General Relativity, Cosmology and Inflation

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Abstract

We start with a brief discussion of general relativity and move to a survey of the basics of relativistic cosmology: we derive a set of equations that governs the large-scale dynamics of the universe (the Friedmann equations), we investigate some simplified solutions of these equations and see how they compare to observations, we discuss the cosmic microwave background radiation and their basic properties, we discuss the horizon and the flatness problems and see how cosmological inflation successfully solves them. And we finish with an investigation of a possible cause of inflation (inflaton field) and how it can give rise to the formation of structure and irregularities in the universe.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Husam Abubakr Alsiddig Abdallah Mohamed, 22 May, 2014

Contents

1. Introduction

In this work, we give a brief introduction to general relativity and its basic mathematical tools. We breifly discuss the notion of space-time (the union of space and time), how to describe its geometry with a metric, how gravity relates to the geometry of space-time and how to quantitatively relate gravity to matter/energy distribution in space-time via the Einstein's field equations of general relativity.

We then survey the basics of relativistic cosmology: we discuss the observed expansion of the universe and see how the constraints of large-scale isotropy and homogeneity can give rise to a set of equations (the Friedmann equations) that model large-scale dynamics of the universe. We derive solutions for these equations in simplified situations of matter-dominated and radiation-dominated universes. We discuss some cosmological observational parameters: Hubble's parameter which relates distance to expansion, the density parameter which determines the geometry of the universe and the deceleration parameter which quantifies the rate of expansion, we discuss the meanings of these parameters and their observed values. We then see how to account for the observed accelerating expansion by introducing the so-called cosmological constant. We also discuss the space-pervading cosmic microwave background (CMB) radiation, its origin and its anomalous minute anisotropies, and we finish our survey by discussing the horizon and flatness problems.

In the third chapter, we discuss how the horizon and flatness problems can be solved by postulating a period of rapid accelerating expansion (inflation), and we conclude the essay with an investigation of a possible cause of inflation (inflaton field) and how quantum fluctuations in that field can explain CMB anisotropies and the formation of small-scale structure in the universe.

2. General relativity and relativistic cosmology

2.1 General Relativity

"Matter tells Spacetime how to curve, and Spacetime tells matter how to move." -John Wheeler. General relativity is Einstein's theory of gravity formulated in 1915, it is a generalisation of special relativity that includes motion in a gravitational field (Stephani, 1982). In general relativity, ordinary space and time are merged together to form a four-dimensional manifold called "space-time". Space-time is endowed with a metric that describes its geometry. In particular, the line element, ds, of this space-time is defined, in terms of the metric $g_{\mu\nu}$, as[1]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.1.1}$$

where dx^μ is an infinitesimal displacement in the μ's direction, and μ runs from 0 to 3 (time and the three spatial directions) (Carroll, 2004).

Gravity thus becomes a curvature in space-time induced by the presence of mass or energy (Carroll, 2004). The relation between this curvature and the distribution of mass present is described by the Einstein's field equations of general relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.1.2}$$

where G, c, $R_{\mu\nu}$, R and T_{ab} are the Newton's gravitational constant, the speed of light, Ricci tensor, Ricci scalar and the energy-momentum tensor, respectively.

In general relativity, material objects, acted on by gravity alone, follow paths in space-time described by the geodesic equation (Carroll, 2004)

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0, \tag{2.1.3}$$

where the metric connection, $\Gamma^\mu_{\alpha\beta}$, is defined as

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma} (\partial_\alpha (g_{\gamma\beta}) + \partial_\beta (g_{\gamma\alpha}) - \partial_\gamma (g_{\alpha\beta})), \tag{2.1.4}$$

The left hand side of Einstein's equations (equation (2.1.2)) represents the geometry of space-time expressed in terms of the metric and its derivatives

$$R_{\mu\nu} = R^\beta_{\mu\beta\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu}, \tag{2.1.5}$$

where $R^\gamma_{\mu\beta\nu}$ is the curvature tensor (or Riemann tensor) given by

[1] When an index appears twice in an expression, it is to be understood that is summed over all its possible values. For example $x^i x_i = x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3$. This is known as Einstein's summation convention.

$$R^{\gamma}_{\mu\beta\nu} = \partial_\beta \Gamma^{\gamma}_{\mu\nu} - \partial_\nu \Gamma^{\gamma}_{\mu\beta} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\gamma}_{\alpha\beta}, \tag{2.1.6}$$

The right hand side of Einstein's equations represents the distribution of mass/energy expressed by the energy-momentum tensor as we shall see in the next section.

From now on, we shall adopt the "natural units" system in which the speed of light and Newton's gravitational constant (and also Planck's constant which we shall encounter later on) are set equal to one. And so we will write Einstein's equations as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}. \tag{2.1.7}$$

We shall also work in spherical co-ordinates system in which

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, r, \theta, \phi). \tag{2.1.8}$$

2.1.1 The energy-momentum tensor.

We shall mostly work with an idealized matter distribution called the perfect fluid. A perfect fluid can be completely characterised by its *isotropic* pressure and rest-frame energy density (Carroll, 2004). The energy-momentum tensor, $T_{\mu\nu}$, of such a fluid is a second rank tensor that contains the physical information about the fluid, to wit, its energy density and pressure. It is defined as

$$T_{\mu\nu} = (\rho + 3p) u_\mu u_\nu - p g_{\mu\nu}, \tag{2.1.9}$$

where ρ, p and u_μ are the energy density[2] of the fluid in its rest-frame, the pressure in the fluid and the four-velocity of the fluid, respectively (Carroll, 2004).

In flat space-time, where the metric is diagonal and its diagonal components are constant, the continuity equation follows

$$\partial_\nu T^{\mu\nu} = 0. \tag{2.1.10}$$

This equation expresses local energy conservation and, in curved space-time, it generalises to

$$\nabla_\nu T^{\mu\nu} = 0, \tag{2.1.11}$$

where ∇_ν is the covariant derivative (D'Inverno, 1992)

$$\nabla_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^{\mu}_{\nu\gamma} T^{\gamma\nu} + \Gamma^{\nu}_{\nu\gamma} T^{\mu\gamma}.$$

2.2 Cosmology

Cosmology is the study of the over-all dynamics of the universe (D'Inverno, 1992). It emerged as a sub-discipline of physics with the advent of general relativity in the mid 1910's (D'Inverno, 1992).

[2]we shall use the word "density" and "energy density" interchangeably in this work.

Cosmology's quest is to answer such fundamental questions like how the universe came into existence, how does it evolve and what is its ultimate fate.

Although any attempt at studying the universe as a whole may seem hopeless when one thinks of all the irregularities in it, it can be made remarkably feasible by adopting a principle of simplicity known as the cosmological principle (D'Inverno, 1992).

2.2.1 The cosmological principle.

The cosmological principle states that local (small-scale) irregularities, like the difference in temperature between the centre of the sun and deep space, or the difference in density between the center of the earth and inter-planetary regions, average out on large scales (cells of side-length of 10^9 or 10^{10} light years or more) such that on these large scales, the universe becomes homogeneous and isotropic about any point in it (D'Inverno, 1992; Carroll, 2004).

More formally, we assume that there is a cosmic time, t, such that a given instant, $t =$ const, defines a three dimensional spatial sub-manifold of space-time that is homogeneous and isotropic about any point in it (D'Inverno, 1992).

There is a good amount of observational evidence that supports this principle. Observations indicate that the distribution of the galaxies is homogeneous to about 30%, observation of radio galaxies reveals an isotropy to even below 5%, and the best support of isotropy is the observation that the Cosmic Microwave Background (CMB)[3] is isotropic to a fraction of a percent (D'Inverno, 1992).

2.3 The expansion of the universe

One of the underpins of today's cosmology is the observation the universe is expanding (Liddle, 2013). This was first discovered by Edwin Hubble in 1929, when he observed that galaxies, although with some deviations, were recessing away from us at velocities proportional to their distance from our galaxy (Stephani, 1982). He found that

$$v = H_0 r, \tag{2.3.1}$$

Where v is the recession velocity of a galaxy and r is its distance from ours, and the proportionality constant, H_0, is known as Hubble's constant (Liddle, 2013). We shall discuss the presently observed value of this constant in a later section.

