A Mathematical Model of the Spread of HIV/AIDS Amongst Injecting Drug Users

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Abstract

In this project, we develop and analyse a mathematical model for the spread of HIV/AIDS amongst injecting drug users. We begin with an introduction and literature review. Following this we examine an early model due to Kaplan (1989) and explain how the model equations are derived. Rather than analyse the Kaplan Model, we instead concentrate on a more general and realistic model due to Greenhalgh and Hay (1997). The Kaplan model will be seen to be a special case of this. We calculate the equilibrium a solutions and determine their stability properties. It is shown that there is a critical threshold parameter R_0, that determines the behaviour of the system. In particular, $R_0 \leq 1$ is a condition for the disease to die out and $R_0 > 1$ is a condition for the disease to persist in a population. We evaluate the effects of control strategies, such as needle-exchange schemes and cleansing of needles.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Godfrey Edward Edward, 22 May 2014

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1. Introduction and Literature Review

The Human Immunodeficiency Virus (HIV) is a slowly replicating retrovirus (a lentvirus) that attacks the human immune system and weakens its ability to fight diseases and infections. Retroviruses are viruses that are encoded in ribonucleic acid (RNA) rather than deoxyribonucleic acid (DNA). The progressive decline in the immune system that this causes in an infected individual leads to the development of Acquired Immunodeficiency Syndrome (AIDS) and ultimately death. According to Dunne (1995), the occurrence of AIDS in humans was first recognized in 1981 among homosexual men in the United States. By 1983, HIV had been identified as the causative agent of AIDS, and by the mid 1980s it had become clear that the virus had spread, largely unnoticed, throughout most of the world.

The symptoms of HIV and AIDS vary depending on the stages of infections. The progress from HIV to AIDS involves the following stages. The acute infections stage normally occurs within 2-4 weeks of an individual becoming infected. During this stage flu-like symptoms such as fever, headache, sore throat and rashes, mouth or genital ulcers and diarrhea are common. Moreover, at this stage there is a large amount of the virus in the body, which means that there is a high probability of the infected individual spreading HIV to others. The second stage, lasting up to 10 years or longer, is the clinical latent stage; which is a period where a virus is living or developing in a person without showing symptoms but an infected individual can still spread HIV to others. The third stage is AIDS, it is a stage of HIV infection that occurs when the immune system has been severely damaged and a person becomes exposed to diseases, some symptoms develop like chronic diarrhea, skin rashes or bumps and other diseases develop like tuberculosis(TB) Osmond (1998).

Individuals living with HIV may progress through these stages at different rates depending on a number of different factors, such as age, genetic make up, health before infection, how soon after infection diagnosis took place, treatment and care after diagnosis. The time between an infection and the development of AIDS is known as the incubation period. This has been reported as being similar, not only for men and women, but also for those infected sexually or via the sharing of drug injecting equipment. As indicated above, the average incubation period from HIV infection to the development of AIDS is estimated to be 10 years. This estimation is for young infected individuals; the incubation period is shorter for infants and older adults Osmond (1998).

HIV and AIDS can be spread in a number of different ways. Examples include homosexual and heterosexual relations with an infected person, transmission via donated blood from an infected person, and transmission to a child from an infected mother. Within the drug injecting community, another common cause of the spread of HIV is through the sharing of needles, and other injection equipments. The major mode of transmission varies in different parts of the world. As pointed out in Arin et al. (2012), in North America and Western Europe HIV is mostly spread via homosexual relations and injecting drug users, while in sub-Saharan African countries, a major factor is heterosexual relations with infected individuals.

However the study on the spread of HIV/AIDS among IDUs has revealed that there is spread of HIV among injecting drug users in the following African countries; Kenya, Mauritius, Libya, Namibia, Tanzania, Malawi, South Africa, Nigeria and Botswana Arin et al. (2012).

In some parts of the world it was illegal to sell injecting equipment, for example in some places of USA, Scort et al (2002). These restrictions on obtaining injecting equipment led many individuals to share their injecting equipment. Sharing of needles occur in different ways. Some IDUs share with their close friends and do not regard this as a risk because they do not view it as sharing. The other form is a shooting gallery. A shooting gallery is a place where addicts sequentially rent the same injecting

equipment. Greenhalgh and Hay (1997) described the two types of shooting gallery as residential and non - residential shooting gallery. The residential shooting gallery comprises the non -residents, long term - residents and short term - residents sharing injecting equipment. While in a non - residential shooting gallery, addicts have to pay money or drugs to use injecting equipment. This situation leads to large groups of addicts sharing injecting equipment and hence contributing to HIV transmission.

Kaplan (1989), developed a first mathematical model for the spread of HIV and AIDS among IDUs through the sharing of needles or syringes. The model revealed that the use of bleaching needles or the distribution of clean injecting equipment could slow or even stop the spread of HIV in a shooting gallery. The model also demonstrated that a disease epidemic is prevented if the reproductive ratio is bellow one. This model is important in understanding the factors affecting the spread of HIV among IDUs and evaluating the effectiveness of the intervention measures.

Kaplan and O'Keefe (1993), later came out with a new study of evaluating the effectiveness of a needle exchange programme in New Haven and Connecticut in the USA. Their findings revealed that the pilot needle exchange program reduced the HIV incidence rate by 33%. These findings also helped to ensure the continued funding of the program and its expansion to Bridgeport and Hartford in Connecticut and further helped to change legislation in other cities like New York and Massachusetts to allow needle exchanges to operate. Before, it had been illegal to posses syringes without a medical prescription and this led IDUs to share syringes because needles were scarce and addicts feared to be arrested for possessing these illegal syringes.

It is estimated that more than 35.5 million people in the world are currently living with HIV, and amongst these are injecting drug users. Injecting drug use contributes to HIV/AIDS transmission mostly in countries of Eastern Europe and Central Asia. It has been reported that about two thirds of all new cases reported with HIV in Ukraine and the Russian Federation in recent years has been due to injecting drug use, UNAIDS (2013).

According to UNODOC (2014), needle sharing is more efficient in transmitting HIV than any other method. For example sharing needles and other injecting equipment is thought to be three times more likely to transmit HIV than sexual intercourse. Thus more effort is needed to eradicate this problem.

HIV is still spreading rapidly among injecting drug users. It is approximated that there are about 15.9 million of drug users worldwide and 3 million of them are HIV positive. The highest number of them are from Eastern Europe, East and South East Asia and Latin America and more than 40% of injecting drug users in these regions are HIV positive. Currently it is estimated that up to 10% of global HIV infections are due to injecting drug users. If we exclude sub Saharan Africa up to 30% of global infections will be due to injecting drug use, Mathers (2011).

People take drugs for different purposes that vary from person to person and from drug to drug. Drugs can be taken in different forms such as drinking, smoking and rubbing, but injection of drugs creates the greatest risk for the spread of HIV. Drug injection involve the re-use of needles which have been infected through prior use by an infected addict. Jacques Normand et al (1997), argues that drug users prefer the injecting method to any other method for taking drugs because they believe that through injection more of the drugs will reach the brain more quickly. This will bring sensations more quickly than any other method. Thus they will spend less amount of money to buy drugs but they will achieve the same effect.

The high risk of drug use to IDUs is due to the way how in which it is prepared and used. For example heroin must be dissolved in water before it can be injected into a person, it is prepared in small vessels called 'cookers'. Drugs in the form of powder are mixed with water and heated into a solution. Addicts

use the same vessels to suck drug solution into their needles through cotton which is used as a filter. Some times cotton that has been used as a filter is saved into the container for future use. If one of the needles used was infectious then a filter will be a source of HIV infections, Greenhalgh and Hay (1997).

Frequency of injecting also has an effect on HIV transmission within an injecting community. There are differences in the rate of injection between drugs. For example, for heroin users it is common to inject 1-3 times a day, while cocaine is commonly injected more than 10 times per day. So cocaine users has more chances of getting HIV as their frequency of injecting is very high.

Due to scarcity of money, sometimes drug users share drugs that have been jointly purchased. Koester and Hoofer (1994) commented on the practice among the drug users called front-loading, where drugs prepared into solution are then divided by squirting a measured portion into another syringe. This action might cause transmission of infected blood. Some drug addicts tend to draw up their own blood into the syringe and reinject it in order to get the full benefit of the drug. This will leave some infected blood in the syringe.

Several mathematical models on the spread of HIV and AIDS among injecting drug users have been developed from the original work done by Kaplan (1989). Kaplan and O'Keefe (1993) did a project to find out the impact of needle exchange by developing a syringe tracking and testing system. Their model suggested that needle exchange reduced the HIV by 33%, Greenhalgh et al. (2009) discussed an improved optimistic three-stage model for the spread of HIV among injecting drug users. From all models it was revealed that there is a threshold effect that determines the behaviour of the model which is called the reproductive number (R_0). If $R_0 < 1$ the means the disease dies out in a population and if $R_0 > 1$, the disease persists in a population.

In this project, we will examine a mathematical model to understand and analyse the spread of both HIV and AIDS through the sharing of syringes. We will describe the original model due to Kaplan (1989). We will also derive and discuss the more realistic model produced by Greenhalgh and Hay (1997) and analyse the behaviour of solutions to their model equations. Also we will evaluate the effect of control strategies such as needle cleaning, needle exchange schemes, and health education of using injecting equipment.

2. Derivation of Kaplan and Greenhalgh model

In this chapter we discuss two mathematical models for the spread of HIV and AIDS among injecting drug users. We examine two systems of ordinary differential equations (ODEs) that have been proposed as models of the spread of HIV and AIDS in a community of Injecting Drug Users (IDUs). We derive the first model according to Kaplan (1989), which enable us to go onto the second model that was developed by Greenhalgh and Hay (1997) and calculate its equilibrium states.

2.1 The Kaplan Model

We begin this chapter by describing a model developed by Kaplan (1989) and then used subsequently by Kaplan and O'Keefe (1993) in connection with a needle exchange program that took place in New Haven, Connecticut in 1990. The purpose of this model was to obtain a mathematical description of the transmission of HIV due to the sharing of drug injecting equipments, such as needles and syringes, in "shooting galleries", where a shooting gallery is the name given to a place where IDUs meet to inject drugs and share equipments.

