Stabilisation Problems with Boundary Controls for Systems of Hyperbolic Conservation Laws

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Abstract

In this research project we address the issue of stabilization problems with boundary controls for a system of hyperbolic conservation laws. We discuss the condition which guarantee the exponential stability of solutions of the linearised systems of hyperbolic conservation laws around the steady-state. We use Lyapunov stability analysis to show that the condition for which exponential stability is satisfied. The analysis is treated on networks of 2×2 systems of hyperbolic conservation laws. Moreover, this is applied to Saint-Venant equations for the model of open-channel with a cascade of n-pools.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Gediyon Yemane Weldegiyorgis, 22 May 2014.

Contents

1. Introduction

A balance law in a one-space variable is a first-order partial differential equations of the form,

$$\partial_t u + \partial_x f(u) = g(u), \tag{1.0.1}$$

where u is a function of x and t while $f(u)$ and $g(u)$ are smooth functions. If the source term is zero, (i.e. g(u) = 0), then the equation,

$$\partial_t u + \partial_x f(u) = 0 \tag{1.0.2}$$

is referred to as a conservation law. The physical interpretation of conservation law on the given bounded interval $[x_1, x_2]$ is described as

$$\begin{aligned}\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)dx &= \int_{x_1}^{x_2} \partial_t u(x,t)dx,\\ &= -\int_{x_1}^{x_2} \partial_x f(u(x,t))dx,\\ &= -[f(u(x_2,t)) - f(u(x_1,t))],\\ &= f(u(x_1,t)) - f(u(x_2,t)).\end{aligned}$$

This means the change of mass of the quantity $u(x,t)$ in $[x_1, x_2]$ at a time t depends on the inflow at x_1 and the outflow at x_2 (Solin, 2006, p. 37). For a Cauchy problem, we also prescribe an initial condition as

$$u(x,0) = u_0(x).$$

We also represent conservation laws as a system,

$$\partial_t U + A(U)\partial_x U = 0,$$

and this system is hyperbolic conservation laws if the matrix $A(U)$ has real eigenvalues.

Consider the physical networks of an irrigation canal or road traffic system. Here, the network means a directed graph with edges and nodes. The physical networks may be described by the system of hyperbolic conservation laws in a one-space variable. There is a control action at the opening gate in an irrigation canal or at a traffic light during road traffic. In this model, the boundary conditions are developed by the algebraic relation at the junctions of the physical networks.

The main goal of this research project is to discuss the condition of exponential stability of systems of hyperbolic conservation laws. The Lyapunov function is used in the process to determine the appropriate controls. In this process, time derivative of the Lyapunov function is shown to be negative if the boundary dissipative condition is satisfied.

In Chapter 2, we give the definition of the system of conservation laws over a specified finite interval. We then look for classical solutions for linear scalar conservation law. We also extend the result to a non-linear case. Here, we apply the method of characteristics for the scalar as well as the system of equations. But, the method holds for a system of strictly hyperbolic conservation laws. Thus, we define matrix decomposition to represent the system into a decoupled system.

For non-linear systems, the decomposition is not trivial. Hence, Riemann invariants are considered. Then, we treat the method of characteristics as a linear case to find the classical solutions. But, the regularity of the solutions are fail for non-linear case because of discontinuity of the solutions. For that, we look for a weak solution.

In Chapter 3, we give the definition of a 2×2 system of hyperbolic conservation laws and the boundary conditions are considered. Then, we consider the definition of the networks of a system. Later, we present a Lyapunov function and discuss the conditions which guarantee the exponential stability analysis of the equilibrium solution.

In Chapter 4, we present the Saint-Venant equations without a source term in a one-space variable for the modelling of a horizontal reach as well as networks of cascade of n-pools. In both cases, we analyse the exponential stability at a steady-state. Moreover, we select a closed-loop system to design the controller.

2. Preliminary Concepts

In this chapter, we shall talk about a system of hyperbolic conservation laws in a one-space variable. Specifically, we will look at the definition, the classical and the weak solutions of a scalar and a system of conservation laws. Furthermore, we will describe the method of finding Riemann invariants of a non-linear system while for a linear system we will use characteristic decomposition. Finally, we will use the wave equation in a one-space variable to illustrate how the system is decoupled.

2.1 System of conservation laws in one-space variable

2.1.1 Definition. (Bressan, 2002, p. 160) Suppose x is an independent space variable in $\mathbb{R}$ and t is an independent time variable $t \in [0, +\infty)$, then a system of conservation laws is a partial differential equation of the form

$$\partial_t U + \partial_x F(U) = 0, \tag{2.1.1}$$

where $U := (u_1, u_2, \ldots, u_n)^T : \mathbb{R} \times [0, +\infty) \to \mathbb{R}^n$ is a conserved quantity and $F := (f_1, f_2, \ldots, f_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ is a flux function.

In this paper, the symbols ∂_x and ∂_t will denote partial derivatives with respect to x and t, respectively.

2.1.2 Definition (Cauchy Problem)**.** (Solin, 2006, p. 37) A Cauchy problem in a system of conservation laws, i.e. the system (2.1.1), with an initial condition which is defined as

$$U(x, 0) = U_0(x), \quad \forall x \in \mathbb{R}, \tag{2.1.2}$$

where U_0 is a given function $U_0 : \mathbb{R} \to \mathbb{R}^n$.

2.2 Classical solutions

2.2.1 Definition. A classical solution of the system (2.1.1) is a C^1 (i.e. a set of smooth functions) function that satisfies the system (2.1.1).

2.2.2 The method of characteristics for the scalar conservation law. We now analyse the classical solutions using the method of characteristics. For simplicity, we are concerned with a scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \tag{2.2.1}$$

which is obtained from equation (2.1.1) in the case when $n = 1$.

Consider a Cauchy problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(x, 0) = u_0(x). \end{cases} \tag{2.2.2}$$

Assume that u is a classical solution of the equation (2.2.1). Then u is constant along the characteristic curve, $t \to \hat{x}(t)$ which is satisfied by the characteristics equation,

$$\frac{d}{dt}\hat{x} = f'(u(\hat{x}(t), t)), \tag{2.2.3}$$

where $f'(u(\hat{x}(t),t))$ is called a signal speed (Lax et al., 1973, p. 5).

2.2.3 Example. Consider advection equation in a one-space variable,

$$\partial_t u + a\partial_x u = 0, \quad a \in \mathbb{R}.$$

The characteristic curve is

$$\hat{x}(t) = at + c,$$

where c is the initial position of the characteristic curve. If u is a classical solution to the advection equation, then u is constant along the the characteristic curves. To verify this

$$\frac{d}{dt}u(\hat{x},t) = \partial_t u + \frac{d}{dt}\hat{x}\partial_x u = \partial_t u + a\partial_x u = 0.$$

It can be also extend to a Cauchy problem

$$\begin{aligned} \partial_t u + f'(u)\partial_x u = 0 \\ u(x,0) = u_0(x), \end{aligned}$$

where u_0 is locally continuously differentiable function and $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ be a classical solution. Then, the solution u is constant along the characteristic curve

$$\hat{x} = f'(u(\hat{x}(0),0))t + c = f'(u_0(c))t + c,$$

where c is an initial position of the characteristic curve.

We now verify that,

$$\frac{d}{dt}u(\hat{x}(t),t) = \partial_t u + \frac{d}{dt}\hat{x}\partial_x u = \partial_t u + f'(u)\partial_x u = 0.$$

2.2.4 Example. Consider a Burgers equation

$$\partial_t u + u\partial_x u = 0,$$

with an initial condition $u(x,0) = u_0(x)$. Then the characteristic curve is the line

$$\hat{x} = u_0(c)t + c.$$

2.2.5 The method of characteristics for systems of conservation laws. We can also apply the method of characteristics for a system of conservation laws provided that the system is strictly hyperbolic, which is defined in Definition 2.2.7 below.

2.2.6 Lemma (Characteristic curve)**.** (Renardy and Rogers, 1993, p. 70) A characteristic curve for the system of conservation laws (2.1.1) with classical solution $U(x,t)$, is a curve $t \to \hat{x}(t)$ such that the matrix

$$\frac{d}{dt}\hat{x}(t)I - \nabla F(U(\hat{x}(t),t)), \tag{2.2.4}$$

has no inverse.

Notice that the gradient $\nabla F(U)$ is called a Jacobian matrix. It follows that the system (2.1.1) can be written as

$$\partial_t U + A(U)\partial_x U = 0, \tag{2.2.5}$$

where $A(U)$ is the Jacobian matrix of F at the state vector U and has entries,

$$A_{ij}(U) = \frac{\partial f_i}{\partial u_j}, \quad i, j = 1, 2, \dots n.$$

We now introduce the definition of hyperbolicity.

2.2.7 Definition (Hyperbolicity)**.** (Solin, 2006, pages 37, 40) The system (2.2.5) is called hyperbolic if the Jacobian matrix $A(U)$ is diagonalizable and has real eigenvalues only. Moreover, if the eigenvalues are distinct, then the system (2.2.5) is called strictly hyperbolic.

Assume that the system (2.2.5) is strictly hyperbolic so that there are n real and distinct eigenvalues,

$$\lambda_1(U) < \dots < \lambda_n(U),$$

with the corresponding basis for right and left eigenvectors, $r_i(U)$ and $l_i(U)$, respectively, such that,

$$\begin{aligned} A(U)r_i(U) &= \lambda_i(U)r_i(U), \\ l_i^T(U)A(U) &= \lambda_i(U)l_i^T(U), \quad i = 1, \dots, n \end{aligned}$$

and the biorthogonal condition

$$l_j(U) \cdot r_k(U) = \begin{cases} 0 & j \neq k, \\ 1 & j = k \end{cases}$$

are satisfied.