The expansion of the universe is not in violation of the cosmological principle. Galaxies being recessing radially away from us does not mean that we are in the privileged location at the center of the expansion. Any observer situated in any galaxy, whichever it may be, will observe that the rest of galaxies are recessing from his in the fashion described by Hubble's law (Mukhanov, 2005). The expansion is homogeneous and isotropic. In fact, we shall see later that the form of Hubble's law is a consequence of homogeneity and isotropy.

To have a better idea of how such an expansion can happen, one can think of the different points on the surface of a balloon that is being blown; for each point, the other ones seem to recess away (Liddle,

[3] see section 2.13.

2013). This analogy captures another important feature of this expansion: galaxies are moving away because it is space, which carries them, that is expanding (Liddle, 2013).

Based on isotropy and homogeneity, we assume that matter in the universe exerts an isotropic pressure[4], and hence we shall model it as a perfect fluid with energy-momentum tensor

$$T_{\mu\nu} = (\rho + 3p)u_\mu u_\nu - p g_{\mu\nu}, \tag{2.3.2}$$

where we stress that pressure and density are functions of time alone, as required by homogeneity and isotropy (D'Inverno, 1992).

2.4 Co-moving coordinates

Having established that the universe is expanding, it is useful to introduce the so-called co-moving co-ordinate system in which the co-ordinates of any given point are time-independent (Liddle, 2013). And so although the universe is expanding, the co-moving coordinates of any given point are fixed. This can be easy to understand if one thinks of the co-moving co-ordinate system as a grid which is itself expanding with the universe (see figure (2.1)).

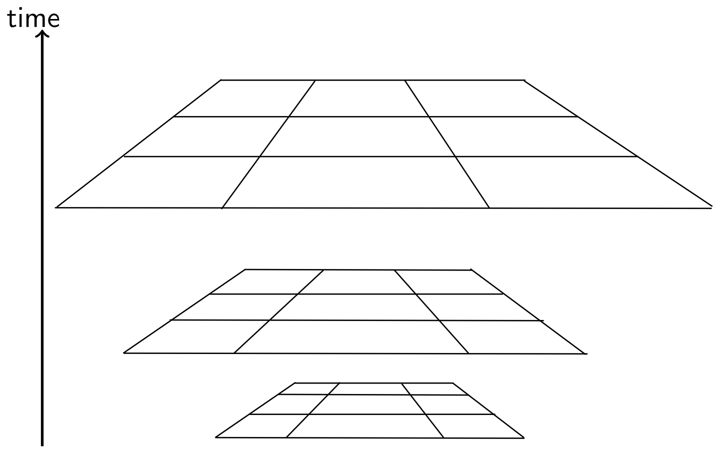

Figure 2.1: Co-moving co-ordinate system as an expanding grid (source of figure (Liddle, 2013))

Thus it follows that the distance between any two nodes (or any two points) in this co-ordinate system, referred to as the co-moving distance, is also fixed, and we shall denote it by σ. It is clear then that the co-moving distance is not necessarily equal to the actual physical distance.

Next, we wish to derive the relation between co-moving distance and physical distance. Let the co-moving distance between any two points on the grid at time $t = t_1$ be σ (see figure (2.2)).

Without loss of generality, let the co-moving distance initially coincide with the physical distance d

$$d(t_1) = \sigma. \tag{2.4.1}$$

[4] See section 2.8.

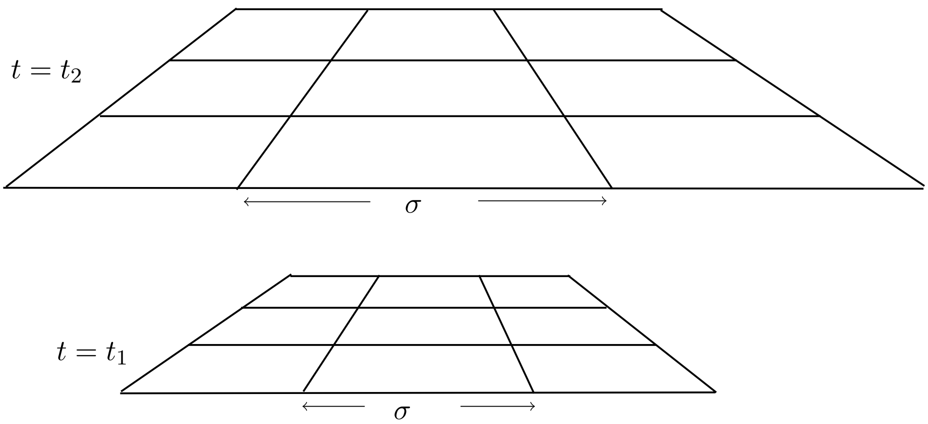

Figure 2.2: Physical distance increases, but co-moving distance remains the same

Because of expansion, at a later time instant, $t = t_2$, the physical distance will be bigger but the co-moving distance, by definition, will be the same. In order for the expansion to be isotropic and homogeneous, the physical distance must increase by a factor that is independent of the physical locations of the two points, and so the expansion factor must be a function of time alone.

$$d(t_2) = a(t_2)\sigma, \tag{2.4.2}$$

where $a(t_2)$ is the expansion factor. The quantity $a(t)$ is known as the scale factor.

2.4.1 Space of constant curvature..

For a space to be homogeneous and isotropic, each point in it must have the same curvature, i.e, it must be a space of constant curvature (D'Inverno, 1992). Such a space has a curvature tensor given by

$$R_{\mu\nu\alpha\beta} = k(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}), \tag{2.4.3}$$

where k is a constant called the curvature. This expression can be contracted as follows[5]

$$g^{\mu\alpha}R_{\mu\nu\alpha\beta} = R_{\nu\beta} = 2kg_{\nu\beta}, \tag{2.4.4}$$

that is

$$R_{\nu\beta} = 2kg_{\nu\beta}, \tag{2.4.5}$$

where $R_{\nu\beta}$ is the Ricci tensor. Also, the isotropy requirement necessitates that the space be spherically symmetric about any point in it, thus the co-moving spatial line element must take the form (D'Inverno, 1992)

$$d\sigma^2 = e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2), \tag{2.4.6}$$

where $\lambda = \lambda(r)$. The non-vanishing components of the Ricci tensor are

$$R_{11} = \frac{\lambda}{r}, \quad R_{22} = \frac{1}{\sin^2(\theta)} \times R_{33} = 1 + \frac{1}{2}re^{-\lambda}\lambda' - e^{-\lambda}, \tag{2.4.7}$$

[5]Contraction is when an index appears both as a subscript and a superscript in the same expression.

and the condition (2.4.5) reduces these two equations to

$$\frac{\lambda'}{r} = 2ke^{\lambda}, \quad 1 + \frac{1}{2}r^{-\lambda}\lambda' - e^{-\lambda} = 2kr^2, \tag{2.4.8}$$

which have a solution

$$e^{-\lambda} = 1 - kr^2. \tag{2.4.9}$$

Therefore, the spatial line element of a space of constant curvature is

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2), \tag{2.4.10}$$

and so, the four dimensional line element of such a space is

$$ds^2 = dt^2 - [a(t)]^2\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)\right). \tag{2.4.11}$$

The curvature, k, can be either positive, zero or negative, which corresponds to closed, flat or open geometries, respectively[6] (D'Inverno, 1992).

2.5 The Friedamnn Equations.

We have so far discussed the field's equations of general relativity

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \tag{2.5.1}$$

and we also explored the cosmological principle and the expansion of the universe that lead us to the Robertson-walker line element

$$ds^2 = dt^2 - [a(t)]^2\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)\right), \tag{2.5.2}$$

and we also saw that the distribution of matter in the universe resembles a perfect fluid, from which we obtained

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \tag{2.5.3}$$

and we also discussed that for every observer at rest in a galaxy, the universe appears isotropic and homogeneous, and so, for such observers, four-velocity takes the form[7]

[6] In a nutshell, a closed geometry is like the surface of a sphere: parallel lines eventually intersect each other and the angles of a triangle add up to more than $180°$, flat geometry is the ordinary Euclidean geometry: parallel lines always remain parallel and the angles of a triangle add up to exactly $180°$, and open geometry is the opposite of closed geometry: parallel lines diverge from each other and the angles of a triangle add up to less than $180°$.

