Quantum Corrections to the Kink Soliton Mass

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Abstract

We us spectral methods from quantum scattering theory to compute the vacuum polarization energy in the background of the kink soliton, which is due to shifts in the single particle energy eigenvalue. Because the energy expression shows that the energy blows up to infinity due to "ultraviolet divergence", we have use the renormalization method to absorb the divergence, that leads to finite energy.

Key words Spectral methods, Kink soliton, Renormalization.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Abdelhamid Mohamed Adam Ali, 22 May 2014

Contents

1. Introduction

The classical solution of the wave equation in the background potential, for example, ϕ^4, potential gives a stable, localized, self-supporting energy density, which is known as a solitary wave or "soliton".

The first study of the soliton was done by John Scott Russell (1834), who discovered what he called the "Wave of Translation", which became later known as solitons. The full discretion of a classical soliton solution can be found in the second chapter.

The concept of localization of the particle in space is inconsistent with the quantum mechanics result, in particular the Heisenberg's uncertainty principle. In finding the solution to this problem, the semi-classical method gives a good result. We approximate the system as a harmonic oscillator in reflectionless potential such as "Pöschl-Teller" potential.

As a result, the only type of interaction is the scattering process, that requires phase shift δ between incident and scattering wavefunctions. Since the phase shift completely depends on the momentum k, for large values of $k \longrightarrow \infty$, the calculation of the phase shift through Born's approximation, can be used to estimate the phase shift, which is very important in the calculation of finite energy eigenvalues.

Renormalization is a method used to overcome the infinity problems, which arise when calculating the quantum corrections to the Kink soliton mass.

2. The classical mass of the kink

2.1 The action

The action can be defined as follows:

$$S = \int d^4x \mathcal{L}\left(\phi, \partial_\mu \phi\right),$$

where

ϕ is the scalar field and $\mathcal{L}$ is the Lagrangian density, which is the function of the field and the field derivatives.

This is defined as follows:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi\right) - V(\phi). \tag{2.1.1}$$

The Lagrangian is invariant under the transformation $\phi \longrightarrow -\phi$, which has a "reflectional" symmetry,[1] where $V(\phi)$ is potential depending on the scalar field ϕ,

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - a^2\right)^2 = -m^2\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{\lambda a^4}{4}, \tag{2.1.2}$$

where

$$m = \sqrt{\frac{\lambda}{2}}a \tag{2.1.3}$$

is the mass. [2]

This potential has two minima when $\phi = \pm a$. This gives the potential two-fold of degeneracy known as the Mexican hat potential "double-well potential" shown in the figure below.

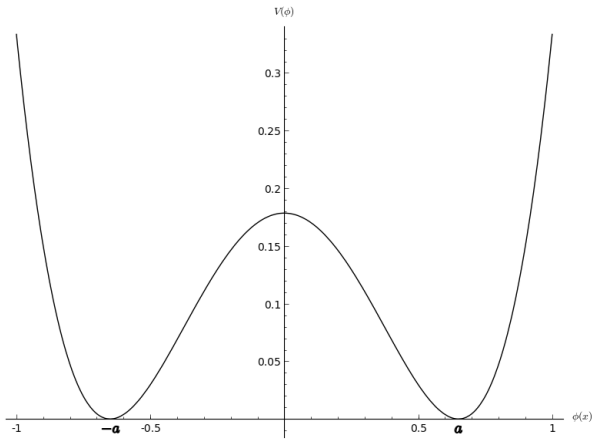

Figure 2.1: The Mexican hat potential.

For (1+1)-dimensional with $\mu = 0, 1$, then the metric $g^{\mu\nu}$ is as follows:

[1] This symmetry is spontaneously broken.

[2] In classical mechanics the coefficient of the quadratic term in the potential is the mass of the particle.

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where g^{μ} is a 2-dimension Minkowski flat matrix.

Then we can obtain $\partial_\mu \partial^\mu$ as follows:

$$\begin{aligned} \partial_\mu \partial^\mu &= \begin{pmatrix} \partial_0 & \partial_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial_0 \\ \partial_1 \end{pmatrix} \\ &= \begin{pmatrix} \partial_0 & \partial_1 \end{pmatrix} \begin{pmatrix} \partial_0 \\ -\partial_1 \end{pmatrix} \\ &= \partial_0^2 - \partial_1^2, \end{aligned}$$

where

$\partial_0 \equiv \frac{\partial}{\partial t}$, and $\partial_1 \equiv \frac{\partial}{\partial x}$.

Therefore,

$$\begin{aligned} \partial_\mu \phi \partial^\mu \phi = \partial_t \phi \partial^t \phi + \partial_x \phi \partial^x \phi \Rightarrow \partial_t \phi \partial_t \phi - \partial_x \phi \partial_x \phi \\ = (\partial_t \phi)^2 - (\partial_x \phi)^2 . \end{aligned}$$

The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} \left((\partial_t \phi)^2 - (\partial_x \phi)^2 \right) - V(\phi).$$

This Lagrangian is a Lorentz invariant, we can find the equation of the motion in the rest frame, then by Lorentz transformation we can find the solution for any other frame.

2.2 The equation of motion

The equations of motion can be derived from the action through the variation derivative of the action,[3]

$$\delta S = \int dx \left(\delta \mathcal{L}(\phi) - \frac{dV}{d\phi} \delta \phi \right) = 0,$$

then the equations of motion,

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\lambda}{4} \frac{d}{d\phi} \left(\phi^2 - a^2 \right)^2 = 0. \tag{2.2.1}$$

The time-independent solution is a static case, so $\frac{\partial 2 \phi}{\partial t^2} = 0$, then

$$- \frac{\partial^2 \phi}{\partial x^2} - \frac{\lambda}{4} \frac{d}{d\phi} \left(\phi^2 - a^2 \right)^2 = 0. \tag{2.2.2}$$

[3] Euler–Lagrange equation, Peskin and Schroeder (1995).

This second-order non-linear differential equation has two kinds of solutions, "trivial and non-trivial". If we compare equation 2.2.2 to Newton's second law of motion, we get

$$\frac{\partial^2 x}{\partial t^2} = \frac{dV(x)}{dx},$$

which describes a unit mass particle under potential $V(x)$. Equation 2.2.2 is known as the equation of motion.

The equation of motion has two kinds of solution, which will be discussed in next the section.

2.3 The Kink solution and the classical mass

The Hamiltonian density can be defined from the relation as follows:

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\dot{\phi} - \mathcal{L},$$

where $\dot{\phi} = \frac{\partial \phi}{\partial t} = 0$, thus it is a "static solution".

Therefore the static solution of equation 2.2.2 leads to stationary potential energy at $\phi(x) = \pm a$.[4] The Hamiltonian density becomes a potential energy density $\mathcal{U}$, which can be written as follows:

$$\begin{aligned} \mathcal{U} &\longrightarrow -\mathcal{L} \\ \mathcal{U} &= \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + V(\phi). \end{aligned} \tag{2.3.1}$$

Thus the energy can be obtained by integration, we obtain

$$E = \int \mathcal{U} \cdot dx \longrightarrow E = \int \left(\frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + V(\phi)\right) \cdot dx. \tag{2.3.2}$$

Now the energy depends on the values of x which are the integration limits.The energy will vanish at any point in space $\phi(x) = \pm a$, these points give the first type of solution to equation 2.2.2, which is known as the trivial solution.

