Pólya's enumeration method

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Abstract

Pólya's enumeration method makes use of ideas from group theory to solve enumeration problems, in particular in situations where symmetries should be taken into account. The aim of this essay is to explain the main concepts of group actions, stabilisers and cycle indices and their diverse applications to colouring, graph counting, etc. A possible highlight would be the proof of Otter's asymptotic formula for the number of nonisomorphic trees of order n.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Ikenna JohnKingsley Ezike, 22 May 2014

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1. Introduction

Generally, the number of ways of colouring n objects using r colours is r^n. The situation changes, however, when we take symmetry into account. Consider the following examples.

1.1 Problem 1

Given a cube which has six faces. Suppose we wish to colour some of the faces of the cube with just one colour up to symmetry, i.e., two colourings are counted as the same if one can be obtained from the other by rotating the cube. In how many ways can this be done? The answer is as follows:

- If we colour all the faces with the one colour, we can achieve this in just 1 way.
- We can colour five faces leaving one face uncoloured, we can achieve this in 1 way.
- We can colour four faces leaving two faces uncoloured, we can achieve this in 2 ways. One way is to leave two opposite faces of the cube uncoloured and colour the remaining 4 faces. Another way is to leave two adjacent faces uncoloured and colour the remaining 4 faces.
- We can colour three faces leaving three faces uncoloured; we can achieve this in 2 ways. One way is to colour two opposite faces and a third face that is adjacent to both. Another way is to colour two adjacent faces and a common adjacent face.
- We can colour two faces leaving four faces uncoloured, we can achieve this in 2 ways. One way is to colour two opposite faces of the cube and leave the remaining 4 faces uncoloured. Another way is to colour two adjacent faces and leave the remaining 4 faces uncoloured.
- We can colour one face leaving five faces uncoloured, we can achieve this in 1 way.
- Leaving all the faces uncoloured is one way.

That gives a total of 10 ways. This example may look trivial, but two natural questions one can study are: what happens when we use another solid figure with a different number of faces and what happens if we decide to do this colouring activity with more than one colour?

1.2 Problem 2

Consider a necklace as in Figure 1.1 with n beads.

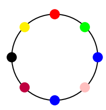
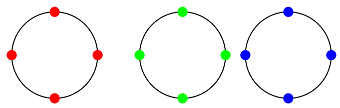


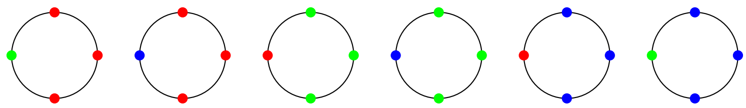
Figure 1.1: Necklace with multiple colours

Suppose that $n = 4$ and that we wish to colour it with three colours, namely red, green and blue. The number of possible colourings is $3^4 = 81$. We say that two colourings are "the same" if one can be obtained from the other by rotating the necklace. The question is how many different colourings can be achieved without double-counting the same colourings. Let us count. We shall classify colourings according to their colour type for which we write "k_1 k_2 k_3". For instance, "2 1 1" means 2 red, 1 green and 1 blue. We shall consider configurations like "2 1 1", "1 2 1", and "1 1 2" as the same type just as "3 0 1", "0 3 1" are of the same type. We classify similar colour types in the following way.

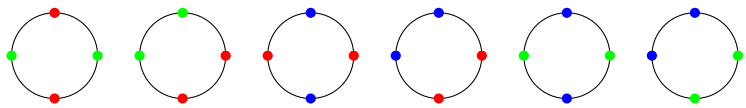
- Type 1: We obtain 3 colourings: "4 0 0" or "0 4 0" or "0 0 4".



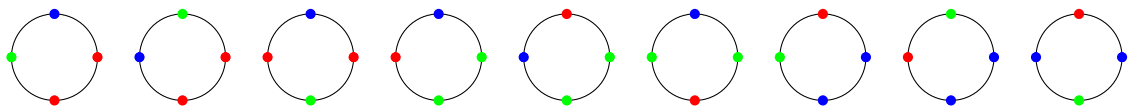
- Type 2: For "3 1 0", one has six colourings.



- Type 3: For "2 2 0", one has six colourings.



- Type 4: For "2 1 1", one has nine colourings.



Therefore, we obtain a total of 24 ways of colouring a four-bead necklace using three colours.

When an object is symmetric, certain changes such as rotation or reflection keep it invariant. Rotations and reflections give rise to groups of permutations (symmetries). The action of these groups induces equivalence classes. In Problem 2 above we obtained 24 equivalence classes. Each of the 24 colourings is a representative of an equivalence class. In this example, we were considering $n = 4$ objects. The general question, "in how many ways can a set of n objects be coloured up to symmetry using r colours" can be rephrased as "how many equivalence classes are there for the set of r^n colourings under the action of a group of permutations". For large n and r colours, instead of explicitly counting possible colourings, which will not be easy, Burnside's lemma provides a counting technique using equivalence classes. Pólya's enumeration theorem gives a general and easy way to solve problems like these. It incorporates Burnside's Lemma and enables one to express the complete generating function for equivalence classes in terms of the cycle index of the underlying group of symmetries. To understand these we need to re-visit some basic concepts.

2. General Introduction to Group Actions

We start by reviewing the definition of groups and give relevant examples. Groups play an important role in the understanding of many principles of modern mathematics. We state the concept of group actions on a set and define orbits and stabilizers. Groups and their actions on a set form the root of counting mathematical objects (sets) taking symmetries into account, which is the main focus of this work. We shall end with Orbit-Stabilizer theory.

2.1 Definition of a Group

A binary operation or law of composition on a non-empty set G is a function which maps the Cartesian product $G \times G$ of G to itself, $G \times G \longrightarrow G$. That means, to each element $(a, b) \in G \times G$, it assigns a unique element $a \circ b \in G$ called the composition of a and b. A group $(G, \circ)$ is a set equipped with a binary operation "$\circ$", which satisfies the following postulates:

- Closure property; $a \circ b \in G, \quad \forall a, b \in G$.
- Associativity property; $a \circ (b \circ c) = (a \circ b) \circ c, \quad \forall a, b, c \in G$.
- Existence of an identity element e such that $e \circ a = a \circ e = a \quad \forall a \in G$.
- Existence of inverses; for each $a \in G$, there exists an inverse element $a^{-1} \in G$, such that $a \circ a^{-1} = a^{-1} \circ a = e$.

Example 2.1.1. (Simple examples of groups.) The set of integers $\mathbb{Z}$ under the usual addition forms a group $(\mathbb{Z}, +)$, but under multiplication, it is not a group. The identity element of $(\mathbb{Z}, +)$ is 0. the inverse of any element $a \in \mathbb{Z}$ is $-a$. Addition is closed and associative in $\mathbb{Z}$. The identity element of $(\mathbb{Z}, \times)$ is 1. However, $(\mathbb{Z}, \times)$ is not a group because of non-existence of inverse elements. The set of real numbers $\mathbb{R}$ under addition $(\mathbb{R}, +)$ and multiplication $(\mathbb{R}\setminus\{0\}, \times)$ forms a group. It is easy to verify that each of $(\mathbb{R}, +)$ and $(\mathbb{R}\setminus\{0\}, \times)$ satisfies the four axioms of a group. The set $GL_n(\mathbb{R})$ of all invertible linear maps ($n \times n$ matrices) over the field $\mathbb{R}$ under matrix multiplication forms a group $(GL_n(\mathbb{R}), \cdot)$ called the general linear group. $GL_n(\mathbb{R})$ is closed and associative since multiplication of $n \times n-$matrices is closed and associative. The identity element is the identity matrix, while the inverse of any $A \in GL_n(\mathbb{R})$ is the inverse matrix A^{-1}.

Remark 2.1.2. If in addition to the four axioms above, we have $a \circ b = b \circ a \quad \forall a, b \in G$, G is called an abelian or a commutative group. Otherwise, it is called a non-abelian group. In Example 2.1.1, $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$, and $(\mathbb{R}\setminus\{0\}, \times)$ are abelian groups while $(GL_n(\mathbb{R}), \cdot)$ is a non-abelian group. It is important to note clearly the operation on the set G which equips it with a particular group structure since a set may have more than one binary operation. For instance in $\mathbb{R}$, $(\mathbb{R}, +)$ and $(\mathbb{R}\setminus\{0\}, \times)$ are groups with different operations and thus different group structures.

Definition 2.1.3. Let G be a group under the binary operation "$\circ$". A subset H of G is said to be a subgroup of G if H is a group under the same binary operation "$\circ$". If H is a subgroup of G, we denote it as $H \leq G$.

The set $\mathbb{Q} \subset \mathbb{R}$ of rational numbers forms a group under the usual addition "$+$". Moreover, $(\mathbb{R}, +)$ is a group under "$+$", therefore $\mathbb{Q} \leq \mathbb{R}$.

Definition 2.1.4. The order of a group G, denoted by $|G|$, is the number of elements in G.

The order of each of the groups mentioned in Example 2.1.1 is infinite. However, the interest of our study focuses on finite groups, specifically permutation groups.

2.2 Permutation Groups

We are familiar with the basic concept in combinatorics that the number of ways to rearrange n objects is $n!$. Given n objects, we may wish to rearrange the objects so that for all $i \in \{1, 2, \ldots, n\}$, the object in ith position moves to the jth position, for some j. This action can be described by a map $\tau : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ given by

$$\tau(i) = j$$

with the properties that $\tau_1(i) \neq \tau_1(j)$ if $i \neq j$, so it is a one-to-one or injective mapping, and for each $k \in \{1, 2, \ldots, n\}$, there exists $i \in \{1, 2, \ldots, n\}$, such that $\tau(i) = k$, so it is a surjective or onto function. Such a map τ is called a permutation. This means that each rearrangement of n objects corresponds to a bijection τ on the set $\{1, 2, \ldots, n\}$.

Definition 2.2.1. The collection of all such bijections τ on $\{1, 2, \ldots, n\}$ forms a group under the composition of functions. It is denoted by S_n. S_n is called the symmetric or permutation group of n elements. Since elements of S_n correspond to ways of rearranging n objects, the order of S_n is $n!$.

Theorem 2.2.2. *S_n is a group.*

Proof. Each element of S_n is a bijective map. The binary operation which equips S_n with its group structure is the composition of functions.