Now the linear and non-linear notions of the system will be defined in Definition 2.2.8 below.

2.2.8 Definition. (Renardy and Rogers, 1993, p. 72-73) A system of conservation laws (2.1.1) is said to be genuinely nonlinear in a region $\Omega \subset \mathbb{R}^n$ if

$$\nabla\lambda_i(U) \cdot r_i(U) \neq 0, \quad i = 1, \dots, n,$$

in Ω and it is linearly degenerate at U if

$$\nabla\lambda_i(U) \cdot r_i(U) = 0, \quad i = 1, \dots, n.$$

Consider a Cauchy problem of a linear system of strictly hyperbolic conservation laws (LeVeque, 1992, p. 58),

$$\partial_t U + A\partial_x U = 0, \quad x \in \mathbb{R}, t \geq 0, \tag{2.2.6}$$
$$U(x, 0) = U_0(x), \quad x \in \mathbb{R}, \tag{2.2.7}$$

where A is a constant $n \times n$ matrix with real and distinct eigenvalues $\lambda_1 < \lambda_2 < \dots < \lambda_n$, and the corresponding constant basis of right and left eigenvectors, r_i and l_i, respectively.

We now want to represent the linear system (2.2.6) as n-linear scalar hyperbolic conservation laws. We will diagonalize a matrix A. Thus,

$$A = PDP^{-1}, \tag{2.2.8}$$

where $D = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ and P is an $n \times n$ matrix whose columns are a basis of right eigenvectors. Then, we multiply the system (2.2.6) by P^{-1} and substitute the diagonalized matrix (2.2.8) to get

$$P^{-1}\partial_t U + P^{-1}PDP^{-1}\partial_x U = 0. \tag{2.2.9}$$

Then, the system (2.2.9) implies that

$$\partial_t(P^{-1}U) + D\partial_x(P^{-1}U) = 0. \tag{2.2.10}$$

If we let $V := P^{-1}U = (v_1, \ldots, v_n)^T$, then the system (2.2.10) implies

$$\partial_t V + D\partial_x V = 0. \tag{2.2.11}$$

Thus, the n scalar conservation laws are

$$\partial_t v_i + \lambda_i \partial_x v_i = 0 \quad i = 1, \ldots, n, \tag{2.2.12}$$

with the initial condition, which is derived as

$$V(x, 0) = P^{-1}U(x, 0) = P^{-1}U_0(x), \qquad x \in \mathbb{R}. \tag{2.2.13}$$

We now specifically define the i-th characteristics curve as

$$\frac{d}{dt}\hat{x}_i(t) = \lambda_i, \quad i = 1, \ldots, n, \tag{2.2.14}$$

where the initial position is $\hat{x}_i(0) = C$. Thus, by using the method of characteristics for a scalar conservation law, the solution of the system (2.2.12) can be obtained as

$$v_i(x, t) = v_i(x - \lambda_i t, 0), \quad i = 1, \ldots, n. \tag{2.2.15}$$

In general, the solution to the system (2.2.6) is

$$U(x, t) = PV(x, t). \tag{2.2.16}$$

Thus,

$$\begin{aligned} U(x, t) &= [r_1 \quad \ldots \quad r_n][v_1(x, t), \ldots, v_n(x, t)]^T \\ &= \sum_{i=1}^{n} v_i(x, t) r_i. \end{aligned}$$

Therefore, by using the equation (2.2.15), we get

$$U(x, t) = \sum_{i=1}^{n} v_i(x - \lambda_i t, 0) r_i. \tag{2.2.17}$$

In the next example we a use wave equation in a one-space variable to illustrate the linear system of hyperbolic conservation laws.

2.2.9 Example (The wave equation)**.** Consider a wave equation in one space variable (LeVeque, 1992, p. 60),

$$\partial_{tt} u = c^2 \partial_{xx} u, \quad \forall x \in \mathbb{R}, \forall t \geq 0, \tag{2.2.18}$$

with a positive constant $c > 0$, called wave speed, with the initial condition

$$\begin{aligned} u(x,0) &= f(x), \\ \partial_t u(x,0) &= g(x), \end{aligned} \tag{2.2.19}$$

where $f(x)$ and $g(x)$ are continuously differentiable functions.

We introduce now the two new variables,

$$u_1 = \partial_x u \qquad \text{and} \qquad u_2 = \partial_t u. \tag{2.2.20}$$

Then the first-partial derivatives of the new variables (2.2.20) with respect to x and t are

$$\begin{cases} \partial_t u_1 = \partial_{xt} u, \\ \partial_x u_1 = \partial_{xx} u \end{cases} \qquad \text{and} \qquad \begin{cases} \partial_t u_2 = \partial_{tt} u, \\ \partial_x u_2 = \partial_{tx} u. \end{cases} \tag{2.2.21}$$

Notice that $\partial_{tt}, \partial_{tx}, \partial_{xt}$ and ∂_{xx} denote the second-partial derivatives. Substituting the equations (2.2.21) into the wave equation (2.2.18), we get a 2×2 system of conservation laws. Thus,

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \partial_x \begin{pmatrix} -u_2 \\ -c^2 u_1 \end{pmatrix} = 0. \tag{2.2.22}$$

Then, the system (2.2.22) can be written in matrix form as,

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}}_{A} \partial_x \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0, \tag{2.2.23}$$

with the initial condition, which is derived from the system (2.2.19) as

$$\begin{aligned} u_1(x,0) &= \partial_x u(x,0) = f'(x), \\ u_2(x,0) &= \partial_t u(x,0) = g(x). \end{aligned} \tag{2.2.24}$$

Thus,

$$\begin{pmatrix} u_1(x,0) \\ u_2(x,0) \end{pmatrix} = \begin{pmatrix} f'(x) \\ g(x) \end{pmatrix}. \tag{2.2.25}$$

The eigenvalues of A are $\lambda_1 = -c$ and $\lambda_2 = c$ with the corresponding basis for the right eigenvectors

$$r_1 = \begin{pmatrix} 1 \\ c \end{pmatrix} \quad \text{and} \quad r_2 = \begin{pmatrix} 1 \\ -c \end{pmatrix}.$$

Moreover, the system (2.2.23) is strictly hyperbolic, since the eigenvalues are real and distinct. We use the solution to a linear system (2.2.17), with the basis of right eigenvectors, r_1 and r_2, to derive the solution to the system (2.2.23) as

$$\begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} = v_1(x+ct,0) \begin{pmatrix} 1 \\ c \end{pmatrix} + v_2(x-ct,0) \begin{pmatrix} 1 \\ -c \end{pmatrix}, \tag{2.2.26}$$

and we use equation (2.2.13) to derive the initial condition

$$\begin{pmatrix} v_1(x,0) \\ v_2(x,0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ c & -c \end{pmatrix}^{-1} \begin{pmatrix} f'(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} \frac{f'(x)}{2} + \frac{g(x)}{2c} \\ \frac{f'(x)}{2} - \frac{g(x)}{2c} \end{pmatrix}. \tag{2.2.27}$$

Substituting the initial conditions to v_1 and v_2 (2.2.27) into the solution (2.2.26), we get

$$u_1(x,t) = \frac{1}{2}\left[f'(x+ct) + \frac{1}{c}g(x+ct) + f'(x-ct) - \frac{1}{c}g(x-ct)\right], \tag{2.2.28}$$

$$u_2(x,t) = \frac{1}{2}\left[cf'(x+ct) + g(x+ct) - cf'(x-ct) + g(x-ct)\right]. \tag{2.2.29}$$

We now use the solutions (2.2.28) and (2.2.29) to find the original solution of the wave equation. Thus, consider one of equations of (2.2.20), then

$$u(x,t) = \int u_1(x,t)dx + h(t), \tag{2.2.30}$$

where h is a continuously differentiable function of t. Then,

$$\partial_t u(x,t) = \int \partial_t u_1(x,t)dx + h'(t).$$

But, from the equation in (2.2.20) and the system (2.2.22), it follows that

$$\begin{aligned} u_2(x,t) &= \int \partial_x u_2(x,t)dx + h'(t), \\ &= u_2(x,t) + h'(t), \end{aligned}$$

which implies that

$$h(t) = k,$$

where k is an arbitrary constant. Hence, we will substitute $h(t)$ into the equation (2.2.30) to get

$$u(x,t) = \int u_1(x,t)dx + k.$$

We now use the initial conditions (2.2.19) and (2.2.25) to get $k = 0$. Therefore, the original solution to the wave equation is

$$u(x,t) = \int u_1(x,t)dx.$$

In general, the solution can be obtained by substituting $u_1(x,t)$ as

$$u(x,t) = \int \frac{1}{2}\left[f'(x+ct) + \frac{1}{c}g(x+ct) + f'(x-ct) - \frac{1}{c}g(x-ct)\right]dx,$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int \left[g(x+ct) - g(x-ct)\right]dx.$$

We now discuss a non-linear system of hyperbolic conservation laws. Consider a non-linear system of strictly hyperbolic conservation laws

$$\partial_t U + A(U)\partial_x U = 0. \tag{2.2.31}$$

Then, we will define the Riemann invariant that we will use to change the coordinate U into a new coordinate, say ξ, so that the system (2.2.31) is changed to n-independent scalar conservation laws similar to the linear case.

2.2.10 Definition (Riemann Invariant)**.** (de Halleux, 2004, p. 16), A function $\xi : \Omega \to \mathbb{R}$ is a Riemann invariant for the system (2.2.31) provided that for every $U \in \Omega$, the vector $\nabla_U \xi(U)$ is a left eigenvector of $A(U) : \exists \lambda : \Omega \to \mathbb{R}$, such that

$$\nabla_U \xi(U) A(U) = D(U) \nabla_U \xi(U), \tag{2.2.32}$$

where $\nabla_U = (\partial_{u_1}, \dots, \partial_{u_n})$ and $D(U) = \text{diag}\{\lambda_1(U), \dots, \lambda_n(U)\}$ (Coron et al., 2007, p. 4).