Next, consider the situation in which different parts of space are represented values. For example, $\phi(x \to \pm\infty) = \pm a$. In this solutions, the function $\phi(t, x)$ has to go from $-a$ to $+a$ as x goes from $-\infty$ to $+\infty$. By continuity of the field, there must be at least one point in space, x_0, such that $\phi(x_0) = 0$. Since $V(0) \neq 0$, there is potential energy in the vicinity of x_0, and the energy of this state is not zero. The solution of the classical equation of motion that interpolates between the different boundary conditions related by transformations is called the kink (Vachaspati, 2010). Therefore,

$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &\longrightarrow 0 \quad \text{when} \quad |x| \longrightarrow \pm\infty \\ V(\phi) &\longrightarrow 0 \quad \text{when} \quad |x| \longrightarrow \pm\infty \end{aligned}\right\} \quad \text{Boundary conditions.} \tag{2.3.3}$$

[4] $\pm$ as result of translation symmetry of ϕ when $x \longrightarrow -x$.

Because the energy E is finite,

$$\left(\frac{\partial\phi}{\partial x}\right)\Big|_{x\approx\infty} = 0 \quad \text{and} \quad V(\phi)|_{x\approx\infty} = 0.$$

The argument is that from the field equation we have $\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - V(\phi)$ is constant, we will use the above equation to show that the constant is zero. Hence,

$$V(\phi) = \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2,$$

$$\frac{d\phi}{dx} = \pm\sqrt{2V(\phi)},$$

$$dx = \pm\frac{d\phi}{\sqrt{2V(\phi)}},$$

$$x - x_0 = \pm\int \frac{d\phi}{\sqrt{2V(\phi)}}, \tag{2.3.4}$$

where x_0 is a constant.

Equation 2.3.4 is a first-order differential equation and can be solved by a separation of variables and the values of ϕ can be obtained if the potential is known.

Now ϕ can be obtained from equation 2.3.4 by substituting $V(\phi)$ from equation 2.1.2 as follows:

$$x - x_0 = \pm\int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{\frac{\lambda}{2}\left(\phi^2 - a^2\right)^2}}.$$

By integration,

$$\begin{aligned} x - x_0 &= \pm\sqrt{\frac{1}{\frac{\lambda}{2}a^2}}\int_0^{\phi(x)} \frac{d\phi}{\phi^2 - a^2} \\ &= \pm\sqrt{\frac{1}{\frac{\lambda}{2}a^2}}\tanh^{-1}\left(\frac{\phi}{a}\right), \end{aligned}$$

then ϕ is,

$$\phi(x) = \pm a\tanh\left(\frac{m}{\sqrt{2}}\left(x - x_0\right)\right), \tag{2.3.5}$$

where $\phi(\infty) \pm a \quad \phi(x_0) = 0.$

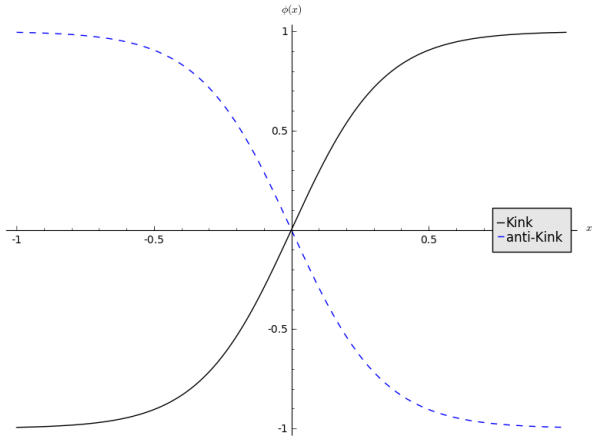

Figure 2.2: The Kink and anti-Kink as function of x.

The solution with a positive coefficient in equation 2.3.5 is called a Kink whilst the negative one is called anti-Kink.

The solution is localized at $x_0 = 0$, where ϕ cuts the x-axis and the localization will change when x_0 changes.

Now the Hamiltonian density $\mathcal{H}$ can be computed according to equation 2.3.1 as follows:

$$\mathcal{H} = \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + 2V(\phi),$$

where

$$\frac{d\phi}{dx} = \pm\sqrt{2V(\phi)}.$$

The Hamiltonian becomes

$$\mathcal{H} = \frac{a^2m^2}{2}\,\mathrm{sech}^2\left(\frac{m}{\sqrt{2}}(x - x_0)\right). \tag{2.3.6}$$

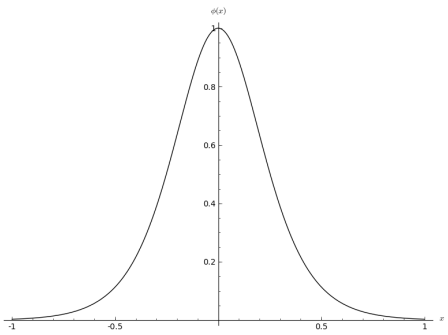

Figure 2.3: The energy density

The energy density, shown in Fig.2.3, is the same for both solutions. It is concentrated within a region around $x = x_0$ and width m^{-1}.

Then the total energy can be evaluated by

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} \mathcal{H} dx \\
&= \left(\frac{am}{2}\right)^2 \int_{-\infty}^{\infty} \operatorname{sech}^2\left(\frac{m}{2}(x - x_0)\right) dx, \\
M_{\text{cl}} &= \frac{2\sqrt{2}m^3}{3\lambda},
\end{aligned} \tag{2.3.7}
$$

where E or M_{cl} is the classical mass of the Kink.

From equation 2.3.7 we can observe that

1. Finite energy, outside the region of $x = x_0$, is equal to zero,
2. The localization of the energy in the space. leads us to consider the Kink solution as a type of particle with its mass given by equation 2.3.7
3. Self-supporting, because the Lagrangian is Lorentz invariant ,consequently, the solution is also Lorentz invariant.

   Now when applying the transform from a static frame x to a moving one $\acute{x}$, which move with velocity v, then $\phi \longrightarrow \acute{\phi}$.

   Note that $c = 1$ [5].

   We apply the Lorentz transformation on the solution x which becomes

   $$\acute{x} = \frac{x - vt}{\sqrt{1 - v^2}}.$$

   The related Kink solution $\phi(t, x)$, which is time dependent, is as follows:

   $$\phi(t, x) = \pm \tanh\left(\frac{m}{2}\left(\frac{(x - x_0) - vt}{\sqrt{1 - v^2}}\right)\right). \tag{2.3.8}$$

   As a result of the time-dependent solution, we get non-zero momentum and it is

   $$\acute{p} = \frac{2\sqrt{2}}{3} m^3 \cdot \gamma v, \tag{2.3.9}$$

   where, $\gamma = \frac{1}{\sqrt{1-v^2}}$.

   The energy can be written as follows:

   $$\acute{E} = \frac{2\sqrt{2}}{3} m^3 \cdot \gamma v. \tag{2.3.10}$$

[5] Natural units notion.