- Closure: The composition of any two bijective map is a bijective maps. Therefore S_n is closed.
- Composition of functions is associative.

$$f \circ (g \circ h)(x) = f \circ (g(h(x)) = f(g(h(x)) = (f \circ g) \circ h(x) \ \forall f, g, h \in S_n$$

- The identity element is the identity map id.
- For each bijective map $f \in S_n$, there is a a bijective inverse map f^{-1} and $(f^{-1} \circ f) = (f \circ f^{-1}) = id$.

□

2.2.3 Notation. For each permutation τ, suppose we know all the values $\tau(i), \quad i \in \{1, 2, \ldots, n\}$. We represent τ as

$$\begin{pmatrix} 1 & \ldots & n \\ \tau(1) & \ldots & \tau(n) \end{pmatrix}.$$

For example, if $\tau(1) = 3, \tau(2) = 1, \tau(3) = 2$, we write $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

2.2.4 Cycle Notation. In a more concise manner, a permutation may be represented using cycle notation. We call $(x_1, x_2, \ldots x_r)$ a cycle with entries $x_1, x_2, \ldots, x_r$ if $\tau(x_1) = x_2, \tau(x_2) = x_3, \ldots, \tau(x_r) = x_1$. For example, if τ is a permutation on $\{1, 2, 3\}$ such that $\tau(1) = 3, \tau(2) = 1, \tau(3) = 2$, we write (132) and call it a cycle with entries $1, 2$ and 3. The permutation τ with $\tau(1) = 1, \tau(2) = 2, \tau(3) = 2$ is written as $(1)\,(32)$. A cycle with one element means the element is fixed. For simplicity, we usually remove cycles with one element. If (243) is a permutation on $\{1, 2, 3, 4, 5\}$, it means that 1 and 5 are fixed, that is $\tau(1) = 1, \tau(5) = 5$ and $\tau(2) = 4, \tau(4) = 3, \tau(3) = 2$, so that $(1)\,(243)\,(5) \equiv (243)$. However, the "do nothing" permutation $\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$ or $(1)(2)\ldots(n)$ is represented by id. The cycle notation is not unique. For example, take a permutation $\tau \in S_5$ such that $\tau(1) = 1$, $\tau(2) = 3$, $\tau(3) = 5$, $\tau(4) = 4$, $\tau(5) = 2$. Then τ can be represented in cycle form as (352) or (235) or (523). Therefore, for uniqueness, we always start a cycle with the smallest of its entries. By this $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix}$ will be denoted by (235). If a permutation has more than one cycle, we write the cycle with the smallest first entry first. For instance, the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$ is denoted as $(123)(45)$.

Definition 2.2.5. The length of a cycle is the number of entries in the cycle. The cycle (132) has length three while the cycle (34) has length two. A cycle of length r is also called an $r-$cycle.

2.2.6 Matrix Notation. A permutation $\tau \in S_n$ can be represented in the form of an $n \times n$ matrix A whose entries A_{ij} are defined by

$$A_{ij} = \begin{cases} 1 & \text{if } \tau(j) = i, \\ 0 & \text{otherwise.} \end{cases}$$

For example, take $(243) \in S_4$, then the matrix corresponding to (243) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

2.2.7 Composition of two Permutations. The composition of any two or more permutations is computed from right to left. For example, let $\tau_1 = (135), \tau_2 = (14523), \tau_3 = (12345)$ be permutations in S_5. To compute $\tau_1 \circ \tau_2 \circ \tau_3$, we proceed cycle by cycle. Open the first cycle c with an entry 1. Next, $\tau_3(1) = 2, \tau_2(2) = 3, \tau_1(3) = 5$, so we enter 5 in c. Next, $\tau_3(5) = 1, \tau_2(1) = 4, \tau_1(4) = 4$, so we enter 4 in c. Next, $\tau_3(4) = 5, \tau_2(5) = 2, \tau_1(2) = 2$, so we enter 2 in c. Next, $\tau_3(2) = 3, \tau_2(3) = 1, \tau_1(1) = 3$, so we enter 3 in c. Next, $\tau_3(3) = 4, \tau_2(4) = 5, \tau_1(5) = 1$, and since 1 has already been entered we close the cycle c. Therefore

$$\tau_1 \circ \tau_2 \circ \tau_3 = (135)(14523)(12345) = (15423).$$

We can also compute compositions of permutations with more than one cycle. For example, take $\tau_4, \tau_5 \in S_5$ with $\tau_4 = (154)(23)$ and $\tau_5 = (153)$. To compute

$$\tau_4 \circ \tau_5 = (154)(23)(153),$$

we open a cycle c, pick the first entry 1 and enter it as the first entry in c. Then, $\tau_5(1) = 5$, $\tau_4(5) = 4$, so we enter 4 in c. Next $\tau_5(4) = 4$, $\tau_4(4) = 1$. Since 1 has already been entered in c, we close the cycle c and open another cycle d. We pick the smallest number not occurring in c, which is 2, and enter it in d. Next, $\tau_5(2) = 2$, $\tau_4(2) = 3$, and so we enter 3 in d. Next, $\tau_5(3) = 1$, $\tau_4(1) = 5$, and so we enter 5 in d. Next, $\tau_5(5) = 3$, $\tau_4(3) = 2$, and since 2 has already been entered in d, we close the cycle d. Therefore

$$\tau_4 \circ \tau_5 = (154)(23)(153) = cd = (14)(235).$$

Alternatively, let A and B be the matrices representing permutations τ_1 and τ_2 respectively.The composition $\tau_1 \circ \tau_2$ is represented by the matrix product AB.

Example 2.2.8.

$$\begin{aligned}
\tau_1 \circ \tau_2 \circ \tau_3 &= (135) \circ (14523) \circ (12345) \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&= (15423).
\end{aligned}$$

It is easy to see that composition of distinct cycles commutes. For instance $(123)(45) = (45)(123)$.

Proposition 2.2.9. Every permutation τ in a S_n can be written as a composition of disjoint cycles.

Definition 2.2.10. A transposition is a 2-cycle which is a cycle of length 2.

In S_4, (12) and (13) are transpositions but (1432) is not. However (1432) can be written as a product of transpositions because of the following theorem.

Proposition 2.2.11. Every permutation in S_n can be written as a composition of transpositions.

The following is a sketch to the proof of this proposition.

Proof. Let S_n be the symmetric group on a set X of n objects and let $\tau \in S_n$. By Proposition 2.2.9, τ can be written as composition of distinct cycles. Suppose $(x_1 x_2 \dots x_k)$ is a cycle of τ with $2 \leq k \leq n$, write

$$(x_1 x_2 \dots x_k) = (x_1 x_k)(x_1 x_{k-1}) \dots (x_1 x_2). \qquad (2.2.1)$$

Following the steps used to compute compositions, it is easy to see that equation (2.2.1) holds. Hence, we can decompose all cycles of τ. □

2.3 Cyclic Groups

The composition $\tau \circ \tau$ of an element τ in the symmetric group S_n with itself generates an element in S_n. Given any $\tau_1 \in S_n$, we want to look at the elements $(\tau_1 \circ \tau_1)$, $(\tau_1 \circ \tau_1 \circ \tau_1)$, $(\tau_1 \circ \tau_1 \circ \tau_1 \circ \tau_1), \ldots$

Definition 2.3.1. Let G be any group and $\alpha \in G$, let m be the least positive integer such that α^m is the identity element (if such an m exists). We set

$$\langle \alpha \rangle = \{\alpha^r : r \in \mathbb{Z},\, 0 \leq r < m\}.$$

Here $\circ$ denotes the group operation, $\alpha^r = \underbrace{\alpha \circ \alpha \circ \ldots \circ \alpha}_{\text{r-times}}$. We call $\langle \alpha \rangle$ a cyclic subgroup of G generated by α, and it is the smallest group containing α (Beardon, 2005).

In particular, if $\alpha = (123 \ldots n) \in S_n$, the composition of $\alpha = (123 \ldots n)$ n times with itself gives the identity permutation. Thus, $\langle (123 \ldots n) \rangle$ is a cyclic permutation group and is denoted by C_n. It has order $|C_n| = n$.

Example 2.3.2. For example, take $(1234) \in S_4$.

$$(1234)^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = (13)(24).$$

Similarly, $(1234)^3 = (1432)$, $(1234)^4 = id$. So,

$$C_4 = \{id, (1234), (13)(24), (1432)\}.$$

The group C_4 can be viewed as the set of rotations of a square. This means that each of its elements acts on the vertices of the square by cycling them around. Simultaneously, it shuffles the edge set $\{12, 23, 34, 14\}$ around as shown in Figure 2.1 below.

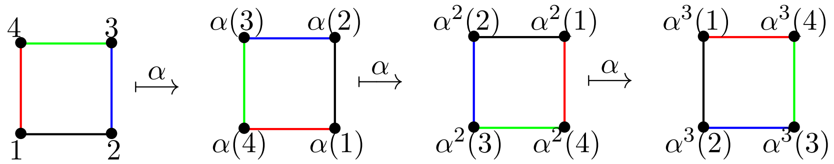


Figure 2.1: C_4 acting on the vertices of a square.

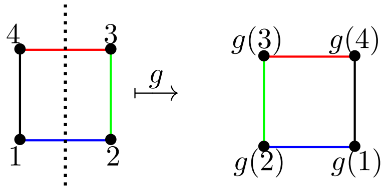
The action of α on the last figure maps it to the original figure, we get $\alpha^4(1) = 1, \alpha^4(2) = 2, \alpha^4(3) = 3, \alpha^4(4) = 4$.

In general C_n is a subgroup of S_n that can be viewed geometrically as the group which contains all possible rotations of a regular polygon P_n with n vertices.