Thus, there are n-distinct Riemann invariants, $\xi_1, \dots, \xi_n$, for the system (2.2.31) such that it is represented by n-scalar conservation laws as

$$\partial_t \xi_i(U) + \lambda_i(U) \partial_x \xi_i(U) = 0, \quad i = 1, \dots, n. \tag{2.2.33}$$

2.2.11 Remark. For existence of Definition 2.2.10 see Lemma 3.8 in Renardy and Rogers (1993), but we shall discuss the verification as described in de Halleux (2004).

Multiply equation (2.2.31) by $\nabla_U \xi(U)$, then we obtain

$$\nabla_U \xi(U) \partial_t U + \nabla_U \xi(U) A(U) \partial_x U = 0.$$

Using the equation (2.2.32), we have

$$\nabla_U \xi(U) \partial_t U + D(U) \nabla_U \xi(U) \partial_x U = 0.$$

Then applying the chain rule, we get

$$\partial_t \xi(U) + D(U) \partial_x \xi(U) = 0. \tag{2.2.34}$$

Furthermore, the i-th characteristic curves are defined as

$$\frac{d}{dt} \hat{x}_i(t) = \lambda_i(U(\hat{x}(t), t)), \quad i = 1, \dots, n.$$

2.3 Weak solutions

In general, the classical solution to the non-linear system of conservation laws can be discontinuous in finite time due to characteristics intersecting. Thus, we introduce a notion of a weak solution.

2.3.1 Definition (Weak solution)**.** (Renardy and Rogers, 1993, p. 78) A function U is called a weak solution to the Cauchy problem (2.1.1) with (2.1.2) if, for each locally integrable function $\phi \in C_0^1(\mathbb{R} \times [0, +\infty))$,

$$\int_0^\infty \int_{-\infty}^\infty [\phi(x,t) \partial_t U(x,t) + \phi(x,t) \partial_x F(U(x,t))] dx dt = 0. \tag{2.3.1}$$

Notice that a compact support set C_0^1 on $\mathbb{R} \times [0, +\infty)$ is defined as

$$C_0^1(\mathbb{R} \times [0, +\infty)) := \{\phi \in C^1(\mathbb{R} \times [0, +\infty)) | \exists r > 0 \quad \text{such that, support}(\phi) \subset B_r(\mathbb{R} \times [0, +\infty))\},$$

and a function ϕ is called locally integrable if for each compact support domain C_0^1, ϕ is integrable. Equation (2.3.1) is equivalently expressed as

$$\int_0^\infty \int_{-\infty}^\infty [U(x,t) \partial_t \phi(x,t) + F(U(x,t)) \partial_x \phi(x,t)] dx dt + \int_{-\infty}^\infty U_0(x) \phi(x,0) dx = 0.$$

2.3.2 Remark. Any classical solution is a weak solution.

In general, we have seen the definition and classification of systems of hyperbolic conservation laws. Moreover, we presented the Riemann invariants for the non-linear system of conservation laws.

So far we have seen systems of hyperbolic conservation laws in infinite space intervals. But, assume we have a situation like irrigation canals or road traffic systems or waterways etc. In all these cases, we observe that space is limited to a finite space interval. Therefore, we consider the boundary conditions.

For a finite space interval, $[0, L]$, we define the boundary conditions of the systems of hyperbolic conservation laws (2.1.1) as

$$B(U(0,t), U(L,t)) = 0, \quad t \in [0, +\infty),$$

where B is a map.

In the next chapter, we shall discuss the stability of the networks of 2×2 systems of hyperbolic conservation laws on a finite spatial variable.

3. Boundary Controls for Systems of Hyperbolic Conservation Laws

In this chapter, we shall discuss boundary controls for a system of hyperbolic conservation laws in one-space variable. We introduce the Lyapunov function to analyse the stability of the solution of a linearised system. We then look for a condition for stability of networks of 2×2 systems of hyperbolic conservation laws.

3.1 2×2 systems of hyperbolic conservation laws

Let t be an independent time variable in the interval $[0, +\infty)$, and x be an independent space variable on a bounded interval $[0, L]$, where L is a positive constant. We now consider 2×2 systems of hyperbolic conservation laws in a one-space variable on a non-empty connected open set $\Omega \subset \mathbb{R}^2$,

$$\partial_t U + \partial_x F(U) = 0, \tag{3.1.1}$$

where $U : [0, L] \times [0, +\infty) \to \Omega$ is a conserved quantity and $F : \Omega \to \mathbb{R}^2$ is a flux function (Coron et al., 2007, p. 3).

In this case, we assume that the solutions to the system (3.1.1) are smooth over $[0, L] \times [0, +\infty)$ and define the initial condition to the system (3.1.1) as

$$U(x, 0) = U_0(x), \quad x \in [0, L]. \tag{3.1.2}$$

Moreover, the boundary conditions to the system (3.1.1) are chosen to be the same as in Coron et al. (2007). Thus,

$$B_0(U(0, t), c_0(t)) = 0 \qquad \text{and} \qquad B_L(U(L, t), c_L(t)) = 0, \quad t \in [0, +\infty), \tag{3.1.3}$$

where the map, $B_0, B_L : \Omega \times \mathbb{R} \to \mathbb{R}$ with the control actions, $c_0, c_L : [0, +\infty) \to \mathbb{R}$.

In the next section, we shall discuss networks of 2×2 systems of hyperbolic conservation laws.

3.2 Networks of 2×2 systems of hyperbolic conservation laws

Consider the stream of water in an irrigation canal or the traffic of vehicles along the road system, which is described as a network in Figure 3.1 below. In both situations, we observe the intersection of flow, which is called confluence in an irrigation canal or a road junction in road systems. We assume the point of intersection as a vertex and the transport of water between pools in irrigation canals or the transport of vehicles on the road in road systems as an edge connecting the two vertices. We now introduce the definition of network.

3.2.1 Definition (Network)**.** (West, 2000, p. 176) A network is a digraph (directed graph) with a non-negative capacity $C(e)$ on each edge and a distinguished source vertex r and sink vertex s. Vertices are also called nodes.

The dynamics of the networks are described as a system of hyperbolic conservation laws for each edge as

$$\partial_t U_i + \partial_x F(U_i) = 0, \quad i = 1, \dots n, \tag{3.2.1}$$

where $U_i := \begin{pmatrix} u_i \\ u_{n+i} \end{pmatrix}$ is conserved quantity, and F is a flux function (Bastin et al., 2008a, p. 5); (Colombo et al., 2008, p. 2).

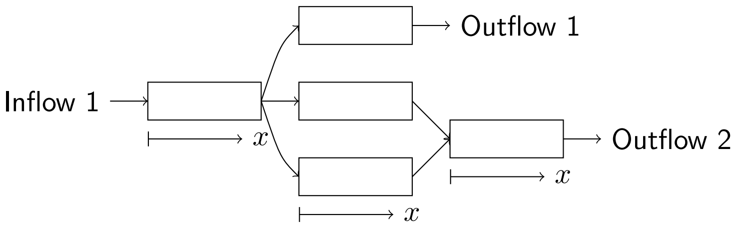

Figure 3.1: Networks: irrigation canal or road system.

In general, the system (3.2.1) can be written in a compact vector form

$$\partial_t U + \partial_x F(U) = 0, \tag{3.2.2}$$

where $U := (u_1, \dots, u_{2n})^T$ and a map $F(U) := (f_1(U_1), \dots, f_1(U_n), f_2(U_1), \dots, f_2(U_n))^T$.

3.2.2 Steady-state. (Bastin et al., 2008a, p. 5) A steady-state solution to the system (3.2.1) is a constant solution

$$U_i(x,t) = U_i^*, \quad i = 1, \dots, n$$

such that the system (3.2.1) is satisfied.

3.2.3 Characteristics form. We can find the Jacobian matrix, A_i $(i = 1, \dots, n)$, such that the system (3.2.1) can be rewritten as

$$\partial_t U_i + A_i(U_i)\partial_x U_i = 0, \quad i = 1, \dots, n. \tag{3.2.3}$$

We assume that the system (3.2.3) is strictly hyperbolic so that each A_i admits two distinct real eigenvalues. Thus, the eigenvalues are

$$\lambda_1(U_1), \dots, \lambda_n(U_n), \lambda_{n+1}(U_1), \dots, \lambda_{2n}(U_n).$$

3.2.4 Riemann Invariants. (Bastin et al., 2008b, p. 1455) We want the system (3.2.3) to be decoupled into $2n$ scalar conservation laws, so that for any steady state solution U_i^*, there are $2n$ Riemann invariants, $\xi_1, \dots, \xi_{2n}$, such that

$$\partial_t \xi + \Lambda(\xi)\partial_x \xi = 0, \tag{3.2.4}$$

where the Riemann coordinate, $\xi := (\xi_1, \dots, \xi_{2n})^T$ and the diagonal matrix,

$$\Lambda(\xi) := \text{diag}\{\lambda_1(\xi_1, \xi_{n+1}), \dots, \lambda_n(\xi_n, \xi_{2n}), \lambda_{n+1}(\xi_1, \xi_{n+1}), \dots, \lambda_{2n}(\xi_n, \xi_{2n})\}.$$

Furthermore, a desired steady state solution to the system (3.2.4) is zero. Thus,

$$\xi_i^* = 0, \quad i = 1, \dots, 2n.$$

3.2.5 Linearisation. The linearised characteristic form around the steady-state is a linear system of hyperbolic conservation laws

$$\partial_t \xi + \Lambda \partial_x \xi = 0, \tag{3.2.5}$$

where $\Lambda := \Lambda(0) = \text{diag}\{\lambda_i : i = 1, \dots, 2n\}$.