[7] Although for example we do move around the sun, this additional velocity of ours is extremely small compared to the recession velocities of galaxies and so setting the spatial parts of four-velocity equal to zero is a legitimate approximation.

$$u_\mu = (1, 0, 0, 0). \tag{2.5.4}$$

These are the three underpins of modern cosmology (D'Inverno, 1992). Combining these results, the field equations lead to the following differential equations for the scale factor (Carroll, 2004)

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}}, \tag{2.5.5}$$

and

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi}{3}\left(\rho + 3p\right)}. \tag{2.5.6}$$

These two are known as the Friedmann equations, and the second one is referred to as the acceleration equation. Also, the conservation of energy equation

$$\nabla^\nu T_{\mu\nu} = 0, \tag{2.5.7}$$

gives a third equation

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p\right) = 0,} \tag{2.5.8}$$

and is known as the fluid equation.

2.6 Hubble's law

Consider any two galaxies (or any two points in space for that matter) whose physical spatial separation at time t_0 is r_0, then, at a later time t, their spatial separation, $r(t)$, is given by[8]

$$r = ar_0, \tag{2.6.1}$$

where $a = a(t)$. Since the universe is expanding, their spatial separation increases at a rate $v = \dot{r}$

$$v = \dot{r} = \dot{a}r_0, \tag{2.6.2}$$

and if we divide $(2.6.2)$ by $(2.6.1)$, we get

$$v = Hr, \tag{2.6.3}$$

where

$$H = \frac{\dot{a}}{a}, \tag{2.6.4}$$

[8]We have assumed, without loss of generality, that $a(t_0) = 1$

and so we see that Hubble's law is a consequence of the isotropy of the expansion as expressed by equation $(2.6.1)$.

We have referred to H before as Hubble's constant, but we must stress that it varies with time, as can be seen by substituting equation $(2.6.4)$ in the first Friedman equation (equation $(2.5.5)$)

$$H^2 = \frac{8\pi}{3}\rho(t) + \frac{k}{(a(t))^2}, \tag{2.6.5}$$

and so we shall use the term "Hubble's constant" to refer to the value of H at the present time which we have denoted by H_0 (Liddle, 2013).

2.7 Light Trajectory and red-shift

We have seen that space-time, when space is homogeneous and isotropic, is characterised by the Robertson-Walker line element

$$ds^2 = dt^2 - [a(t)]^2 \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \right). \tag{2.7.1}$$

In general relativity, light rays travel through paths for which $ds = 0$, and therefore for which

$$dt^2 = [a(t)]^2 \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \right). \tag{2.7.2}$$

And since space is homogeneous and isotropic, we would expected these paths to be of constant θ and ϕ (Brawer, 1995), that is, for light rays trajectories[9]

$$\frac{dt}{a(t)} = \frac{dr}{\sqrt{1 - kr^2}}. \tag{2.7.3}$$

Now, suppose that a light ray is emitted from the origin (which could be any point since space is isotropic and homogeneous (Brawer, 1995)). Let the first wave front be emitted at time t_e and received later at a co-moving distance r at time t_r, then it follows that

$$\int_{t_e}^{t_r} \frac{dt}{a(t)} = \int_0^r \frac{dr}{\sqrt{1 - kr^2}}. \tag{2.7.4}$$

Now, let the second wave front be emitted at time $t_e + dt_e$ and received at time $t_r + dt_r$. Therefore

$$\int_{t_e+dt_e}^{t_r+dt_r} \frac{dt}{a(t)} = \int_0^r \frac{dr}{\sqrt{1 - kr^2}}. \tag{2.7.5}$$

Now, since the left hand sides of equations $(2.7.4)$ and $(2.7.5)$ are the same (co-moving distances do not change), then

[9] The positive root corresponds to a recessing light ray, while the negative corresponds to an approaching light ray (D'Inverno, 1992).

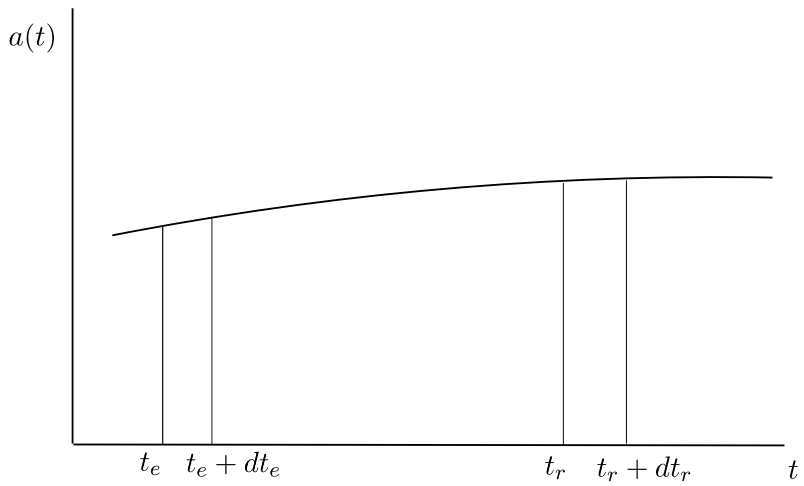

Figure 2.3: $a(t)$ as a slowly varying function of time.

$$\int_{t_e+dt_e}^{t_r+dt_r} \frac{dt}{a(t)} - \int_{t_e}^{t_r} \frac{dt}{a(t)} = 0, \tag{2.7.6}$$

and if we assume that the scale factor is a slowly varying function of time (see figure (2.3)), then

$$\int_{t_e+dt_e}^{t_r+dt_r} \frac{dt}{a(t)} - \int_{t_e}^{t_r} \frac{dt}{a(t)} = \frac{dt_r}{a(t_r)} - \frac{dt_e}{a(t_e)} = 0. \tag{2.7.7}$$

Therefore

$$\frac{dt_e}{a(t_e)} = \frac{dt_r}{a(t_r)}. \tag{2.7.8}$$

dt_e and dt_r are also related to the frequencies of the emitted and received light by

$$\nu_e = \frac{1}{dt_e}, \qquad \nu_r = \frac{1}{dt_r}, \tag{2.7.9}$$

where ν_e and ν_r are the frequencies of emitted and received light, respectively. We now define the red-shift parameter z as

$$z = \frac{\lambda_r - \lambda_e}{\lambda_e}, \tag{2.7.10}$$

where λ_e and λ_r are the wavelenghts of emitted and received light waves, respectively. And since[10]

$$\lambda = \frac{1}{\nu}, \tag{2.7.11}$$

we can write (Brawer, 1995)

[10]Remember the speed of light is set equal to 1.

$$1 + z = \frac{a(t_r)}{a(t_e)}. \tag{2.7.12}$$

2.8 The material constituents of the universe.

The material constituents of the universe can be subdivided into two groups.

• Matter: by this we shall mean all "particles" whose kinetic energies are much smaller than their masses. Examples for these would be stars and galaxies, which interact with one another mainly via gravitational attraction, and very rarely collide with each other, and hence this constituent is characterised by an internal pressure $p = 0$ (Liddle, 2013).

• Radiation: this we define as all the material objects whose kinetic energies are very large compared to their masses. From the theory of radiation, the pressure and density of this constituent are related by the equation of state (Liddle, 2013)

$$p = \frac{1}{3}\rho, \tag{2.8.1}$$

where ρ is radiation energy density.

• Dark Matter: The observed rotation of galaxies does not conform to what is predicted given the observed amounts of visible matter in them, a discrepancy known as "galaxy rotation curve anomaly". This deviation is believed to be due to the presence of an invisible type of matter in them called Dark Matter. Gravitationally, dark matter behaves like matter and radiation, i.e., it attracts. The true nature of this dark matter is as yet unknown (Liddle, 2013).

2.9 Solutions for the Freidmann equations

Having obtained a general idea about the constituents of the universe, we are now in position to obtain solutions for the Friedmann equations for these constituents.