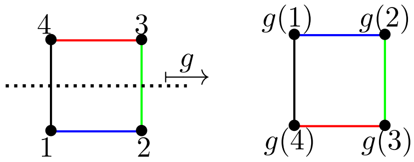
2.4 Dihedral Groups

The dihedral group is another subgroup of S_n. It can be interpreted geometrically as the group which consists of permutations represented by rotations and reflections of a regular polygon P_n with n vertices. For example, the dihedral group D_4 consists of

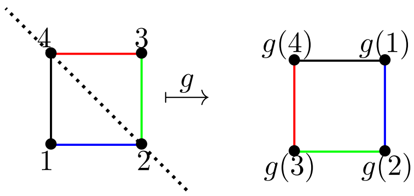
- the 4 rotations in C_4,
- the permutation obtained by reflecting about the vertical axis of the square as $(12)(34)$,



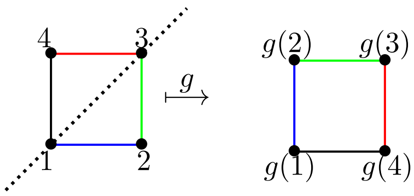
- the permutation obtained by reflecting about the horizontal axis of the square as $(14)(23)$,



- the permutation obtained by reflecting about one of the diagonal axes as (13),



- and the permutation obtained by reflecting about the other diagonal axis as (24).



In general the order of D_n is $2n$.

Remark 2.4.1. $C_n \leq D_n \leq S_n$.

2.5 Alternating Groups

Every permutation can be represented as a product of transpositions which are cycles of length two (see Proposition 2.2.11). For instance in S_6, $(16)(253) = (16)(23)(25)$. Such representations are not unique because $(16)(253)$ can also be $(16)(253) = (16)(45)(23)(45)(25)$ or $(16)(253) = (16)(23)(46)(46)(25)$. It turns out that the number of transpositions in any such decomposition is either always even or always odd. There cannot be different decompositions of the same permutation such that one of them contains an even number of transpositions, and the other one an odd number. See (Judson, 2010, Theorem 5.6, p. 83). If the number of transpositions in a decomposition of a permutation is even, the permutation is said to be an even permutation, otherwise it is called an odd permutation.

Definition 2.5.1. The collection of all even permutations of S_n forms a group called the alternating group denoted by A_n. The order of A_n is $\frac{n!}{2}$. It is a subgroup of S_n.

Example 2.5.2. The alternating group $A_4 \leq S_4$ consists of

$$A_4 = \{id, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}.$$

2.6 Group Actions on a Set

Group actions on a set are the key ingredient in Burnside's Lemma and Pólya's enumeration principle.

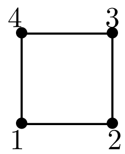
Definition 2.6.1. Let X be any set and G be a group. Then, a left action of G on the set X is a map from $G \times X$ to X, given by $(g, x) \longrightarrow gx$, where $g \in G$ and $x \in X$, that satisfies

- $(gh)x = g(hx) \quad \forall x \in X, h, g \in G$, and
- $ex = x \; \forall x \in X$, where e is the identity element.

We note that in the above definition, there is no restriction on the nature of the set X. However, group actions are more interesting when X has some relationship with G and even more interesting if $X = G$. For instance, we shall later consider the action of the permutation group S_n on the set X, where X is all the possible labellings of a graph. In this case, both X and S_n consist of permutations.

Example 2.6.2. Let $G = GL_n(\mathbb{R})$ and $X = \mathbb{R}^n$. $GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by left multiplication. The identity matrix $I \in GL_n(\mathbb{R})$ is the identity element of the group, so that for $v \in \mathbb{R}^n$ we have $Iv = v$. Also, if A, B are $n \times n$ matrices, then $(AB)v = A(Bv)$. So the two properties of group actions are satisfied.

Example 2.6.3. Consider the vertices of a square labelled $1, 2, 3$ and 4 below.



Take $X = \{1, 2, 3, 4\}$ and $G = D_4 = \{id, (13), (24), (1432), (1234), (12)(34), (14)(23), (13)(24)\}$. Each permutation in D_4 acts on the vertices which are the elements of X either by reflecting or by rotating the square. For instance, (13) is a reflection along the diagonal joining 4 and 2. It sends the vertex labelled 1 to 3, 3 to 1, 2 to 2 and 4 to 4. It is easy to verify that D_4 acting on X satisfies the two axioms of a group action on a set.

Example 2.6.4. Let G be a group and suppose that $X = G$. If H is a subgroup of G, then H acts on G by conjugation, the map $H \times G \longrightarrow G$ is defined by

$$(h, g) \; = hgh^{-1} \quad \text{for} \quad h \in H \quad \text{and} \quad g \in G.$$

- $(e, g) = ege^{-1} = g$ for $g \in G$, where e is the identity in G.

- For $h_1, h_2 \in H$,

$$\begin{aligned}(h_1h_2, g) &= h_1h_2g(h_1h_2)^{-1}\\ &= h_1(h_2gh_2^{-1})h_1^{-1}\\ &= (h_1, (h_2, g)).\end{aligned}$$

2.7 Orbit and Stabilizer

Definition 2.7.1. Let G be a group acting on a set X, and let $x \in X$. The set of all invariant element $x \in X$ under the action of an element $g \in G$, denoted by X_g, is

$$X_g = \{x \in X : gx = x\}.$$

The invariant set under the action of the identity element e of a group acting on X is $X_e = X$.

Definition 2.7.2. The orbit of x denoted by $\operatorname{orb}(x)$ is defined by

$$\operatorname{orb}(x) = \{gx \in X : g \in G\}.$$

Definition 2.7.3. The stabilizer of x denoted by $\operatorname{stab}(x)$ is defined by

$$\operatorname{stab}(x) = \{g \in G : gx = x\}.$$

Proposition 2.7.4. Let G be a group acting on a set X. For every $x \in X$, $\operatorname{stab}(x)$ is a subgroup of G.

Proof. Let $g_1, g_2 \in \operatorname{stab}(x)$, so that $g_1(x) = x$ and $g_2(x) = x$. Hence $g_1g_2(x) = g_1(x) = x$, which implies $g_1g_2 \in \operatorname{stab}(x)$. Thus $\operatorname{stab}(x)$ is closed. The associativity property holds in $\operatorname{stab}(x)$ since $\operatorname{stab}(x)$ is a subset of G. The identity element $e \in G$ is in $\operatorname{stab}(x)$ since $e(x) = x$. Lastly, for any $g \in \operatorname{stab}(x)$ we have that $g(x) = x$. This implies $x = g^{-1}g(x) = g^{-1}(x)$, thus g^{-1} is in $\operatorname{stab}(x)$. □

The stabilizer $\operatorname{stab}(x)$ is sometimes called the isotropy subgroup.

Theorem 2.7.5. (Orbit-Stabilizer) *Let S be a permutation group acting on a set X. Then for every $x \in X$,*

$$|S| = |\operatorname{stab}(x)||\operatorname{orb}(x)|,$$

where $|\operatorname{orb}(x)|$ and $|\operatorname{stab}(x)|$ are the cardinally of the sets $\operatorname{orb}(x)$ and $\operatorname{stab}(x)$ respectively.

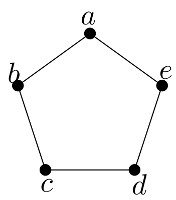
Proof. We prove this by showing that for each $\tau \in S$, τ can be written as a composition of a permutation $s \in \operatorname{stab}(x)$ with a permutation r in a particular set P, where $|P| = |\operatorname{orb}(x)|$. Suppose the elements of $\operatorname{orb}(x)$ are $x_1, x_2, \ldots, x_n$. For each x_i there exists $\tau_i \in S$ such that $\tau_i(x) = x_i$. Then, let $P = \{\tau_1, \tau_2, \ldots, \tau_n\}$. Now, choose τ arbitrarily in S, and note that $\tau(x) = x_i \in \operatorname{orb}(x)$ for some i. Thus,

$$\begin{aligned}&\tau(x) = x_i = \tau_i(x)\\ \implies\ &\tau_i^{-1} \circ \tau(x) = x\\ \implies\ &\tau_i^{-1} \circ \tau \in \operatorname{stab}(x).\end{aligned}$$

But $\tau = \tau_i \circ (\tau_i^{-1} \circ \tau)$. Hence, we have written τ in the required form.

Uniqueness: Suppose τ can be written as $\tau = \tau_i \circ \alpha = \tau_j \circ \beta$ for some τ_i, τ_j in P and $\alpha, \beta \in \operatorname{stab}(x)$. Then $\tau_i(\alpha(x)) = \tau_i(x) = x_i$ and $\tau_j(\beta(x)) = \tau_j(x) = x_j$. This means that $i = j$. Therefore $\alpha = \beta$. So τ is represented in a unique form. □

Example 2.7.6. We recall the dihedral group D_n that is described by rotations and reflections of a regular $n-$vertex polygon. Let $X = \{a, b, c, d, e\}$ be the vertices of the pentagon below.



Let $G = D_5 = \{id, c_{72}, c_{144}, c_{216}, c_{288}, r_a, r_b, r_c, r_d, r_e\}$. Here, c_k means rotation by k degree and r_x denotes reflection along the axis that fixes x. Then

$$X_{id} = X, \quad X_{c_k} = \emptyset\, \forall\, k, \quad \text{and} \quad X_{r_x} = \{x\}\, \forall\, x \in X.$$

$$\text{orb}(a) = \text{orb}(b) = \text{orb}(c) = \text{orb}(d) = \text{orb}(e) = X.$$

$$\text{stab}(a) = \{id, r_a\}, \text{stab}(b) = \{id, r_b\}, \text{stab}(c) = \{id, r_c\}, \text{stab}(d) = \{id, r_d\}, \text{stab}(e) = \{id, r_e\}.$$

Example 2.7.7. Let $X = \{1, 2, 3, 4, 5\}$ be a set of 5 objects and let G be the set of permutations defined on X given by

$$G = \{id, (123), (132), (45), (123)(45), (132)(45)\}.$$

G is a group since it satisfies the four axioms of groups. For the action of G on X, we have

$$X_{id} = \{1, 2, 3, 4, 5\},\ X_{(123)} = \{4, 5\}, X_{(132)} = \{4, 5\},\ X_{(45)} = \{1, 2, 3\},\ X_{(123)(45)} = \emptyset = X_{(132)(45)}.$$

$$\text{orb}(1) = \text{orb}(2) = \text{orb}(3) = \{1, 2, 3\},\ \text{orb}(4) = \text{orb}(5) = \{4, 5\}.$$

$$\text{stab}(1) = \text{stab}(2) = \text{stab}(3) = \{id,\ (45)\},\ \text{stab}(4) = \text{stab}(5) = \{id,\ (123), (132)\}.$$

Example 2.7.8. Consider the 4×4 chessboard in Figure 2.2. Let X be the set containing all possible colourings with 3 colours (red, white and blue) under the action of the dihedral group D_4. Let $x \in X$ represent the chosen colour combination.