From the dispersion relation we can compute $\acute{M}$,

$$\begin{aligned}
\acute{M}^2 &= \acute{E}^2 - \acute{p}^2 \\
&= \left(\frac{2\sqrt{2}}{3}m^3\right)^2 \left(\gamma^2\left(1-v^2\right)\right) \\
&= \left(\frac{2\sqrt{2}}{3}m^3\right)^2 \left(\frac{1}{(1-v^2)}\left(1-v^2\right)\right) \\
\acute{M}^2 &= \left(\frac{2\sqrt{2}}{3}m^3\right)^2 . \qquad (2.3.11)
\end{aligned}$$

The above equation shows that the Kink solution gives the Dispersionless "Self-supporting" energy configuration.

2.4 The properties of the Knik solution

1. Boost invariant, as result of equations 2.3.9 and 2.3.9,
2. Invariant under time translation because it is time-independent, "the static situation".
3. Non-invariant under space translation, from Figure 2.3 the energy is localized around the origin; if these solutions are shifted from the origin, this creates another solution, therefore is not invariant.
4. The parity $x \longrightarrow -x$ and $x_0 \longrightarrow -x_0$,

$$\phi(-x) = -a\tanh\left(\frac{m}{2}(x-x_0)\right) \equiv \text{anti-Kink solution.}$$

3. The quantum fluctuation of the Kink

As we see in the previous chapter, that the solution of equation (2.3.5), represents a localize particles in space. Therefore it has zero momentum, this result contradict Heisenberg's uncertainty principle. The details of the problem will be discussed in this chapter, by studying fluctuation in the Kink solution around the minima of the potential as harmonic oscillator approximation.

3.1 Harmonic oscillator approximation

The static solution $x(t) = \pm a$ where $V(-a) = V(a)$, can be described by Figure (2.1). Since the potential is invariant under $x \longrightarrow -x$, the potential has two minima at a where

$$\left.\frac{\partial^2 V(x)}{\partial x^2}\right|_{\pm a} > 0.$$

Hence the solution in this classical ground state is stable.

At a, the total energy can be written as

$$E^{\mathsf{cl}}_{\mathsf{Total}} = V(a).$$

Note that this solution is static and non-consistent with the uncertainty principle because the momentum $p = 0$ and the position $x = a$ are known. Therefore the particle must have a fluctuation around $x = a$ that leads to a new ground state such that

$$E_0 = E^{\mathsf{cl}}_0 + \Delta_0 = V(a) + \Delta_0,$$

where Δ_0, is the quantum correction due to zero-point energy.

To include the quantum correction in the Kink solution around a, we can approximate the potential around a by using the harmonic potential Figure (2.1), then by using Taylor expansion around a,

$$V(x) = V(a) + \frac{1}{2}\omega^2 (x-a)^2 + \frac{1}{3!}\lambda_3 (x-a)^3 + \frac{1}{4!}\lambda_4 (x-a)^4 + \ldots,$$

where ω, λ are constants, ω give by the derivation of the potential around a. Then, from the solution of the harmonic oscillator the expectation value is

$$E_n = V(a) + \left(n + \frac{1}{2}\right)\hbar\omega,$$

where n are the occupation numbers, that are required to choose λ as sufficiently small, such that

$$\lambda_r \left\langle (x-a)^r \right\rangle < \omega^2 \left\langle (x-a)^2 \right\rangle,$$

where λ_r is anharmonics constants $r = 3, 4, 5, \ldots$.

Therefore, the energy for any states can written as

$$E_n = V(a) + \left(n + \frac{1}{2}\right)\hbar\omega + O(\lambda).$$

For the ground state,

$$E_0 = V(a) + \frac{1}{2}\hbar\omega + O(\lambda), \tag{3.1.1}$$

equation (3.1.1) describes the lower localised energy state, it can be considered as a weak coupling approximation.

3.2 The vacuum state of the Kink

Now let us study a small fluctuation in wavefunction $\phi(t,x)$, which is normally known as "excite states" $\eta(t,x)$ around the Kink solution, "the vacuum state" ϕ_{vac} then,

$$\eta(t,x) = \phi(t,x) - \phi_{\text{vac}}. \tag{3.2.1}$$

Now the Lagrangian equation (2.1.1) in terms of $\eta(t,x)$, and ϕ_{vac} is

$$\mathcal{L} = \mathcal{L}_{\text{cl}} + \mathcal{L}_{\text{quad}} + \mathcal{L}_{\text{int}},$$
$$\mathcal{L}_{\text{cl}} = -\int dx V\left(\phi_{\text{vac}}\right),$$

where $\mathcal{L}_{\text{cl}}$ is the classical vacuum, Lagrangian its energy and corresponding energy is given by (2.3.7).

The Lagrangian of interaction is given by

$$\mathcal{L}_{\text{quad}} = \int dx \left[\frac{1}{2}\left(\partial_x\right)^2 - V''\left(\phi_{\text{vac}}\right)\right].$$

$\mathcal{L}_{\text{quad}}$ contains the quadratic terms in η, and the interaction term $\mathcal{L}_{int}$, which contains terms that are cubic and higher order in η, thus can be treated as a perturbation. There are no linear terms in η because we have expanded about a minimum of V. We can diagonalize $\mathcal{L}_{\text{quad}}$ by changing the variables.(Giulini et al., 1996)

The excited states can be described by plane weave $e^{i(kx\pm\omega t)}$. At asymptotic limits, these excite states or "scattering states" can be defined by solving the equation of motion for the solutions to the fluctuation wavefunction around the Kink background.

Now the solution of $\eta(t,x)$ for scattering states can be written as follows:

$$\eta(t,x) = f(x)e^{-i\omega t},$$

where the real and the imaginary parts are in consideration.

Now, by applying the superposition principle in the eigenstates η, which is the orthonormal states, then

$$\eta = \sum_n \left[c_n(t) f_x + \text{h.c}\right].$$

Because η is a real field, we can add h.c which is the Hermitian conjugate for the complex part of η. Then, the equation of motion will be

$$-\frac{\partial^2\eta(t,x)}{\partial t^2} - \frac{\partial^2\eta(t,x)}{\partial x^2} + U(x)\eta(t,x) = 0,$$

using the fact that $\frac{\partial^2\eta(t,x)}{\partial t^2} = \omega^2\, e^{-i\omega t} f$ and dividing both sides by $e^{-i\omega t}$, and by considering, $U(x) = V'' = \frac{\partial^2 V}{\partial\phi^2}\Big|_{\phi_{\text{vac}}}$, we have

$$\omega_n^2 f = \left[-\frac{\partial^2}{\partial x^2} + \frac{\partial^2 V}{\partial\phi^2}\Big|_{\phi_{\text{vac}}}\right] f_n(x), \tag{3.2.2}$$

where $\phi_{\text{vac}} = \frac{m}{\sqrt{\lambda}}$.