The diagonal matrix Λ has non-zero distinct real diagonal entries. The physical meaning of these entries is characteristic velocities of the moving of waves to the left and rightward directions as described in Figure 3.2 below.

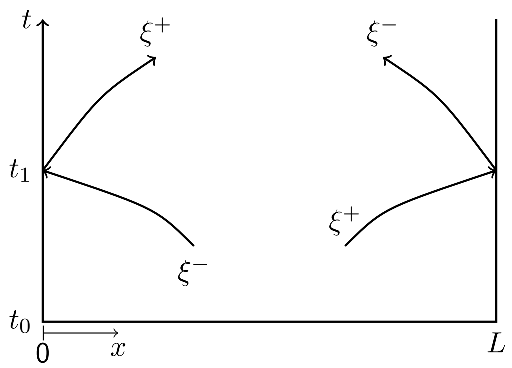

Figure 3.2: The moving waves to the left and rightward directions.

Therefore, without loss of generality, we can redefine the diagonal matrix Λ as

$$\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_{2n}\}, \tag{3.2.6}$$

where $\lambda_i > 0$ $(i = 1, 2, \dots, n)$ and $\lambda_i < 0$ $(i = n+1, \dots, 2n)$. Then, the Riemann coordinate is

$$\xi = ({\xi^+}^T, {\xi^-}^T), \tag{3.2.7}$$

where $\xi^+ = (\xi_1, \dots, \xi_n)$ and $\xi^- = (\xi_{n+1}, \dots, \xi_{2n})$. Moreover, the diagonal matrix (3.2.6) can be written as

$$\Lambda = \text{diag}\{\Lambda^+, -\Lambda^-\}, \tag{3.2.8}$$

where $\Lambda^+ = \text{diag}\{\lambda_1, \dots, \lambda_n\}$ and $\Lambda^- = \text{diag}\{|\lambda_{n+1}|, \dots, |\lambda_{2n}|\}$. Thus, the system (3.2.5) can be rewritten as

$$\partial_t \begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix} + \begin{pmatrix} \Lambda^+ & 0 \\ 0 & -\Lambda^- \end{pmatrix} \partial_x \begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix} = 0. \tag{3.2.9}$$

We are dealing with analysing the exponential stability of a steady-state solution (i.e. $\xi^* \equiv 0$) of the system (3.2.9), with the linear boundary conditions, which is described as

$$\begin{pmatrix} \xi^+(0,t) \\ \xi^-(L,t) \end{pmatrix} = \underbrace{\begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}}_{K} \begin{pmatrix} \xi^+(L,t) \\ \xi^-(0,t) \end{pmatrix}, \tag{3.2.10}$$

and an initial condition of the form

$$\xi(x,0) = \xi^0(x), \quad \forall x \in (0, L). \tag{3.2.11}$$

We now look for the definition of a solution to the Cauchy problem, (3.2.9) with (3.2.10) and (3.2.11).

3.2.6 Definition. (Diagne et al., 2012, p. 110) Suppose $\xi^0 \in L^2((0,L);\mathbb{R}^{2n})$. Then a map $\xi : (0,L) \times [0,+\infty) \to \mathbb{R}^{2n}$ is a solution of a Cauchy problem, (3.2.9) with (3.2.10) and (3.2.11), if $\xi \in C^0(L^2((0,L);\mathbb{R}^{2n}),[0,+\infty))$ is such that, for every $\psi = (\psi_+^T, \psi_-^T)^T \in C^1(L^2([0,L];\mathbb{R}^{2n}),[0,+\infty))$ with compact support and satisfying

$$\begin{pmatrix}\psi_+(L,t)\\ \psi_-(0,t)\end{pmatrix} = \begin{pmatrix}(\Lambda^+)^{-1}K_{00}^T\Lambda^+ & (\Lambda^+)^{-1}K_{01}^T\Lambda^- \\ (\Lambda^-)^{-1}K_{10}^T\Lambda^+ & (\Lambda^-)^{-1}K_{11}^T\Lambda^-\end{pmatrix}\begin{pmatrix}\psi_+(0,t)\\ \psi_-(L,t)\end{pmatrix},$$

we have

$$\int_0^{+\infty}\int_0^L (\partial_t\psi^T + \partial_x\psi^T\Lambda)\xi dx dt + \int_0^L \psi^T(x,0)\xi^0(x)dx = 0.$$

i.e. ψ^T satisfying a weak solution condition.

Under this definition, we now discuss the uniqueness of the solution in Diagne et al. (2012), which is described as, for every initial condition $\xi^0 \in L^2((0,L);\mathbb{R}^{2n})$ the Cauchy problem, (3.2.9) with (3.2.10) and (3.2.11), has a unique solution. Furthermore, for any $T > 0$, there exists $C(T) > 0$ such that, for every $\xi^0 \in L^2((0,L);\mathbb{R}^{2n})$, the solution to the Cauchy problem, (3.2.9) with (3.2.10) and (3.2.11) satisfies

$$\|\xi(\cdot,t)\|_{L^2((0,L);\mathbb{R}^{2n})} \le C(T)\|\xi^0\|_{L^2((0,L);\mathbb{R}^{2n})}, \quad \forall t \in [0,T].$$

We now introduce the definition of exponential stability of a linear system of hyperbolic conservation laws.

3.2.7 Definition (Exponentially stable)**.** (Bastin et al., 2008b, p. 1456) The linear hyperbolic system (3.2.9), with the linear boundary condition (3.2.10), is exponentially stable (in L^2-norm) if there exist $v > 0$ and $C > 0$ such that, for every initial condition

$$\xi(x,0) = \xi^0(x) \in L^2((0,L);\mathbb{R}^{2n}), \tag{3.2.12}$$

the solution to the Cauchy problem, (3.2.9) with (3.2.10) and (3.2.11) satisfies

$$\|\xi(\cdot,t)\|_{L^2((0,L);\mathbb{R}^{2n})} \le Ce^{-vt}\|\xi^0\|_{L^2((0,L);\mathbb{R}^{2n})}. \tag{3.2.13}$$

In the next section, we shall discuss the Lyapunov stability analysis of the linearised systems of hyperbolic conservation laws.

3.3 Lyapunov stability analysis of the linearised system

In this section, we shall look for sufficient conditions for exponential stability of the linearised systems of hyperbolic conservation laws using Lyapunov function.

We now rewrite a Cauchy problem, (3.2.9) with (3.2.10) and (3.2.11) as

$$\partial_t\xi + \Lambda\partial_x\xi = 0, \quad \forall t \in [0,+\infty), x \in (0,L), \tag{3.3.1}$$
$$K_0\xi(0,t) + K_1\xi(L,t) = 0, \quad \forall t \in [0,+\infty), \tag{3.3.2}$$
$$\xi(x,0) = \xi^0(x), \quad x \in (0,L). \tag{3.3.3}$$

with

$$K_0 := \begin{pmatrix} I & -K_{01} \\ 0 & -K_{11} \end{pmatrix}, \quad K_1 := \begin{pmatrix} -K_{00} & 0 \\ -K_{10} & I \end{pmatrix}.$$

Then, the candidate Lyapunov function is defined as (Bastin et al., 2008a, p. 7),

$$V = \int_0^L \xi^T P(x) \xi dx, \tag{3.3.4}$$

where a diagonal matrix,

$$P(x) := \text{diag}\{p_i e^{-\sigma_i \mu x} : i = 1, \dots, 2n\},$$

for positive real numbers $\mu > 0$ and $p_i > 0$ and $\sigma_i := \text{sign}(\lambda_i)$. The time derivative of the Lyapunov function along the solution of the Cauchy problem,(3.3.1) with (3.3.2) and (3.3.3) is

$$\begin{aligned} \dot{V} &= \int_0^L \partial_t(\xi^T P(x)\xi) dx, \\ &= \int_0^L \left[\partial_t(\xi^T) P(x) \xi + \xi^T P(x) \partial_t(\xi)\right] dx. \end{aligned}$$

But, by using (3.3.1), we obtain

$$\begin{aligned} \dot{V} &= \int_0^L \left[-\partial_x(\xi^T) \Lambda^T P(x) \xi - \xi^T P(x) \Lambda \partial_x(\xi)\right] dx, \\ &= \int_0^L \left[-\partial_x(\xi^T) P(x) \Lambda \xi - \xi^T P(x) \Lambda \partial_x(\xi)\right] dx, \\ &= -\int_0^L \partial_x \left[\xi^T P(x) \Lambda \xi\right] dx + \int_0^L \xi^T P'(x) \Lambda \xi dx, \end{aligned}$$

where in the last line, we have used,

$$\partial_x \left[\xi^T P(x) \Lambda \xi\right] = \partial_x(\xi^T) P(x) \Lambda \xi + \xi^T P(x) \Lambda \partial_x(\xi) + \xi^T P'(x) \Lambda \xi.$$

But,

$$P'(x) = \text{diag}\{-\sigma_i p_i \mu e^{-\sigma_i \mu x} : i = 1, \dots, 2n\}.$$

Thus,

$$\begin{aligned} P'(x)\Lambda &= -\text{diag}\{\sigma_i p_i \lambda_i \mu e^{-\sigma_i \mu x} : i = 1, \dots, 2n\}, \\ &= -\mu(\text{diag}\{p_i |\lambda_i| e^{-\sigma_i \mu x} : i = 1, \dots, 2n\}), \\ &= -\mu P(x) |\Lambda|, \end{aligned}$$

where $|\Lambda| := \text{diag}\{|\lambda_i| : i = 1, \dots, 2n\}$. Hence,

$$\begin{aligned} \dot{V} &= -\int_0^L \partial_x \left[\xi^T P(x) \Lambda \xi\right] dx - \int_0^L \xi^T \mu P(x) |\Lambda| \xi dx, \\ &= -\left[\xi^T Q_0(x) \xi\right]_0^L - \int_0^L \xi^T \mu Q_1(x) \xi dx, \\ &= -\left[\xi^T(L,t) Q_0(L) \xi(L,t) - \xi^T(0,t) Q_0(0) \xi(0,t)\right] - \int_0^L \xi^T \mu Q_1(x) \xi dx, \end{aligned}$$

where

$$Q_0(x) := P(x)\Lambda \quad \text{and} \quad Q_1(x) := P(x)|\Lambda|.$$

Thus, the system (3.3.1), with the boundary condition (3.3.2), is exponentially stable if $\dot{V}$ is a negative definite quadratic form. The following theorem gives conditions which guarantee the exponential stability of the solutions of the system (3.3.1) with the boundary condition (3.3.2).

