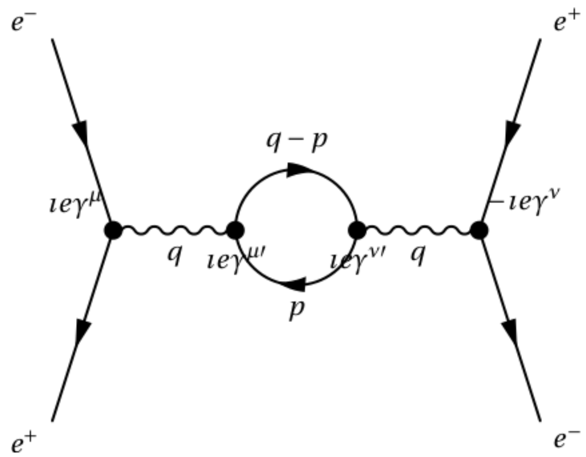


Figure 4.1: Feynman diagram showing the scattering process.

Applying section 3.3 and 3.4 to Figure 4.1, we have the invariant amplitude as

$$\begin{aligned}\mathfrak{M} = &-\imath\left(\imath e \overline{u}_{f_1}\gamma^\mu u_{i_1}\right)\left(\frac{-\imath g_{\mu\mu'}}{q^2}\right)\\ &\times\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\left[\left(\imath e\gamma^{\mu'}\right)_{\alpha\beta}\frac{\imath\left(\not{p}+m\right)_{\beta\lambda}}{p^2-m^2+\imath\epsilon}\left(\imath e\gamma^{\nu'}\right)_{\lambda\tau}\frac{\imath\left(\not{q}-\not{p}+m\right)_{\tau\alpha}}{(q-p)^2-m^2+\imath\epsilon}\right]\\ &\times\left(\frac{-\imath g_{\nu'\nu}}{q^2}\right)\left(-\imath e\overline{u}_{f_2}\gamma^\nu u_{i_2}\right),\end{aligned} \tag{4.1.1}$$

where the $-\imath\epsilon$ prescription of the photon propagator is explicit. In section 3.1 we found that the second-order perturbation leads to a modification in the vertex factors as seen in (3.1.13); using this fact and the virtual disintegration of the photon into an electron-positron pair as shown in Figure 4.1 the photon propagator is modified as indicated in Figure 4.2.

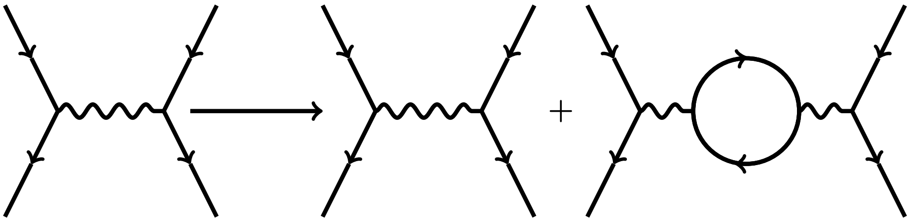


Figure 4.2: Modification of the photon propagator.

The modified photon propagator is given as

$$\frac{-\imath g_{\mu\nu}}{q^2} \to \frac{-\imath g_{\mu\nu}}{q^2} + \left(\frac{-\imath g_{\mu\mu'}}{q^2}\right) - \imath I^{\mu'\nu'}\left(\frac{-\imath g_{\nu'\nu}}{q^2}\right),$$
$$\to \frac{-\imath g_{\mu\nu}}{q^2} + \frac{-\imath}{q^2}\left(-\imath I_{\mu\nu}\right)\frac{-\imath}{q^2}, \tag{4.1.2}$$

where $I_{\mu\nu}$ is referred to as the vacuum polarization tensor. We have seen that if an external potential is introduced it interacts with the charge-current of the electron through the exchange of photons (polarizes the vacuum). The Fourier transform of this interaction process is given as