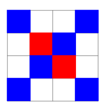
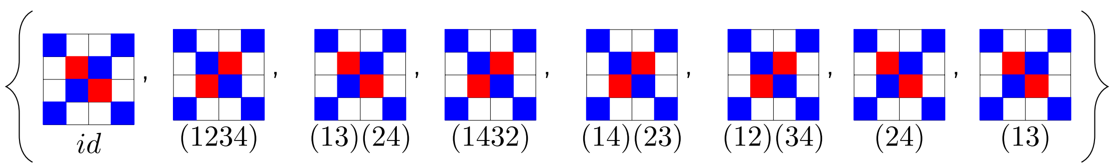


Figure 2.2: 4×4 chessboard.

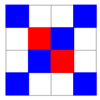
The images of x under the action of D_4 are:

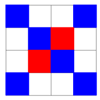


The stabilizer of x under D_4 is $\mathrm{stab}(x) = \{id, (13)(24), (24), (13)\}$.

The orbit $\mathrm{orb}(x)$ is

$$\mathrm{orb}(x) = \left\{ \quad , \quad \right\}.$$





3. Pólya's Enumeration Method

I will explain the concept of symmetry in a way that anyone even without mathematical intuition will understand. An object A is symmetric if when you close your eyes, and another person makes changes (rotation or reflection) to A and upon opening your eyes again, you cannot tell what changes have been made. In Figure 3.1, the triangle is symmetric by $120°$ rotation but the square is not symmetric by $120°$ rotation. However, the square is symmetric by $90°$ rotation unlike the triangle. Therefore, when we speak of an object A being symmetric, it means we keep in mind the existence of certain changes that keep either parts of A, or A itself, invariant. These changes can be seen as a group action on A. The group here is a subgroup of the symmetric group. A group acting on a set induces orbits or equivalence classes. Burnside's Lemma provides a simple way to count the number of equivalence classes of a set under the action of a group of symmetries.

(a) A triangle.　　(b) A square.

Figure 3.1: Symmetric objects.

3.1 Burnside's Lemma

Let G be a group acting on a set X, and let $x, y \in X$. Then x is said to be G-equivalent to y under the action of G if there exists $g \in G$ such that $gx = y$. We denote this relation by $x \sim y$.

Proposition 3.1.1. Let G be a group acting on a set X. Then the G-equivalence is an equivalence relation on X.

Proof.

- Reflexive: for any $x \in X$, there exists the identity element e of G, such that $ex = x$. So $x \sim x$.
- Symmetric: For $x, y \in X$, assume $x \sim y$, so there exists a $g \in G$ such that we have $gx = y$. This implies $g^{-1}y = x$, where $g^{-1} \in G$. Thus $y \sim x$.
- Transitivity: Suppose $x \sim y$ and $y \sim z$, so that there exist $g_1, g_2 \in G$ such that $g_1x = y$ and $g_2y = z$. This means $z = g_2y = g_2(g_1x) = (g_2g_1)x$, where $g_2g_1 \in G$. Thus, $x \sim z$.

□

So the action of a group on a set X partitions X into equivalence classes. Recall the definition of $\mathrm{orb}(x)$. It turns out that if $x \in X$, the equivalence class of x is exactly $\mathrm{orb}(x)$. So the orbits are the equivalence classes and the number of disjoint orbits is also the number of equivalence classes.

Lemma 3.1.2. (Burnside) The number of equivalence classes N of a finite set X under the action of a permutation group G of finite order is given by

$$N = \frac{1}{|G|}\sum_{g\in G}|X_g|,$$

where $X_g \subset X$ is the invariant set under the action of $g \in G$.

Proof. Let $\omega_1, \omega_2, \ldots, \omega_N$ be the orbits of X, where N is the number of orbits. Then

$$\begin{aligned}\frac{1}{|G|}\sum_{g\in G}|X_g| &= \frac{1}{|G|}\sum_{g\in G}\sum_{x\in X}|x\in X : gx = x| \\ &= \frac{1}{|G|}\sum_{x\in X}\sum_{g\in G}|x\in X : gx = x| \\ &= \frac{1}{|G|}\sum_{x\in X}|\operatorname{stab}(x)|. \end{aligned} \tag{3.1.1}$$

We observe that,

1. if $x, y \in \operatorname{orb}(z)$, then by Theorem 2.7.5,

$$|\operatorname{stab}(x)| = \frac{|G|}{|\operatorname{orb}(z)|} = |\operatorname{stab}(y)|.$$

2.

$$\begin{aligned}\sum_{x\in\operatorname{orb}(z)}|\operatorname{stab}(x)| &= \underbrace{\frac{|G|}{|\operatorname{orb}(z)|} + \cdots + \frac{|G|}{|\operatorname{orb}(z)|}}_{|\operatorname{orb}(z)|-\text{number of times}} \\ &= |G|.\end{aligned}$$

Hence,

$$\sum_{x\in X}|\operatorname{stab}(x)| = N \times |G|.$$

Therefore by equation (3.1.1),

$$\begin{aligned}\frac{1}{|G|}\sum_{g\in G}|X_g| &= \frac{1}{|G|}\times N \times |G| \\ &= N.\end{aligned}$$

□

Example 3.1.3. In Example 2.7.7, the number N of equivalence classes is, by Burnside's Lemma,

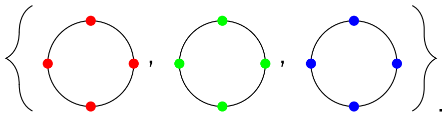
$$\begin{aligned}N &= \frac{1}{|G|}\sum_{g\in G}|X_g| \\ &= \frac{1}{6}\left(|X_{id}| + |X_{(123)}| + |X_{(132)}| + |X_{(45)}| + |X_{(132)(45)}| + |X_{(123)(45)}|\right) \\ &= \frac{1}{6}(5+2+2+3+0+0) = 2\end{aligned}$$

Example 3.1.4.

Let us use Burnside's Lemma to solve the second problem in our introduction. We are colouring a $4-$bead necklace with 3 colours. Since we are considering rotations, the group acting on the set X of $81(=3^4)$ possible colourings is $C_4 = \{id, (1234), (13)(24), (1432)\}$. We have to count the colourings fixed by each element of C_4.

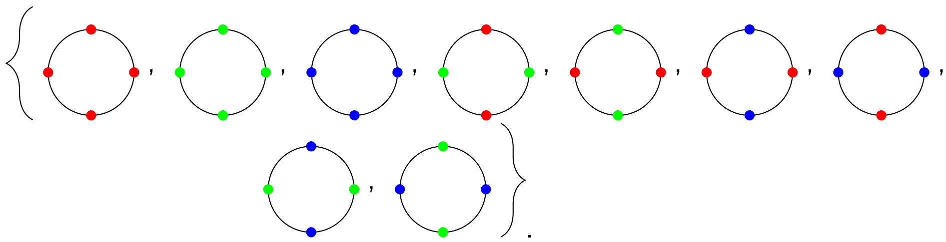
- $X_e = X$, so that $|X_{id}| = 81$.
- (1234) is a rotation by $90°$. The invariant set $X_{(1234)}$ under the action of (1234) is

$X_{(1234)} =$



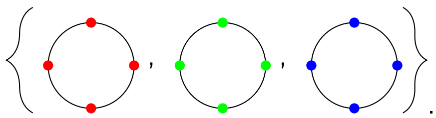
- $(13)(24)$ is a rotation by $180°$. The invariant set $X_{(13)(24)}$ under the action of $(13)(24)$ is

$X_{(13)(24)} =$



- (1432) is a rotation by $270°$. The invariant set $X_{(1432)}$ under the action of (1432) is

$X_{(1432)} =$



So by Burnside's Lemma, the number N of equivalence classes is

$$\begin{aligned} N &= \frac{1}{|C_4|}\left(|X_{id}| + |X_{(1234)}| + |X_{(13)(24)}| + |X_{(1432)}|\right) \\ &= \frac{1}{4}(81+3+9+3) = \frac{96}{4} = 24. \end{aligned}$$

This is as we obtained by explicit counting in Chapter 1.

Observation: We notice that the permutation (1234) has one cycle and the size of the invariant set under the action of (1234) is 3^1. $(13)(24)$ has two cycles and the size of the invariant set under the action of $(13)(24)$ is $9 = 3^2$. Finally, (1432) has one cycle and the size of the invariant set under the action of (1432) is 3^1. In the next section, we will explain why this is the case. This simple observation provides us an easy way to count the number of invariant sets under the action of any permutation. This is why Burnside's Lemma is useful. In order to count the number of equivalence classes, instead of calculating all orbits we simply use this method to calculate invariant sets and use Burnside's Lemma to get the number of equivalence classes.

3.2 Cycle Indices

Suppose we wish to determine the number of ways to colour n objects using up to r colours, up to symmetries on the objects described by a group G. If a colouring is invariant under the action of a permutation τ in G, then every object belonging to one specific cycle of τ must have the same colour. Thus if τ has k disjoint cycles, the number of colourings invariant under the action of τ is $|C_\tau| = r^k$.

Example 3.2.1. If we want to colour a six-bead necklace up to symmetry under the action of the cyclic group C_6 (i.e., up to rotation), we obtain the sizes of the invariant sets as:

$g \in C_6$	X_g
id	3^6
(123456)	3^1
(135)(246)	3^2
(14)(25)(36)	3^3
(153)(264)	3^2
(165432)	3^1

Let G be a subgroup of the symmetric group and $\tau \in G$. Let $\ell_k(\tau)$ denote the number of k-cycles in the decomposition of τ into disjoint cycles. We associate a monomial $M_\tau(x_1, x_2, \dots, x_n)$ with τ which is given by

$$M_\tau = M_\tau(x_1, x_2, \dots, x_n) = \prod_{k=1}^{n} x_k^{\ell_k(\tau)}. \tag{3.2.1}$$

To illustrate this, let us consider the dihedral group.