For a universe dominated by matter (referred to as matter-dominated universe) for which $p = 0$, the fluid equation equation becomes (Liddle, 2013)

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0, \tag{2.9.1}$$

which has a solution

$$\rho \propto \frac{1}{a^3}, \tag{2.9.2}$$

and so we can see that the density of matter falls as the third power of a, which is to be expected, since a^3 corresponds to volume (Liddle, 2013).

For a universe dominated by radiation (referred to as radiation-dominated universe) for which $p = \frac{\rho}{3}$, the fluid equation becomes

$$\dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0, \tag{2.9.3}$$

and upon separating variables and integrating, we get

$$\rho \propto \frac{1}{a^4}, \tag{2.9.4}$$

and so radiation density falls as the fourth power of a. The additional power, as compared to matter density, can be understood as follows: the energy of radiation is given by Planck's formula

$$E = \nu = \frac{1}{\lambda}, \tag{2.9.5}$$

but from $(2.7.10)$ and $(2.7.12)$, we see that

$$\lambda \propto a, \tag{2.9.6}$$

and hence radiation density scales as the fourth power (Liddle, 2013).

Assuming zero curvature, the first Friedmann equation (equation $(2.5.5)$) can be solved for matter-dominated universe (equation $(2.9.2)$) and radiation-dominated universe (equation $(2.9.4)$) to give the scale factor as a function of time

$$a(t) \propto t^{\frac{2}{3}} \qquad \text{for matter-dominated}, \tag{2.9.7}$$

$$a(t) \propto t^{\frac{1}{2}} \qquad \text{for radiation-dominated}. \tag{2.9.8}$$

2.10 Observational parameters

We have thus far considered situations where the universe is matter-dominated or radiation-dominated, and of course there can well be situations whereby the universe is occupied by a mixture of the two[11]. But there is nothing in our equations that permits us to calculate the actual situation of the universe, that is, the actual magnitudes of densities of matter or radiation, the current magnitude of the scale factor or Hubble's constant for example. This is a task that is left to observation (Liddle, 2013), and, in the next few sections, we shall look at some observational parameters and their current observational status.

2.10.1 Hubble's constant H_0.

We stress again that by "Hubble's constant" we refer to the present value of the Hubble's parameter, H_0. Hubble's law reads

$$\dot{r} = Hr. \tag{2.10.1}$$

[11] The study of such situations is however beyond the scope of this essay

It turns out that, despite the simplicity of this law, there are two major difficulties involved in attempting to determine the value of H_0: the first one is that, although the velocities of galaxies can be determined to a good degree of accuracy by methods of red-shift, we must remember that the cosmological principle is a simplicity principles that applies on very large scales, and, therefore, galaxies may exhibit (and indeed they do) irregular motions in addition to their expansive motions, with velocities known as the peculiar velocities. Fortunately, these peculiar velocities become very small compared to velocities due to expansion for galaxies far away from us, that is, at distances large enough for the cosmological principle to be applicable (D'Inverno, 1992). And so the expansive velocities of such galaxies, which are large compared to the peculiar velocities, can be determined with good accuracy by methods of red-shift, and this raises the second difficulty: how to measure such great distances?. This can be done by comparing the relative luminosities of special types of stars (referred to as "standard candles") which are assumed to have the same absolute luminosity everywhere in the universe, "This is the cosmological equivalent of saying that if one light bulb looks a quarter as bright as another, then form the inverse square law is must be twice as far away - fine as long as you believe that all light bulbs have precisely the same brightness" (Liddle, 2013). The value of Hubble's constant determined by this method is[12]

$$H_0 = 100h \quad \text{km s}^{-1}\text{Mpc}^{-1}, \tag{2.10.2}$$

where $h = 0.72 \pm 0.08$ is a parameter that quantifies the uncertainty in this value (Liddle, 2013) .

Hubble's constant can be useful for giving a rough estimation of the age of the universe. If we run the clock backwards, how long will it take galaxies, given their current velocities, to run into each other?, the answer for that is

$$t = \frac{r}{v} = H_0^{-1} = 9.77h^{-1} \times 10^9 \quad \text{years}, \tag{2.10.3}$$

this is roughly the time that took the scale factor to attain its current value and hence is roughly the age of the universe (Liddle, 2013). More accurate calculations and measurements show that the age of the universe is (Ade et al., 2013)

$$t = (13.798 \pm 0.037) \times 10^9 \quad \text{years}, \tag{2.10.4}$$

or

$$t = (4.254 \pm 0.013) \times 10^{17} \quad \text{seconds}. \tag{2.10.5}$$

2.10.2 The density parameter Ω.

From the Friedmann equation

$$H^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2}, \tag{2.10.6}$$

we see that if

$$\rho = \frac{3H^2}{8\pi}, \tag{2.10.7}$$

[12] 1 Mpc $= 3.08 \times 10^{22}$ meters.

then the curvature, k, becomes zero. This value of the density is known as the critical density, and is denoted by ρ_c

$$\rho_c = \frac{3H^2}{8\pi}, \tag{2.10.8}$$

and we must stress that it is not a constant, but rather a function of time, since H varies with time (Liddle, 2013). Given the value of H_0 above, the present value of ρ_c is (Liddle, 2013).

$$\rho_c = \frac{3H_0^2}{8\pi} = 1.88h^2 \times 10^{-26} \quad \text{kg m}^{-3}. \tag{2.10.9}$$

It is useful to define the density parameter, $\Omega(t)$, as

$$\Omega(t) = \frac{\rho}{\rho_c}, \tag{2.10.10}$$

where ρ is the actual density of the universe, which is also a function of time. In terms of this parameter, the Friedman equation becomes

$$H^2 = H^2\Omega - \frac{k}{a^2}, \tag{2.10.11}$$

and by re-arranging terms

$$\Omega - 1 = \frac{k}{a^2H^2}. \tag{2.10.12}$$

From this expression, we see that

$$k = \begin{cases} \text{positive} & \text{(closed geometry)} & \text{if } \Omega > 1 \\ \text{zero} & \text{(flat Geometry)} & \text{if } \Omega = 1 \\ \text{negative} & \text{(open geometry)} & \text{if } \Omega < 1 \end{cases}$$

We will come back to the present value of Ω later on when we discuss Inflation.

2.11 The deceleration parameter q

This is a parameter that quantifies how the rate of expansion changes with time. It is defined as follows

$$q_0 = -\frac{\ddot{a}(t_0)}{a(t_0)}\frac{1}{H_0^2}. \tag{2.11.1}$$

It was a great surprise when measurements showed that the actual value of this parameter is negative, which means that the rate of expansion is actually accelerating (Liddle, 2013). This is inexplicable in terms of our models where the rate of expansion should decelerate under the action of gravity[13]. We will see in the next section how this problem can be solved by introducing the so-called cosmological constant into our equations (Liddle, 2013).

[13] see that acceleration equation (equation (2.5.6))

2.12 The Cosmological constant

We have seen that our considerations could not account for an accelerating expansion. This situation, at least on the mathematical level, can be easily remedied by introducing a constant term into the acceleration equation, so that it becomes (Liddle, 2013)

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) + \frac{\Lambda}{3}, \tag{2.12.1}$$

where Λ is known as the cosmological constant, and its division by 3 is just a matter of convenience. And so if Λ is positive and is big enough, then the right hand side of the acceleration equation can become positive, which corresponds to a negative deceleration parameter (equation $(2.11.1)$) and an accelerating expansion.