$$e^2 I_{\mu\nu} A^\nu(q),$$

where A^ν is the Fourier transform of the four-dimensional potential. From section 2.4 we found that the induced current must satisfy the continuity equation. Hence, for a real photon with $q^2 = 0$ we have

$$q^\mu I_{\mu\nu}(q) A^\nu(q) = 0.$$

Since $A^\nu(q)$ is arbitrary we have

$$q^\mu I_{\mu\nu}(q) = 0. \tag{4.1.3}$$

Now, since the gauge change in momentum space $A_\mu(q) \to A_\mu(q) + q_\mu \chi(q)$ does not alter the results of any physical amplitude, the vacuum polarization tensor must satisfy

$$q^\mu I_{\mu\nu}(q) = 0 = I_{\mu\nu}(q) q^\nu. \tag{4.1.4}$$

From Feynman rules the vacuum polarization tensor (the middle line of (4.1.1) with $\mu' = \mu$ and $\nu' = \nu$) becomes

$$-\imath I_{\mu\nu}(q) = (-1)^1 e^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{Tr}\left\{\gamma_\mu\left(\not{p}+m\right)\gamma_\nu\left(\not{q}-\not{p}+m\right)\right\}}{\left[p^2 - m^2 + \imath\epsilon\right]\left[(q-p)^2 - m^2 + \imath\epsilon\right]},$$
$$I_{\mu\nu}(q) = -\imath e^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{Tr}\left\{\gamma_\mu\left(m+\not{p}\right)\gamma_\nu\left(m+\not{p}-\not{q}\right)\right\}}{\left[m^2 - p^2 - \imath\epsilon\right]\left[m^2 - (q-p)^2 - \imath\epsilon\right]}, \tag{4.1.5}$$

where the trace is as a results of the Dirac indices in (4.1.1). Now from (4.1.5) we have that

$$\mathrm{Tr}\left\{\gamma_\mu\left(m+\not{p}\right)\gamma_\nu\left(m+\not{p}-\not{q}\right)\right\} = m^2\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\right) + m\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\not{p}\right) - m\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\not{q}\right)$$
$$+ m\mathrm{Tr}\left(\gamma_\mu\not{p}\gamma_\nu\right) + \mathrm{Tr}\left(\gamma_\mu\not{p}\gamma_\nu\not{p}\right) - \mathrm{Tr}\left(\gamma_\mu\not{p}\gamma_\nu\not{q}\right). \tag{4.1.6}$$

From equation (2.3.2) and (2.3.4) of section 2.3 we have that the second, third and forth term of (4.1.6) goes to zero and

$$\mathrm{Tr}\left(\gamma_\mu\not{p}\gamma_\nu\not{q}\right) = p^\alpha q^\beta \mathrm{Tr}\left(\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\right)$$
$$= 4\left(p_\mu q_\nu - p\cdot q g_{\nu\mu} + p_\nu q_\mu\right). \tag{4.1.7}$$

Using the analogy in (4.1.7) and equation (2.3.1) of section 2.3 we have (4.1.5) as

$$I_{\mu\nu}(q) = -\imath 4e^2 \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\left[m^2 g_{\mu\nu} + p_\mu (p-q)_\nu + p_\nu (p-q)_\mu - g_{\mu\nu} p\cdot(p-q)\right]}{[m^2 - p^2 - \imath\epsilon]\,[m^2 - (p-q)^2 - \imath\epsilon]}. \tag{4.1.8}$$

To ensure that the quadratic divergence $\int \frac{\mathrm{d}^4p}{p^2}$ satisfies the gauge invariance we regularize this integral by making use of our discussion in section 3.4 and introduce what we call renormalization scale $\left(M^{4-d}\right)$ to adjust the energy dimensions. Then (4.1.8) becomes

$$I_{\mu\nu}(q) = -4\imath e^2 M^{4-d} \int \frac{\mathrm{d}^dp}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{N_{\mu\nu}}{[m^2 - p^2 + 2pqx - q^2x - \imath\epsilon]^2}. \tag{4.1.9}$$