$D_4 = \{id, (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24)\}$. The identity element $id = (1)(2)(3)(4)$ has four 1-cycles. Therefore we associate the monomial x_1^4 to id. (1234) has one 4-cycle, therefore the monomial $M_{(1234)}$ associated with (1234) is x_4^1. $(13)(24)$ has two 2-cycles, thus $M_{(13)(24)} = x_2^2$. (1432) has one 4-cycle, thus $M_{(1432)}$ associated with (1432) is x_4^1. $(12)(34)$ has two 2-cycles, thus $M_{(12)(34)} = x_2^2$. $(14)(23)$ has two 2-cycles, thus $M_{(14)(23)} = x_2^2$. $(13) = (13)(2)(4)$ has one 2-cycle and two 1-cycles, thus $M_{(13)} = x_1^2 x_2^1$. $(24) = (24)(1)(3)$ has one 2-cycle and two 1-cycles, thus $M_{(24)} = x_1^2 x_2^1$. We note that in order to compute the monomial M_τ associated with a permutation τ, we will have to include omitted 1-cycles.

Definition 3.2.2. The cycle index of a group G is obtained by averaging the monomials M_τ over all permutations τ in G. It is denoted by $P_G(x_1, x_2, \dots, x_n)$ and given by

$$P_G(x_1, x_2, \dots, x_n) = \frac{1}{|G|} \sum_{\tau \in G} M_\tau. \tag{3.2.2}$$

The x_i $(1 \leq i \leq n)$ in P_G are indeterminates.

Example 3.2.3. The elements of C_4 are $\{(1)(2)(3)(4), (1234), (13)(24), (1432)\}$. Therefore its cycle index is

$$\begin{aligned} P_{C_4} &= \frac{1}{4}\left(M_{(1)(2)(3)(4)} + M_{(1234)} + M_{(13)(24)} + M_{(1432)}\right) \\ &= \frac{1}{4}\left(x_1^4 + x_4 + x_2^2 + x_4\right) \\ &= \frac{1}{4}\left(x_1^4 + x_2^2 + 2x_4\right). \end{aligned}$$

Example 3.2.4. The cycle index of $D_4 = \{id, (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24)\}$ is as follows:

$$\begin{aligned} P_{D_4} &= \frac{1}{|D_4|}\left(M_{(1)(2)(3)(4)} + M_{(1234)} + M_{(13)(24)} + M_{(1432)} + M_{(14)(23)} + M_{(12)(34)} + M_{(24)(1)(3)} + M_{(13)(2)(4)}\right) \\ &= \frac{1}{8}\left(x_1^4 + x_4 + x_2^2 + x_4 + x_2^2 + x_2^2 + x_1^2x_2 + x_1^2x_2\right) \\ &= \frac{1}{8}\left(x_1^4 + 3x_2^2 + 2x_1^2x_2 + 2x_4\right). \end{aligned}$$

By Burnside's Lemma (Lemma 3.1.2), the number of ways to colour n objects using r colours (up to symmetry) under the action of a group of symmetries G is $P_G(r, r, \ldots, r)$. So, in Example (3.2.3), the cycle index of C_4 becomes

$$P_{C_4} = \frac{1}{4}\left(r^4 + r^2 + 2r\right).$$

Again, with $r = 3$, $P_{C_4} = \frac{1}{4}\left(3^4 + 3^2 + 2 \times 3\right) = 24$, as obtained previously.

It is important to note that in the 4-bead problem, we considered two colourings to be the same if one can be obtained from the other by rotation. Thus, we solved the problem by considering the action of the cyclic group C_4 on the set of 81 possible colourings. On the other hand, we might consider two colourings to be the same if one can be obtained from the other either by rotation or reflection. In this case, an intelligent guess is that the number of ways to colour a 4-bead necklace with three colours up to symmetry will be fewer. To count the number of equivalence classes we have to consider the action of the dihedral group D_4 on the set of 81 possible colourings. By Example 3.2.4, the cycle index of D_4 in one variable r is

$$P_{D_4} = \frac{1}{8}\left(r^4 + 3r^2 + 2r^3 + 2r\right).$$

Substituting $r = 2$, we obtain $P_{D_4} = \frac{1}{8}(81 + 27 + 54 + 6) = \frac{168}{8} = 21 < 24$ as expected.

3.3 Cycle Index for Specific Groups of Permutations

To evaluate the cycle index of any group of permutations, we need to know the cycle structure of each member of the group. This may not be easy when the order of the group is large. Below are explicit formulas for calculating the cycle index for four major groups of permutations, namely S_n, A_n, C_n and D_n.

A set $\{A_1, A_2, \ldots, A_n\}$ of non-empty sets is a partition of a set X if $\bigcup_{i=1}^n A_i = X$ and for $i \neq j$, $A_i \cap A_j = \emptyset$. Let S_n be the symmetric group acting on a set X of n elements and let $\tau \in S_n$. The cycles in the decomposition of τ define a partition of X if we consider the cycles of τ as the sets of the partition, and so

$$1\ell_1 + 2\ell_2 + \cdots + n\ell_n = n, \tag{3.3.1}$$

where ℓ_k is the number of cycles of length k in τ. Let $\mathbf{y} = (\ell_1, \ell_2, \cdots, \ell_n)$ be a partition of n. Define $d(\mathbf{y})$ to be the number of permutations in S_n that has cycle decomposition given by $\mathbf{y}$, then

$$d(\mathbf{y}) = \frac{n!}{\prod_{k=1}^n k^{\ell_k}\ell_k!}.$$

For example, consider $S_3 = \{id, (123), (132), (1)(23), (12)(3), (13)(2)\}$. $\mathbf{y} = (1, 1, 0)$ is a possible partition of 3, and

$$d(1,1,0) = \frac{3!}{(1^1 1!)(2^1 1!)} = 3.$$

Theorem 3.3.1. *The cycle index of S_n is given by*

$$P_{S_n} = \frac{1}{n!} \sum_{\mathbf{y}} d(\mathbf{y}) \prod_{k=1}^{n} x_k^{\ell_k},$$

where $\mathbf{y}$ runs over all partitions of n (i.e., solutions to equation 3.3.1). See (Harary and Palmer, 1973, equation (2.2.5), p. 36).

Example 3.3.2.

$$\begin{aligned} P_{S_n} &= \frac{1}{3!} \sum_{\mathbf{y}} d(\mathbf{y}) \prod_{k=1}^{3} x_k^{\ell_k} \\ &= \frac{1}{6} \left(x_1^3 + 3x_1^1 x_2^1 + 2x_3^1 \right). \end{aligned}$$

Theorem 3.3.3. *The cycle index of A_n is given by*

$$P_{A_n} = P_{S_n} + P_{S_n}(x_1, -x_2, x_3, -x_4, \ldots).$$

See (Harary and Palmer, 1973, equation (2.2.6), p. 36).

Example 3.3.4.

$$\begin{aligned} P_{A_3} &= \frac{1}{6} \left(x_1^3 + 3x_1 x_2 + 2x_3 \right) + \frac{1}{6} \left(x_1^3 - 3x_1 x_2 + 2x_3 \right) \\ &= \frac{1}{3} \left(x_1^3 + 2x_3 \right). \end{aligned}$$

Definition 3.3.5. The Euler totient function $\phi(n)$ of any positive integer n is defined to be the number of positive integers less than or equal to n that are relatively prime to n, that is

$$\phi(n) = |\{x \in \{1, 2, \ldots, n\} : gcd(x, n) = 1\}|.$$

For example,

$$\phi(1) = 1, \quad \phi(2) = 1, \quad \phi(3) = |\{1, 2\}| = 2, \quad \phi(4) = |\{1, 3\}| = 2, \quad \phi(8) = |\{1, 3, 5, 7\}| = 4.$$

Theorem 3.3.6. *The cycle index of C_n is given by*

$$P_{C_n} = \frac{1}{n} \sum_{k/n} \phi(k) x_k^{n/k}.$$

See (Harary and Palmer, 1973, equation (2.2.10), p. 36).

Example 3.3.7.

$$\begin{aligned} P_{C_4} &= \frac{1}{4} \sum_{k/4} \phi(k) x_1^{4/k} \\ &= \frac{1}{4} \left(\phi(1) x_1^4 + \phi(2) x_2^{4/2} + \phi(4) x_4^{4/4} \right) \\ &= \frac{1}{4} \left(x_1^4 + x_2^2 + 2x_4 \right). \end{aligned}$$

Theorem 3.3.8. *The cycle index of D_n is*

$$P_{D_n} = \frac{1}{2}P_{C_n} + \begin{cases} \frac{1}{4}\left(x_2^{n/2} + x_1^2 x_2^{n/2-1}\right) & \text{if } n \text{ is even,} \\ \\ \frac{1}{2}x_1 x_2^{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

See (Harary and Palmer, 1973, equation (2.2.11), p. 37).

Example 3.3.9.

$$\begin{aligned} P_{D_5} &= \frac{1}{2}P_{C_5} + \frac{1}{2}x_1 x_2^{\frac{5-1}{2}} \\ &= \frac{1}{2}\left(\frac{1}{5}(x_1^5 + 4x_5)\right) + \frac{1}{2}x_1 x_2^{\frac{5-1}{2}} \\ &= \frac{1}{10}\left(x_1^5 + 5x_1x_2^2 + 4x_5\right). \end{aligned}$$

3.4 Generating Functions

Consider a set $X = \{x_1, x_2, \ldots, x_n\}$ of n objects. Let $\mathcal{P}$ be the set of all its possible subsets (power set).

We associate $\emptyset$ with 1, and each non empty subset $A \subset \mathcal{P}$ is associated with a monomial M_A defined by

$$M_A = \prod_{x \in A} x.$$

For instance, $A = \{x_1\}$ is associated with $M_A = x_1$, $B = \{x_1, x_2\}$ is associated with $M_B = x_1x_2$, and so on. Then taking the sum over all monomials we get

$$F_{\mathcal{P}}(x_1, x_2, \ldots, x_n) = \sum_{A \in \mathcal{P}} M_A. \tag{3.4.1}$$

Equation (3.4.1) can be seen as an algebraic representation of the patterns in the subsets in $\mathcal{P}$. Each monomial term represents a pattern, and has coefficient 1. If $n = 1$, we get $F_P(x_1) = (1 + x_1)$. If $n = 2$, we have

$$A_1 = \emptyset,\ A_2 = \{x_1\},\ A_3 = \{x_1, x_2\},\ A_4 = \{x_2\}.$$

Then

$$\begin{aligned} F_P(x_1, x_2) &= 1 + x_1 + x_1x_2 + x_2 \\ &= (1 + x_1)(1 + x_2). \end{aligned}$$

In general, for any n

$$F_P(x_1, x_2, \ldots, x_n) = (1 + x_1)(1 + x_2)\cdots(1 + x_n).$$

If $x_1 = x_2 = \cdots x_n = x$, then we have

$$F_{\mathcal{P}} = (1 + x)^n. \tag{3.4.2}$$

Next, we give a formal definition of generating functions.