The paper by Kaplan was the first in which an attempt was made to model the spread of HIV caused by the sharing of drug injecting equipments, and one of the main assumptions of the model is that sharing of needles and syringes can only take place through attendance at a shooting gallery. Kaplan made various simplifying assumptions that led to a model in the form of an autonomous system of ODEs that can be analysed using techniques from ODE theory. In particular, the number of IDUs and shooting galleries are both taken to be constant. Note that the assumption that the IDUs population size remains constant means that any infected addict who leave the population, (due, for example, to death, the onset of AIDS, treatment with methadone or migration), is immediately replaced by a susceptible addict. In other words, the removal and recruitment rates are assumed to be the equal.

Moreover, the only way that an addict can become infected with HIV is through needle sharing. Each addict selects a shooting gallery at random and during the visit to the gallery will inject only once. If the addict using the injection equipment is infected, then the equipment becomes infectious. An uninfected addict using an infectious equipment may, through the act of injecting, cleanse or flush the equipment but will then be considered to be exposed to HIV.

Based on these assumptions, Kaplan introduced the following model variables and parameters.

(i) There are n IDUs, where n is large, and m shooting galleries (this is equivalent to the assumption that there are m drug injection kits in circulation).

(ii) The fractions of infected addicts and infectious syringes at time t are represented by $\pi(t)$ and $\beta(t)$ respectively.

(iii) Addicts visit shooting galleries in accordance with a Poison process, with rate parameter λ, independently of the actions of other addicts.

(iv) There is a constant probability of θ that a susceptible (uninfected) addict using infectious equipment will flush the equipment.

(v) There is a constant probability of α that a susceptible addict becomes infected through the use of an infectious equipment.

Using these assumptions, and the variables and parameters introduced in the above list, Kaplan derived the corresponding model equations by arguing as follows. Consider initially the evolution in time of $\beta(t)$. Recall that an addict chooses a shooting gallery at random. As there are n addicts who inject in a shooting gallery at a rate λ and there are m galleries, we set

$$\gamma \qquad = \frac{n}{m} = \frac{\text{Number of addicts}}{\text{Number of shooting gallery}} = \text{gallery ratio.}$$

Then, at any particular shooting gallery, the arrival rate (or injection rate) is $\lambda\gamma$.

Since the fraction of infected addicts at time t is $\pi(t)$, then the probability of an infected addict arriving at a particular shooting gallery during the interval $[t, t+\Delta t)$ is given by $\lambda\gamma\pi(t)\Delta t+o(\Delta t)$. Consequently, the increase in infectious equipment at time $t + \Delta t$ arising from non-infectious equipment becoming infectious during $[t, t + \Delta t)$ is

$$\lambda\gamma\pi(t)(1 - \beta(t))\Delta t + o(\Delta t).$$

If we now consider the case when the equipment at a particular gallery is infectious at time t, then there are the following three possibilities during the interval $[t, t+\Delta t)$ for it to remain infectious at time $t + \Delta t$.

(i) No addict uses the equipment during the time $(t, t + \Delta t)$;

this possibility occurs with a probability of $1 - \lambda\gamma\Delta t + o(\Delta t)$.

(ii) An infected addict uses equipment with a probability of $\lambda\gamma\pi(t)\Delta t + o(\Delta t)$.

(iii) An uninfected addict uses the equipment but does not flush the equipment with a probability of $\lambda\gamma(1 - \pi(t))(1 - \theta)\Delta t + o(\Delta t)$.

Thus

$$\begin{aligned}\beta(t + \Delta t) &= (1 - \beta(t))\lambda\gamma\pi(t)\Delta t + \beta(t)(1 - \lambda\gamma\Delta t) + \lambda\gamma\pi(t)\beta(t)\Delta t + \lambda\gamma(1 - \pi(t)(1 - \theta)\Delta t\beta(t) + o(\Delta t)\\ &= \Big(\lambda\gamma\pi(t) - \lambda\gamma\pi(t)\beta(t) + \beta(t) - \lambda\gamma\beta(t) + \lambda\gamma\pi(t)\beta(t) + \lambda\gamma\beta(t)\left((1 - \pi(t))(1 - \theta)\right)\Big)\Delta t + o(\Delta t).\end{aligned}$$

Now we subtract $\beta(t)$ and divide by Δt on both sides and let $\Delta t \to 0$ to obtain

$$\lim_{\Delta t\to 0}\left[\frac{\beta(t + \Delta t) - \beta t}{\Delta t}\right] = \lambda\gamma\pi(t) - \lambda\gamma\beta(t)(1 - (1 - \pi(t))(1 - \theta)).$$

Thus

$$\frac{d\beta}{dt} = \lambda\gamma\pi(t) - \lambda\gamma\beta(t)(1 - (1 - \pi(t))(1 - \theta)). \tag{2.1.1}$$

We now consider the evolution of $\pi(t)$.

Let $I(t) = n\pi(t)$ be the number of infected addicts at time t. Then, the number of new infections that occur during the time interval $[t, t + \Delta t)$ is approximately equal to,

$$[n - I(t)]\lambda\beta(t)\alpha\Delta t.$$

This means that each of the $(n - I(t))$ uninfected addicts will visit a particular gallery with a probability $\lambda\Delta t$, will become exposed with a probability of $\beta(t)$ and infected with a probability of α.

Approximately $I(t)\mu\Delta t$ of infected addicts leave the population during the period $[t, t + \Delta t)$ (due to death, AIDS, treatment by Methadone).

Thus the number of infected addicts in the population at time $t + \Delta t$ is given by:

$$I(t + \Delta t) = I(t) + [(n - I(t))\lambda\beta(t)\alpha - I(t)\mu]\Delta t,$$

and, since $I(t) = n\pi(t)$, this leads to;

$$n\pi(t + \Delta t) = n\pi(t) + n[(1 - \pi)\lambda\beta(t)\alpha - I(t)\mu]\Delta t. \tag{2.1.2}$$

Subtracting $n\pi(t)$ from each side of the equation (2.1.2), dividing throughout by $n\Delta t$ and letting Δt tend to zero, we obtain the following,

$$\frac{d\pi(t)}{dt} = \lim_{\Delta t \to 0}\left[\frac{\pi(t + \Delta t) - \pi, (t)}{\Delta t}\right] = (1 - \pi)\lambda\beta(t)\alpha - \pi(t)\mu, \tag{2.1.3}$$

Therefore the equations,

$$\frac{d\beta}{dt} = \lambda\gamma\pi(t) - \lambda\gamma\beta(t)(1 - (1 - \pi(t))(1 - \theta)) = f_1(\beta, \pi), \tag{2.1.4}$$

$$\frac{d\pi}{dt} = (1 - \pi)\lambda\beta(t)\alpha - \pi(t)\mu = f_2(\beta, \pi), \tag{2.1.5}$$

represent the Kaplan model for the spread of HIV among injecting drug users. Equation (2.1.4) shows the rate of change of the proportion of infectious needles in the needles population and hence also describes the rate of change in the probability of an addict using infectious equipment. While equation (2.1.5) describes the rate of change of the HIV infected fraction of the addict population.

The basic reproductive number(Ratio) R_0.

The basic reproductive number R_0 is defined as the expected number of secondary infections caused by a single infected individual or needle entering a disease free population. If $R_0 < 1$ we expect each of the infected individuals(needles) to produce less than one infected individual in the entire period of infection. When $R_0 > 1$, it means each of the infected individuals or needles introduced in the susceptible population will produce more than one infected individual.

We can find R_0 from the equations (2.1.4) and (2.1.5) developed above, by considering the steady states for HIV infections. At the steady state, $\beta(t)$ and $\pi(t)$ remain constant at their saturation values of β^* and π^* respectively.

From equation (2.1.4) and (2.1.5), we can write

$$f_1(\beta^*, \pi^*) = \lambda\gamma\pi^* - \lambda\gamma\beta^*(1 - (1 - \pi^*)(1 - \theta)) = 0, \tag{2.1.6}$$

$$f_2(\beta^*, \pi^*) = (1 - \pi^*)\lambda\beta^*\alpha^* - \pi^*\mu = 0. \tag{2.1.7}$$

From (2.1.6) we deduce that

$$\beta^* = \frac{\pi^*}{\theta + (1 - \theta)\pi^*}. \tag{2.1.8}$$

Substituting equation (2.1.8) into equation (2.1.7) we obtain,

$$\pi^* = \frac{\lambda\alpha - \mu\theta}{\lambda\alpha + \mu(1-\theta)}. \tag{2.1.9}$$

Since epidemics can sustain when $\pi^* > 0$, then equation (2.1.9) is valid when $\lambda\alpha > \mu\theta$.

So the system has two equilibrium points as $(\beta^*, \pi^*) = (0, 0)$ this is the equilibrium point where the disease dies out and $(\beta^*, \pi^*) = \left(\frac{\pi^*}{\theta + (1-\theta)\pi^*}, \frac{\lambda\alpha - \mu\theta}{\lambda\alpha + \mu(1-\theta)}\right)$, is the equilibrium point where the disease persists in the population.

Note that the saturation prevalence of π^* increases with λ and α, and decreases with μ and θ.

From equation (2.1.9), Kaplan defined the reproductive number R_0 as,

$$R_0 = \frac{\lambda\alpha}{\mu\theta}. \tag{2.1.10}$$

Equation (2.1.10) can be interpreted as follows. The uninfected population will visit a shooting gallery with a rate of λ over a time span of $\frac{1}{\mu}$. Infected addicts infect $\frac{\lambda}{\mu}$ kits of injection equipments. On average $\frac{1}{\theta}$ uninfected addicts will be exposed to each of these infected kits. Each exposure leads to a new HIV infection with a probability of α.

Kaplan showed that R_0 has a biological meaning. He pointed out that a single infected addict can produce approximately R_0 new infections in a population of susceptible addicts. Thus an epidemic can only occur if the initially infected addicts can produce more than one secondary infection, that is if $R_0 > 1$ and for $R_0 \leq 1$ we expect the disease to die out. Therefore R_0 is a key parameter that determines the behaviour of the system.