By shifting the origin we introduce $p = k + qx$ and we have

$$I_{\mu\nu}(q) = -4\imath e^2 M^{4-d} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^dp}{(2\pi)^d} \frac{N'_{\mu\nu}}{[m^2 - k^2 - q^2x(1-x) - \imath\epsilon]^2}, \tag{4.1.10}$$

where

$$\begin{aligned} N'_{\mu\nu} = m^2 g_{\mu\nu} &+ 2k_\mu k_\nu + (q_\mu k_\nu + k_\mu q_\nu)(2x-1) \\ &- 2q_\mu q_\nu x(1-x) - g_{\mu\nu}(k+qx)(k-q(1-x)). \end{aligned} \tag{4.1.11}$$

From (3.4.7) and (3.4.8) we have

$$\begin{aligned} I_{\mu\nu}(q) &= -4\imath e^2 M^{4-d} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^dk}{(2\pi)^d} \frac{g_{\mu\nu}\left(m^2 + \left(\frac{2}{d}-1\right)k^2 + q^2x(1-x)\right) - 2q_\mu q_\nu x(1-x)}{[m^2 - k^2 - q^2x(1-x) - \imath\epsilon]^2}, \\ &= \frac{4e^2 M^{4-d}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)} \int_0^1 \mathrm{d}x \left\{g_{\mu\nu}\left(m^2 + q^2x(1-x)\right) - 2q_\mu q_\nu x(1-x)\right\} \left(\frac{1}{m^2 - q^2x(1-x)}\right)^{2-\frac{d}{2}} \\ &\quad - \frac{d}{2}\left(\frac{2}{d}-1\right) \frac{4e^2 M^{4-d}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(1-\frac{d}{2}\right)}{\Gamma(2)} \int_0^1 \mathrm{d}x g_{\mu\nu} \left(\frac{1}{m^2 - q^2x(1-x)}\right)^{1-\frac{d}{2}}, \\ &= \frac{4e^2 M^{4-d}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)} \int_0^1 \mathrm{d}x \left(\frac{1}{m^2 - q^2x(1-x)}\right)^{2-\frac{d}{2}} \Bigg[g_{\mu\nu}\left(m^2 - q^2x(1-x)\right) \\ &\qquad \left\{1 - \frac{d}{2}\left(\frac{2}{d}-1\right)\frac{\Gamma\left(1-\frac{d}{2}\right)}{\Gamma\left(2-\frac{d}{2}\right)}\right\} + \left(g_{\mu\nu}q^2 - q_\mu q_\nu\right)2x(1-x)\Bigg]. \end{aligned} \tag{4.1.12}$$

But from the properties of the Gamma function we have that $\Gamma\left(2-\frac{d}{2}\right) = \left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)$ and this reduces the coefficient of $g_{\mu\nu}$ to zero and thus ensures the condition (4.1.3) and (4.1.4). So we have

$$I_{\mu\nu}(q) = \frac{4e^2 M^{4-d}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)} \int_0^1 \mathrm{d}x \left(\frac{1}{m^2 - q^2x(1-x)}\right)^{2-\frac{d}{2}} \left(g_{\mu\nu}q^2 - q_\mu q_\nu\right)2x(1-x). \tag{4.1.13}$$

That is, we have that $I_{\mu\nu}$ satisfies the equation

$$I_{\mu\nu}(q) = \left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) I(q^2). \tag{4.1.14}$$

This ensures that equation (4.1.4) holds for $I_{\mu\nu}(q)$. Here $I(q^2)$ is referred as the vacuum polarization scalar function of q^2 and is given as

$$I(q^2) = \frac{4e^2 M^{4-d}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)} \int_0^1 \frac{2x(1-x)}{\left[m^2 - q^2 x(1-x)\right]^{2-\frac{d}{2}}} \mathrm{d}x,$$
$$= M^{4-d} \frac{\alpha}{\pi} \frac{\Gamma\left(3-\frac{d}{2}\right)}{\left(2-\frac{d}{2}\right)(4\pi)^{\frac{d}{2}-2}} \int_0^1 \frac{2x(1-x)}{\left[m^2 - q^2 x(1-x)\right]^{2-\frac{d}{2}}} \mathrm{d}x. \tag{4.1.15}$$