Definition 3.4.1. The generating function for an infinite sequence $(a_0, a_1, a_2, a_3, \cdots)$ is the formal power series

$$F(x) = a_0 + a_1 x + a_2 x^3 + a_3 x^3 + \ldots,$$

Example 3.4.2. Returning to Equation (3.4.2), if we consider the binomial expansion of $(1+x)^n$, we get

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n. \quad (3.4.3)$$

Equation (3.4.3) can be regarded as the generating function for the sequence a_k defined by

$$a_k = \begin{cases} \binom{n}{k} & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

It is important to note that in this case some terms in Equation (3.4.1) which represent subsets with the same number of elements will combine to a term $a_k x^k$ in equation (3.4.3), where a_k represents the number of subsets whose monomial representations combine to form $a_k x^k$. In other words, a_k is the number of ways to choose k elements from a set of n elements, i.e, $a_k = \binom{n}{k}$. Again we see that substituting $x = 1$ in equation (3.4.3) gives

$$\sum_{k=1}^{n} \binom{n}{k} = 2^n,$$

which is the total number of possible choices that can be made.

3.5 Pólya's Enumeration Theorem

The goal of this section is to discuss Pólya's enumeration theorem. We shall state the theorem, illustrate it with examples and prove it.

By Burnside's Lemma, we have seen that the number of ways to colour a set X of 4 objects up to symmetry using two colours is 6, if we take any two colourings to be equivalent (the same up to symmetry) when one can be obtained from the other by rotation. This means that under the action of C_4, using two colours to colour a set X of 4 objects, there are six equivalence classes or 6 categories of distinguishable colourings out of the $2^4 = 16$ possibilities. The equivalence classes are

$$\begin{aligned} Q_1 &= \{gggg\}, \\ Q_2 &= \{gggr, ggrg, grgg, rggg\}, \\ Q_3 &= \{ggrr, grrg, rggr, rrgg\}, \\ Q_4 &= \{grgr, rgrg\}, \\ Q_5 &= \{grrr, rgrr, rrgr, rrrg\}, \\ Q_6 &= \{rrrr\}. \end{aligned}$$

Suppose we write $r^k g^s$ for a colouring with k red and s green objects. We notice power patterns in each of the equivalence classes. For Q_1, we obtain the pattern g^4. For Q_2, we obtain $g^3 r$. For Q_3, we obtain $g^2 r^2$. For Q_4, we obtain $r^2 g^2$. For Q_5, we obtain gr^3. For Q_6, we obtain r^4. If we take the sum over all patterns, we get

$$\text{SUM} = g^4 + g^3 r + g^2 r^2 + r^2 g^2 + rg^3 + r^4.$$

The patterns in Q_3 and Q_4 are essentially the same pattern, we can write

$$\text{SUM} = g^4 + g^3 r + 2g^2 r^2 + rg^3 + r^4. \tag{3.5.1}$$

The sum in equation (3.5.1) is a pattern inventory function. It has important information contained in the coefficients of the terms. The coefficient of the term g^4 is 1. This tells us that there is one equivalence class with 4 green objects. The coefficient of the term $g^3 r$ is 1, this tells us that there is only one equivalence class with 3 green and 1 red objects. On the other hand, the coefficient of the term $g^2 r^2$ is two. This tells us that there are two equivalence classes with 2 green and 2 red objects, and so on.

We recall that the number of equivalence classes gives the number of distinguishable colourings. This implies that if we need to colour 4 objects with two colours (red and green), but require precisely 2 red and 2 green objects, there are two distinguishable ways to achieve it, while if we require precisely 1 green and 3 red objects, there is only one way to achieve it since the coefficient of gr^3 is 1, and so on.

In general, the question to ask is: suppose we need to determine the number of equivalence classes of colourings of n objects using the set $\{c_1, c_2, \ldots, c_r\}$ of r colours, where each c_j $(1 \leq j \leq r)$ occurs a prescribed number of times, in how many ways can we achieve this up to symmetry? For instance, suppose we need to colour the vertices of an octagon using three colours (red, blue and green) but require precisely 3 blue, 3 red and 2 green vertices. Or, in our 4-bead necklace problem, we decide to colour the beads with precisely 2 red, 1 green and 1 blue bead, how many distinguishable colourings can be obtained? To answer this question, we need to determine the generating function for pattern inventories.

An easy way to obtain the right answer to problems like these was first discovered by the Hungarian mathematician George Pólya. He provided in his theorem known as Pólya's Enumeration Theorem (PET), a technique on how to compute generating functions for patterns using cycle indices.

Theorem 3.5.1. *Suppose we wish to colour n objects with r colours up to symmetry. Let $P_G(x_1, x_2, \ldots, x_n)$ be the cycle index for the corresponding group of symmetries $G \leq S_n$ acting of the set X of r^n possibilities. Then, the generating function $F_X(c_1, c_2, \ldots, c_r)$ for the number of equivalence classes on the set X is obtained by substituting*

$$x_k = c_1^k + c_2^k + \ldots + c_r^k, \quad 1 \leq k \leq n,$$

into the cycle index P_G. That is,

$$F_X\left(c_1, c_2, \ldots, c_r\right) = P_G\left(\sum_{i=1}^{r} c_i^1, \sum_{i=1}^{r} c_i^2, \ldots, \sum_{i=1}^{r} c_i^n\right).$$

Before we prove this theorem, let us look at some examples.

Example 3.5.2. Let us evaluate Pólya's generating function for the number of equivalence classes if we want to colour 4 objects using 2 colours (red and green), under rotational symmetry. From example 3.2.3, the cycle index fo C_4 is

$$P_{C_4}(x_1, x_2, x_3, x_4) = \frac{1}{4}\left(x_1^4 + x_2^2 + 2x_4\right).$$

Since there are 2 colours c_1 and c_2, by Pólya's theorem, we make the substitution

$$x_1 = c_1 + c_2, \quad x_2 = c_1^2 + c_2^2, \quad x_3 = c_1^3 + c_2^3, \quad x_4 = c_1^4 + c_2^4,$$

or more conveniently by using the names of the colours r for red and g for green, we can use the substitution

$$x_1 = r + g, \quad x_2 = r^2 + g^2, \quad x_3 = r^3 + g^3, \quad x_4 = r^4 + g^4$$

into $P_{C_4}(x_1, x_2, x_3, x_4)$. So we obtain the generating function $F_C(r, g)$ as

$$\begin{aligned} F_X(r, g) &= \frac{1}{4}\left((r+g)^4 + (r^2+g^2)^2 + 2(r^4+g^4)\right) \\ &= \frac{1}{4}\left(r^4 + 4r^3g + 6r^2g^2 + 4rg^3 + g^4 + r^4 + 2r^2g^2 + g^4 + 2r^4 + 2g^4\right) \\ &= r^4 + r^3g + 2r^2g^2 + rg^3 + g^4, \quad \text{as obtained in equation (3.5.1)}. \end{aligned}$$

The number of equivalence classes is the sum over all coefficients of the terms in the generating function. That is, $1 + 1 + 2 + 1 + 1 = 6$.

Again, let us investigate how Pólya's general technique helps to solve the second problem we considered in the introductory section.

Example 3.5.3. We want to evaluate the generating function for the number of equivalence classes under the action of the cyclic group C_4 on the set of 3^4 possible colourings, where we are using three colours: red, green and blue, to colour 4 objects. The cycle index of the cyclic group C_4 is

$$P_{C_4}(x_1, x_2, x_3, x_4) = \frac{1}{4}\left(x_1^4 + x_2^2 + 2x_4\right).$$

Since in this case we are using a set of three colours, by Pólya's theorem we set

$$x_1 = c_1 + c_2 + c_3, \quad x_2 = c_1^2 + c_2^2 + c_3^2, \quad x_3 = c_1^3 + c_2^3 + c_3^3, \quad x_4 = c_1^4 + c_2^4 + c_3^4,$$

or more conveniently using r for red, g for green and b for blue, we can use the substitution

$$x_1 = r + g + b, \quad x_2 = r^2 + g^2 + b^2, \quad x_3 = r^3 + g^3 + b^3, \quad x_4 = r^4 + g^4 + b^4.$$

Substituting these x_i into the cycle index of C_4, we obtain the generating function,

$$\begin{aligned} F_X(r, g, b) = b^4 + b^3g + 2\,b^2g^2 + bg^3 + g^4 + b^3r + 3\,b^2gr + 3\,bg^2r + g^3r + 2\,b^2r^2 + 3\,bgr^2 + 2\,g^2r^2 \\ + br^3 + gr^3 + r^4. \end{aligned} \tag{3.5.2}$$

We observe that the powers of each term give the different possible colour combinations and the sum over all exponents in each term is 4. For instance bg^2r is a possible colour combination where we used 1 blue, 2 green and 1 red. This implies that the number of equivalence classes that will be obtained if we require precisely 3 blue and 1 green is one since the coefficient of b^3g is one. The number of equivalence classes that will be obtained if we require precisely 2 blue, 1 red and 1 green is three since three is the coefficient of the term b^2gr and so on. Therefore, for the general question, the number of equivalence classes we shall obtain if we use three colours to colour 4 objects up to symmetry is the sum over all the coefficients of the terms,

$$1+1+2+1+1+1+3+3+1+2+3+2+1+1+1=24.$$

Alternatively, we can simply evaluate $F_X(1,1,1) = 1{+}1{+}2{+}1{+}1{+}1{+}3{+}3{+}1{+}2{+}3{+}2{+}1{+}1{+}1 = 24$.