3.3.1 Theorem. *(Bastin et al., 2008a, p. 8) The system (3.3.1) with the boundary condition (3.3.2) is exponentially stable if there exist $\mu > 0$ and $p_i > 0$ $(i = 1, \ldots, 2n)$ such that,*

C1: The boundary quadratic form $\xi^T(L,t)Q_0(L)\xi(L,t) - \xi^T(0,t)Q_0(0)\xi(0,t)$ is positive definite under the constraint of the linear boundary condition $K_0\xi(0,t) + K_1\xi(L,t) = 0, \quad \forall t \in [0, +\infty)$ along the solutions of the system, (3.2.9) with (3.2.10) and (3.2.11);

C2: The matrix $\mu Q_1(x)$ is positive definite for all $x \in (0, L)$.

3.3.2 Remark. Boundary conditions that satisfy condition *C1* are called *Dissipative Boundary Conditions*. Moreover, *C1* is satisfied if, and only if, the leading principal minors of order $> 4n$ of the matrix

$$\begin{pmatrix} 0 & K_0 & K_1 \\ -K_0^T & -Q_0(0) & 0 \\ -K_1^T & 0 & Q_0(L) \end{pmatrix},$$

are strictly positive [Appendix A.1 and A.2].

3.3.3 Remark. Condition *C2* is clearly satisfied, since $Q_1(x)$ is a diagonal matrix of strictly positive diagonal entries.

3.3.4 Definition. (Bastin et al., 2008a, p. 8) For a positive integer p, let D_p denote the set of diagonal $p \times p$ real matrices with strictly positive diagonal entries. Then the norm over the matrix K

$$\rho(K) := \text{inf}\{\|\Delta K \Delta^{-1}\|, \Delta \in D_{2n}\}, \tag{3.3.5}$$

where $\|\cdot\|$ denote the usual matrix 2-norm [Appendix A.1].

3.3.5 Theorem. *(Bastin et al., 2008a, p. 8) If $\rho(K) < 1$, then the linear hyperbolic system (3.2.9) with the linear boundary condition (3.2.10) is exponentially stable.*

Proof. Consider the system (3.2.9), then in this case, we derive the candidate Lyapunov function (3.3.4). Thus, we take the matrix $P(x)$ from equation (3.3.4),

$$\begin{aligned} P(x) &= \text{diag}\{p_i e^{-\sigma_i \mu x} : i = 1, \ldots, 2n\}, \\ &= \text{diag}\{p_1 e^{-\mu x}, \ldots, p_n e^{-\mu x}, p_{n+1} e^{\mu x}, \ldots, p_{2n} e^{\mu x}\}, \\ &= \text{diag}\{P_0 e^{-\mu x}, P_1 e^{\mu x}\}, \end{aligned}$$

where $P_0, P_1 \in D_n$ and $\mu > 0$. Then, the candidate Lyapunov function is simplified as

$$\begin{aligned} V &= \int_0^L \xi^T P(x) \xi dx, \\ V &= \int_0^L \begin{pmatrix} \xi^{+T} & \xi^{-T} \end{pmatrix} \begin{pmatrix} P_0 e^{-\mu x} & 0 \\ 0 & P_1 e^{\mu x} \end{pmatrix} \begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix} dx, \\ V &= \int_0^L \left[(\xi^{+T} P_0 \xi^+) e^{-\mu x} + (\xi^{-T} P_1 \xi^-) e^{\mu x} \right] dx. \end{aligned}$$

The time derivative of V is obtained as

$$\begin{aligned}
\dot{V} &= \int_0^L \partial_t \left[({\xi^+}^T P_0 \xi^+) e^{-\mu x} + ({\xi^-}^T P_1 \xi^-) e^{\mu x} \right] dx, \\
&= \int_0^L \left[\partial_t({\xi^+}^T) P_0 \xi^+ e^{-\mu x} + {\xi^+}^T P_0 \partial_t(\xi^+) e^{-\mu x} \right] dx + \int_0^L \left[\partial_t({\xi^-}^T) P_1 \xi^- e^{\mu x} + {\xi^-}^T P_1 \partial_t(\xi^-) e^{\mu x} \right] dx, \\
&= \int_0^L \left[-\partial_x({\xi^+}^T) {\Lambda^+}^T P_0 \xi^+ e^{-\mu x} - {\xi^+}^T P_0 \Lambda^+ \partial_x(\xi^+) e^{-\mu x} \right] dx \\
&+ \int_0^L \left[\partial_x({\xi^-}^T) {\Lambda^-}^T P_1 \xi^- e^{\mu x} + {\xi^-}^T P_1 \Lambda^- \partial_x(\xi^-) e^{\mu x} \right] dx, \\
&= \int_0^L \left[-\partial_x({\xi^+}^T) P_0 \Lambda^+ \xi^+ e^{-\mu x} - {\xi^+}^T P_0 \Lambda^+ \partial_x(\xi^+) e^{-\mu x} \right] dx \\
&+ \int_0^L \left[\partial_x({\xi^-}^T) P_1 \Lambda^- \xi^- e^{\mu x} + {\xi^-}^T P_1 \Lambda^- \partial_x(\xi^-) e^{\mu x} \right] dx.
\end{aligned}$$

But,

$$\begin{aligned}
&- \partial_x({\xi^+}^T) P_0 \Lambda^+ \xi^+ e^{-\mu x} - {\xi^+}^T P_0 \Lambda^+ \partial_x(\xi^+) e^{-\mu x} + \partial_x({\xi^-}^T) P_1 \Lambda^- \xi^- e^{\mu x} + {\xi^-}^T P_1 \Lambda^- \partial_x(\xi^-) e^{\mu x} \\
&= - \left[\partial_x({\xi^+}^T P_0 \Lambda^+ \xi^+ e^{-\mu x}) - \partial_x({\xi^-}^T P_1 \Lambda^- \xi^- e^{\mu x}) \right] - \mu \left[({\xi^+}^T P_0 \Lambda^+ \xi^+) e^{-\mu x} + ({\xi^-}^T P_1 \Lambda^- \xi^-) e^{\mu x}) \right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\dot{V} &= - \int_0^L \left[\partial_x({\xi^+}^T P_0 \Lambda^+ \xi^+ e^{-\mu x}) - \partial_x({\xi^-}^T P_1 \Lambda^- \xi^- e^{\mu x}) \right] dx \\
&- \int_0^L \mu \left[({\xi^+}^T P_0 \Lambda^+ \xi^+) e^{-\mu x} + ({\xi^-}^T P_1 \Lambda^- \xi^-) e^{\mu x}) \right] dx, \\
&= - \left[{\xi^+}^T P_0 \Lambda^+ \xi^+ e^{-\mu x} \right]_0^L + \left[{\xi^-}^T P_1 \Lambda^- \xi^- e^{\mu x} \right]_0^L - \int_0^L \xi^T \mu P(x) |\Lambda| \xi dx.
\end{aligned}$$

Therefore, simply the time derivative of V is written as

$$\dot{V} = \dot{V}_1 + \dot{V}_2,$$

where

$$\begin{aligned}
\dot{V}_1 &= - \left[{\xi^+}^T P_0 \Lambda^+ \xi^+ e^{-\mu x} \right]_0^L + \left[{\xi^-}^T P_1 \Lambda^- \xi^- e^{\mu x} \right]_0^L, \\
\dot{V}_2 &= - \int_0^L \xi^T \mu P(x) |\Lambda| \xi dx.
\end{aligned}$$

We now want to show that the boundary condition (3.2.10) is dissipative; for that, we need to select the appropriate P_0, P_1 and μ such that the quadratic form $\dot{V}_1$ is negative definite. By using the boundary condition (3.2.10), for simplicity, we denote

$$\begin{pmatrix} \xi^+(0,t) \\ \xi^-(0,t) \end{pmatrix} := \begin{pmatrix} \xi_0^+ \\ \xi_0^- \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \xi^+(L,t) \\ \xi^-(L,t) \end{pmatrix} := \begin{pmatrix} \xi_L^+ \\ \xi_L^- \end{pmatrix},$$

we obtain

$$\begin{aligned}
\dot{V}_1 =& - \left[\xi^{+T} P_0 \Lambda^+ \xi^+ e^{-\mu x}\right]_0^L + \left[\xi^{-T} P_1 \Lambda^- \xi^- e^{\mu x}\right]_0^L, \\
=& - \left(\xi_L^{+T} P_0 \Lambda^+ \xi_L^+ e^{-\mu L}\right) + \left(\xi_0^{+T} P_0 \Lambda^+ \xi_0^+\right) + \left(\xi_L^{-T} P_1 \Lambda^- \xi_L^- e^{\mu L}\right) - \left(\xi_0^{-T} P_1 \Lambda^- \xi_0^-\right), \\
=& - \left(\xi_L^{+T} P_0 \Lambda^+ \xi_L^+ e^{-\mu L} + \xi_0^{-T} P_1 \Lambda^- \xi_0^-\right) + \left(\xi_0^{+T} P_0 \Lambda^+ \xi_0^+\right) + \left(\xi_L^{-T} P_1 \Lambda^- \xi_L^- e^{\mu L}\right), \\
=& - \left(\xi_L^{+T} P_0 \Lambda^+ \xi_L^+ e^{-\mu L} + \xi_0^{-T} P_1 \Lambda^- \xi_0^-\right) \\
&+ \left(\xi_L^{+T} K_{00}^T + \xi_0^{-T} K_{01}^T\right) P_0 \Lambda^+ \left(K_{00} \xi_L^+ + K_{01} \xi_0^-\right) \qquad (3.3.6) \\
&+ \left(\xi_L^{+T} K_{10}^T + \xi_0^{-T} K_{11}^T\right) P_1 \Lambda^- \left(K_{10} \xi_L^+ + K_{11} \xi_0^-\right) e^{\mu L}. \qquad (3.3.7)
\end{aligned}$$