The expression $I(q^2)$ is well-defined for $d < 4$, and its physical significance is obtained in the limit as $d \to 4$. We now evaluate this by writing it as a sum of an infinite constant and a finite constant, that is

$$I(q^2) = I(0) + \left(I(q^2) - I(0)\right),$$
$$= I(0) + I'(q^2). \tag{4.1.16}$$

where

$$I(0) = \frac{\alpha}{\pi} \frac{2\Gamma\left(1+\frac{\epsilon}{2}\right)}{\epsilon\,(4\pi)^{-\frac{\epsilon}{2}}} \int_0^1 2x(1-x) \left(\frac{M}{m}\right)^{\epsilon} \mathrm{d}x \tag{4.1.17}$$

with $\epsilon = 4 - d$. Taking the limit as $\epsilon \to 0$ of (4.1.17) and making use of the identity

$$\lim_{\epsilon \to 0} A^{\epsilon} = 1 + \epsilon \log A + \mathcal{O}(\epsilon^2), \tag{4.1.18}$$

and the properties of gamma function we have

$$I(0) = \frac{2\alpha}{\pi} \frac{\Gamma\left(1+\frac{\epsilon}{2}\right)}{\epsilon\,(4\pi)^{-\frac{\epsilon}{2}}} \int_0^1 2x(1-x) \left(\frac{M}{m}\right)^{\epsilon} \mathrm{d}x,$$
$$= \frac{2\alpha}{3\pi} \left(1 - \frac{\epsilon}{2}\gamma + \frac{\epsilon}{2}\log(4\pi) + \mathcal{O}(\epsilon^2)\right) \left(\frac{1}{\epsilon} + \log\left(\frac{M}{m}\right)\right),$$
$$= \frac{2\alpha}{3\pi} \left(\frac{1}{\epsilon} - \frac{\gamma}{2} + \frac{1}{2}\log(4\pi) + \log\left(\frac{M}{m}\right) + \mathcal{O}(\epsilon^2)\right),$$
$$= \frac{\alpha}{3\pi} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) + \log\left(\frac{M}{m}\right)^2\right). \tag{4.1.19}$$

Also we have

$$I'(q^2) = \frac{\alpha}{\pi} \frac{4\Gamma\left(1+\frac{\epsilon}{2}\right)}{\epsilon\,(4\pi)^{-\frac{\epsilon}{2}}} \int_0^1 x(1-x) \left[\frac{M^{\epsilon}}{\left[m^2 - q^2 x(1-x)\right]^{\frac{\epsilon}{2}}} - \left(\frac{M}{m}\right)^{\epsilon}\right] \mathrm{d}x. \tag{4.1.20}$$

Making use of the identity in (4.1.18) and the properties of the gamma function we have

$$I'(q^2) = \frac{\alpha}{\pi}\frac{4}{\epsilon} \left(1 - \frac{\epsilon}{2}\gamma + \frac{\epsilon}{2}\log(4\pi) + \mathcal{O}(\epsilon^2)\right) \int_0^1 x(1-x)$$
$$\times \left[1 + \epsilon \log\left(\frac{M}{\left[m^2 - q^2 x(1-x)\right]^{\frac{1}{2}}}\right) - 1 - \epsilon \log\left(\frac{M}{m}\right)\right] \mathrm{d}x,$$
$$= \frac{4\alpha}{\pi} \left(1 - \frac{\epsilon}{2}\gamma + \frac{\epsilon}{2}\log(4\pi) + \mathcal{O}(\epsilon^2)\right) \int_0^1 x(1-x) \log\left(\frac{m^2}{m^2 - q^2 x(1-x)}\right) \mathrm{d}x,$$
$$\to -\frac{2\alpha}{\pi} \int_0^1 x(1-x) \log\left[1 - \frac{q^2}{m^2} x(1-x)\right] \mathrm{d}x, \tag{4.1.21}$$