The Kaplan model discussed in this section revealed that interventions such as breaching needles or distributing clean injecting equipments will slow or stop the spread of HIV in shooting galleries: Also Kaplan and O'Keeffe in their findings of effectiveness of a pilot needle exchange in New Haven, Connecticut and USA, revealed that the long term prevalence of HIV in addicts could be reduced by up to 33% by applying needle exchange. These results supported the continued funding of the programme and its expansion to other cities like Bridgeport and Hartford. Furthermore it facilitated the change of Connecticut legislation to allow needle exchange to take place.

2.2 The Greenhalgh and Hay Model

In 1997, Greenhalgh and Hay modified some of the assumptions made in the Kaplan model and then proceeded to develop a more realistic model.

They introduced some changes to allow addicts with different HIV status to attend the shooting galleries at different rates, while in the Kaplan model, all addicts attended at the same rate irrespective of their status. The arrival rate at the shooting gallery of those who do not know their HIV status will be higher than those who know that they are infected with HIV, since those who tested positive might have received some counselling so they might either stop visiting galleries altogether, or reduce their visiting rate.

Greenhalgh and Hay also introduced different transmission probabilities for flushed and unflushed syringes. They considered the possibility of an infectious needle to be flushed or cleared of infectious blood during the injection process and thus ending up as non-infectious.

In Kaplan's Model it was assumed that susceptible addicts using infectious needle may leave a needle uninfected with a fraction of mθ. This situation may not always be true because contamination may also depend on the amount of blood present in the needle and the amount of virus contained in the blood.

Model Parameters

(i) 'p' is the proportion of infected addicts who know that they have been infected. These addicts visit the shooting gallery at a lower rate of λ_2. Susceptible addicts and those addicts who do not know if they have been infected visit a shooting gallery at a higher rate of λ_1.

(ii) ξ is the fraction of all addict who bleach their needles after use.

(iii) α is the fraction of HIV susceptible addicts infected due to the use of contaminated needles.

(iv) Thus if a susceptible addict uses an infected needle, then:

P_1 is the probability that the needle is flushed and the addict is infected;

P_2 is the probability that the needle is flushed and the addict remains uninfected;

P_3 is the probability that addicts become infected without the needle being flushed;

P_4 is the probability that addicts remain uninfected and the needle is not flushed.

P_1, P_2, P_3, P_4 are probabilities assumed to be positive and so we have

$$P_1 + P_2 + P_3 + P_4 = 1. \tag{2.2.1}$$

Finally, Greenhalgh and Hay extended the Kaplan model to include the chance whereby infected addicts may not always leave the needle contaminated even before the needle has been cleaned.

So if an infected addict uses;

(a) a clean needle, then the needle will be infected with a probability of $1 - \phi_1$,

(b) an infected needle, the injection process will flush the needle with a probability of θ_1.

The quantities ϕ_1 and θ_1 are related to the flushing chance θ introduced by Kaplan. Each of these parameters represent the fraction of HIV - infected addicts using clean needles, infected addicts using contaminated needles and HIV - susceptible addicts using infected needles. This model still contains some of the assumptions made by Kaplan. For example we still assume the population of drug users to be homogeneous and addicts choose shooting galleries at random.

Model formulation

By using the assumptions and parameters discussed above, Greenhalgh and Hay arrived at a more realistic model as follows.

Let $\beta(t)$ and $\pi(t)$ be the proportions of infected needles and addicts respectively. If there are $i(t)$ infected needles and $I(t)$ infected addicts at time t, consider the number of contaminated needles at time $t + \Delta t$.

$n(1-p\pi(t))$ addicts visit shooting galleries at a rate of λ_1 and n$\pi(t)p$ addicts visit at a rate of λ_2. Each addict will choose one of the m shooting galleries at random.

So the arrival rate at a particular shooting gallery is $[\lambda_1(1-p\pi(t))+\lambda_2 p\pi(t)]\gamma$, where $\gamma = \dfrac{n}{m}$, as in the Kaplan model.

Thus the number of infected needles that are not visited by any addicts in the interval $[t, t+\Delta t)$ is

$$[1-(\lambda_1(1-p\pi(t))+\lambda_2 p\pi(t))\gamma\Delta t]\, i(t) + o(\Delta t).$$

The arrival rate of contaminated addicts at a particular shooting gallery is $[\lambda_1(1-p)+\lambda_2 p]\pi(t)\gamma$.

If a contaminated addict uses the equipment, the equipment will be infectious before cleaning by a fraction of $(1-\beta)(1-\phi_1)+\beta(1-\theta_1)$.

So the number of needles left infected after being used by infected individuals during the time interval $[t, t+\Delta t)$ is

$$m\left(\lambda_1(1-p\pi(t))+\lambda_2 p\pi(t)\right)\pi(t)\gamma(1-\xi)\left(1-\beta(t)\theta_1-(1-\beta(t))\phi_1\right)\Delta t + o(\Delta t).$$

Susceptible addicts visit a shooting gallery at a rate of $\lambda_1\gamma(1-\pi(t))$. If a susceptible addict uses infectious equipment, then the equipment is left infectious after cleaning with a fraction of $(1-P_1-P_2)(1-\xi)$ of the time. So the number of needles left infectious after being visited by a susceptible addict in the time interval $[t, t+\Delta t$ is

$$\lambda_1\gamma(1-\pi(t))i(t)(1-P_1-P_2)(1-\xi)\Delta t + o(\Delta t).$$

Therefore on noting that $\beta(t) = \dfrac{i(t)}{m}$, we obtain

$$\begin{aligned}
i(t+\Delta t) &= m\beta(t+\Delta t),\\
m\beta(t+\Delta t) &= m\beta(t)[1-(\lambda_1(1-p\pi(t))+\lambda_2 p\pi(t))\gamma\Delta t]\\
&\quad + m[\lambda_1(1-p)+\lambda_2 p)\pi(t)\gamma(1-\xi)\left(1-\beta(t)\theta_1-(1-\beta(t))\phi_1\right)]\Delta t\\
&\quad + \lambda_1\gamma(1-\pi(t))m\beta(t)(1-P_1-P_2)(1-\xi)\Delta t + o(\Delta t).
\end{aligned}$$

On subtracting $m\beta(t)$ from each side, dividing throughout by $m\Delta t$ and then taking the limit as $\Delta t \to 0$, we obtain the following ODE.

$$\begin{aligned}
\beta'(t) &= \lim_{\Delta t\to 0}\left[\frac{\beta(t+\Delta t)-\beta(t)}{\Delta t}\right],\\
&= [\lambda_1(1-p)+\lambda_2 p]\pi(t)\gamma(1-\xi)\left(1-\beta(t)\theta_1-(1-\beta(t))\phi_1\right)\\
&+ \lambda_1\gamma(1-\pi(t))\beta(t)(1-P_1-P_2)(1-\xi) - [\lambda_1(1-p)+\lambda_2 p]\pi\gamma\beta(t) - \lambda_1(1-\pi)\gamma\beta(t),\\
&= \big[\lambda_1(1-p)+\lambda_2 p\big]\pi\gamma\big((1-\xi)\left[1-\beta\theta_1-(1-\beta)\phi_1\right]-\beta\big) - \lambda_1(1-\pi)\gamma\beta\big[1-(1-\xi)(1-P_1-P_2)\big].
\end{aligned}$$

Now we want to find the number of infected addicts at time $t+\Delta t$. There are $n-I(t)$ susceptible addicts at time t with a proportion of $\lambda_1\Delta t + o(\Delta t)$ who use needles during the small time interval

$[t, t+\Delta t)$. Among these addicts, a fraction of $\beta(t)(P_1+P_3)$ use infected needles and are infected. so the number of new infections that occur during the time $[t, t+\Delta t)$ is

$$[n - I(t)]\lambda_1\beta(t)(P_1+P_3)\Delta t + o(\Delta t).$$

Since a $\mu I(t)\Delta t + o(\Delta t)$ of the infected addicts stop sharing injecting equipment during the time $[t, t+\Delta t)$, we obtain the following,

$$I(t+\Delta t) = I(t) + [n - I(t)]\lambda_1\beta(t)(P_1+P_3)\Delta t - \mu I(t)\Delta t + o(\Delta t), \tag{2.2.2}$$

and since $I(t)$=$n\pi(t)$, this leads to,

$$n\pi(t+\Delta t) = n\pi(t) + n[1-\pi(t)]\lambda_1\beta(t)(P_1+P_3) - \mu\pi(t)\Delta t]\Delta t + o(\Delta t).$$

On subtracting $n\pi(t)$ from each side, dividing throughout by $n\Delta t$ and then taking the limit as $\Delta t \longrightarrow 0$, we obtain the following ODE.

$$\frac{d\pi}{dt} = \lim_{\Delta t\to 0}\left[\frac{\pi(t+\Delta t)-\pi(t)}{\Delta t}\right] = (1-\pi))\lambda_1\beta(P_1+P_3) - \mu\pi. \tag{2.2.3}$$

The above calculation can be summarized as follows

$$\frac{d\beta}{dt} = \pi(\sigma - \tau\beta) - (1-\pi)\rho\beta = f(\beta,\pi), \tag{2.2.4}$$

$$\frac{d\pi}{dt} = (1-\pi)v\beta - \mu\pi = g(\beta,\pi), \tag{2.2.5}$$

Where

$$\begin{aligned}
\sigma &= [\lambda_1(1-p) + \lambda_2 p]\gamma(1-\xi)(1-\phi_1),\\
\tau &= [\lambda_1(1-p) + \lambda_2 p]\gamma[1-\phi_1(1-\xi) + \theta_1(1-\xi)],\\
\rho &= \lambda_1\gamma\left[1-(1-\xi)(1-P_1-P_2)\right],\\
\nu &= \lambda_1(P_1+P_3).
\end{aligned} \tag{2.2.6}$$

Thus σ, τ, ρ and ν are positive with $\sigma \le \tau$. Equations (2.2.4) and (2.2.5) will enable us to find out the behaviour of the system and whether the disease persists or not.