Example 3.5.4. Similarly, let us evaluate the generating function for the number of equivalence classes under the action of the dihedral group D_4, on the set of 3^4 possible colourings, where we are using three colours (red, green and blue), to colour 4 objects. The cycle index of the dihedral group D_4 is

$$P_{D_4}(x_1,x_2,x_3,x_4) = \frac{1}{8}\left(x_1^4 + 3x_2^2 + 2x_1^2x_2 + 2x_4\right).$$

Using r for red, g green and b for blue, we make the substitution by Pólya's theorem, namely,

$$x_1 = r+g+b, \quad x_2 = r^2+g^2+b^2, \quad x_3 = r^3+g^3+b^3, \quad x_4 = r^4+g^4+b^4,$$

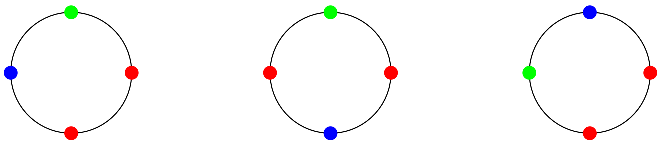
into the cycle index of D_4, to obtain the generating function, which is

$$F_X(r,g,b) = b^4 + b^3g + 2\,b^2g^2 + bg^3 + g^4 + b^3r + 2\,b^2gr + 2\,bg^2r + g^3r + 2\,b^2r^2 + 2\,bgr^2 + 2\,g^2r^2 + br^3 + gr^3 + r^4. \tag{3.5.3}$$

Therefore, the number of equivalence classes is

$$F_X(1,1,1) = 1+1+2+1+1+1+2+2+1+2+2+2+1+1+1 = 21.$$

We notice that there are three non-equivalent necklaces with 2 red-coloured beads, 1 green-coloured bead and 1 blue-coloured bead when taking into account only rotational symmetry (that is under the action of C_4) while there are only two non-equivalent necklaces under the action of both rotational and reflective symmetries (D_4). Figure 3.2 shows three possible colourings that can be obtained from colouring four beads with three colours that are all of the form r^2gb.



(a) (b) (c)

Figure 3.2: Colourings with the power pattern r^2gb

Up to rotational symmetry, Figures 3.2a, 3.2b, and 3.2c belong to three different equivalence classes, no one can be obtained by rotating the other. Hence, they generate three equivalence classes. On the other hand, if we take reflection into account as well, since Figure 3.2c can be obtained by reflecting Figure 3.2a, they are in the same equivalence class while (b) belongs to a different class. Hence, we have only two non-equivalent necklaces of this type.

Proof. (PET). Let X denote the set of possible colourings of n objects using r colours. Let $\mathbf{v} = (n_1, n_2, \ldots, n_r)$ be a vector of non-negative integers of length r whose components sum to n, and let $X_{\mathbf{v}}$ denote the subset of X where exactly n_i objects have the colour c_i, for each i. Let $X_{\mathbf{v},\tau}$ denote the invariant set of $X_{\mathbf{v}}$ under the action of a permutation τ.

If a permutation τ in G does not disturb a particular colouring then any k objects belonging to the same k-cycle of τ must have the same colour. So, either each of the k objects has colour c_1 or each has c_2 or each has c_3, etc. In terms of generating functions, we represent these choices as

$$c_1^k + c_2^k + \ldots + c_r^k.$$

Thus, if we substitute $x_k = \sum_{i=1}^{r} c_i^k$ into the monomial M_τ defined in equation (3.2.1) which is associated with τ, M_r will now represent the generating function for invariant colourings. Therefore, $|X_{\mathbf{v},\tau}|$ is the coefficient of $c_1^{n_1} c_2^{n_2} \cdots c_r^{n_r}$ in $M_\tau\left(\sum c_i, \sum c_i^2, \ldots, \sum c_i^n\right)$. Let $\mathbf{c}^{\mathbf{v}}$ denote $c_1^{n_1} c_2^{n_2} \cdots c_r^{n_r}$. Then, summing over all permissible vectors, we obtain

$$\sum_{\mathbf{v}} |X_{\mathbf{v},\tau}| \mathbf{c}^{\mathbf{v}} = M_\tau \left(\sum_{i=1}^{r} c_i, \sum_{i=1}^{r} c_i^2, \ldots, \sum_{i=1}^{r} c_i^n \right). \tag{3.5.4}$$

Now we sum both expressions in equation (3.5.4) over all $\tau \in G$ and divide by $|G|$. On the left hand side, we have

$$\begin{aligned} \frac{1}{|G|} \sum_{\tau \in G} \sum_{\mathbf{v}} |X_{\mathbf{v},\tau}| \mathbf{c}^{\mathbf{v}} &= \sum_{\mathbf{v}} \left(\frac{1}{|G|} \sum_{\tau \in G} |X_{\mathbf{v},\tau}| \right) \mathbf{c}^{\mathbf{v}} \\ &= \sum_{\mathbf{v}} a_{\mathbf{v}} \mathbf{c}^{\mathbf{v}}. \end{aligned}$$

By Burnside's Lemma, this is the generating function $F_X(c_1, \ldots, c_r)$ (or pattern inventory) where $a_{\mathbf{v}}$ represents the number of non-equivalent colourings of the n objects with the colour c_i occurring precisely n_i times.

On the right hand side, we obtain the cycle index of G evaluated for $x_i = \sum_i c_i^k$. Hence,

$$\begin{aligned} F_X(c_1, \ldots, c_r) &= \frac{1}{|G|} \sum_{\tau \in G} M_\tau \left(\sum_{i=1}^{r} c_i, \sum_{i=1}^{r} c_i^2, \ldots, \sum_{i=1}^{r} c_i^n \right) \\ &= P_G \left(\sum_{i=1}^{r} c_i, \sum_{i=1}^{r} c_i^2, \ldots, \sum_{i=1}^{r} c_i^n \right). \end{aligned}$$

□

3.6 The Cube Problem

Let us now resolve the cube problem in our introduction using Pólya's enumeration theorem. We want to find the number of ways of colouring a cube with just one colour up to symmetry. In order words we want to count the equivalence classes. Two colourings are considered the same (equivalent) if one can be obtained from the other by rotating the cube. To solve this problem by PET, first we need to determine the cycle structures for every element in the group of permutations on the set of faces of the cube and then construct its cycle index.

Let us label the cube with top side 1, bottom side 3, right side 2, left side 4, front side 5 and back side 6. There are three types of rotational symmetries of the cube as shown in Figure 3.3.

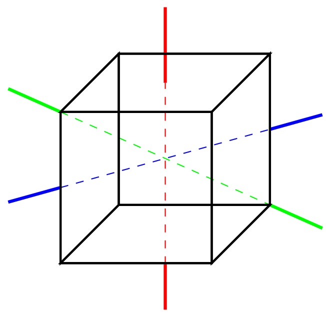


Figure 3.3: Axes of symmetry of a cube.

- The rotations about axes joining two opposite faces (the red axis). There are three such axes. For each of the axes, we can rotate the cube by $90°$, $180°$, or $270°$. These will generate the following permutations,

Rotations by $90°$	Rotations by $180°$	Rotations by $270°$
$(1)(2645)(3)$	$(1)(24)(3)(56)$	$(1)(2546)(3)$
$(1536)(2)(4)$	$(13)(2)(4)(56)$	$(1536)(2)(4)$
$(1234)(5)(6)$	$(13)(24)(5)(6)$	$(1432)(5)(6)$.

- Rotations about the diagonal axes joining opposite corners (the green axis). There are four such axes. For each of the axes, we can rotate the cube by $120°$ or $240°$. These will generate the following permutations,

$$(145)(263),\ (146)(253),\ (154)(236),\ (164)(235),\ (152)(364),\ (126)(346),\ (125)(346),\ (162)(354).$$

- The rotations about the axes joining midpoints of opposite edges (the blue axis). There are six such axes. For each of the axes. For each of the axes, we can rotate the cube by $180°$. These will generate the following permutations,

$$(14)(23)(56),\ (13)(26)(45),\ (12)(34)(56),\ (16)(24)(35),\ (13)(25)(46),\ (15)(24)(36).$$

We can verify that these permutations together with the identity permutation $(1)(2)(3)(4)(5)(6)$ form a group. Let Z denote this group. Then considering the cycle structure for each element of Z, we obtain the cycle index of Z as

$$P_Z(x_1, x_2, x_3, x_4) = \frac{1}{24}\left(x_1^6 + 3x_1^2x_2^2 + 6x_1^2x_4 + 6x_3^2 + 8x_2^3\right).$$

Let C and W denote colouring and not colouring a face of the cube respectively. Since we can either colour a face or leave it uncoloured, the generating function is $W + C$, therefore by PET we substitute

$$x_1 = W + C,\ x_2 = W^2 + C^2,\ x_3 = W^3 + C^3,\ x_4 = W^4 + C^4,$$

into the cycle index P_Z to get

$$F_Z(W, C) = C^6 + C^5W + 2\,C^4W^2 + 2\,C^3W^3 + 2\,C^2W^4 + CW^5 + W^6.$$

Hence the number of equivalence classes is

$$F_Z(1, 1) = 10, \quad \text{as obtained previously.}$$

Remark 3.6.1. Pólya's theorem provides an easy way to find the number of distinguishable colourings up to symmetry. In general, the main task is to determine the corresponding group of symmetries G acting of the set of possible colourations and then evaluate the cycle index $P_G(x_1, x_2, \ldots, x_n)$ for the group. Here, we have seen an illustration of this under some groups of symmetries. Following the steps, one can obtain the cycle index of any group of symmetries. The secondary task is to expand the expression obtained by substituting for x_i in the cycle index, and hence find the required coefficients.

4. Graphical Enumeration

A graph $G(V, E)$ is an ordered pair (V, E) where V is a non-empty set and E is a set of pairs of elements of the V. Members of V and E are referred to as vertices and edges of G respectively. The order of a graph $G(V, E)$ is $|V|$ and the size of the graph is $|E|$. An (n, m) graph is a graph of order n and size m. Throughout this discussion, we will consider simple graphs, which are graphs $G(V, E)$ where E contains distinct unordered pairs.

Given n elements, there are $\binom{n}{2}$ ways to pair the elements. Our goal here is to determine the counting polynomial

$$F(z) = \sum_{m=0}^{\binom{n}{2}} a_m z^m,$$

for the number of (n, m) graphs.

Definition 4.0.2. Two graphs $G(V, E)$ and $G'(V', E')$ are said to be isomorphic if there exists a bijection, $\theta : V \to V'$ such that $\{a, b\} \in E \Leftrightarrow \{\theta(a), \theta(b)\} \in E'$.