Comparing equation (3.2.2) to the Schrödinger equation of a harmonic oscillator, the energy eigenvalue can be written as follows:

$$E_{N_n} = \sum_{N_n}\left(N_n + \frac{1}{2}\right)\omega_n + O(\lambda), \tag{3.2.3}$$

where N_n are occupation numbers on mode n [1] and $\omega_n = \left(k_n^2 + 2m^2\right)^{\frac{1}{2}}$; $O(\lambda)$ is some approximation related to the high-order terms greater than two in derivative of the potential.

$$E_{N_n} = \sum_{N_n}\left(N_n + \frac{1}{2}\right)\left(k_n^2 + 2m^2\right)^{\frac{1}{2}} + O(\lambda). \tag{3.2.4}$$

When $N = 0$, $n \neq 0$ and this state represents the vacuum state of the Kink,

$$E_{vac} = \frac{1}{2}\sum_n \left(k_n^2 + 2m^2\right)^{\frac{1}{2}} + O(\lambda),$$

$$E_{vac} = \frac{1}{2}\sum_n \omega_n + O(\lambda). \tag{3.2.5}$$

Equation (3.2.5) describes the vacuum state as being accessible for all normal mode n.

Equation (3.2.4) shows the momentum of the Kink in $p_n = k_n$,[2] where k is a wave vector, which is related to the occupation number through the size of the system. A (1+1) dimension problem can be solved as a particle in a box system in a one dimension, thus the wavefunction obeys the boundary conditions of a particle in a box,

$$k_n L = n\pi, \tag{3.2.6}$$

where L is box length.

[1] (Graham et al., 2009)
[2] Note that the momentum in natural unit

3.3 The Bound states in Kink solution background

The bound states of Kink and anti-Kink can be considered as a resonance phenomenon between the natural excitation frequency of the Kink (shape mode) and the frequency of oscillation of the bound Kink or anti-Kink.

The solution of the equation depends on the potential, which is invariant under $x \longrightarrow -x$, so we choose $U(x)$ to be,

$$U(x) = l\,(l+1)\,\mathrm{sech}^2\,x, \tag{3.3.1}$$

where l is a positive integer $l = 0, 1, 2, \cdots$.

This kind of potential is known as Pöschl-Teller, which has the following properties:

1. Describe the bound and scattering states,
2. Including bound states at zero energy,
3. Reflectionless scattering states.

$$\omega_n^2 f_n(x) = \left[-\frac{\partial^2}{\partial x^2} + l\,(l+1)\,\mathrm{sech}^2\left(\frac{m}{\sqrt{2}}x\right)\right] f_n(x), \tag{3.3.2}$$

where ω_n is the frequency in the state n. We are particular interested in the case $l = 2$, which parametrizes the Kink background.

Equation (3.3.2) describes two different types of solutions, "Quantum states", the discrete also known as"bound" states, and the continuum "scattering" one. The solution of equation (3.3.2) can be found in Appendix A 2, (Vachaspati, 2010) and (Morse and Feshbach, 1953).

In fact there are two bound states, for $n = 0, 1$ which is the Kink sector.

3.3.1 The Translation mode. The first bound state is when $\omega_0 = 0$, which is known as a translation mode,

$$\omega_0 = 0, \tag{3.3.3}$$

and the corresponding eigenstates are,

$$\eta_0 = \sqrt{\frac{3m}{8}}\,\mathrm{sech}^2\left(\frac{m}{\sqrt{2}}x\right). \tag{3.3.4}$$

3.3.2 The shape mode. and the 2^{nd} one has the shape mode,

$$\omega_1 = \sqrt{\frac{3}{2}}m, \tag{3.3.5}$$

with corresponding eigenstates,

$$\eta_1 = \sqrt{\frac{3m}{4}}\tanh\left(\frac{m}{\sqrt{2}}x\right)\mathrm{sech}\left(\frac{m}{\sqrt{2}}x\right). \tag{3.3.6}$$

3.4 The Scattering states

There are a continuum of states starting when $\sqrt{\frac{3}{2}}m < \omega < \infty$,

where ω can be described by k, in this case $k \geq 2m$, then,

$$\omega_k = \sqrt{\frac{k^2}{2} + m^2}. \tag{3.4.1}$$

The corresponding eigenstates are

$$\eta_{k>2}(x) = e^{ikx}\left(3m^2\tanh^2\left(\frac{m}{\sqrt{2}}x\right) - 1 - k^2 - i3mk\tanh\left(\frac{m}{\sqrt{2}}x\right)\right). \tag{3.4.2}$$

Because the potential is reflectionless the incident wave is fully transmitted, and the reflection coefficient will vanish. So, the only kind of interaction that can take place is scattering interaction, the transmitted wave undergoes phase shifting, and the states become scattering states due to the presence of potential $U(x)$.

From equation (3.4.2), the asymptotic behaviour leads to

$$\eta_k(x) = e^{ikx}\left(2m^2 - k^2 - i3km\right).$$

The phase can be computed in different ways [3]. However, we will use the simplest approach, which is the relation between the angle and the component of any complex number.

$$\tan(\delta) = \frac{\text{The imaginary part of z}}{\text{The real part of z}}.$$

Let $z \in \mathbb{C}$ then we can write $z = 2m^2 - k^2 + 3ikm$,

$$\delta(k) = -2\tan^{-1}\left(\frac{3km}{2m^2 - k^2}\right). \tag{3.4.3}$$

[3] Appendix A 2

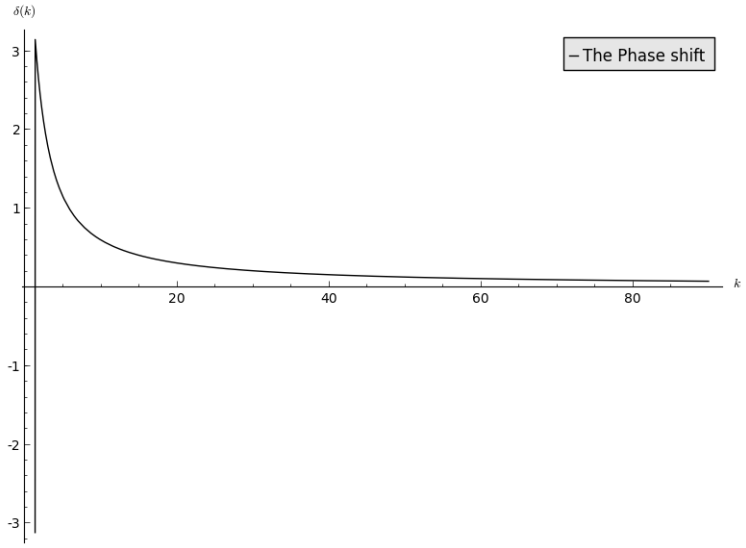

Figure 3.1: The phase shift for lager value of k

where $\delta(k)$ is the phase shift, which depends on k, not x.

Therefore the scattering wavefunction works asymptotically, when $\pm x = \pm\frac{L}{2} \longrightarrow \pm\infty$ becoming,

$$\eta_k(x) = e^{i\left(kx \pm \frac{1}{2}\delta(k)\right)}. \tag{3.4.4}$$

In fact the scattering potential (3.3.1), is also real, this implies that the imaginary part of the wavefunction is ignored.