2.3 Equilibrium results

The equilibrium points of the system of differential equations are the points at which the first derivatives are equal to zero. For the system (2.2.4) and (2.2.5) we obtain,

$$f(\beta^*,\pi^*) = \pi^*(\sigma - \tau\beta^*) - (1-\pi^*)\rho\beta^* = 0. \tag{2.3.1}$$

$$g(\beta^*,\pi^*) = (1-\pi^*)v\beta^* - \mu\pi^* = 0. \tag{2.3.2}$$

It follows from equation (2.3.2) that,

$$(1-\pi^*)\nu\beta^* = \mu\pi^* \Rightarrow \frac{\pi^*}{1-\pi^*} = \frac{\nu\beta^*}{\mu}, \tag{2.3.3}$$

and from equation (2.3.1), that,

$$\frac{\pi^*}{1-\pi^*} = \frac{\rho\beta^*}{\sigma-\tau\beta^*}. \tag{2.3.4}$$

Comparing equations (2.3.3) and (2.3.4) leads to

$$\frac{\pi^*}{1-\pi^*} = \frac{\nu\beta^*}{\mu} = \frac{\rho\beta^*}{\sigma-\tau\beta^*}. \tag{2.3.5}$$

(a) From equation (2.3.5), if $\beta^* = 0$,then $\pi^* = 0$.Which is the trivial zero equilibrium.

(b) For an endemic equilibrium, we are looking for a non-zero equilibrium. Solving (2.3.5) for β^* gives

$$\begin{aligned} \frac{\nu\beta^*}{\mu} &= \frac{\rho\beta^*}{\sigma-\tau\beta^*} \\ \Rightarrow \beta^* &= \frac{\sigma}{\tau}\left(1-\frac{\rho\mu}{\nu\sigma}\right). \end{aligned} \tag{2.3.6}$$

We use $\beta^* = \dfrac{\mu\pi^*}{\nu-\pi^*\nu}$, obtained from (2.3.2), to substitute into equation (2.3.1) in order to get the value of π^*

$$\begin{aligned} &\pi^*(\sigma-\tau\beta^*) - (1-\pi^*)\rho\beta^* = 0, \\ \Rightarrow &\pi^*(\sigma-\tau(\frac{\mu\pi^*}{\nu-\pi^*\nu})) - (1-\pi^*)\rho\frac{\mu\pi^*}{\nu-\pi^*\nu} = 0, \\ \Rightarrow &\sigma(\nu-\nu\pi^*) - \tau\mu\pi^* - (1-\pi^*)\rho\mu = 0, \\ \Rightarrow &\sigma\nu - \sigma\nu\pi^* - \tau\mu\pi^* - \rho\mu + \pi^*\rho\mu = 0, \\ \Rightarrow &\pi^* = \frac{\sigma\nu-\rho\mu}{\sigma\nu+\tau\mu-\rho\mu}. \end{aligned} \tag{2.3.7}$$

So the the equilibrium point where the disease dies out is $(\beta^*, \pi^*) = (0, 0)$ and the endemic equilibrium is as follows

$$\beta^* = \frac{\sigma}{\tau}\left(1-\frac{\rho\mu}{\sigma\nu}\right), \quad \pi^* = \frac{\sigma\nu-\rho\mu}{\mu\tau+\sigma\nu-\rho\mu}. \tag{2.3.8}$$

For the endemic equilibrium to exist we require $\pi^* > 0$ and $\beta^* > 0$ and thus $\nu\sigma > \rho\mu$. Therefore we define $R_0 = \dfrac{\nu\sigma}{\rho\mu}$ as the main threshold parameter that determines the behaviour of the system. When $R_0 \leq 1$ it is anticipated that the disease dies out and when $R_0 > 1$ the disease takes off.

3. Analysis of the Model

In this section, we determine the stability properties of the equilibria found in section (2.3). Our analysis is based on the system of equations (2.2.4) and (2.2.5). We examine both the local asymptotic stability and global asymptotically stability. The local asymptotic stability can be explained as; given an initial state $\beta(0), \pi(0)$ which is sufficiently close to the steady state, the resulting solution emanating from $(\beta(0), \pi(0))$ will converge to the steady state as $t \to \infty$. However we cannot quantify "sufficiently close". On the other hand, if we know that a steady state is globally asymptotically stable, then we can identify all the physically acceptable initial states $(\beta(0), \pi(0))$ that give solutions converging to the steady state. So given a specific $(\beta(0), \pi(0))$, we know exactly what is going to happen as $t \to \infty$. Since $0 \leq \beta \leq 1$ and $0 \leq \pi \leq 1$ from the definition of these variables we examine this system in the physically relevant region $D = [0,1] \times [0,1] = \{(\beta, \pi) : 0 \leq \beta \leq 1, 0 \leq \pi \leq 1\}$.

3.1 Local Stability Analysis

We perform a local stability analysis to find out the behaviour of the system in the long term. Locally stability can be defined as solution $(\beta(t), \pi(t))$ that start near the equilibrium point (β^*, π^*) must remain near an equilibrium point for all the time. The equilibrium point (β^*, π^*) is locally asymptotically stable, if all solutions starting near the equilibrium tend towards the equilibrium as time become large.

Theorem 3.1.1. *If $R_0 < 1$, then the disease free equilibrium point is locally asymptotically stable, and when $R_0 > 1$ the disease free equilibrium point is unstable.*

Proof. We have:

$$f(\beta, \pi) = \pi(\sigma - \tau\beta) - (1 - \pi)\rho\beta,$$
$$g(\beta, \pi) = (1 - \pi)\nu\beta - \mu\pi,$$

then the Jacobian matrix is given by

$$J(\beta, \pi) = \begin{bmatrix} \frac{\partial f}{\partial \beta} & \frac{\partial f}{\partial \pi} \\ \frac{\partial g}{\partial \beta} & \frac{\partial g}{\partial \pi} \end{bmatrix} = \begin{bmatrix} -\pi\tau - \rho + \pi\rho & \sigma - \tau\beta + \rho\beta \\ \nu(1 - \pi) & -\upsilon\beta - \mu \end{bmatrix}. \tag{3.1.1}$$

Concequently

$$J(0,0) = \begin{bmatrix} -\rho & \sigma \\ \nu & -\mu \end{bmatrix}. \tag{3.1.2}$$

therefore $\mathrm{tr}(J(0,0)) = -(\rho + \mu)$ and $\det(J(0,0)) = \rho\mu - \nu\sigma$,

If $R_0 = \dfrac{\upsilon\sigma}{\rho\mu} < 1$,then $\nu\sigma < \rho\mu$, and have $\mathrm{tr}(J(0,0)) < 0$ and $\det(J(0,0)) > 0$. Consequently for $R_0 < 1$, the equilibrium is locally asymptotically stable. This means that the disease dies out in both population of needles and addicts.

Similarly if $R_0 = \dfrac{\nu\sigma}{\rho\mu} > 1$, then $\nu\sigma > \rho\mu$. This means that $\mathrm{tr}(J(0,0))$ is still negative but $\det(J(0,0))$ is now negative. Hence the disease free equilibrium point is unstable, this means that the disease will persists. □

Theorem 3.1.2. *The endemic steady state* (β^*, π^*) *given by* (2.3.8)*, is locally asymptotically stable if* $R_0 > 1$.

Proof. From equation (3.1.1), the Jacobian matrix at the endemic steady state (β^*, π^*)

$$J(\beta^*, \pi^*) = \begin{bmatrix} -\rho + \pi^*(\rho - \tau) & \sigma + \beta^*(\rho - \tau) \\ \nu(1 - \pi^*) & -\nu\beta^* - \mu \end{bmatrix}. \tag{3.1.3}$$

As in the previous proof, to show that (β^*, π^*) is locally asymptotically stable, it is sufficient to verify that, $\mathrm{tr}(J(\beta^*, \pi^*)) < 0$ and $\det(J(\beta^*, \pi^*)) > 0$.

Now

$$\begin{aligned} \mathrm{tr}(J(\beta^*, \pi^*)) &= -\rho + \pi^*(\rho - \tau) - \nu\beta^* - \mu \\ &= -\rho + \pi^*\rho - \pi^*\tau - \nu\beta^* - \mu \\ &= -\rho(1 - \pi^*) - \tau\pi^* - \nu\beta^* - \mu < 0, \end{aligned} \tag{3.1.4}$$

and

$$\begin{aligned} \det(J(\beta^*, \pi^*)) &= (-\rho + \pi^*(\rho - \tau))(-\nu\beta^* - \mu) - \nu(1 - \pi^*)(\sigma + \beta^*(\rho - \tau)) \\ &= \rho\nu\beta^* + \rho\mu - \pi^*\beta^*\nu\rho - \pi^*\rho\mu + \pi\beta^*\tau\nu + \pi^*\tau\mu - \nu(1 - \pi^*)(\sigma + \beta^*(\rho - \tau)) \\ &= \rho\nu\beta^*(1 - \pi^*) + \rho\mu(1 - \pi^*) + \pi^*\tau\mu + \pi^*\beta^*\tau\nu - \nu(1 - \pi^*)(\sigma + \beta^*(\rho - \tau)) \\ &= \rho\mu + \pi^*\tau\mu + \pi * \beta\tau\nu - \nu(1 - \pi^*)(\sigma + \beta^*(\rho - \tau)) \end{aligned} \tag{3.1.5}$$

where we have used the equilibrium equation,

$$(1 - \pi^*)\nu\beta^* - \mu\pi^* = 0$$

we can also use the equilibrium equations (2.3.1) and (2.3.2) to write,

$$\beta^*(\rho - \tau) + \sigma = \frac{\rho\beta^*}{\pi^*},$$

$$(1 - \pi^*) = \frac{\mu\pi}{\beta^*},$$

and on substituting into equation (3.1.5), we obtain

$$\begin{aligned} \det(\beta^*, \pi^*) &= \rho\mu + \pi^*\tau\mu + \pi^*\beta^*\tau\mu - \frac{\mu\pi^*}{\beta^*}\left(\frac{\rho\beta^*}{\pi^*}\right), \\ \det &= \pi\tau\mu + \pi^*\beta^*\tau\nu > 0, \end{aligned}$$