4.1 Counting Labelled Graphs

A labelled graph of order n has its vertices labelled, say, $1, 2, \ldots, n$. Two labelled graphs are considered the same if there is an isomorphism from G to G' that preserves the labels. The number of non-isomorphic labelled graph with n points is $2^{\binom{n}{2}}$. This is because, in any of such graph, each of the $\binom{n}{2}$ possible pairs is either present or not, so that, the total number of choices to be made is $2^{\binom{n}{2}}$.

4.2 Counting Unlabelled Graphs

Two unlabelled graphs G and G' are said to be distinct (non-isomorphic) if there is no permutation on the vertices mapping the edges of G to the edges of G'. The group of permutations on the vertices of a graph of order n is precisely the symmetric group S_n. The action of S_n on the set of vertices of a graph naturally induces a group of permutations acting on the set of edges of the graph. The induced group denoted by $S_n^{(2)}$ is called a pair group and is defined as follows.

Definition 4.2.1. Let $G(V, E)$ be a graph of order n, and let S_n be the permutation group on V. A pair permutation τ', is a permutation defined on E given by

$$\tau'(\{x, y\}) = \{\tau(x), \tau(y)\},$$

where $x, y \in V$ and $\tau \in S_n$.

The orbits under the action of $S_n^{(2)}$ on the set of edges correspond exactly to non-isomorphic labelled graphs of order n. Therefore, to count the number of non-isomorphic unlabelled graphs of order n, we need to determine the cycle index $P_{S_n^{(2)}}$, of $S_n^{(2)}$ and then use PET to count the equivalence classes. As Harary and Palmer explained in their book (Harary and Palmer, 1973), substituting

$$x_k = 1 + z^k,$$

into the cycle index $P_{S_n^{(2)}}$, yields the desired counting polynomial. We illustrate this with the following example.

Example 4.2.2. Determining the number of non-isomorphic unlabelled graphs of order 4.

Let $V = \{1, 2, 3, 4\}$.

The set $V^{(2)}$ of possible unordered pairs are $V^{(2)} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$.

Next, we find the members of $S_4^{(2)}$. For each $\tau \in S_4$ we look for the induced permutation $\tau' \in S_4^{(2)}$ such that

$$\tau'(\{x, y\}) = \{\tau(x), \tau(y)\}.$$

Pick $\tau = (34)$, then

$$\begin{aligned}
\tau'(\{1,2\}) &= \{\tau(1), \tau(2)\} = \{1, 2\},\\
\tau'(\{1,3\}) &= \{\tau(1), \tau(3)\} = \{1, 4\},\\
\tau'(\{1,4\}) &= \{\tau(1), \tau(4)\} = \{1, 3\},\\
\tau'(\{2,3\}) &= \{\tau(2), \tau(3)\} = \{2, 4\},\\
\tau'(\{2,4\}) &= \{\tau(2), \tau(4)\} = \{2, 3\},\\
\tau'(\{3,4\}) &= \{\tau(3), \tau(4)\} = \{4, 3\}.
\end{aligned}$$

So that, the induce pair permutation is $\tau' = \begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ 12 & 14 & 13 & 24 & 23 & 34 \end{pmatrix} \equiv (12)(13\ 14)(23\ 24)(34)$.

Pick $\tau = (1342)$, then

$$\begin{aligned}
\tau'(\{1,2\}) &= \{\tau(1), \tau(2)\} = \{3, 1\},\\
\tau'(\{1,3\}) &= \{\tau(1), \tau(3)\} = \{3, 4\},\\
\tau'(\{1,4\}) &= \{\tau(1), \tau(4)\} = \{3, 2\},\\
\tau'(\{2,3\}) &= \{\tau(2), \tau(3)\} = \{1, 4\},\\
\tau'(\{2,4\}) &= \{\tau(2), \tau(4)\} = \{1, 2\},\\
\tau'(\{3,4\}) &= \{\tau(3), \tau(4)\} = \{4, 2\}.
\end{aligned}$$

So that, the induced pair permutation is $\tau' = \begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ 13 & 34 & 23 & 14 & 12 & 24 \end{pmatrix} \equiv (12\ 13\ 34\ 24)(14\ 23)$.

We continue in this way, and then obtain the summary of all $\tau' \in S_4^{(2)}$ induced by $\tau \in S_4$ in the following table.

τ	τ'	τ	τ'	τ	τ'
(1)(2)(3)(4)	(12)(13)(14)(23)(24)(34)	(12)	(12)(13 23)(14 24)(34)	(13)	(12 13)(14 34)(23)(24)
(14)	(12 24)(13 34)(14)(23)	(23)	(12 13)(24 34)(14)(23)	(24)	(12 14)(23 34) (13)(24)
(34)	(12)(13 14)(23 24) (34)	(13)(24)	(12 34)(14 23)(13)(24)	(123)	(12 23 13) (14 24 34)
(124)	(12 24 14) (13 24 34)	(132)	(12 13 23) (14 34 24)	(134)	(12 23 34) (13 34 14)
(142)	(12 14 24) (13 34 23)	(143)	(12 24 23) (13 14 34)	(234)	(12 13 14) (23 34 24)
(243)	(12 14 13) (23 24 34)	(1234)	(12 23 34 14) (13)(24)	(1243)	(12 24 34 13) (14)(23)
(1324)	(12 34) (13 23 24 14)	(1342)	(12 23 34 24) (14 23)	(1423)	(12 34)(13 14 24 23)
(1432)	(12 14 34 23)(13 24)	(12)(34)	(12)(13 24)(14 23)(34)	(14)(23)	(12 34)(13 24)(12)(23)

Table 4.1: Cycles structure of elements in S_4 and $S_4^{(2)}$.

So the cycle index of $S_4^{(2)}$ is

$$P_{S_4^{(2)}} = \frac{1}{24}\left(x_1^6 + 9x_1^2x_2^2 + 8x_3^2 + 6x_2x_4\right).$$

By PET, substituting $x_k = 1 + z^k$, for $k = 1, 2, 3, 4$ into $P_{S_4^{(2)}}$, we obtain the counting polynomial as

$$F(z) = 1 + z + 2z^2 + 3z^3 + 2z^4 + z^5 + z^6.$$

Therefore, the number of non-isomorphic unlabelled graphs is $F(1) = 11$. Below we list all such graphs which verifies the result.

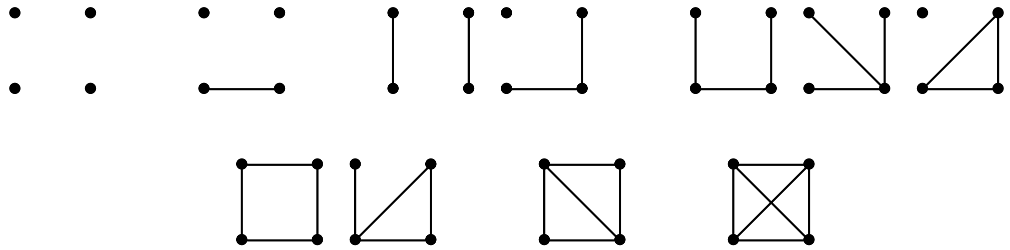


Figure 4.1: Non-Isomorphic graphs of order four.

From Table 4.1, the cycle index of S_4 is

$$P_{S_4} = \frac{1}{24}\left(x_1^4 + 6x_1^2x_2 + 8x_1x_3 + 3x_2^2 + 6x_4\right),$$

which differs from the cycle index of $S_4^{(2)}$. In general S_n^2 is a subgroup of $S_{\binom{n}{2}}$, and is isomorphic to S_n. But, S_n and S_n^2 do not necessarily have the same cycle index, because the actions differ.

A cycle of length $2k+1$ induces k pair of cycles of length $2k+1$. A cycle of length $2k$ induces $k-1$ pair cycles of length $2k$ and one pair cycle of length k. Two cycles of length ℓ and m induces $gcd(\ell, m)$ pair cycles of length $lcm(\ell, m)$. We end with the following proposition which gives explicit formula for computing the cycle index of the pair group.

Proposition 4.2.3. The cycle index of the pair group is

$$P_{S_n^{(2)}} = \frac{1}{p!} \sum_{\substack{j_1, j_2, j_3 \\ j_1+2j_2+3j_3+\cdots+nj_n}} \prod_{k\geq 1} \frac{p!}{k^{j_k} j_k} \prod_{k\geq 0} x_{2k+1}^{kJ_{2k+1}} \prod_{k\geq 1} \left(x_k x_{2k}^{k-1}\right)^{j_{2k}} \prod_{\ell\geq 1} x_\ell^{\ell\binom{j_\ell}{2}} \prod_{m>\ell\geq 1} x_{lcm(\ell,m)}^{gcd(\ell,m)j_\ell j_m}.$$

(Harary and Palmer, 1973, equation (4.1.9), p. 84).

5. Conclusion

Counting objects in the presence of symmetries appears tricky. Symmetries of objects arise due to certain groups of permutations acting on them. Under these actions, objects with similar features are grouped in the same orbit, so that the number of distinguishable objects is exactly the number of orbits. To effectively count objects therefore, we need to clearly decide what it means for two objects to be similar.

Burnside's Lemma and PET discussed here enable us to count orbits, i.e, distinct objects. However, PET is more complete and general; it also tells us the number of orbits with prescribed features. Using the PET technique, the main task is to understand how the underlying group of permutations act on the objects and then determine its cycle index. We have seen explicit formulas for computing cycle indices of relevant groups of permutations. Two isomorphic groups do not necessarily have the same cycle indices as we have seen between S_4 and $S_4^{(2)}$. The second task is to substitute appropriate generating function into the cycle index, expand the expression and find the desired coefficients. The coefficients fully describe the number of orbits with prescribed features.

The group G of symmetries of a cube has order 24. In Section 3.6, we solved the problem of finding distinguishable ways of colouring the six faces of the cube up to symmetry using r colours. We achieved this by considering the induced permutation group, named Z, acting on the set of faces of the cube. Simultaneously, G also induces groups of permutations on the set of edges and the set of vertices of the cube. These groups will have different cycle structures since the number of vertices is different from the number of edges. Hence, we can also find the number of distinguishable ways of colouring the edges of a cube using r colours or the colouring the vertices using r colours.

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