Then, the corresponding energy eigenvalues are a compensation of the classical Kink solution and that is a quantum correction of the scattering state which can be written as follows:

$$E_{k_n} = V\left[\phi_k\right] + \sum_{k_n=0}^{\infty}\left(k_n + \frac{1}{2}\right)\omega_n + O(\lambda), \tag{3.4.5}$$

but $V\left[\phi_k\right] = \frac{2\sqrt{2}}{3}m^3$ is classical, when one considers the ground state, translation and shape modes, then

$$E_{k_n} = \frac{2\sqrt{2}}{3}m^3 + \frac{1}{2}\sum_{k=0}\left(k_n + \frac{1}{2}\right)\omega_{N_n} + O(\lambda). \tag{3.4.6}$$

For the vacuum states, the contribution of classical potential $V\left[\phi_{\text{vac}}\right] = 0$,

$$E_{\text{vac}} = \frac{1}{2}\sum_{n}\omega_n^0 + O(\lambda). \tag{3.4.7}$$

The expressions of the energy in equations (3.4.6) and (3.4.7) contain an infinite sum, which gives infinite values of energy. So these expressions are modified to overcome the infinite sum problem.

4. Quantum corrections of the Mass of The Kink Soliton

In this chapter we will compute the quantum corrections to the mass of the Kink soliton using renormalization.

4.1 Renormalization

Renormalization is a technique used to overcome the "infinity" problem arising when calculating the matrix element in quantum field theory.

This problem comes as a result of the definition of the space-time as a continuum in quantum mechanics, which explains the large error in theoretical calculations of the physical parameters, compared to the real values. For instance, in non-renormalise technique the calculation of the theoretical particle mass increase dramatically to infinity, which is contradict the real value which is finite, strictly speaking constant.

This technique was firstly used in quantum electrodynamics (QED) to make sense of infinite integrals in perturbation theory.

We will apply this technique and find the renormalised expression of mass by follow these steps:

1. Computing the bare Lagrangian, which depend on the bare mass m_0, the bare coupling constant λ_0 and ultraviolet. This step has the corresponding Feynman diagram using the regulator.
2. Computing the physical mass and coupling constant which also depend on $m_0,\ \lambda_0$.
3. Compensating the transform from $m_0 \longrightarrow m$ etc. by counterterms in the Lagrangian.

4.1.1 The continuum phase shift. Before renormlisation, the energy eigenvalues needed to compute the continuous phase shift, which make the calculation more easily.

Putting the system in a large box of size L. Then, the boundary condition is $\phi(L) = 0$. That allows to count the states via asymptotic form,

$$\phi(x) \approx \sin\left(kx + \delta(k)\right) \tag{4.1.1}$$

so that the number of states (n) is determined by

$$k_n L + \delta(k_n) = n\pi. \tag{4.1.2}$$

The change of the phase shift as function of the wave number and occupation number for difference states can be computed as follows:

Considering the next state $n+1$ (4.1.2) becomes,

$$k_{(n+1)} L + \delta(k_{n+1}) = (n+1)\pi, \tag{4.1.3}$$

then, subtracting (4.1.2) from (4.1.1) we get

$$\left(k_{(n+1)} - k_n\right) L + \delta(k_{(n+1)}) - \delta(k_{(n)}) = \pi, \tag{4.1.4}$$

defining $\Delta k = k_{(n+1)} - k_n$ and $\Delta\delta(k) = \delta(k_{(n+1)}) - \delta(k_{(n)})$. Now by dividing both sides by $\Delta k = k_{(n+1)} - k_n$ and π, we have

$$\frac{1}{\pi}\left(L + \frac{\Delta\delta(k)}{\Delta k}\right) = \frac{1}{\Delta k}, \tag{4.1.5}$$

where $\frac{1}{\Delta k}$ is the density of the state, defined as the separation between two neighbouring states. When the size of the box $L \longrightarrow \infty$, $\frac{1}{\Delta k} \longrightarrow 0$, directly. Then, (4.1.5) becomes

$$\frac{1}{\pi}\frac{d\delta(k)}{dk} = \Delta\rho(k), \tag{4.1.6}$$

where $\Delta\rho(k) = \rho(k) - \rho_0(k)$ is the change in the density of the state due to the existence of a potential, in this situation, the Pöschl-Teller potential. Here $\rho_0(k) \approx \frac{L}{\pi}$ is the density of state without interaction.

Then we can write the over all states as integration

$$\sum_n \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} dk. \tag{4.1.7}$$

4.2 Renormalization Calculation

As a result of the quantization of the Kink solution, equations (3.2.5) and (3.4.7) are included as essential problems, which can be stated briefly in two points:

1. Contain the discrete bound and continuum scattering states.
2. The sum is not finite, while the energy must be finite.

The difference between the free and interacting energy eigenstates, (where we ignore the approximate function $O(\hbar)$), is

$$\Delta E = E - E_{\text{vac}} \Rightarrow \sum_k \frac{1}{2}\omega_k - \sum_k \frac{1}{2}\omega_k^0. \tag{4.2.1}$$

Here ΔE is the energy change due to vacuum fluctuation, which is a central part of the interaction with background potential. Hence, ΔE depends on the background potential.

We convert the sum (4.2.1) into integral over momentum according to (4.1.7),

$$\Delta E = \sum_j \frac{1}{2}\omega_j + \frac{1}{\pi}\int_0^{\infty} \frac{1}{2}\left(k^2 + 2m^2\right)^{\frac{1}{2}} \frac{d\delta(k)}{dk} dk. \tag{4.2.2}$$

The integration in equation (4.2.2) is diverges,

The phase shift written in equation (3.4.3), when k is large, can be approximated to

$$\delta(k) \approx \tan^{-1}\left(\frac{3m}{k}\right), \tag{4.2.3}$$

then,

$$\frac{d\delta(k)}{dk} = \frac{6\,m}{k^2\left(\frac{9\,m^2}{k^2} + 1\right)}. \tag{4.2.4}$$

Using a binomial expansion to simplify,

$$\left(k^2+2m^2\right)^{\frac{1}{2}} \approx k\left(\frac{m^2}{k^2}+1\right)^{\frac{1}{2}}. \tag{4.2.5}$$

We recall $\omega=\sqrt{k^{2+2m^2}}$, so the integral becomes

$$\int_0^\infty \omega\frac{d\delta(k)}{dk}dk \quad = \int_0^\infty -\tfrac{6}{k^2}k\left(\tfrac{m^2}{k^2}+1\right)dk, \tag{4.2.6}$$

$$= 6\int_0^\infty \left(k+\tfrac{m^2}{k}\right)\tfrac{1}{k^2}dk, \tag{4.2.7}$$

$$= 6\int_0^\infty \tfrac{1}{k}dk + 6m^2\int_0^\infty \tfrac{1}{k^3}dk. \tag{4.2.8}$$

Since k is very large then

$$\int_0^\infty \omega\frac{d\delta(k)}{dk}dk = 6m^2\int_0^\infty \frac{1}{k}dk. \tag{4.2.9}$$

The phase shifts localized potential asymptotically behave $\frac{1}{k}$ at large k, that leads to being logarithmically divergent in the integral.

Now we have UV divergence, which is due to the short distance effects.

To cancel the divergence, we introduce regularization, which is a process of addition of a counterterm such that it has an opposite sign of the term that causes the divergence.