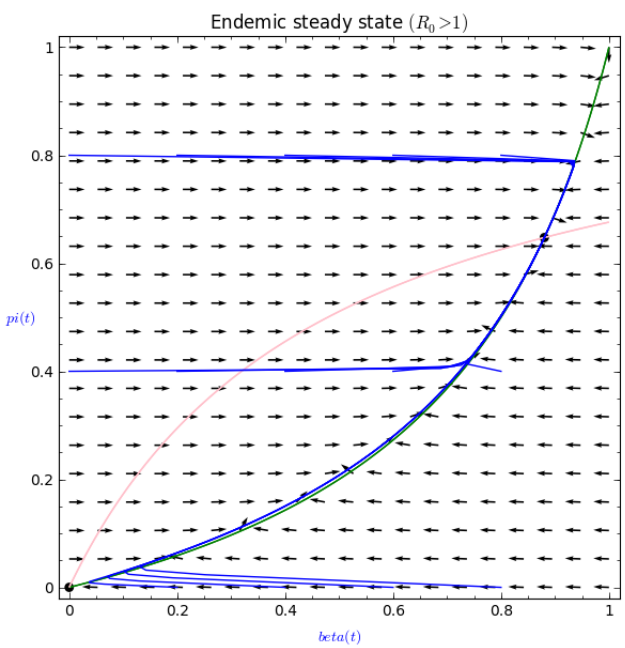
as required.

Since $\mathrm{tr}(J(\beta^*, \pi^*) < 0)$ and $\det(J(\beta^*, \pi)) > 0$, then the endemic steady state (β^*, π^*), given by (2.3.8), is locally asymptotically stable if $R_0 > 1$. This means that the disease persists in the population of both addicts and needles. □

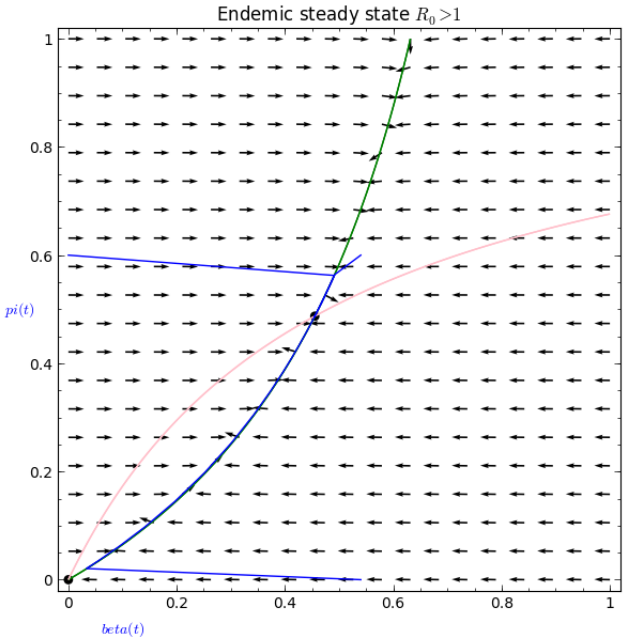
Plots of phase portraits

We create phase portraits to confirm the local stability results and motivate further mathematical investigations of global stability. We use real parameters obtained by Greenhalgh and Hay and summarised

in the Table 3.1. We investigate the stability of the unique endemic equilibrium when $R_0 > 1$ and disease free equilibrium when $R_0 < 1$. The first two phase portraits, (a) and (b) show if we start with some infected needles $\beta(0) > 0$ and infected addicts $\pi(0) > 0$, the system will approach a unique endemic equilibrium state (β^*, π^*). The plot shows that all trajectories converge to the endemic equilibrium state. The third plot, (c) shows the situation for $R_0 < 1$, when trajectories converge to the disease free equilibrium as time become large. Thus for $R_0 < 1$, the disease free equilibrium is stable at $(\beta, \pi) = (0, 0)$

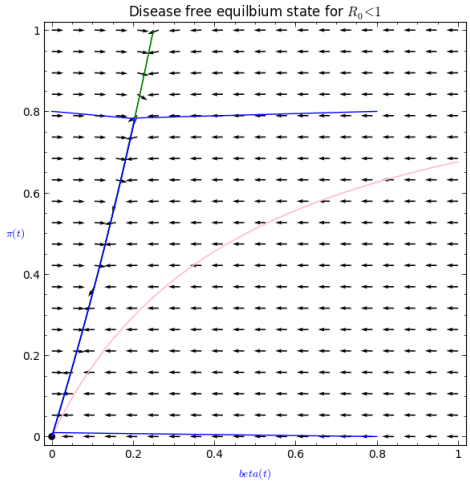


(a) when $\xi = 0$ $R_0 = 8.36$



(b) $\xi = 0.25$, $R_0 = 3.58$

Figure 3.1: simulation results for endemic steady state



(a) when $\xi = 0.75$ $R_0 = 0.64$

Figure 3.2: simulation result for disease free equilibrium steady state

3.2 Global Stability Analysis

We begin by proving the global stability of the disease free equilibrium.

Theorem 3.2.1. If $R_0 = \frac{\sigma\nu}{\rho\mu} < 1$, then the equilibrium where the disease dies out, that is $(\beta^*, \pi^*) = (0, 0)$, is globally stable.

Proof. To prove the global stability of the disease free equilibrium for $R_0 < 1$, we need to show that $\pi(t) \to 0$ and $\beta(t) \to 0$ as $t \to \infty$, where $(\beta(0), \pi(0))\epsilon D \times D$.

The model equations,

$$\frac{d\beta}{dt} = \sigma\pi - \rho\beta + \pi\beta(\rho - \tau), \tag{3.2.1}$$

$$\frac{d\pi}{dt} = \nu\beta - \mu\pi - \nu\pi\beta, \tag{3.2.2}$$

can be combined into one equation by defining $u = \beta + k\pi$, where $k \geq 0$.Now

$$\frac{du}{dt} = \frac{d\beta}{dt} + k\frac{d\pi}{dt}, \tag{3.2.3}$$

and On substituting equations (3.2.1) and (3.2.2) into equation (3.2.3) we obtain

$$\begin{aligned}\frac{du}{dt} &= \sigma\pi - \rho\beta + \pi\beta(\rho - \tau) + k(\nu\beta - \mu\pi - \nu\pi\beta).\\ &= (\sigma - k\mu)\pi + (k\nu - \rho)\beta + \pi\beta(\rho - \tau - \nu k).\end{aligned} \tag{3.2.4}$$

Suppose $k = \frac{\rho}{\nu} - \epsilon$, where ϵ is small and positive, substituting k in (3.2.4) leads to

$$\begin{aligned}\frac{du}{dt} &= \left(\sigma - (\frac{\rho}{\nu} - \epsilon)\mu\right)\pi + \left((\frac{\rho}{\nu} - \epsilon)\nu - \rho\right)\beta - \pi\beta\left(\nu(\frac{\rho}{\nu} - \epsilon) - \rho + \tau\right)\\ &= (\sigma - \frac{\rho\mu}{\nu} + \epsilon\mu)\pi + (\rho - \epsilon\nu - \rho)\beta - \pi\beta(\rho - \epsilon\nu - \rho + \tau)\\ &= (\sigma - \frac{\rho\mu}{\nu} + \epsilon\mu)\pi - \epsilon\nu\beta - \pi\beta(\tau - \epsilon\nu)\\ &= \left[\sigma(1 - \frac{\rho\mu}{\nu\sigma}) + \epsilon\mu\right]\pi - \epsilon\nu\beta - \pi\beta(\tau - \epsilon\nu)\\ &= \left[\sigma(1 - \frac{1}{R_0}) + \epsilon\mu\right]\pi - \epsilon\nu\beta - \pi\beta(\tau - \epsilon\nu).\end{aligned}$$

We choose $\epsilon > 0$ to be sufficiently small so that $k > 0$, $\tau - \epsilon\nu > 0$ and $\sigma(1 - \frac{1}{R_0}) + \epsilon\mu = -\eta < 0$.

Then,

$$\frac{du}{dt} \leq -\eta\pi - \epsilon\nu\beta = -\frac{\eta}{k}k\pi - \epsilon\nu\beta.$$

$$\text{If} \quad \frac{\eta}{k} < \epsilon\nu \quad \text{then} \quad -\frac{\eta}{k} > -\epsilon\nu \quad \text{and so}$$

$$\frac{du}{dt} \leq -\frac{\eta}{k}k\pi - \frac{\eta}{k}\beta = -\frac{\eta}{k}(k\pi + \beta) = -\frac{\eta}{\nu}u$$

Similarly,

$$\frac{\eta}{k} > \epsilon\nu \Rightarrow \quad \frac{-\eta}{k} < -\epsilon\nu$$
$$\Rightarrow \frac{du}{dt} \leq -\epsilon\nu k\pi - \epsilon\nu\beta = -\epsilon\nu(k\pi + \beta) = -\epsilon\nu u$$

that is

$$\frac{du}{dt} \leq -\omega u, \text{ where } \omega = \min(\epsilon\nu, \frac{\eta}{k}) > 0. \tag{3.2.5}$$

Using differential equation in (3.2.5) we have

$$\int_0^t \frac{u'(s)}{u(s)} ds \leq -\int_0^t \psi ds = -\omega t$$

thus,

$$\ln\left[\frac{u(t)}{u(s)}\right] \leq -\psi t$$

Hence

$$u(t) \leq u(0)e^{-\psi t} \to 0 \ \ as \ \ t \ \ \to \infty.$$

Therefore $0 \leq u(t) \leq u(0)e^{-\psi t}$, meaning that $u(t) \to 0$ as $t \to \infty$, so both $\pi(t)$ and $\beta(t)$ tend to zero as $t \to \infty$. Thus the disease disappears in both population of addicts and needles as time become large. For $R_0 = 1$ this proof is not possible since we cannot choose an $\epsilon > 0$ to satisfies the required conditions. □