By assumption, $\rho(K) < 1$, then there exist diagonal matrices, $D_0, D_1 \in D_n$ and $\Delta := \text{diag}\{D_0, D_1\}$ such that

$$\|\Delta K \Delta^{-1}\| < 1. \qquad (3.3.8)$$

Now the matrices P_0 and P_1 are selected such that $P_0 \Lambda^+ = D_0^2$ and $P_1 \Lambda^- = D_1^2$ and introduce the new coordinate

$$z := (z_0, z_1)^T,$$

where $z_0 := D_0 \xi_L^+$ and $z_0 := D_1 \xi_0^-$. Then, using the inequality (3.3.8) for expression, we get

$$\begin{aligned}
&(\xi_L^{+T} K_{00}^T + \xi_0^{-T} K_{01}^T) P_0 \Lambda^+ (K_{00} \xi_L^+ + K_{01} \xi_0^-) \\
&+ (\xi_L^{+T} K_{10}^T + \xi_0^{-T} K_{11}^T) P_1 \Lambda^- (K_{10} \xi_L^+ + K_{11} \xi_0^-) \\
&= \|\Delta K \Delta^{-1} z\|^2 \\
&< \|z\|^2 = \xi_L^{+T} P_0 \Lambda^+ \xi_L^+ + \xi_0^{-T} P_1 \Lambda^- \xi_0^-. \qquad (3.3.9)
\end{aligned}$$

Then, by using the inequality (3.3.9) into $\dot{V}_1$ with sufficiently small $\mu > 0$, we obtain

$$\dot{V}_1 < - \left(\xi_L^{+T} P_0 \Lambda^+ \xi_L^+ e^{-\mu L} + \xi_0^{-T} P_1 \Lambda^- \xi_0^-\right) + \xi_L^{+T} P_0 \Lambda^+ \xi_L^+ + \xi_0^{-T} P_1 \Lambda^- \xi_0^-.$$

It implies that,

$$\dot{V}_1 < 0.$$

Thus, the quadratic form, $\dot{V}_1$, is negative definite.

Furthermore, for any $\mu > 0$, the quadratic form $\dot{V}_2$ can be derived as

$$\begin{aligned}
\dot{V}_2 &= - \int_0^L \xi^T \mu P(x) |\Lambda| \xi dx, \\
&< - \int_0^L \xi^T \mu \gamma P(x) \xi dx, \\
&< -\mu\gamma V = -\alpha V,
\end{aligned}$$

where $\gamma = \min\{|\lambda_i| : i = 1, \ldots, 2n\}$ and $\alpha = \mu\gamma > 0$. Therefore,

$$\dot{V} = \dot{V}_1 + \dot{V}_2 < -\alpha V.$$

It follows that the solutions of the system (3.2.9) with the boundary condition (3.2.10) exponentially converge to 0 in L^2-norm. □

3.3.6 Remark. To show that the exponential stability of the solutions of the system (3.2.9) is equivalent to finding the appropriate boundary condition that satisfies the condition $\rho(K) < 1$. We apply this ideal in Chapter 4, for the linearised system of Saint-Venant equations.

In this chapter, we have briefly explained the condition for exponential stability of steady-state solution of a linearised systems of hyperbolic conservation laws, which was, $\rho(K) < 1$. But, it can be extended to the non-linear hyperbolic system. The notations defined in Section 3.3 can be also used for a non-linear hyperbolic system around the origin (i.e. $\xi^* \equiv 0$) as

$$\partial_t \begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix} + \begin{pmatrix} \Lambda^+(\xi) & 0 \\ 0 & -\Lambda^-(\xi) \end{pmatrix} \partial_x \begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix} = 0, \tag{3.3.10}$$

with the non-linear boundary conditions of the form

$$\begin{pmatrix} \xi^+(0,t) \\ \xi^-(L,t) \end{pmatrix} = K(\xi) \begin{pmatrix} \xi^+(L,t) \\ \xi^-(0,t) \end{pmatrix}, \tag{3.3.11}$$

where the non-linear map $K : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. The discussion for exponentially stable of the non-linear hyperbolic system is left for further study.

In the next chapter, we shall apply the exponential stability analysis for a linearised hyperbolic system to stabilise the linearised system of Saint-Venant equations.

4. Application to the Linearised System of Saint-Venant Equations

In this chapter, we shall present an open-channel water flow model. This model is represented by the Saint-Venant equations. Then, we will discuss the condition of exponential stability of solutions of a linearized system of Saint-Venant equations. We shall also look at the networks of the channel made up of n pools. We begin our discussion by considering the model on one compartment of the channel (i.e. a horizontal reach or a pool).

4.1 Saint-Venant equations

The flow we consider in the open-channel is unsteady which means the depth and velocity of water vary with time. In 1871, Saint-Venant developed the equations (4.1.1) shown below. He employed the principles of mass balance and momentum balance to derive the equations.

We now consider a one-dimensional horizontal reach, which is described in Figure 4.1 below. In this model, we assume the friction of water with the wall of the horizontal reach is neglected and a slope of the bed is zero. Moreover, we are concerned with a rectangular cross-section and there is a control action at the outflow. This action operate by an automated gate, which can be seen in Figure 4.1, at $x = L$. Finally, we assume the flow is sub-critical, see Section 4.1.4 below. Thus, the dynamics of the system is described by Saint-Venant equations as in de Halleux; G. Bastin (2002) and it is simplified as

$$\partial_t \begin{pmatrix} H \\ V \end{pmatrix} + \partial_x \begin{pmatrix} VH \\ gH + \frac{V^2}{2} \end{pmatrix} = 0, \quad t \in [0, +\infty), x \in [0, L], \tag{4.1.1}$$

where $H(x,t)$ denotes the water depth (at a point x and time t) and $V(x,t)$ is a horizontal water velocity (at a point x and time t), while the gravitational constant is g and the reach length is L.

The water flow rate with a unit width, say Q, is described as

$$Q(x,t) = V(x,t)H(x,t). \tag{4.1.2}$$

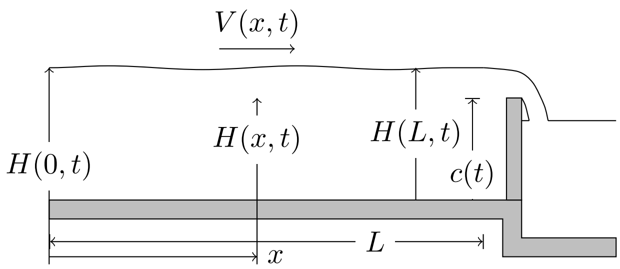

Figure 4.1: The horizontal reach: $V(x,t)$ is horizontal water velocity, $H(x,t)$ is water depth, $c(t)$ is a gate opening control action and L is length of the reach.

4.1.1 Steady-state. A steady-state solution to the system (4.1.1) is a constant state (H^*, V^*) with respect to position and time such that the system (4.1.1) is satisfied.

4.1.2 Linearisation. We want to analyse the linear system of Saint Venant equations. Hence, it is necessary to linearise the system. We now define new variables h and v with the state variables H and V, respectively, around the steady state H^* and V^* such that

$$h(x,t) := H(x,t) - H^* \qquad \text{and} \qquad v(x,t) := V(x,t) - V^*, \tag{4.1.3}$$

and the system (4.1.1) is satisfied. To proceed with the linearisation, we derive the first-partial derivatives of the new variables with respect to x and t as

$$\partial_t h = \partial_t H, \quad \partial_t v = \partial_t V, \quad \partial_x h = \partial_x H \quad \text{and} \quad \partial_x v = \partial_x V. \tag{4.1.4}$$

We now substitute the new variables (4.1.3) and the partial derivatives (4.1.4) into the system (4.1.1) to get

$$\begin{aligned} &\partial_t h + V^* \partial_x h + H^* \partial_x v + \partial_x vh = 0, \\ &\partial_t v + g\partial_x h + V^* \partial_x v + \frac{1}{2}\partial_x v^2 = 0. \end{aligned} \tag{4.1.5}$$

Then, we neglect the non-linear terms from the system of equations (4.1.5) to obtain

$$\begin{aligned} &\partial_t h + V^* \partial_x h + H^* \partial_x v = 0, \\ &\partial_t v + g\partial_x h + V^* \partial_x v = 0. \end{aligned} \tag{4.1.6}$$

4.1.3 Characteristic form. The system (4.1.6) can be rewritten in vector-matrix form

$$\partial_t \begin{pmatrix} h \\ v \end{pmatrix} + \underbrace{\begin{pmatrix} V^* & H^* \\ g & V^* \end{pmatrix}}_{A} \partial_x \begin{pmatrix} h \\ v \end{pmatrix} = 0, \quad t \in [0, +\infty), x \in [0, L]. \tag{4.1.7}$$

We now assume the system (4.1.7) is strictly hyperbolic, then the constant matrix, A, has two distinct real eigenvalues,

$$\lambda_1 = V^* + \sqrt{gH^*} \quad \text{and} \quad -\lambda_2 = V^* - \sqrt{gH^*}. \tag{4.1.8}$$

We use the method of finding the new variables, say ξ_1 and ξ_2, as described in Section 2.2.5. Thus, the new variables are

$$\xi_1(x,t) = v(x,t) + h(x,t)\sqrt{\frac{g}{H^*}}, \tag{4.1.9}$$

$$\xi_2(x,t) = v(x,t) - h(x,t)\sqrt{\frac{g}{H^*}}. \tag{4.1.10}$$

Equivalently, we define

$$h(x,t) = \frac{\xi_1(x,t) - \xi_2(x,t)}{2}\sqrt{\frac{H^*}{g}}, \tag{4.1.11}$$

$$v(x,t) = \frac{\xi_1(x,t) + \xi_2(x,t)}{2}, \tag{4.1.12}$$

where these are obtained by subtracting and adding the new variables (4.1.9) and (4.1.10), respectively. It follows that at a steady-state, $\xi_i^* = 0, (i = 1, 2)$, since $h = 0$ and $v = 0$ for H^* and V^*, respectively.