The cosmological constant is associated with the energy density of a hypothetical type of energy known as "dark Energy" that, hypothetically, pervades all space (Ryden, 2003). The discussion of the physical nature of this constant is beyond the scope of this work, however it shall prove useful to us to define a density parameter for this constant by

$$\Omega_\Lambda = \frac{\Lambda}{3H^2}, \tag{2.12.2}$$

which is a function of time. And in terms of this new parameter and the density parameter, Ω, which we defined previously, we can re-write the Friedmann equation as

$$\Omega + \Omega_\Lambda - 1 = \frac{k}{a^2H^2}, \tag{2.12.3}$$

and so, by taking the cosmological constant into account, the condition for a flat space ($k = 0$) becomes (Liddle, 2013)

$$\Omega_{tot} = \Omega + \Omega_\Lambda = 1. \tag{2.12.4}$$

2.13 Cosmic Microwave background (CMB)

A startling evidence on the isotropy of the universe was the discovery of the cosmic microwave background (CMB) in 1964. This is an electromagnetic radiation, mostly in the microwave domain, with temperature $T = 2.725 \pm 0.001$K, that showers the earth from outer-space at a constant and isotropic rate. This radiation was actually predicted by the so-called Hot Big Bang theory, according to which the universe came into existence in a "cataclysmic event" sometime in the past (Hypothetically, at time $t = 0$)[14] and has been expanding ever since. (Liddle, 2013).

[14] One should be careful when setting $t = 0$, because this corresponds to a zero scale factor and hence infinite density and, as we shall see, infinite temperature. This is known as the initial singularity, and is not fully understood (Liddle, 2013).

A detailed discussion of the Hot Big Bang theory is beyond the scope of this work, but, for reasons that will appear in the next chapter, we shall discuss briefly the origin of CMB radiation.

Since the energy density of radiation, as described by the Stephan-Boltzmann law, is related to absolute temperature T by

$$\rho \propto T^4, \tag{2.13.1}$$

and since

$$\rho \propto \frac{1}{a^4}, \tag{2.13.2}$$

we immediately obtain

$$T \propto \frac{1}{a}, \tag{2.13.3}$$

and so we see that, as we move backwards in time, the universe becomes hotter and hotter, and also denser and denser, and so what we are looking at is a very dense and hot early universe.

Another key element towards understanding the origin of CMB is the process of Nucleosynthesis (also a part of the Hot Big Bang theory) according to which the early universe was pervaded by baryons, electrons and photons, and that baryons and electrons bounded together as the universe expanded and cooled to form Hydrogen and Helium (the light elements). And so, in the circumstances of the hot and dense early universe, photons must have had very high energies and short mean-free-paths. And since Hydrogen atoms have ionization energies of 13.6 ev, these photons must have kept hydrogen ionized all the time, i.e, they must have kicked out electrons from Hydrogen atoms very frequently (Liddle, 2013).

But as the universe expanded and cooled, photons' mean-free-paths increased and their energies dropped below the ionization threshold of Hydrogen and consequently they were allowed to travel freely through space without being scattered by Hydrogen atoms. This release of photons is known as decoupling (the decoupling of radiation from matter). And as photons travelled through space, they lost even more energy because of red-shift, till they reached us in the form of microwave radiation (Liddle, 2013).

Calculations show that the temperature of this radiation when it was first released at decoupling was $T \approx 3000$ K (Liddle, 2013). This is useful to calculate the red-shift this radiation has undergone while travelling in space. If we substitute equation $(2.13.3)$ in the red-shift formula

$$1 + z = \frac{a(t_r)}{a(t_e)}, \tag{2.13.4}$$

we get

$$1 + z = \frac{T_e}{T_r}, \tag{2.13.5}$$

where T_e and T_r are the temperature of radiation when it was emitted and received, respectively. And so by plugging the numbers above, we get

$$z \approx 1080. \tag{2.13.6}$$

Since CMB was the first radiation to travel freely through space, regions it was emitted from are the farthest we can observe, since nothing can travel faster than light. These regions, which are of course isotropically distributed, constitute the what is called the surface of last scattering (where CMB photons were scattered by Hydrogen for the last time) . Decoupling is believed to have taken place $300,000$ years after the big bang (Liddle, 2013). This lead us to define the radius of observable universe, d_o, the farthest distance we can actually observe at the present. If we assume that the universe has been matter dominated since decoupling, then the radius of the observable universe is[15]

$$d_o = a(t) \int_{t_d}^{t} \frac{dt}{a(t)} = t^{\frac{2}{3}} \int_{t_d}^{t} t^{-\frac{2}{3}} dt, \tag{2.13.7}$$

where t_d is the time of decoupling and t is present time (the age of the universe), and so

$$d_o \approx 3t \sim 10^{26}\text{meters}. \tag{2.13.8}$$

An important observation about CMB is that it is not perfectly isotropic, but rather exhibits small fluctuations "about one part in one hundred thousand" (Liddle, 2013). These small fluctuations are believed to be what have led later to the formation of galaxies (Liddle, 2013). We will come back to these later on.

[15]see section 2.7.

3. The Hot Big Bang shortcomings and Inflation.

We have seen in the previous chapter how successful was the Hot Big Bang theory at predicting the origin of CMB, and also we alluded to nucleosynthesis that successfully explains the formation of light elements, which is also a part of the Hot Big Bang theory. But despite all these successes, this theory could not explain some very puzzling observations (Liddle, 2013). We shall discuss these anomalies[1] in the following sections and see how they can be solved by supplementing the hot big bang theory with Inflation (Liddle, 2013).

3.1 The flatness problem.

We have seen that the Friedmann equation can be written in the form

$$\Omega_{tot} - 1 = \frac{k}{a^2H^2}, \tag{3.1.1}$$

and so if the expansion has been decelerating because of gravity, then

$$\ddot{a} < 0 \longrightarrow \frac{d(\dot{a})}{dt} < 0 \longrightarrow \frac{d(aH)}{dt} < 0, \tag{3.1.2}$$

or more explicitly (Brawer, 1995)

$$\left|\frac{\Omega_{tot} - 1}{\Omega_{tot}}\right| \propto a^2 \quad \text{(for radiation-dominated)}, \tag{3.1.3}$$

$$\left|\frac{\Omega_{tot} - 1}{\Omega_{tot}}\right| \propto a \quad \text{(for matter-dominated)}, \tag{3.1.4}$$

i.e., provided it wasn't initially 1, Ω_{tot} would have been pushed further and further from it. However, observations suggest that $|\Omega_{tot} - 1| < 0.01$ (Spergel et al., 2006), which means that space at the present is close to being flat, and so, given the age of the universe, Ω_{tot} must have been extremely close to 1 in the early universe. In fact, detailed calculations can show that it must have been roughly (Ryden, 2003)

$$|\Omega_{tot} - 1| \leq 10^{-60}. \tag{3.1.5}$$

And so the question arises: among all the values Ω_{tot} might have had in the past, why this peculiarly close to 1 value?, even if its value has been one, then why this value?, why not any other?. The horizon problem is a need for an explanation of this finely tuned value (Liddle, 2013).

[1]They are actually three but we will only discuss two of them. The missing one is monopole particles abundance.

3.2 The horizon problem.

We have seen how cosmic microwave radiation has the same temperature regardless of which direction it is coming from. This raises a serious problem: in order for the different parts of the surface of last scattering to emit radiation at the same temperature, they must be in thermal equilibrium, which means that they must be in causal contact. But a simple calculation can show that this is not always possible. Consider for example cosmic microwave radiation arriving at the earth from a part of the surface of last scattering that is 10 billion light years away from us. This means that light has been travelling to us since 10 billion years. But there will also be light coming to us from another part which is 10 billion light years in the exact opposite direction, and so these two parts could not be in casual contact, for in this case, light must travel for 20 billion years from one of them to the other, which is a time greater than the age of the universe itself (Brawer, 1995).

There would be no problem if these parts of the universe were in contact in the past and established thermal equilibrium then, for then they would still be in thermal equilibrium even after they moved apart (Liddle, 2013). To investigate this possibility, we define a particle horizon, d_H, as the distance that light could have travelled since the beginning of time (the big bang) till a time t

$$d_H(t) = a(t) \int_0^t \frac{dt}{a(t)}. \tag{3.2.1}$$

Now, in order for these two parts to emit radiation at the same temperature, they must at least be in each others' horizon by the time of the emission, so that they exchange energy and establish thermal equilibrium (Liddle, 2013). The distance of separation between them at the time of emission, which we denote by d_{sep}, is given by[2]

$$d_{sep}(t_e) = 2a(t_e) \int_{t_e}^{t_r} \frac{dt}{a(t)}, \tag{3.2.2}$$

where t_e and t_r are the times of emission and reception, respectively. And so if $d_H(t_e) < d_{sep}(t_e)$, then they could not have been in contact (Brawer, 1995). We shall do this calculation assuming a zero curvature, because we have seen that the early universe must have been very flat, and a matter-dominated universe for which $a(t) \propto t^{\frac{1}{2}}$, so

$$\frac{d_{sep}(t_e)}{d_H(t_e)} = \left((1+z)^{\frac{1}{2}} - 1\right), \tag{3.2.3}$$

and when plugging in the value of z we calculated,[3] we get

$$\frac{d_{sep}(t_e)}{d_H(t_e)} = 2((1+1080)^{\frac{1}{2}} - 1) \approx 64 \gg 1. \tag{3.2.4}$$

[2]That is, twice the distance between any of them and what has become the location of the earth.