Theorem 3.2.1. *If $R_0 = \frac{\sigma\sigma}{\rho\mu} > 1$ and $\tau > \rho$ the endemic equilibrium (β^*, π^*), given by (2.3.8), is globally stable.*

Proof. To prove this result, we use the method of Lyapunov functions, D.W.Jordan and P.Smith. An appropriate Lyapunov functions is constructed as follows. It is convenient to introduce new variables $\hat{\beta}$ and $\hat{\pi}$ defined by,

$$\hat{\beta}(t) = \beta(t) - \beta^*, \ \ \hat{\pi}(t) = \pi(t) - \pi^*. \tag{3.2.6}$$

Then, using equations (3.2.1),(3.2.2) and (3.2.6) we have,

$$\frac{d\beta}{dt} = \frac{d\hat{\beta}}{dt} = \sigma(\hat{\pi} + \pi^*) - \rho(\hat{\beta} + \beta^*) + (\hat{\pi} + \pi^*)(\hat{\beta} + \beta^*)(\rho - \tau)$$
$$= \sigma\hat{\pi} + \sigma\pi^* - \rho\hat{\beta} - \rho\beta^* + \left(\hat{\pi}\hat{\beta} + \hat{\pi}\beta^* + \pi^*\hat{\beta}^* + \pi^*\beta^*\right)(\rho - \tau).$$

Now, since (β^*, π^*) is an equilibrium point it follows that, $\sigma\pi^* - \rho\beta^* + \pi^*\beta^*(\rho - \tau) = 0$.

Consequently,

$$\frac{d\beta}{dt} = \frac{d\hat{\beta}}{dt} = \sigma\hat{\pi} - \rho\hat{\beta} + (\rho - \tau)\hat{\beta}\pi + (\rho - \tau)\beta^*\hat{\pi}. \tag{3.2.7}$$

Similarly,

$$\frac{d\pi}{dt} = \frac{d\hat{\pi}}{dt} = (1-(\hat{\pi}+\pi^*))\nu(\hat{\beta}+\beta^*) - \mu(\hat{\pi}+\pi^*)$$
$$= \nu\hat{\beta} - \nu\beta^* - \nu(\hat{\pi}\hat{\beta} + \hat{\pi}\beta^* + \pi^*\hat{\beta} + \pi^*\beta^*) - (\mu\hat{\pi} - \mu\pi^*),$$

Since (β^*, π^*) is an equilibrium then, $\nu\beta^* - \mu\pi^* - \nu\beta^*\pi^* = 0$.

Therefore,

$$\frac{d\pi}{dt} = \frac{d\hat{\pi}}{dt} = \nu\hat{\beta} - \mu\hat{\pi} - \nu(\hat{\pi}\beta + \pi^*\hat{\beta}). \tag{3.2.8}$$

In matrix form these equations can be written as

$$\begin{pmatrix}\beta(t)\\ \pi(t)\end{pmatrix}' = \begin{pmatrix}\hat{\beta}(t)\\ \hat{\pi}(t)\end{pmatrix}' = V(\beta,\pi)\begin{pmatrix}\hat{\beta}(t)\\ \hat{\pi}(t)\end{pmatrix}, \tag{3.2.9}$$

$$= \begin{pmatrix}-\rho + (\rho-\tau)\pi & \sigma + (\rho-\tau)\beta^*\\ \nu(1-\pi^*) & -(\mu+\nu\beta)\end{pmatrix}\begin{pmatrix}\hat{\beta}(t)\\ \hat{\pi}(t)\end{pmatrix} \tag{3.2.10}$$

where,

$$V(\beta,\pi) = \begin{pmatrix}-\rho + (\rho-\tau)\pi & \sigma + (\rho-\tau)\beta^*\\ \nu(1-\pi^*) & -(\mu+\nu\beta)\end{pmatrix}. \tag{3.2.11}$$

In particular,

$$V_0 = V(0,0) = \begin{pmatrix}-\rho & \sigma + (\rho-\tau)\beta^*\\ \nu(1-\pi^*) & -\mu\end{pmatrix}. \tag{3.2.12}$$

We shall denote the elements in $V(\beta,\pi)$ by $v_{11}, v_{12}, v_{21}\ and\ v_{22}$, with elements in $V(0)$ denoted by $v_{11}^0, v_{12}^0, v_{21}^0\ and\ v_{22}^0$.

Since $\rho < \tau$ we have, for all $(\beta,\pi) \in [0,1]\times[0,1]$,

$$v_{11} \le v_{11}^0,\ \ v_{22} \le v_{22}^0,\ \ v_{12} = v_{12}^0,\ \ v_{21} = v_{21}^0. \tag{3.2.13}$$

We now work with the constant matrix V_0 and show that a diagonal matrix

$$W = \begin{pmatrix}w_1 & 0\\ 0 & w_2\end{pmatrix},\ \text{with} \qquad w_1, w_2 > 0,$$

can be found such that $(WV_0 + V_0^T W)$ is a negative semi-definite matrix so that

$$y^T(WV_0 + V_0^T W)y \le 0\ \ \forall y = \begin{pmatrix}y_1\\ y_2\end{pmatrix}. \tag{3.2.14}$$

First note that

$$v_{11}^0 v_{22}^0 - v_{12}^0 v_{21}^0 = \rho\mu - \nu(1-\pi^*)(\sigma + (\rho-\tau)\beta^*),$$

and, from the equilibrium equations (2.3.1) and (2.3.2),

$$\nu(1-\pi^*) = \frac{\mu\pi^*}{\beta^*} \text{ and } \sigma + \beta^*(\rho - \tau) = \frac{\rho\beta^*}{\pi^*}.$$

Consequently,

$$v_{11}^0 v_{22}^0 - v_{12}^0 v_{21}^0 = \rho\mu - \rho\mu = 0. \tag{3.2.15}$$

Now,

$$WV_0 + V_0^T W = \begin{pmatrix} 2v_{11}^0 w_1 & v_{12}^0 w_1 + v_{21}^0 w_2 \\ v_{12}^0 w_1 + v_{21}^0 w_2 & 2v_{22}^0 w_2 \end{pmatrix}$$

and so,

$$y^T(WV_0 + V_0^T W)y = 2v_{11}^0 w_1 y_1^2 + 2v_{22}^0 w_1 y_2^2 + 2(v_{12}^0 w_2 + v_{21}^0 w_1)y_1 y_2$$

.

Choosing $w_1 = v_{12}^0 > 0$ and $w_2 = v_{21} > 0$ results in,

$$\begin{aligned} y^T(WV_0 + V_0^T W)y &= 2v_{11}^0 v_{12}^0 y_1^2 + 2v_{22}^0 v_{21}^0 y_2^2 + 4v_{12}^0 v_{21}^0 y_1 y_2 \\ &= -2\left(-v_{11}^0 v_{12}^0 y_1^2 - v_{22}^0 v_{21}^0 y_2^2 - 2v_{12}^0 v_{21}^0 y_1 y_2\right) \\ &- 2\left(\sqrt{-v_{11}^0 v_{12}^0}\, y_1 - \sqrt{-v_{22}^0 v_{21}^0}\, y_2\right)^2 \leq 0 \ \ \forall y_1 y_2. \end{aligned}$$

Where we have used (3.2.15) to deduce that,

$$v_{11}^0 v_{12}^0 v_{22}^0 v_{21}^0 = (v_{12}^0 v_{21}^0)^2. \tag{3.2.16}$$

Hence, with $w_1 = v_{12}^0,\ \ w_2 = v_{21}^0,$ the matrix $WV_0 + V_0^T W$ is semi-definite as required.

The next step is to use the matrix W to define a Lyapunov function via,

$$h(\hat\beta, \hat\pi) = (\hat\beta, \hat\pi)\begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}\begin{pmatrix} \hat\beta \\ \hat\pi \end{pmatrix} \tag{3.2.17}$$

$$= w_1\hat\beta^2 + w_2\hat\pi^2 \geq 0, \ \ \forall(\hat\beta, \hat\pi), \tag{3.2.18}$$

Let $(\beta(t), \pi(t))$ be the solution of the system (2.2.4) and (2.2.5), then

$$\begin{aligned} \frac{d}{dt}(h(\hat\beta(t), \hat\pi(t)) &= 2w_1\hat\beta(t)\hat\beta'(t) + 2w_2\hat\pi(t)\hat\pi'(t) \\ &= (\hat\beta(t), \hat\pi(t))W\begin{pmatrix} \hat\beta'(t) \\ \hat\pi'(t) \end{pmatrix} + (\hat\beta'(t), \hat\pi'(t))W\begin{pmatrix} \hat\beta(t) \\ \hat\pi(t) \end{pmatrix} \\ &= (\hat\beta(t), \hat\pi(t))WV(\beta, \pi)\begin{pmatrix} \hat\beta(t) \\ \hat\pi(t) \end{pmatrix} + (\hat\beta(t), \hat\pi(t))V^T(\beta, \pi)W\begin{pmatrix} \hat\beta(t) \\ \hat\pi(t) \end{pmatrix} \\ &= (\hat\beta(t), \hat\pi(t))\left[WV(\beta, \pi) + V^T(\beta, \pi)W\right]\begin{pmatrix} \hat\beta(t) \\ \hat\pi(t) \end{pmatrix}, \end{aligned}$$

where we have used the result

$$\begin{pmatrix}\beta'(t)\\ \pi'(t)\end{pmatrix} = \begin{pmatrix}\hat{\beta}'(t)\\ \hat{\pi}'(t)\end{pmatrix} = V(\beta,\pi)\begin{pmatrix}\hat{\beta}(t)\\ \hat{\pi}(t)\end{pmatrix}.$$

Now we can write

$$V(\beta,\pi) = V_0 + \begin{pmatrix}(\rho-\tau)\pi & 0\\ 0 & -\nu\beta\end{pmatrix}$$

and a simple calculation then show that

$$WV(\beta,\pi) + V^T(\beta,\pi)W = WV_0 + V_0^T W + 2\begin{pmatrix}(\rho-\tau)\pi w_1 & 0\\ 0 & -\nu\beta w_2\end{pmatrix}.$$