Here, we introduce the counterterm which is proportional to the potential $V(x)$ and necessarily is local in the Lagrangian (2.2.2) as $cV(x)$ and the coefficient c depends on the regulator. The contribution of the counterterm to the energy is $-\int_{-\infty}^{\infty} cV(x)dx$, then the divergence in the quantum calculation disappears by this contribution.

The vacuum energy has Feynman diagrams expansion under UV divergence, which we will use to regularise the above identified divergence. The phase shift formula also has expansion in powers of the potential background, which is the Born series.

The identification order of the Feynman diagrams and the Born series to phase shift is added back to Feynman diagrams.

The divergence of the Feynman diagrams will be compensated by the counterterms.

To compute the vacuum energy in this situation, for which the potential is linear, using the Born approximation,

$$\Delta E_A^1 = \int_0^\infty \omega\frac{d\delta^1(k)}{dk}dk, \tag{4.2.10}$$

where

$$\delta_{\mathrm{A}}^1(k) = -\frac{1}{k}\int_0^\infty V(x)\sin^2(kx)dx, \tag{4.2.11}$$

where δ_{A}^1 is a first Born approximation from anti-symmetric channel.

Note that there is no bound state in the Born approximation. The contribution of a symmetric channel can be obtained by replacing equation (4.2.11) $\sin^2(kx)$ by $\cos^2(kx)$.

Now equation (4.2.11) shows a kind of divergence, when k is larger then $\delta^1(k)$, so we can approximate $\sin^2(kx) = \frac{1}{2}$.

Our calculation relies on its validity as an approximation.

$$\Delta E_{\mathsf{RE}} = \Delta E - \Delta E^1 = \frac{1}{2}\sum_n \omega_n + \int_0^\infty \omega \frac{d}{dk}\left(\delta(k) - \delta^{(1)}(k)\right) dk, \tag{4.2.12}$$

where

$$\Delta E^1 = \Delta E_A^1 + \Delta E_S^1.$$

A Born approximation yields a better result at large k, so this subtraction absorbs the logarithmic divergence, which makes the integral converge.

Now we are able to integrate by parts and use Levinson's theorem;

for integration by parts,

$$\int udv = uv - \int vdu.$$

Let

$$u = \left(k^2 + m^2\right)^{\frac{1}{2}} \Rightarrow \frac{du}{dk} = \frac{k}{\omega} \quad , \omega = \left(k^2 + m^2\right)^{\frac{1}{2}},$$

and

$$dv = \frac{d}{dk}\left(\delta(k) - \delta^{(1)}(k)\right) \Rightarrow v = \left(\delta(k) - \delta^{(1)}(k)\right),$$

then,

$$\int_0^\infty \omega \frac{d}{dk}\left(\delta(k) - \delta^{(1)}(k)\right) dk = \omega\left(\delta(k) - \delta^{(1)}(k)\right)\Big|_0^\infty - \int_0^\infty \frac{k}{\omega}\left(\delta(k) - \delta^{(1)}(k)\right) dk.$$

Now recall the binomial expansion from (4.2.5) to simplify the Born series for UV divergence has no bound states. The lower integration bound can be evaluated with Levinson's theorem.

$$\Delta E_{\mathsf{RE}} = \frac{1}{2\pi}\int_0^\infty \omega \frac{d}{dk}\left(\delta(k) - \delta^{(1)}(k)\right) dk + \frac{1}{2}\sum_n m,$$

where ΔE_{RE} is renormalised energy. When we include the analogous calculation of the symmetric channel, the total quantity to be subtracted is

$$\Delta E^1 = \frac{1}{2\pi}\int_{-\infty}^\infty \omega \left(\frac{d\delta_A^{(1)}}{dk} + \frac{d\delta_S^{(1)}}{dk}\right) dk, \tag{4.2.13}$$

where $\delta_S^{(1)}$, is the symmetric channel phase shift, which is equal to

$$\delta_S^{(1)} = -\frac{1}{2k}\int_0^\infty V(x)\cos(kx)dx,$$

from the Born approximation.

Now the total contribution in the energy can be written as

$$\Delta E^{(1)} = -\frac{1}{4\pi}\int_0^\infty dk \frac{k}{\omega}\int_{-\infty}^{\infty} V(x)dk, \tag{4.2.14}$$

which is indeed just (divergent) constant times the integral over the potential. Hence, this $-\int_0^\infty V(x)dx$ divergent contribution to the energy can indeed be absorbed by a local counterterm in the Lagrangian.

The counterterm can be chosen in such a way that the vacuum polarization energy has no linear contribution in the potential,

which necessarily depend on the Kink solution vacuum polarization expression. The full bound states which contains the Kink vacuum polarization energy should be expresses in a different way,

$$\frac{1}{2}\sum_k \omega_k - \frac{1}{2}\sum_k m.$$

According to Levinson's theory, the binding energies,

$$\Delta E = \frac{1}{2}\sum_i (\omega_i - m) + \frac{1}{2\pi}\int dk\,(\omega_i - m)\frac{d}{dk}\left(\delta_S(k) - \delta_S^1(k) + \delta_A(k) - \delta_A^1(k)\right), \tag{4.2.15}$$

but,

$$\delta_S^1(k) + \delta_A^1(k) = -\frac{1}{k}\int_{-\infty}^{\infty} V(x)\left(\cos^2(kx) + \sin^2(kx)\right)dx = -\frac{1}{k}\int_{-\infty}^{\infty} V(x)dx. \tag{4.2.16}$$

The expectation value in quantum mechanics can be defined as

$$\langle V\rangle = -\int_{-\infty}^{\infty} V(x)dx = 3m^2 \tag{4.2.17}$$

Because the first-order Feynman diagram is completely local, it exactly cancels the counterterm in the no-tadpole renormalization scheme, and the renormalized vacuum polarization energy becomes (Graham et al., 2009)

$$\Delta E_{RE} = \frac{1}{2}\sum_i (\omega_i - m) - \frac{1}{2\pi}\int_0^\infty dk\,(\omega(k) - m)\left(\delta_A(k) + \delta_S(k) + \frac{\langle V\rangle}{k}\right). \tag{4.2.18}$$

$$\Delta E_{RE} = \frac{1}{2}\sum_i (\omega_i - m) - \frac{1}{2\pi}\int_0^\infty dk\frac{k}{\omega(k)}\left(\delta(k) + \frac{\langle V\rangle}{k}\right). \tag{4.2.19}$$

Note that the integration in the above equation does not diverge. Using the phase shift from (4.2.3)

$$\delta_A = \delta_S = \delta = -2\arctan\left(\frac{3k}{2-k^2}\right), \tag{4.2.20}$$

and the bound state energies (3.3.3) and (3.3.5), we can do all integrals and sums.