[3]see page 16.

3.3 Inflation

Inflation was proposed by Alan Guth in 1981 as a solution to the flatness and horizon problems[4](Ryden, 2003). It is a period characterised by an accelerating scale factor

$$\ddot{a} > 0. \tag{3.3.1}$$

How can an accelerating scale factor be brought about?. Consider what happens to the first Friedmann equation if the energy density is constant and is much larger than the curvature term

$$H^2 = \frac{8\pi}{3}\rho. \tag{3.3.2}$$

But since

$$H = \frac{\dot{a}}{a}, \tag{3.3.3}$$

therefore

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi}{3}\rho}, \tag{3.3.4}$$

from which

$$a(t) \propto \exp\left(\sqrt{\frac{8\pi\rho}{3}}\,t\right), \tag{3.3.5}$$

or

$$\boxed{a(t) \propto e^{Ht}} \tag{3.3.6}$$

i.e, the scale factor grows exponentially. And so we see that during the inflationary era, Hubble's parameter is constant, and the universe expands exponentially. According to some calculations, inflation started nearly 10^{-36} seconds after the big bang, and ended nearly 10^{-34} seconds after it, and that H was approximately 10^{36} second^{-1} (Ryden, 2003).

It is useful to define the number of e-foldings

$$N = H(t_e - t_s), \tag{3.3.7}$$

where t_s is the time when inflation starts and t_e is the time when it ends. And so during inflation the scale factor increases by a factor of e^N.

3.4 Solving the Big Bang problems

3.4.1 The flatness problem. Under the condition of an accelerating scale factor, we see that

$$\ddot{a} > 0 \longrightarrow \frac{d(\dot{a})}{dt} > 0 \longrightarrow \frac{d(aH)}{dt} > 0, \tag{3.4.1}$$

[4]As well as the monopole abundance problem which we haven't discussed.

and so the right hand side of the equation

$$\Omega_{tot} - 1 = \frac{k}{a^2 H^2}, \tag{3.4.2}$$

is always decreasing during inflation, thus driving Ω_{tot} closer and closer to 1. In fact, during a perfect exponential expansion, we can see that

$$\Omega_{tot} - 1 \propto e^{-2Ht}, \tag{3.4.3}$$

and so, given the numbers above, Ω_{tot} can be driven so extraordinarily close to 1 that all the expansion following inflation till the present is not sufficient to drive it significantly away from it (Liddle, 2013).

3.4.2 The horizon problem.

This problem can be readily solved in terms of an inflationary accelerating scale factor. Because of the rapid exponential growth of the scale factor, regions which were once in contact for long enough to establish thermal equilibrium could be blown apart to distances of separation even larger than the observable universe (Liddle, 2013). To see this, consider the size of the horizon just before inflation

$$d_H(t_s) = t_s^{\frac{1}{2}} \int_0^{t_s} t^{-\frac{1}{2}} dt. \tag{3.4.4}$$

By plugging the numbers above, we get $d_H(t_s) \sim 10^{-28}$ meters. The radius of observable universe now is

$$d_o \sim 10^{26} \text{ meters.} \tag{3.4.5}$$

At the end of inflation, the scale factor was smaller by a factor of 10^{-27}, and so by that time the observable universe had a radius d_e

$$d_e = a(t_e) d_o \sim 0.1 \text{ meters}, \tag{3.4.6}$$

and so, just before inflation, the observable universe had a radius of

$$d_s = a(t_s) d_e = e^{-100} \times 10 \sim 10^{-45} \text{ meters}, \tag{3.4.7}$$

which is much less than the horizon distance at that time given above (Ryden, 2003). It is can be easily shown that the horizon problem is resolved as long as the number of e-foldings[5] is approximately greater than 62 (Liddle, 2013).

[5]See expression $(3.3.6)$.

4. What caused inflation?

A look at equation $(3.3.5)$ suggests that inflation should last forever since ρ is a constant, but the hypothesis of inflation requires that it starts at a certain time t_s and ends at a certain time t_e. A constant ρ obviously can't bring that about. And so we impose a condition upon ρ that it is only nearly a constant during inflation, but then afterwards drops to a negligible value signalling the end of inflation. We shall model this nearly constant energy density as the energy density of a scalar field ϕ (known as inlflaton) that it is slowly varying during inflation (slow roll), and that it is big enough during that period so that it dominates the energy density of the universe then (see figure (4.1)). The action

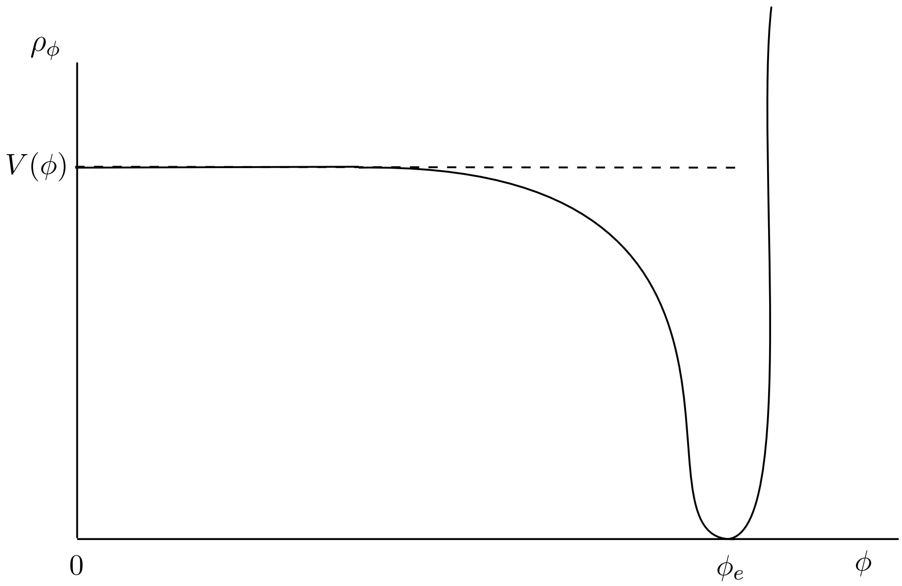

Figure 4.1: Energy density of inlflaton is nearly a constant but is dropping as ϕ "slowly rolls" towards the minima at ϕ_e (source of figure (Ryden, 2003)).

of such a field is given by[1]

$$S = \int dt d^3x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right), \tag{4.0.1}$$

where $V(\phi)$ is the potential energy of the field (Baumann). The energy density obtained from this lagrangian density is[2]

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{4.0.2}$$

and the pressure

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{4.0.3}$$

The fluid equation for this density and pressure is (Ryden, 2003)

[1]See the appendix.

[2]The energy density is homogeneous, i.e., $\phi = \phi(t)$.