Finally, the new representation of the system (4.1.7) can be derived by substituting (4.1.11) and (4.1.12) into (4.1.6) and (4.1.7), respectively, to get

$$\begin{aligned} &\partial_t \xi_1 + \lambda_1 \partial_x \xi_1 = 0, \\ &\partial_t \xi_2 - \lambda_2 \partial_x \xi_2 = 0. \end{aligned} \tag{4.1.13}$$

4.1.4 Sub-critical flow. (Bastin et al., 2009, p. 176) The sub-critical condition at the steady state flow (i.e. the flow rate that does not depend on time) is

$$(V^*)^2 < gH^*. \tag{4.1.14}$$

The implication of the condition is that

$$-\lambda_2 = V^* - \sqrt{gH^*} < 0 < V^* + \sqrt{gH^*} = \lambda_1, \quad -\lambda_2 < 0 < \lambda_1.$$

In the next section, we shall talk about Lyapunov stability analysis for a single pool.

4.2 Lyapunov stability analysis for a single pool

We now apply the Lyapunov stability analysis described in Section 3.3 to the linearised system of Saint-Venant equations. We consider a simple boundary condition, which is defined as

$$\begin{pmatrix} \xi_1(0,t) \\ \xi_2(L,t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix}}_{K} \begin{pmatrix} \xi_1(L,t) \\ \xi_2(0,t) \end{pmatrix}, \quad \forall t \in [0, +\infty), \tag{4.2.1}$$

where k_1 and k_2 are control-tuning parameters and the initial condition is also defined as

$$\begin{pmatrix} \xi_1(x) \\ \xi_2(x) \end{pmatrix} = \begin{pmatrix} \xi_1^0(x) \\ \xi_2^0(x) \end{pmatrix} \in L^2((0,L);\mathbb{R}^2), \quad \forall x \in [0, L]. \tag{4.2.2}$$

Consider a Cauchy problem, (4.1.13) with (4.2.1) and (4.2.2), then we look for a condition for exponential stability of a steady-state solution, $\xi^* \equiv 0$, by using the Lyapunov stability analysis discussed in Section 3.3. Thus, it is enough to find the values of k_1 and k_2 such that sufficient condition, $\rho(K) < 1$ is satisfied. Then, the convergence analysis is treated in a similar way as the proof of Theorem 3.3.5. Hence, we now derive the sufficient condition. That is equivalently given as

$$\|\Delta K \Delta^{-1}\| < 1,$$

where the matrix, Δ is taken from the proof of Theorem 3.3.5, which is

$$\Delta = \begin{pmatrix} \sqrt{p_1\lambda_1} & 0 \\ 0 & \sqrt{p_2\lambda_2} \end{pmatrix},$$

where $p_1, p_2 > 0$ and $0 < \lambda_2 < \lambda_1$. We want to show that

$$\|\Delta K \Delta^{-1}\| = \left\| \begin{matrix} 0 & k_1\sqrt{\frac{p_1\lambda_1}{p_2\lambda_2}} \\ k_2\sqrt{\frac{p_2\lambda_2}{p_1\lambda_1}} & 0 \end{matrix} \right\| < 1.$$

Therefore, we use the usual matrix 2-norm $\|\cdot\|$, to obtain

$$\begin{aligned} \|\Delta K \Delta^{-1}\| &= \max\left\{\sqrt{\sigma} : (\Delta K \Delta^{-1})^T \Delta K \Delta^{-1} - \sigma I \quad \text{is singular}\right\}, \\ &= \max\left\{ |k_1|\sqrt{\frac{p_1\lambda_1}{p_2\lambda_2}}, \quad |k_2|\sqrt{\frac{p_2\lambda_2}{p_1\lambda_1}} \right\} < 1, \end{aligned}$$

if, and only if,

$$|k_1| < \sqrt{\frac{p_2\lambda_2}{p_1\lambda_1}} \quad \text{and} \quad |k_2| < \sqrt{\frac{p_1\lambda_1}{p_2\lambda_2}}.$$

In general, the sufficient condition $\rho(K) < 1$ in this case is equivalent to the condition

$$|k_1 k_2| < 1.$$

In the next section, we shall discuss exponential stability analysis for a cascade of n-pools in a network.

4.3 Boundary feedback control for a channel with a cascade of n-pools

In this section, we shall talk about the networks of an open-channel made up of n-pools. We then analyse the exponential stability of the solutions of the linearised Saint-Venant equations for this network.

We now consider the flow of water along successive pools, which is described in Figure 4.2 below. This flow is forced by the power of gravity and it passes through different automated overflow gates. Therefore, it helps to adjust the flow rate. We make use of assumptions considered for a single pool. Thus, the model for n-pools is described by the Saint-Venant equations as

$$\partial_t \begin{pmatrix} H_i \\ V_i \end{pmatrix} + \partial_x \begin{pmatrix} H_i V_i \\ \frac{1}{2}V_i^2 + gH_i \end{pmatrix} = 0, \quad i = 1, \ldots, n. \tag{4.3.1}$$

For each pool, we define the water flow rate with a unit width, say Q_i as

$$Q_i(x,t) = V_i(x,t)H_i(x,t), \quad i = 1, \ldots, n. \tag{4.3.2}$$

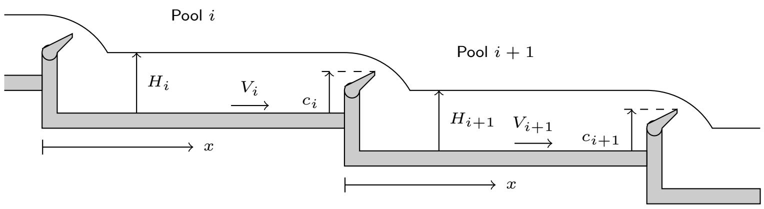

Figure 4.2: Lateral view of successive pools of an open-water channel with overflow gates: V_i is horizontal water velocity of pool i, H_i is water depth of pool i, c_i is a gate opening control action of pool i.

In this model, $2n$ boundary conditions are considered and these are classified into three categories as in Bastin et al. (2009)

1) There are $n-1$ boundary conditions at the position connecting the two pools. In this case the flow that exits pool i is equal to the flow that enters pool $i+1$. Thus,

$$H_i(L,t)V_i(L,t) = H_{i+1}(0,t)V_{i+1}(0,t), \quad i = 1, \ldots, n-1. \tag{4.3.3}$$

2) There are n-boundary conditions that take the gate operations into consideration. Thus,

$$H_i(L,t)V_i(L,t) = k_G\sqrt{[H_i(L,t) - c_i(t)]^3}, \tag{4.3.4}$$

where k_G is a positive constant coefficient and $c_i(t)$ is control action.

3) The last boundary condition will be considered as a constant inflow rate, say $Q_0(t)$,

$$Q_0(t) = WH_1(0,t)V_1(0,t), \tag{4.3.5}$$

where W is width.

4.3.1 Steady-state. A steady state solution to the system (4.3.1) is a constant H_i^* and V_i^* $(i = 1,\dots,n)$ such that the system (4.3.1) is satisfied. Moreover, the sub-critical condition to the model is

$$(V_i^*)^2 < gH_i^*, \quad i = 1,\dots,n. \tag{4.3.6}$$

4.3.2 Boundary control design. The characteristic state variables of the system (4.3.1), with respect to the steady-state, are

$$\begin{aligned} \xi_i &= (V_i - V_i^*) + (H_i - H_i^*)\sqrt{\frac{g}{H_i^*}}, \\ \xi_{n+i} &= (V_i - V_i^*) - (H_i - H_i^*)\sqrt{\frac{g}{H_i^*}}, \quad i = 1,\dots,n, \end{aligned} \tag{4.3.7}$$

and the boundary conditions are defined as

$$\xi_{n+i}(L,t) = -k_i\xi_i(L,t), \quad i = 1,\dots,n, \tag{4.3.8}$$

where k_i refers to control-tuning parameters. From the characteristic variables (4.3.7), we can observe that, at steady-state solutions, H_i^* and V_i^*, we get

$$(\xi_i^*, \xi_{n+i}^*) \equiv (0,0).$$

We now use the boundary conditions (4.3.4) and (4.3.8) together with the characteristic variables (4.3.7) to get the simplified expression for the control actions. Thus, from the equation (4.3.4), we obtain

$$c_i(t) = H_i(L,t) - \left(\frac{H_i(L,t)}{k_G}V_i(L,t)\right)^{\frac{2}{3}}, \tag{4.3.9}$$

and

$$V_i(L,t) = V_i^* + \left(\frac{1-k_i}{1+k_i}\right)\left[(H_i(L,t) - H_i^*)\sqrt{\frac{g}{H_i^*}}\right], \tag{4.3.10}$$

by using equations (4.3.7) and (4.3.8). We combine the two equations (4.3.10) and (4.3.9) to get

$$c_i(t) = H_i(L,t) - \left[\frac{H_i(L,t)}{k_G}\left(V_i^* + \frac{1-k_i}{1+k_i}\left[(H_i(L,t) - H_i^*)\sqrt{\frac{g}{H_i^*}}\right]\right)\right]^{\frac{2}{3}}. \tag{4.3.11}$$

Therefore, we can observe that we need only to know the depth, $H_i(L,t)$, at the gates to obtain the control actions and adjust the flow rate.