To compute the integral in equation (4.2.18), use the value of $\langle V \rangle$ from equation (4.2.17)

$$= \int_0^\infty \frac{-3m^2}{2\pi} \left(k^2 + m^2\right)^{\frac{-1}{2}} dk. \tag{4.2.21}$$

Now by binomial expansion, factorize

$$\left(k^2 + 2m^2\right)^{\frac{1}{2}} = m\left(1 + \frac{k^2}{m^2}\right).$$

Then substitute the factorization in (4.2.21),

$$-\frac{3m}{4\pi} \int_0^\infty \frac{dk}{\left(1 + \frac{k^2}{m^2}\right).} \tag{4.2.22}$$

The integral in equation (4.2.22) can evaluated using $\beta(m, n)$, and $\Gamma(n)$, see special functions [1], in this case we have

$$\beta(1, -0.5) = \frac{\Gamma(1)\Gamma(-0.5)}{\Gamma(+0.5)}.$$

The integral in (4.2.18) becomes

$$\frac{1}{2\pi} \int_0^\infty \frac{k}{\omega} \left(\frac{\langle V \rangle}{2k} - \frac{\langle V \rangle}{k}\right) dk = -\frac{3m}{2\pi}, \tag{4.2.23}$$

and the part becomes

$$\begin{aligned} \frac{1}{2}\sum_i^2 \sqrt{k^2 + m^2} - m =& \frac{1}{2}\sum_i^2 \sqrt{\frac{im^2}{2^2} + m^2} - m = \frac{1}{2}\left[\sqrt{\frac{m^2}{4} + 2m^2 +} - m\right] \\ =& \frac{1}{2}\left[\sqrt{\frac{m^2}{4} + 2m^2} - m\right] = \frac{3}{4}m - m = \frac{1}{4}m. \end{aligned} \tag{4.2.24}$$

The Kink mass with order quantum correction is obtained as

$$\Delta E_{RE} = \left(\frac{1}{4} - \frac{3}{2\pi}\right) m \approx 0.2274648m.$$

[1] Methods of theoretical physicsMorse and Feshbach (1953)

5. Conclusion

We have used the spectral method of scattering to compute the quantum corrections of the bosonic vacuum polarization energy in Kink background, which depends on the binding energy states. Based on this, we can compute other physical quantities such as the wavelength, the frequency etc.

For our future work we wish to verify the physical charge of the Kink existence and predict the magnitude of physical charge. To do this, we will need to modify the field in order to obey the gauge principle.

Appendix A. Derrick Theory

A.1 Derrick Theory

There are no time independent finite energy solutions in space dimension $d > 1$.

Proof.

$$E = \int d^d x \left[\frac{1}{2} (\nabla \phi)^2 + U(\phi) \right].$$

Let $\phi = \phi_G(x)$ be the solution,

$$\phi \equiv \phi_G(\alpha, x),$$

$$E_k = \int d^d x \left(\nabla \phi(x, \alpha) \right)^2.$$

Let

$$\bar{x} = \alpha x \Longrightarrow d^d = \frac{d^d \bar{x}}{\alpha^d},$$

$$\nabla \equiv \frac{d}{d\bar{x}} = \frac{d}{dx_i},$$

therefore the energy becomes

$$\bar{E} = \frac{1}{\alpha^d} \int d^d \bar{x} \alpha^2 \nabla \phi_G(\bar{x}),$$

then,

$$\bar{E} = \alpha^{2-d} E = K_F + V_F,$$

$$\frac{dE}{d\alpha} = (2-d)\alpha^{1-\alpha} K_G - dx\alpha^{-(d+1)} V = 0. \tag{A.1.1}$$

Multiply equation A.1.1 by α^{d+1},

$$\frac{dE}{d\alpha} \Longrightarrow (2-d)\alpha^2 K_G - dV_G = 0,$$

$$\alpha^2 = \frac{dV_G}{(2-d)K_G}.$$

So α can be at on of this condition:

$$d > 2 \Rightarrow 2 - d < 0 \quad \text{or} \quad d = 2\alpha \longrightarrow \infty \quad \text{or} \quad d = d \Rightarrow 0, \alpha \in \mathbb{R}$$

$\phi_G(x)$ will be a solution if,

$$\left. \frac{dE}{d\alpha} \right|_{\alpha=1} = 0,$$

and

$$\frac{d^2E}{d\alpha^2} > 0,$$

if,

1. $\alpha = 1$.

   From equation (A.1.1),

   $$\frac{dE}{d\alpha}|_{d=1} = 0 \quad \Rightarrow \quad K_G = V_G,$$

   this case gives the Kink solution.

2. When $d = 2 \Rightarrow \alpha = 1$ $V_G = 0$ this case has to be the classical vacuum.

3. When $d > 2$, equation (A.1.1) becomes the sum of two negative terms, then $\frac{dE}{d\alpha} = 0$ if and only if, both of K_G, V_G equal zero.

□

The results in 1, 2 and 3, shown there no time independent finite energy solutions in space dimension $d > 1$.

Appendix B. Reflectionless potential and phase shift

B.1 Reflectionless potential

The Pöschl-Teller potential Jaffe (2009)

$$V_\ell(x) = -\ell(\ell+1)\,\mathrm{sech}^2\,x, \tag{B.1.1}$$

where ℓ is any positive integer ($\ell = 0, 1, 2, \ldots$).

Now we substitute this potential in the Schrödinger equation to compute the "bound and scattering" states of the system by using various values of ℓ.

$$\Big[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - V_0\,\mathrm{sech}^2(x)\Big]\psi_\ell(x) = E_\ell\psi(x), \tag{B.1.2}$$

where

$$\psi_{(\ell)}(k,x) = e^{ikx}. \tag{B.1.3}$$

Define $k^2 = E$, so for $V_0(x) = \ell(\ell+1)\,\mathrm{sech}^2\,x$,

$$\mathcal{H}_\ell\;\psi(x) = \big[p^2 - \ell(\ell+1)\,\mathrm{sech}^2\,x\big]\psi(x) = k^2\psi(x), \tag{B.1.4}$$

where $p = -i\frac{\mathrm{d}}{\mathrm{d}x}$.

If $k^2 \leq 0$, then it is related to bound states, and must be normalizable wavefunctions,

or $k^2 > 0$ then it is related to scattering.

We can define the scattering as a function of an incident and scattering wavefunction.

Then the system can be defined through harmonic oscillator operators as

$$a_\ell = p - i\ell\tanh x \tag{B.1.5}$$

$$a_\ell^\dagger = p + i\ell\tanh x. \tag{B.1.6}$$

Recall the commutation relation, $[x,p] = i$, then we conclude that

$$\mathcal{A}_\ell \equiv a_\ell^\dagger a_\ell = p^2 + \ell^2 - \ell(\ell+1)\,\mathrm{sech}^2\,x, \tag{B.1.7}$$

$$\mathcal{B}_\ell \equiv a_\ell a_\ell^\dagger = p^2 + \ell^2 - \ell(\ell-1)\,\mathrm{sech}^{2x}\,. \tag{B.1.8}$$

From the relation between the operators $\mathcal{A}_\ell$, $\mathcal{B}_\ell$, and $\mathcal{H}_\ell$,

and by comparing eqs. (B.1.4) and eqs (B.1.8),

$$\mathcal{A}_\ell = \mathcal{H}_\ell + \ell^2 \tag{B.1.9}$$

$$\mathcal{B}_\ell = \mathcal{H}_{\ell-1} + \ell^2\;. \tag{B.1.10}$$

Suppose ψ is an eigenstate of $\mathcal{A}_\ell$,

$$\mathcal{A}_\ell \psi(x) = \alpha \psi(x). \tag{B.1.11}$$

When the operator $\mathcal{B}_\ell$ acts in the eigenstate $\phi(x)$, it gives α, which is same eigenvalue of $\mathcal{A}_\ell$ and we can show that as follows:

$$\begin{aligned} a_\ell \mathcal{A}_\ell \psi(x) &= \alpha a_\ell \psi(x) && \text{(B.1.12)} \\ &= \left\{ a_\ell a_\ell^\dagger \right\} a_\ell \psi(x) = \mathcal{B}_\ell \psi(x). && \text{(B.1.13)} \end{aligned}$$

The zero eigenstate has a

because $a_\ell \psi(x)_\ell = 0$. So ψ is an eigenstate of $\mathcal{A}_\ell$ with an eigenvalue $\alpha = 0$, which has no corresponding eigenstate of $\mathcal{B}_\ell$.