as $\epsilon \to 0$. Now in the limit of small momentum transfer $|q^2| \ll m^2$ we have

$$\log\left[1 - \frac{q^2}{m^2}x(1-x)\right] \sim \frac{-q^2x(1-x)}{m^2}, \tag{4.1.22}$$

and (4.1.21) reduces to

$$I'(q^2) \sim \frac{2\alpha}{\pi}\frac{q^2}{m^2}\int_0^1 x^2(1-x)^2\mathrm{d}x = \frac{\alpha}{15\pi}\frac{q^2}{m^2}. \tag{4.1.23}$$

4.2 Charge Renormalization

We have seen in section 4.1 that the virtual disintegration of the photon into an electron-positron pair modifies the photon propagator as seen in Figure 4.2. We can write (4.1.2) as

$$\frac{-\imath g_{\mu\nu}}{q^2} + \frac{-\imath}{q^2}\left(-\imath I_{\mu\nu}\right)\frac{-\imath}{q^2} = \frac{-\imath}{q^2}\left[g_{\mu\nu} - \frac{I_{\mu\nu}}{q^2}\right]. \tag{4.2.1}$$

The effects of this modification on the scattering amplitude amounts to

$$\begin{aligned}
j_\mu\frac{-\imath}{q^2}j^\nu &\to j_\mu\frac{-\imath}{q^2}j^\nu + j_\mu\frac{-\imath}{q^2} - \imath I_{\mu\nu}\frac{-\imath}{q^2}j^\nu,\\
e^2\left(\overline{u}_{f_1}\gamma_\mu u_{i_1}\right)\frac{-\imath}{q^2}\left(\overline{u}_{f_2}\gamma^\nu u_{i_2}\right) &\to e^2\left(\overline{u}_{f_1}\gamma_\mu u_{i_1}\right)\frac{-\imath}{q^2}\left[g_{\mu\nu} - \frac{I_{\mu\nu}(q)}{q^2}\right]\left(\overline{u}_{f_2}\gamma^\nu u_{i_2}\right),\\
&= e^2\left(\overline{u}_{f_1}\gamma_\mu u_{i_1}\right)\frac{-\imath}{q^2}\left\{\frac{g_{\mu\nu}q^2 - \left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)\left[I(0) + I'(q^2)\right]}{q^2}\right\}\left(\overline{u}_{f_2}\gamma^\nu u_{i_2}\right),\\
&= e^2\left(1 - I(0)\right)\left(\overline{u}_{f_1}\gamma_\mu u_{i_1}\right)\frac{-\imath}{q^2}\left(\overline{u}_{f_2}\gamma^\mu u_{i_2}\right)\\
&\qquad + \imath e^2 I'(q^2)\left(\overline{u}_{f_1}\gamma_\mu u_{i_1}\right)\frac{1}{q^2}\left(\overline{u}_{f_2}\gamma^\mu u_{i_2}\right).
\end{aligned} \tag{4.2.2}$$

In the third line of equation (4.2.2) we have make use of the fact that the continuity equation in a momentum space gives

$$\begin{aligned}
q_\mu j^\mu &= \left(p_f - p_i\right)\left(\overline{u}_f\gamma^\mu u_i\right),\\
&= \overline{u}_f\left(\not{p}_f - \not{p}_i\right)u_i = \overline{u}_f\left(\not{p}_f - m\right)u_i,\\
&= \overline{u}_f\left(m - m\right)u_i = 0,
\end{aligned}$$