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{4.0.4}$$

And if the energy density of this field dominates, then the first Freidmann equation and the acceleration equation can be written as

$$H^2 = \frac{8\pi}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{4.0.5}$$

and

$$\frac{\ddot{a}}{a} = -\frac{8\pi}{3}\left(\dot{\phi}^2 - V(\phi)\right), \tag{4.0.6}$$

respectively. We see that if ϕ is sufficiently slowly varying

$$\dot{\phi}^2 \ll V(\phi), \tag{4.0.7}$$

then, equation $(4.0.2)$ becomes

$$\rho_\phi \approx V(\phi) \approx \text{const}, \tag{4.0.8}$$

and equation $(4.0.5)$ becomes

$$H^2 \approx \frac{8\pi}{3}V(\phi) \approx \text{const}, \tag{4.0.9}$$

which corresponds to exponential expansion [3]

$$a(t) \propto \exp\left(\sqrt{\frac{8\pi V(\phi)}{3}}\, t\right). \tag{4.0.10}$$

Slow roll parameters

It is useful to define the slow roll parameter (Baumann)

$$\epsilon = \frac{8\pi}{3}\frac{\dot{\phi}^2}{H^2}, \tag{4.0.11}$$

in terms of which the acceleration equation can be written as

$$\frac{\ddot{a}}{a} = H^2(1-\epsilon), \tag{4.0.12}$$

and so the expansion accelerates as long as $\epsilon < 1$. By looking at equations $(4.0.7)$ and $(4.0.9)$, we see that during inflation

$$\epsilon \ll 1. \tag{4.0.13}$$

[3] see section 3.3

Also, by looking at $(4.0.4)$, we see that if ϕ is to remain slowly varying for a sufficiently long time then (Baumann)

$$\ddot{\phi} \ll 3H\dot{\phi}, \tag{4.0.14}$$

and so we define the second slow roll parameter

$$\eta = \frac{\ddot{\phi}}{3H\dot{\phi}}. \tag{4.0.15}$$

Therefore, during inflation, we must have

$$\eta \ll 1, \tag{4.0.16}$$

and so inflation ends when $\epsilon = 1$ (zero acceleration) and $\eta \approx 1$ (rapidly changing energy density) (Baumann).

4.1 Inflaton and the origin of structure.

We have seen how inflation can be accounted for by introducing a scalar field, inflaton, whose energy density dominates in the early universe. It turns out that this can also explain the formation of CMB anisotropies as well as the formation of small-scale structure (i.e. stars, galaxies, etc.).

We have treated ϕ so far as being perfectly homogeneous. In particular, we postulated that inflation ends at a given time instant t_e in which $\phi = \phi_e$. Quantum mechanically, however, the uncertainty principle dictates that ϕ cannot be sharply defined, but rather must have a non-zero variance, and so inflation in different points in space ends at different times (since inflaton will have different values in different points of space) (Baumann). This leads to the formation of inhomogeneities in the energy density of inflaton as well as space-time inhomogeneities[4] which ultimately results in CMB anisotropies and the formation of irregularities (structure) (Baumann).

This discussion thus permits us to express ϕ as a combination of two parts: a spatially homogeneous part $\bar{\phi}(t)$, and a fluctuating part $\delta\phi(t, x)$

$$\phi = \bar{\phi}(t) + \delta\phi(t, x). \tag{4.1.1}$$

In other words, inflaton consists of a homogeneous part $\bar{\phi}(t)$ with fluctuations, $\delta\phi(t, x)$, superimposed on it (Baumann). Our program now is to calculate these inflaton fluctuations by giving ϕ a quantum mechanical treatment.

4.1.1 Classical fluctuations.

First, we give inflaton fluctuations a classical treatment, i.e, we *postualte* their existence and then see how they evolve.

We also make some simplifying assumptions. We assume that the expansion is perfectly exponential (Baumann)

[4]Energy density fluctuations inevitably lead to space-time inhomogeneities since the two are coupled by Einstein's field equations (Baumann).

$$a(t) \propto e^{Ht}, \tag{4.1.2}$$

and we introduce conformal time τ (Liddle and Lyth, 2000)

$$\tau = -\frac{1}{Ha} \quad \longrightarrow \quad d\tau = \frac{dt}{a(t)}. \tag{4.1.3}$$

From equation (4.1.2) we see that the scale factor is zero when $t = -\infty$, in other words, the big bang now takes place at $t = -\infty$ (Baumann). This corresponds to a conformal time $\tau = -\infty$. We will also find it convenient to express inflaton fluctuations as (Baumann)

$$\delta\phi(\tau, x) = \frac{f(\tau, x)}{a(\tau)}, \tag{4.1.4}$$

and thus equation (4.1.1) takes the form

$$\phi(\tau, x) = \bar{\phi}(\tau) + \frac{f(\tau, x)}{a(\tau)}. \tag{4.1.5}$$

Now we return to our quest: calculating inflaton fluctuations. In terms of conformal time, the inflaton action takes the form

$$S = \int d\tau d^3x \left(\frac{1}{2}a^2(\phi'^2 - (\nabla\phi)^2) - a^4 V(\phi)\right), \tag{4.1.6}$$

where a prime denotes differentiation with respect to conformal time (Baumann). We can also perform the following expansions

$$\phi' = \bar{\phi}' + \frac{f'a - fa'}{a^2}, \tag{4.1.7}$$

$$V(\phi) = V(\bar{\phi}) + \frac{f}{a}V'(\bar{\phi}) + \ldots, \tag{4.1.8}$$

where $a = a(\tau)$ and $f = f(\tau, x)$. And thus we can expand equation (4.1.6) to second order in f as

$$S = \bar{S} + S^{(1)} + S^{(2)}, \tag{4.1.9}$$

where

$$\bar{S} = \int d\tau d^3x \left(\frac{1}{2}a^2(\bar{\phi}' - a^4 V(\bar{\phi}))\right), \tag{4.1.10}$$

$$S^{(1)} = \int d\tau d^3x \left(a\bar{\phi}' f' - a'\bar{\phi}' f - a^3 V'(\bar{\phi}) f\right), \tag{4.1.11}$$

$$S^{(2)} = \frac{1}{2}\int d\tau d^3x \left(f'^2 - (\nabla f)^2 - 2\mathcal{H} f f' + (\mathcal{H}^2 - a^2 V'(\bar{\phi}))f^2\right), \tag{4.1.12}$$

and $\mathcal{H} = \frac{a'}{a}$.

$\bar{S}$ is just the action of the spatially homogeneous part of the field, what we are interested in are $S^{(1)}$ and $S^{(2)}$. As for $S^{(1)}$

$$S^{(1)} = \int d\tau d^3x \left(a\bar{\phi}' f' - a'\bar{\phi}' f - a^3 V'(\bar{\phi}) f\right), \tag{4.1.13}$$

and by integrating the first term by parts and then dropping the boundary term, we find

$$S^{(1)} = -\int d\tau d^3x a \left(\bar{\phi}'' + 2H\bar{\phi}' + a^2 V'(\bar{\phi})\right) f, \tag{4.1.14}$$

which is proportional to $\bar{S}$'s equation of motion and is thus zero (Baumann). As for S^2, after integrating the f' term by parts, we get (Baumann)

$$S^{(2)} = \frac{1}{2}\int d\tau d^3x \left(f'^2 - (\nabla f)^2 + \left(\frac{a''}{a} - a^2 V''(\bar{\phi})\right) f^2\right). \tag{4.1.15}$$

Now we pause for a bit and have a look at equation $(4.0.4)$: differentiating with time respect to time t and setting $\dddot{\phi} = 0$ (slow-roll), we see that

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} + V''\dot{\phi} = 0, \tag{4.1.16}$$

and by re-arranging terms, we obtain

$$\frac{\dot{H}}{3H^2} + \frac{\ddot{\phi}}{3H\dot{\phi}} + \frac{V''}{9H^2} = 0, \tag{4.1.17}$$

however, during inflation, H is nearly a constant and hence

$$\dot{H} \approx 0 \quad \longrightarrow \quad \frac{\dot{H}}{3H^2} \ll 1, \tag{4.1.18}$$

and

$$\frac{\ddot{\phi}}{3H\dot{\phi}} = \eta \ll 1, \tag{4.1.19}$$

and so we see that

$$\frac{V''}{3H^2} \ll 1, \tag{4.1.20}$$

during inflation. This result can be useful in the following way: during inflation

$$\frac{a''}{a} \approx 2a^2H^2, \tag{4.1.21}$$

by comparing this to equation $(4.1.20)$, we see that

$$\frac{a''}{a} \gg a^2 V'', \tag{4.1.22}$$

and hence we can drop the last term in S^2 and thus we obtain (Baumann)

$$S^{(2)} \approx \int d\tau d^3x \frac{1}{2}\left((f')^2 - (\nabla f)^2 + \frac{a''}{a} f^2\right). \tag{4.1.23}$$

The equation of motion we obtain from this action is

$$\boxed{f'' - \nabla^2 f - \frac{a''}{a} f = 0}, \tag{4.1.24}$$

and is known as the Mukhanov-Sasaki equation.