Since $WV_0 + V^T W$ is a negative semi-definite, we know that

$$(\hat{\beta},\hat{\pi})\left(WV_0 + V_0^T W\right)\begin{pmatrix}\hat{\beta}\\ \hat{\pi}\end{pmatrix} \leq 0, \ \ \forall(\hat{\beta},\hat{\pi}). \tag{3.2.19}$$

Also,

$$= (\hat{\beta},\hat{\pi})\left(WV_0 + V^T W + 2\begin{pmatrix}w_1(\rho-\tau)\pi & 0\\ 0 & -w_2\nu\beta\end{pmatrix}\right)\begin{pmatrix}\hat{\beta}\\ \hat{\pi}\end{pmatrix} \tag{3.2.20}$$

$$= (\hat{\beta},\hat{\pi})\left(WV_0^T + V^T W\right)\begin{pmatrix}\hat{\beta}\\ \hat{\pi}\end{pmatrix} + \underbrace{2w_1(\rho-\tau)\pi\hat{\beta}^2 - 2w_2\nu\beta\hat{\pi}^2}_{\leq 0} \tag{3.2.21}$$

$$\leq 0 \ \ \forall(\hat{\beta,\pi}). \tag{3.2.22}$$

Consequently $h(\hat{\beta}(t),\hat{\pi}(t))$ is non increasing along trajectories $(\hat{\beta}(t),\hat{\pi}(t))$, and it follows from the theory of weak Lyapunov functions D.W.Jordan and P.Smith, that $(0,0)$ is a stable equilibrium for the translated equations involving $\hat{\beta}$ and $\hat{\pi}$, that is (β^*,π^*) is stable equilibrium for the solution $(\beta(t),\pi(t))$. However since h$(\hat{\beta,\pi})$ is not strict Lyapunov function, it is not possible at this stage to deduce global asymptotic stability. To prove global asymptotic stability we verify that h$(\hat{\beta}(t),\hat{\pi}(t)) \to 0$ as $t \to \infty$ for this we require the following lemma.

We continue to work with the translated equations and we want to show that (0,0) is asymptotically stable for all initial values $(\hat{\beta}(0),\hat{\pi}(0))$ in the "translated" domain D. □

Lemma 3.2.2. Let $(\beta(0),\pi(0)) = (\beta_0,\pi_0 \in D,$ with $(\beta_0,\pi_0) \neq (0,0)$. Then there exists $\epsilon > 0$ such that $\beta^2(t) + \pi^2(t) \geq \epsilon^2 \ \ \forall t$.

Proof. We examine two cases. First, where $0 \leq \beta_0 \leq \beta^*, \ \ 0 \leq \pi_0 \leq \pi^*,$ and then where $\beta_0 > \beta^*$ or $\pi_0 > \pi^*$.

(a) $0 \leq \beta_0 \leq \beta^*$ and $0 \leq \pi_0 \leq \pi^*$, $(\beta(0), \pi(0)) \neq (0,0)$.

Since $(h\hat{\beta}(t), \hat{\pi}(t))$ is non -increasing along trajectories $(\hat{\beta}(t), \hat{\pi}(t))$ and

$$h(\hat{\beta}(0), \hat{\pi}(0)) = w_1(\beta_0 - \beta^*)^2 + w_2(\pi_0 - \pi^*)^2 = k_1$$

represents an ellipse with centre (β^*, π^*), it follows that the trajectory $(\beta(t), \pi(t))$, stays within this ellipse. Moreover, as either $\beta_0 > 0$ or $\pi_0 > 0$, we have

$$k_1 < k_2 = w_1\beta^{*2} + w_2\pi^{*2} = h(0,0).$$

By continuity of h, there must exist $\epsilon > 0$ such that $h(\beta, \pi) > k_1$ for $\beta^2 + \pi^2 < \epsilon$ and so $(\beta(t), \pi(t))$ never enters the circle.

(b) $(\beta_0, \pi_0) \in D$ with $\beta_0 > \beta^*$ or $\pi_0 > \pi^*$.

Let $\delta = \min(\frac{1}{2}\beta^*, \frac{1}{2}\pi^*)$ and consider the circle $\beta^2 + \pi^2 = \delta^2$. If the trajectory emanating from (β_0, π_0) never enters this circle then the results is then true. Otherwise at some time t_1, the trajectory must meet the boundary of the circle at a point (β_1, π_1), where $0 \leq \beta_1 \leq \beta^*$ and $0 \leq \pi_1 \leq \pi^*$.

Since β_1 and π_1 cannot both be zero, it follows that, for all $t \geq t_1$ the trajectory will remain within the ellipse that has centre (β^*, π^*) and passes through (β_1, π_1). We can argue as in (a) to arrive at the stated result.

□

Lemma (3.2.2) can now be used to prove the global asymptotic stability of (β^*, π^*).

Theorem 3.2.2. Let $R_0 = \frac{\sigma\nu}{\rho\mu} > 1$, $\tau > \rho$ and $(\beta(0), \pi(0)) = (\beta_0, \pi_0) \in D$ with $(\beta_0, \pi_0) \neq (0,0)$.

Then

$$(\beta(t), \pi(t)) \to (\beta^*, \pi^*) \text{ as } t \to \infty.$$

Proof. From the Lemma (3.2.2), there exists $\epsilon > 0$ such that $\beta^2(t) + \pi^2(t) \geq \epsilon$ for all t and therefore, on taking $\epsilon_1 \leq \sqrt{\frac{\epsilon}{2}}$, we must have $\beta(t) > \epsilon_1$ or $\pi(t) \geq \epsilon_1 \ \forall t$. We shall choose ϵ_1, so that it is also less than $\epsilon < \delta = \min(\frac{\beta^*}{2}, \frac{\pi^*}{2})$. Recall from (3.2.19) and (3.2.21) that

$$\frac{d}{dt}h(\hat{\beta}(t), \hat{\pi}(t)) \leq -2(\tau - \rho)\pi(t)w_1\hat{\beta}^2(t) - 2\nu\beta(t)w_2\hat{\pi}^2(t).$$

There are three cases to consider. For convenience we define $H'(t) = h(\hat{\beta}(t), \hat{\pi}(t))$.

(i) $\pi(t) \geq \epsilon_1$ and $\beta(t) < \epsilon_1$.

In this case

$$\begin{aligned} H'(t) &\leq -2(\tau - \rho)\epsilon_1 w_1\hat{\beta}^2(t) - 2\nu\epsilon_1 w_2\hat{\pi}^2(t) \\ &\leq -2\epsilon_1 \min(\tau - \rho, \nu)(w_1\hat{\beta}(t)^2 + w_2\pi^2(t)) \\ &= -\phi_0 h(\hat{\beta}(t), \hat{\pi}(t)), \end{aligned}$$

where $\phi_0 = 2\epsilon_1 \min(\tau - \rho, \nu) > 0$.

(ii) $\pi(t) \geq \epsilon_1$ and $\beta(t) geq \epsilon_1$

If $\beta(t) < \epsilon_1 < \min(\frac{\beta^*}{2}, \frac{\pi^*}{2})$, then $\hat{\beta}(t) = \beta(t) - \beta^* < -\frac{1}{2}\beta^*$.

Consequently,

$$H'(t) \leq -\frac{1}{2}\beta^{*2}(\tau - \rho)w_1\epsilon_1 = -\phi_1.$$

Since $h(0,0) > 0$ and $H'(t) \leq h(\hat{\beta}(0), \hat{\pi}(0))$, it follows that

$$H'(t) \leq -\phi_1 \frac{H(t)}{H(0)},$$

where $\phi_1 > 0$.

(iii) $\pi(t) < \epsilon_1$ and $\beta(t) \geq \epsilon_1$,

Arguing as in (ii), leads to, hence,

$$H'(t) \leq -\phi_2 \frac{H(t)}{H(0)},$$

where $\phi_2 = \frac{1}{2}\nu\pi^*\epsilon_1 > 0$.

It follows that,

$$H'(t) \leq -\omega H(t) \forall t$$

where,

$$\omega = \min(\phi_0, \frac{\phi_1}{H(0)}, \frac{\phi_2}{H(0)}) > 0$$

and therefore

$$0 \leq H(t) \leq H(0)e^{-\omega t} \ \forall t.$$

This establishes that, $H(t) = h(\hat{\beta}(t), \hat{\pi}(t)) = w_1(\beta(t) - \beta^*)^2 + w_2(\pi(t) - \pi^*)^2$ converges to 0 as $t \to \infty$ and this can only happen if $\beta(t) \to \beta^*$ and $\pi(t) \to \pi^*$ as $t \to \infty$. This complete the proof of global asymptotic stability of (β^*, π^*).

□

3.3 Simulations

We now perform some simulations to confirm and illustrate the analytical results. Graphs of solutions are plotted using python code. We show graphical representations of our results for both cases $R_0 < 1$ and $R_0 > 1$. We use realistic parameters values given by Greenhalgh and Hay (1997). These values are summarised in table 3.1. We also examine the value of ξ to illustrate the effect of cleansing needles on endemic prevalence of HIV. Other values are summarised in the table below.

We calculate values of σ, ρ, τ,and ν using equation (2.2.6) by substituting the parameters listed above.

Parameters	Values per year
λ	52.195 per year
p	0.0
ϕ_1	0.0
θ_1	0.0
θ	0.25
α	0.01
γ	1.0
μ	0.25 per year

Table 3.1: parameter values

(a) When $\xi = 0$, we obtain $\sigma = 52.195,\ \tau = 52.195,\ \rho = 13.04875,\ \nu = 0.52195.$

$$R_0 = \frac{\nu\sigma}{\rho\mu} = 8.36.$$

Also we calculate the endemic steady states as follows;

$$\beta^* = \frac{\sigma}{\tau}\left(1 - \frac{\rho\mu}{\nu\sigma}\right) = \frac{\sigma}{\tau}\left(1 - \frac{1}{R_0}\right) = 0.88.$$
$$\pi^* = \frac{\nu\sigma - \rho\mu}{\mu\tau + \nu\sigma - \rho\mu} == 0.65.$$

Thus the endemic steady state is (β^*, π^*)=(0.88,0.65).