using (2.5.1). This equation shows that for small q^2 the modified photon propagator differs by a factor of $(1 - I(0))$ from the original propagator. This factor is known as the "wave function renormalization constant" for the photon, and because of gauge invariance it is strongly related to charge renormalized and contributes as a correction factor to the coupling constant. That is, instead of having the coupling constant as $\alpha = \dfrac{e^2}{4\pi}$, what is observed is rather $\alpha = \dfrac{e^2(1 - I(0))}{4\pi}$. We have that the observable physical charge absorbs the singularity in $I(0)$. This observable charge is referred to as the renormalized charge and it is given by the $\overline{\mathrm{MS}}$ scheme as

$$\begin{aligned} e_R^2 &= e^2\left(1 - I(0)\right), \\ &= e^2\left(1 - \frac{\alpha}{3\pi}\left(\frac{2}{\epsilon} - \gamma + \log(4\pi) + \log\left(\frac{M}{m}\right)^2\right)\right). \end{aligned} \tag{4.2.3}$$

We can write (4.2.2) as

$$\begin{aligned} \left[e^2\left(1 - I(0)\right) - e^2 I'(q^2)\right]\left(\overline{u}_{f_1}\gamma_\mu u_{i_1}\right)\frac{-\imath}{q^2}\left(\overline{u}_{f_2}\gamma^\mu u_{i_2}\right) &= \left[e_R^2 - e^2 I'(q^2)\right]\left(\overline{u}_{f_1}\gamma_\mu u_{i_1}\right)\frac{-\imath}{q^2}\left(\overline{u}_{f_2}\gamma^\mu u_{i_2}\right), \\ &= e_R^2\left[1 - I'(q^2)\right]\left(\overline{u}_{f_1}\gamma_\mu u_{i_1}\right)\frac{-\imath}{q^2}\left(\overline{u}_{f_2}\gamma^\mu u_{i_2}\right), \\ &= \frac{-\imath e_R^2}{q^2}\left[1 - \frac{\alpha_R}{15\pi}\frac{q^2}{m^2} + \mathcal{O}(\alpha_R^2)\right] \\ &\qquad \times \left(\overline{u}_{f_1}\gamma_\mu u_{i_1}\right)\left(\overline{u}_{f_2}\gamma^\mu u_{i_2}\right), \end{aligned}$$

with $\alpha_R = \frac{1}{137}$.

4.3 The Modified Coulomb Potential

In section 4.2 we found that the correction factor of a free electron due to the scalar "vacuum fluctuations" for $|q|^2 \ll m^2$ is of the form

$$\frac{-\imath e_R^2}{q^2}\left[1 - \frac{\alpha_R}{15\pi}\frac{q^2}{m^2} + \mathcal{O}(\alpha_R^2)\right].$$

In coordinate space, the interaction between the electron and the static nucleus of charge Ze_R with $q^2 = |\mathbf{q}|$ gives a potential of the form

$$V(\mathbf{r}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{\imath \mathbf{q}\cdot\mathbf{r}} \frac{-Ze_R^2}{|\mathbf{q}|^2\left[1 - I'(-|\boldsymbol{q}|^2)\right]}. \tag{4.3.1}$$

Expanding $I'(-|\mathbf{q}|^2)$ in the limits of small momentum transfer $|q^2| \ll m^2$, we have

$$\begin{aligned} V(\mathbf{r}) &= \int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{\imath \mathbf{q}\cdot\mathbf{r}} \frac{-Ze_R^2}{|\mathbf{q}|^2\left(1 - \frac{\alpha|\mathbf{q}|^2}{15\pi m^2}\right)}, \\ &= -\int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{\imath \mathbf{q}\cdot\mathbf{r}} \frac{Ze_R^2}{|\mathbf{q}|^2}\left[1 + \frac{\alpha|\mathbf{q}|^2}{15\pi m^2}\right], \\ &= -\int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{\imath \mathbf{q}\cdot\mathbf{r}} \frac{Ze_R^2}{|\mathbf{q}|^2} - \int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{\imath \mathbf{q}\cdot\mathbf{r}} \frac{\alpha Z e_R^2}{15\pi m^2}. \end{aligned} \tag{4.3.2}$$