To make the analysis of fluctuations easier, they can be decomposed into simpler components (fourier modes) characterised by different wave numbers k where each mode, f_k, is given by

$$f_k(\tau) = \int \frac{d^3x}{(2\pi)^{\frac{3}{2}}} f(\tau, x) e^{-ik.x}, \tag{4.1.25}$$

and so we obtain the Mukhanov-Sasaki equation for each fourier mode

$$f''_k + \left(k^2 - \frac{a''}{a}\right) f_k = 0. \tag{4.1.26}$$

And since

$$\frac{a''}{a} \approx \frac{1}{\tau^2}, \tag{4.1.27}$$

we can write

$$\boxed{f_k'' + \left(k^2 - \frac{1}{\tau^2}\right) f_k = 0}, \tag{4.1.28}$$

which the equation of motion of a harmonic oscillator with time-dependent frequency (Baumann). It has a general solution of the form

$$f_k(\tau) = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + \beta \frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right), \tag{4.1.29}$$

where α and β are constants to be fixed by initial conditions. To see how can this be done, we note that in the distant past ($\tau \longrightarrow -\infty$) all the modes had time independent frequencies (Baumann)

$$f_k'' + k^2 f_k \approx 0, \tag{4.1.30}$$

this equation has a solution

$$f_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}, \tag{4.1.31}$$

and so we require equation $(4.1.29)$ to reduce to equation $(4.1.31)$ in the limit $\tau \longrightarrow -\infty$, and hence $\alpha = 1$ and $\beta = 0$ (Baumann).

4.2 Quantum fluctuations.

Now we can proceed to quantize inflaton fluctuations. First, we promote f and its momentum conjugate π [5]

$$\pi = \frac{\partial \mathcal{L}}{\partial f'} = f', \tag{4.2.1}$$

to quantum operators. And thus they satisfy the canonical commutation relation

$$[\hat{f}(t,x), \hat{\pi}(t,y)] = i\delta(x-y), \tag{4.2.2}$$

and $f(x,\tau)$ generalises to

$$\hat{f}(t,x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \left(f_k(t)\hat{a}_k e^{ikx} + f_k^*(t) a_k^\dagger e^{-ikx} \right), \tag{4.2.3}$$

where $\hat{a}_k$ and its Hermitian conjugate $\hat{a}_k^\dagger$ are time independent operators such that

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = i\delta(k-k'), \tag{4.2.4}$$

and $f_k(t)$ and its complex conjugate $f_k^*(t)$ are two linearly independent solutions of the Mukhanov-Sasaki equation (Baumann).

And so we now calculate the variance of $f = a\delta\phi$ in the ground state of a harmonic oscillator

$$\langle 0|f^2(\tau,0)|0\rangle = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k'}{(2\pi)^{\frac{3}{2}}} \langle 0|(f_k\hat{a}_k + f_k^*\hat{a}_k^\dagger)(f_{k'}\hat{a}_{k'} + f_{k'}^*\hat{a}_{k'}^\dagger|0\rangle \tag{4.2.5}$$

$$= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k'}{(2\pi)^{\frac{3}{2}}} f_k f_{k'}^* \langle 0|[\hat{a}_k, \hat{a}_{k'}^\dagger]|0\rangle \tag{4.2.6}$$

$$= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} |f_k(\tau)|^2 \tag{4.2.7}$$

$$= \int d\ln(k) \frac{k^3}{2\pi^2} |f_k(\tau)|^2. \tag{4.2.8}$$

And here we define the dimensionless power spectrum as

$$\Delta_f^2(k,\tau) = \frac{k^3}{2\pi^2}|f_k(\tau)|^2, \tag{4.2.9}$$

and from which

$$\Delta_{\delta\phi}^2 f(k,\tau) = a^{-2}\Delta_f^2(k,\tau) = a^{-2}\frac{k^3}{2\pi^2}|f_k(\tau)|^2. \tag{4.2.10}$$

This is a measure of the power (amplitude squared) fluctuations carry (see figure 4.2).

[5] See appendix

Figure 4.2

4.3 Curvature perturbations power spectrum

We have thus far ignored the perturbations in space time, i.e, we used an unperturbed Robertson-Walker metric in our calculations. It can be shown that this is valid under certain conditions (Baumann) and that, under these conditions, inflaton fluctuations lead to curvature perturbations quantified by a power spectrum (Baumann)

$$\Delta_{\mathcal{R}}^2(k) = \frac{\Delta_{\delta\phi}^2}{2\epsilon}. \tag{4.3.1}$$

However a detailed derivation of this formula is beyond the scope of this essay.

These fluctuations in energy density and in space-time are what lead to CMB anisotropies and later to the formation of galaxies and all the irregularities in the universe, but that's a story too long for this short essay to cover, and so this is as far as we get.

5. Conclusion

In this essay, we briefly discussed general relativity, we discussed the large-scale isotropy and homogeneity of the universe (the cosmological principle), the observed expansion of the universe and saw that matter distribution in it can be modelled as a perfect fluid, and then we derived a set of equations that relate this expansion to matter distribution (the Friedmann equations). We also discussed the observational status of the rate of expansion (Hubble's parameter), the shape of the universe (the density parameter), and we saw that the expansion is accelerating (the deceleration parameter). We discussed the cosmic microwave background (CMB) phenomenon, its prediction by the hot big bang theory and its puzzling anisotropies. We then reviewed major shortcomings of the hot big bang theory (the flatness and the horizon problems), and saw how they can be solved by supplementing it with cosmological inflation. And we finished by a discussion of a possible cause of inflation (inflaton field) and saw how it can naturally lead to the formation of structure and CMB anisotropies.

For future work, we recommend a more extensive study of the nature of the cosmological constant that causes the observed acceleration, of the properties of CMB, the nature and properties of inflaton field (particularly, how it can be incorporated in the scheme of particle Physics) as well as a detailed discussion of the formation of structure and CMB anisotropies.

Appendix A. Lagrangian formalism of fields

In classical Mechanics, the dynamics of a system can be can be encoded in a Lagrangian L

$$L = L(q_1 \dots q_n, \dot{q_1} \dots \dot{q_n}), \tag{A.0.1}$$

where $q_i = q_i(t)$ is the co-ordinate of the ith particle. The action S of such a system is defined as

$$S = \int dt L, \tag{A.0.2}$$

and the equation of motion of the ith particle can be obtained by applying the principle of least action

$$\delta S = 0 \quad \longrightarrow \quad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial(\dot{q_i})}\right) = 0, \tag{A.0.3}$$

and is known as the Euler-Lagrange equation of motion (Carroll, 2004)

An analogous representation can be used to describe the dynamics of a field ϕ

$$\phi = \phi(x^\mu). \tag{A.0.4}$$

However in this case it is convenient to work with Lagrangian density $\mathcal{L}$ where

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi), \tag{A.0.5}$$

and

$$L = \int d^3x \mathcal{L}, \tag{A.0.6}$$

where d^3x is the spatial volume element. And thus action takes the form

$$S = \int dt d^3x \mathcal{L}, \tag{A.0.7}$$

and the Euler-Lagrange equation follows also from the principle of least action (Peskin and Schroeder, 1995).

$$\delta S = 0 \quad \longrightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0. \tag{A.0.8}$$

Also, the energy momentum tensor of the field is related to the lagrangian density by

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \delta^{\mu\nu} \mathcal{L}. \tag{A.0.9}$$

Another quantity of interest to us is the conjugate momentum of the field π

$$\pi = \frac{\partial \mathcal{L}}{\partial(\dot{\phi})} = \dot{\phi}, \tag{A.0.10}$$

where the last equality is for a scalar field. The reason why this quantity is of interest is because, for a quantum field, the field and its conjugate momentum are mutually exclusive, and this gives rise to uncertainties in the magnitude of the field at any given point in space-time (Peskin and Schroeder, 1995).

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