(b) When ξ=0.25, we obtain $\sigma = 39.1462,\ \tau = 52.195,\ \nu = 0.52195,$ and $\rho = 22.8353125$

$$R_0 = \frac{\nu\sigma}{\rho\mu} = 3.58,$$
$$\beta^* = \frac{\sigma}{\tau}\left(1 - \frac{\rho\mu}{\nu\sigma}\right) = \frac{\sigma}{\tau}\left(1 - \frac{1}{R_0}\right) = 0.54$$
$$\pi^* = \frac{\nu\sigma - \rho\mu}{\mu\tau + \nu\sigma - \rho\mu} = 0.53.$$

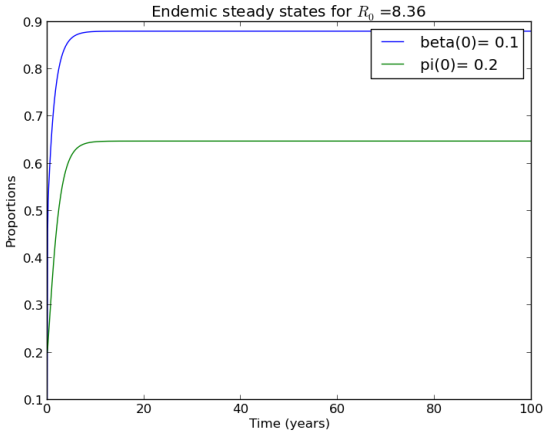
Thus the endemic steady steady state is (β^*, π^*) =(0.54,0.53).

(c) When ξ=0.75, we have σ=13.04875, ρ=42.4084375,ν = 0.52195, τ and = 52.195.

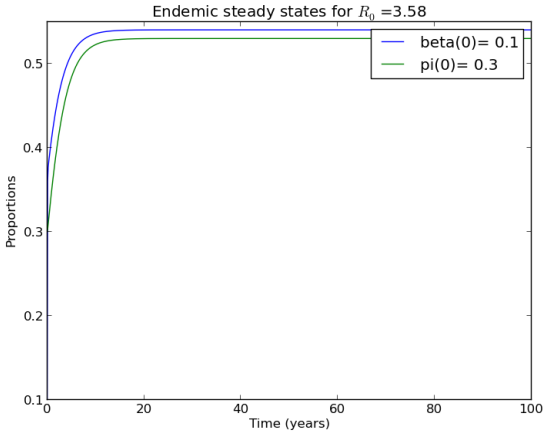
$$R_0 = \frac{\nu\sigma}{\rho\mu} = 0.64.$$
$$\beta^* = \frac{\sigma}{\tau}\left(1 - \frac{\rho\mu}{\nu\sigma}\right) = \frac{\sigma}{\tau}\left(1 - \frac{1}{R_0}\right) = -0.14,$$
$$\pi^* = \frac{\nu\sigma - \rho\mu}{\mu\tau + \nu\sigma - \rho\mu} = -0.41.$$

Thus (β^*, π^*) = (-0.14,-0.41). Since it is negative, this steady state is not possible, thus for $R_0 < 1$ we have only one steady state which is $(0, 0)$

Simulation results

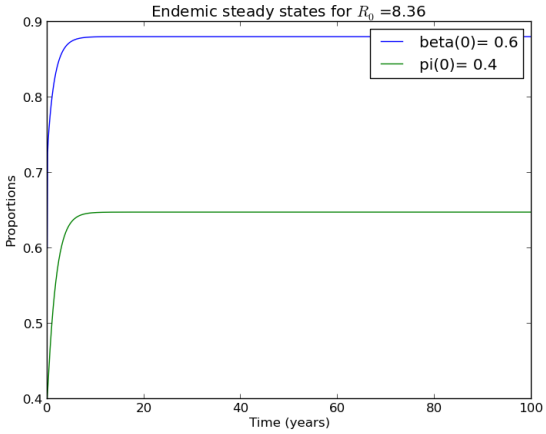


(a) when $\xi = 0$ $R_0 = 8.36$

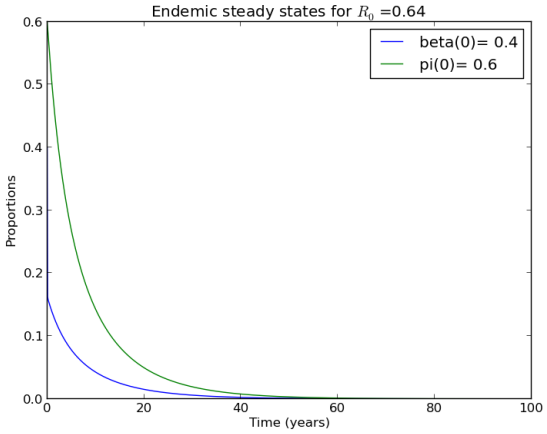


(b) $\xi = 0.25$, $R_0 = 3.58$

Figure 3.3: simulation results for endemic steady state



(a) when $\xi = 0$ $R_0 = 8.36$



(b) $\xi = 0.75$, $R_0 = 0.64$

Figure 3.4: Simulation results for (a)endemic equilibrium and (b)free equilibrium state

The two plots from Figure 3.2 illustrate the situation when $R_0 > 1$. We observe from the first plot, where $R_0 = 8.36$, that the solution $(\beta(t), \pi(t))$ that has initial value $(0.1, 0.2)$ converges to the disease endemic equilibrium point $(0.88, 0.65)$, which is the endemic equilibrium point obtained from our analysis. However the proportion of infected needles is higher than the proportion of infected addicts. Note that $\xi = 0$ means that there are no addict who clean their injecting equipment after use.

We conducted a further simulation for $R_0 = 3.58$ with initial values $(\beta(0), \pi(0)) = (0.1, 0.3)$. The solution converges to the disease endemic equilibrium point at $(0.54, 0.53)$, this value corresponding with the endemic equilibrium we obtained in our analytical calculations. These simulations illustrate the case $R_0 > 1$. Provided the disease is initially present in either the population of addicts or needles then the disease will tend to the unique endemic equilibrium.

The first plot in Figure 3.3 with $R_0 > 1$ illustrates the effects of using different initial values to attain

endemic equilibrium. We use the same parameters as those used in Figure 3.2 (a), but we started with different initial values. The Results show that although we started with different initial values, solutions will converge to the same endemic equilibrium state as we obtained in Figure 3.2 (a). This experiment shows that the endemic long term behaviour is independent of the initial conditions.

The second plot in Figure 3.3 illustrates the condition $R_0 < 1$. We observe that when $R_0 = 0.64$ the solution converges to the equilibrium where the disease is dying out, This simulation shows that about 64% of addicts must clean their injection equipment after use to reduce or eliminate HIV in the long term. So the increase of ξ, the fraction of individuals who clean their injection equipments after use, reduces R_0 and this leads to eradication of disease in a population of addicts and needles in the long term. It is observed from Figure 3.3 (b), that the disease disappears from the population of addicts and needles after 50 years. These results are in agreement with our analytical findings.

4. Conclusion and Recommendations

In this project we have discussed two models for the spread of HIV/AIDS amongst injecting drug users. We began with an introduction and literature review. We examined a system of ODEs that was proposed by Kaplan (1989), as a model of the spread of HIV and AIDS in a population of IDUs. From this, we moved onto the second model developed by Greenhalgh and Hay (1997). The mathematical analysis that we presented was based on this second more general and realistic model. A major assumption in this model is that, the only one way addicts can get HIV is through the use of injecting equipment .

Several assumptions that had been made by Kaplan when formulating his model were relaxed by Greenhalgh and Hay. According to Kaplan, all addicts move to the shooting gallery at the same rate. However this assumption might not always be true because some addicts who know that they are infected may lower their rate of attending a shooting gallery because they might have started receiving counselling. As in the Kaplan model, Greenhalgh and Hay also incorporated the flushing of needle equipment in their model,Kaplan had introduced possibilities such as, when an infected syringe is used by uninfected addicts, the injection process will clean the syringe with a probability of θ and an addict who is exposed to HIV becomes infected with a probability of α. However he he did not discuss in detail the probabilities that exist in this situation. Thus Greenhalgh and Hay extended Kaplan's model by making general assumptions on these joint probabilities, but these assumption did not affect the results. Previously, it was assumed that injection equipment became contaminated if it is used by infected addicts. This assumption is not always true because the contamination process depends also on other factors such as the amount of blood contained in the needle and the amount of virus contained in the blood. Therefore this assumption was modified by Greenhalgh and Hay to include the chance that injection equipment is not always left infected.

We derived the differential equations for the spread of disease as proposed by Greenhalgh and Hay by letting π and β be the proportions of infected addicts and needles respectively. We calculated equilibrium stability analysis to find out the behaviour of the system. We demonstrated that there is a key threshold parameter R_0 that determines the behaviour of the system. If $R_0 < 1$ and $\tau > \rho$, there is a unique disease free equilibrium which is stable; biologically this means disease has been eliminated from the population of both addicts and needles. For $R_0 > 1$ the disease free equilibrium is unstable and there is also a unique endemic equilibrium which is locally stable. This means disease persists in the population of addicts and needles. If we start away from the disease free equilibrium, that is $\beta(0) > 0, \pi(0) > 0$ and $\tau > \rho$, the system will approach the unique endemic equilibrium and the condition $\tau > \rho$ is satisfied for the Kaplan model.

Hence we have described how global stability results can be proved for the realistic model.In this project it is assumed that addicts clean their injection equipment after use, and those who know that they are infected reduce their movement to the shooting galleries.This might reduce the speed at which the disease is spreading and the equilibrium level of the disease.

This model can enable us to establish the control strategies for the elimination of HIV/AIDS, such as reducing needle sharing, encouraging addicts to clean their needles after use and counselling HIV infected addicts to reduce or stop sharing injecting equipment.

As part of future work this model can be extended to include the three -stage infectious period, which also has been done by Greenhalgh et al. (2009).

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