But from the properties of the delta function and taking the Fourier transform of the coordinate space $\frac{1}{|\mathbf{r}|}$ we have

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{\imath \mathbf{q}\cdot\mathbf{r}} = \delta(\mathbf{r}) \quad \text{and} \quad \frac{1}{4\pi|\mathbf{r}|} = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{\imath \mathbf{q}\cdot\mathbf{r}} \frac{1}{|\mathbf{q}|^2}. \tag{4.3.3}$$

Using (4.3.3), equation (4.3.2) becomes

$$V(r) = -\frac{Ze_R^2}{4\pi r} - \frac{Z\alpha_R^2}{15\pi m^2}\delta(\mathbf{r}) = -\frac{Ze_R^2}{4\pi r} - \frac{Ze_R^4}{60\pi^2 m^2}\delta(\mathbf{r}). \tag{4.3.4}$$

5. Conclusion and Discussion

The modification of the Coulomb potential that arises from the vacuum polarization was first done by (Uehling, 1935). Basing his argument on the Dirac "hole" theory, he explained that the creation and annihilation of electron-positron pairs induces a charge, and that the polarization of the vacuum is as a result of the induced charge; which subsequently leads to deviations from the Coulomb potential. In his calculations, he ignored the infinite constant introduced by (Oppenheimer, 1930) and considered convergent expressions introduced by Heisenberg. Treating the electromagnetic field as a perturbation, he found the modified Coulomb potential as

$$V(r) = -\frac{\alpha}{r} \cdot \frac{\alpha}{4\sqrt{\pi}} \frac{e^{-2mr}}{(mr)^{\frac{3}{2}}}. \tag{5.0.1}$$

He predicted that the deviations are observable and may be detected in the scattering of high energy particles; thus they contribute to the Lamb shift of the hydrogen atom.

In this essay we have developed a systematic approach in computing this modified Coulomb potential. We have introduced the concept of invariant amplitude which arises from the perturbation theory by discussing Feynman rules based on the Feynman and Stückelberg interpretation of the positron theory.

In our calculation, to absorb the infinite constant polarization we make use of the Feynman integrals and introduced a dimensional regularization M^ϵ which we called the renormalization scale. Based on this, the induced charge caused by the creation and annihilation of the electron-positron pairs become what is known as the charge renormalization. Using the value of the charge renormalization we got the value of the Coulomb potential plus a modification as

$$V(r) = -\frac{Ze_R^2}{4\pi r} - \frac{Ze_R^4}{60\pi^2 m^2}\delta(\mathbf{r}), \tag{5.0.2}$$

where e_R is the charge.

The first term of (5.0.2) is as a result of the scattering of the electron with a very small momentum transfer where the charge e_R is the measured electron charge in any long-range electromagnetic interaction. Now, for scattering with large momentum transfer, the electron penetrates the virtual cloud of electron-positron pairs and thereby increases its effective interaction strength. This effect is represented by the δ function in the second term of (5.0.2), which corresponds to the additional attractive force between the electron and nucleus and it is equivalent to (5.0.1). The energy shift of this modified Coulomb potential as computed by (Uehling, 1935, p.61) is given as

$$\Delta E_{nl} = -\frac{e_R^4}{60\pi^2 m^2}|\psi_{nl}(0)|^2\delta_{l0} = -\frac{8\alpha_R^3}{15\pi n^3}R_y\delta_{l0}, \tag{5.0.3}$$

where $R_y = \dfrac{m\alpha_R^2}{2}$ is the Rydberg constant. This energy shift contributes $-27\mathrm{MHz}$ to the total Lamb shift of $+1057 \pm 0.2\mathrm{MHz}$ of the hydrogen atom. Without this modification, the experimental datum would not be matched.

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