Eq.(B.1.10) shows the Hamiltonians, $\mathcal{H}_\ell$: $\mathcal{H}_{\ell-1}$ and $\mathcal{H}_\ell$, so $\mathcal{H}$ has a common spectrum for all eigenstates except when$\phi(x)$ is a single state.

B.2 Eigenstates and Eigenenergies

Consider the eigenstates and phase shift for several values of ℓ.

B.2.1 $\ell = 2$. The spectral property of the Hamiltonian $\mathcal{H}$ makes $\mathcal{H}_\in$ match with $\mathcal{H}_\infty$ in all excited states. The Hamiltonian $\mathcal{H}_\ell$, when $\ell = 2$,

$$\mathcal{H}_2 = p^2 - 6\,\mathrm{sech}^2\, x\ . \tag{B.2.1}$$

In addition, the energy eigenvalues corresponding to the eigenstates can be obtained when the operator $a_2^\dagger$ acting on the state $\phi_1(x)$, is

$$\begin{aligned} \psi_2^{(1)}(x) &= x a_2^\dagger \phi_1(x) && \text{(B.2.2)} \\ &\propto (p + 2i\tanh x)\,\mathrm{sech}\, x && \text{(B.2.3)} \\ &\propto \tanh x/\cosh x. && \text{(B.2.4)} \end{aligned}$$

Equation (B.2.4) shows that the first excited state wavefunction is non-invariant under a parity transformation in a one-dimensional potential.

Now the wavefunctions of scattering states are constructed by following a procedure analogous to the $\ell = 1$ case. In short,

$$\begin{aligned} \psi_2(k,x) &= x a_2^\dagger \psi_1(k) && \text{(B.2.5)} \\ &= (p + 2i\tanh x)(k + i\tanh x) e^{ikx} && \text{(B.2.6)} \\ &= (1 + k^2 + 3ik\tanh x - 3\tanh^2 x) e^{ikx}. && \text{(B.2.7)} \end{aligned}$$

The scattering and reflection process can be described as the a one-dimension according to boundary condition,

$$\lim_{x\to-\infty}\psi(k,x) = e^{ikx} + R(k)e^{-ikx} \tag{B.2.8}$$

$$\lim_{x\to\infty}\psi(k,x) = T(k)e^{ikx}, \tag{B.2.9}$$

where R, T are reflection and transmitting "scattering" coefficients respectively.However for the reflectionless process $R = 0$, therefore

$$R_2(k) = 0 \quad \text{Reflectionless process} \tag{B.2.10}$$

$$T_2(k) = \frac{(k+i)(k+2i)}{(k-i)(k-2i)} \tag{B.2.11}$$

$$= \exp 2i\big(\tan^{-1}(1/k) + \tan^{-1}(2/k)\big), \tag{B.2.12}$$

where $\tan^{-1}(1/k), \tan^{-1}(2/k)$ are the phase shift due to the potential.

we can write the general formula of the phase shift as,

$$\delta = \sum_{i=1}^{n} \tan^{-1}\left(\frac{im}{n}\right),$$

and corresponding momentum of the bound states at

$$k_n^2 = \frac{im^2}{n^2} \qquad i = 0, 1, 2, \ldots, n \tag{B.2.13}$$

Appendix C. Scattering Differentiation Equation

C.1 General solution of scattering differentiation equation

We have often encountered a differential equation of the type

$$-\frac{d^2\psi}{dx^2}+\left[\epsilon-\nu\cosh 2-\nu\sinh 2\mu\tanh x+\nu\cosh^2 x\mu\operatorname{sech}^2 x\right]\psi, \tag{C.1.1}$$

where μ,ν are parameters and ϵ is the eigenvalue. This the general way to differential equation like 3.3.2 the Schrödinger problem.

The solution can be written in terms of a, b which are parameters,

$$a=\frac{1}{2}\left(\nu e^{2\mu}-\epsilon\right)^{\frac{1}{2}}-\frac{1}{2}\left(\nu e^{-2\mu}-\epsilon\right)^{\frac{1}{2}}, \tag{C.1.2}$$

$$b=\frac{1}{2}\left(\nu e^{2\mu}-\epsilon\right)^{\frac{1}{2}}+\frac{1}{2}\left(\nu e^{-2\mu}-\epsilon\right)^{\frac{1}{2}}, \tag{C.1.3}$$

let ϕ is an eigenvector define as follows:

$$\psi=\exp\left[-ax\right]\operatorname{sech}^b xG(x),$$

the equation for G becomes,

$$G"-2\left(a+b\tanh x\right)G'+\left(\nu\cosh^2\mu-b(b+1)\right)\operatorname{sech}^2 xG=0,$$

G is a function which, subject to the derivation x.assume

$$u=\frac{1}{2}[1-\tanh x],$$

$$u(1-u)\frac{d^2G}{du^2}+[a+b+1-2(b-1)u]\frac{dG}{du}, \tag{C.1.4}$$

Equation C.1.4 is hypergeometric equation and the general solution can be written as,

$$F=AG_1+BG_2, \tag{C.1.5}$$

where A and B are constants of integration and,

$$G_1=G(\alpha,\beta,\gamma,u),$$

$$G_2=u^{1-\gamma}G(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,u),$$

where

$$\alpha=\quad b-\tfrac{1}{2}\sqrt{\nu\cosh^2\mu+\tfrac{1}{4}}, \tag{C.1.6}$$

$$\beta=\quad b+\tfrac{1}{2}\sqrt{\nu\cosh^2\mu+\tfrac{1}{4}}, \tag{C.1.7}$$

$$\gamma=\qquad a+b+1, \tag{C.1.8}$$

and, γ is assumed to not be an integer.

Now we can study the solution for extreme point is space such as $x = \pm\infty$. When $x \longrightarrow \infty$ $u = 0$ is obtained by therefore in equation C.1.5 $B = 0$. While when $x = -\infty \Longrightarrow u = 1$ in this case we required to find solution .

The eigenstates when $\mu = 0$ can be written as the following discrete values of $b > 0$

$$b_n = \sqrt{\nu + \frac{1}{4}} - \left(n + \frac{1}{2}\right), \tag{C.1.9}$$

where $n = 0, 1, 2, \ldots, N$ N with N determined by $b_{N+1} \geq 0$. The discrete eigenvalues of ϵ follow from the definition in equation C.1.3

$$\epsilon_n = (2n+1)\sqrt{\nu + \frac{1}{4}} - \left(n^2 + \frac{1}{2}\right).$$

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