4.3.3 Closed-loop stability analysis. In a closed-loop system, there is a correction for error between the inflow and outflow, which is called feedback (Bernhoff, 2008). We now set under which condition a boundary feedback control system (4.3.1) is exponentially stable so that the system is stabilised. Thus, we consider a closed-loop system with a constant inflow rate

$$Q_0(t) = Q^*. \tag{4.3.12}$$

The characteristic form of the linearised system of Saint-Venant equations for n-pools is described as

$$\begin{aligned} &\partial_t \xi_i(x,t) + \lambda_i \partial_x \xi_i(x,t) = 0, \\ &\partial_t \xi_{n+i}(x,t) - \lambda_{n+i} \partial_x \xi_{n+i}(x,t) = 0, \quad i = 1, \dots, n, \end{aligned} \tag{4.3.13}$$

where $-\lambda_{n+i} = V_i^* - \sqrt{gH_i^*}$ and $\lambda_i = V_i^* + \sqrt{gH_i^*}$ such that $0 < \lambda_{n+i} < \lambda_i$.

We now write the equations in (4.3.13) as the characteristic variables, (ξ^+, ξ^-), as the system (3.2.9) with the boundary conditions (3.2.10) as

$$\xi^+(0,t) = K_{00}\xi^+(L,t) + K_{01}\xi^-(0,t), \tag{4.3.14}$$
$$\xi^-(L,t) = K_{10}\xi^+(L,t) + K_{11}\xi^-(0,t). \tag{4.3.15}$$

Before we proceed, to determine the condition for exponential stability, we take the matrices from Bastin et al. (2009), which are derived using the boundary conditions (4.3.3) and (4.3.8) to get an $n \times n$ matrix with entries

$$K_{00}[i+1, i] = \frac{(\lambda_i - k_i \lambda_{n+i})}{\lambda_{i+1}} \sqrt{\frac{H_i^*}{H_{i+1}^*}},$$

and 0 elsewhere, and calculations from boundary conditions (4.3.3) and (4.3.5) give a diagonal matrix

$$K_{01} = \text{diag}\{-\frac{\lambda_{n+i}}{\lambda_i} : i = 1, \dots, n\}.$$

The boundary conditions (4.3.15) can be rewritten as

$$\xi^-(L,t) = K_{10}\xi^+(L,t),$$

where $K_{10} = \text{diag}\{-k_i : i = 1, \dots, n\}$. Thus, by equation (4.3.15), we get $K_{11} = 0$.

Now, using these matrices and following the proof of Theorem 3.3.5, we look for the control tuning parameters. Thus, we begin by introducing a diagonal matrix, Δ as in the proof of Theorem 3.3.5, which is $\Delta = \text{diag}\{D_0, D_1\}$, with $D_0 = \sqrt{P_0\Lambda^+}$ and $D_1 = \sqrt{P_1\Lambda^-}$ where $P_0 = \text{diag}\{p_i > 0 : i = 1, \dots, n\}$, $P_1 = \text{diag}\{p_{n+i} > 0 : i = 1, \dots, n\}$, $\Lambda^+ = \text{diag}\{\lambda_1, \dots, \lambda_n\}$ and $\Lambda^- = \text{diag}\{\lambda_{n+1}, \dots, \lambda_{2n}\}$. Then, the sufficient condition, $\rho(K) < 1$ is satisfied if the inequality (3.3.8) is satisfied. Thus, consider the matrix,

$$\begin{aligned} \Delta K \Delta^{-1} &= \begin{pmatrix} \sqrt{P_0\Lambda^+} & 0 \\ 0 & \sqrt{P_1\Lambda^-} \end{pmatrix} \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix} \begin{pmatrix} (\sqrt{P_0\Lambda^+})^{-1} & 0 \\ 0 & (\sqrt{P_1\Lambda^-})^{-1} \end{pmatrix}, \\ &= \begin{pmatrix} \sqrt{P_0\Lambda^+} K_{00} (\sqrt{P_0\Lambda^+})^{-1} & \sqrt{P_0\Lambda^+} K_{01} (\sqrt{P_1\Lambda^-})^{-1} \\ \sqrt{P_1\Lambda^-} K_{10} (\sqrt{P_0\Lambda^+})^{-1} & \sqrt{P_1\Lambda^-} K_{11} (\sqrt{P_1\Lambda^-})^{-1} \end{pmatrix}, \\ &= \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}, \end{aligned}$$

where an $n \times n$ matrix with entries

$$A_{00}[i+1,i] = \frac{(\lambda_i - k_i\lambda_{n+i})}{\lambda_{i+1}}\sqrt{\frac{H_i^*}{H_{i+1}^*}}\sqrt{\frac{p_{i+1}\lambda_{i+1}}{p_i\lambda_i}},$$

and 0 elsewhere and

$$A_{01} = \text{diag}\left\{-\sqrt{\frac{p_i\lambda_{n+i}}{p_{n+i}\lambda_i}} : i = 1,\dots,n\right\},$$

$$A_{10} = \text{diag}\left\{-k_i\sqrt{\frac{p_{n+i}\lambda_{n+i}}{p_i\lambda_i}} : i = 1,\dots,n\right\},$$

$$A_{11} = 0.$$

We assume that the parameters p_i can be selected such that $p_{i+1} = \epsilon p_i (i = 1,\dots,n)$. Then, for the special case $\epsilon = 0$, we get $A_{00} = 0$. Thus,

$$\|\Delta K \Delta^{-1}\| = \max\left\{|k_i|\sqrt{\frac{p_{n+i}\lambda_{n+i}}{p_i\lambda_i}}, \quad \sqrt{\frac{p_i\lambda_{n+i}}{p_{n+i}\lambda_i}}\right\} < 1,$$

if, and only if,

$$|k_i|\sqrt{\frac{p_{n+i}\lambda_{n+i}}{p_i\lambda_i}} < 1 \quad \text{and} \quad \sqrt{\frac{p_i\lambda_{n+i}}{p_{n+i}\lambda_i}} < 1,$$

or

$$|k_i| < \sqrt{\frac{p_i\lambda_i}{p_{n+i}\lambda_{n+i}}} \quad \text{and} \quad \sqrt{\frac{p_i}{p_{n+i}}} < \sqrt{\frac{\lambda_i}{\lambda_{n+i}}}.$$

In this case, it follows that for sufficiently small $\epsilon > 0$, the inequality (3.3.8) is satisfied, which equivalently means that the control-tuning parameters,

$$|k_i| < \sqrt{\frac{p_i\lambda_i}{p_{n+i}\lambda_{n+i}}} < \sqrt{\frac{\lambda_i^2}{\lambda_{n+i}^2}} = \frac{|\lambda_i|}{|\lambda_{n+i}|}. \tag{4.3.16}$$

In general, the physical meaning of the inequality (4.3.16) is that the control-tuning parameters for each pool must be less than the ratio of the the largest to the smallest characteristic velocity of the moving waves to the left and rightward directions.

Hence, if the parameters $p_i > 0$ are selected such that $p_{n+i} = \epsilon p_i$ $(i = 1,\dots,n-1)$ for sufficiently small $\epsilon > 0$, then the sufficient condition, $\rho(K) < 1$, is satisfied.

Therefore, by using Theorem 3.3.5, we can conclude that the solutions of the linearised system of Saint-Venant equations in a network converge to 0 in L^2-norm.

In the next example, we shall summarise the exponential stability analysis of the linearised system of Saint-Venant equations.

4.3.4 Example. We now construct a channel made up of identical n-pools, we use the data from Section 3.5 in de Halleux; G. Bastin (2002). Thus, the pools length $L_i = 5000m$ and the width is $40m$ with the steady-state

$$Q_i^* = 10m^3/s \quad \text{and} \quad H_i^* = 4m,$$

and taking the gravitational constant as $10m/s$.

Putting these values into equation (4.3.2), we obtain the horizontal water velocity, $V_i^* = 0.0625m/s$ at a steady-state. Then, the characteristic curves can be found from equations $-\lambda_{n+i} = V_i^* - \sqrt{gH_i^*}$ and $\lambda_i = V_i^* + \sqrt{gH_i^*}$ as

$$\lambda_i = 6.387m/s \quad \text{and} \quad \lambda_{n+i} = 6.262m/s.$$

Finally, we use inequality (4.3.16) to conclude that, the convergence of the steady-state solution to 0, is guaranteed if $|k_i| < 1.01996 \ \forall i = 1, \ldots, n$.

In this chapter, we have discussed the open-channel model represented by the Saint-Venant equations. In this model we have considered that the flow is sub-critical, the slope of the bed is zero, as well as the friction of water between the wall of the pool was being neglected. Under these assumptions, we linearised the equations around the steady-state solutions. The system was coupled so that we have decoupled the system and analysed the exponential stability of the solutions of the system at the origin for a single pool and network of pools. For the analysis, we have used a closed-loop stability analysis. As a result of the discussion, we have discovered the control actions obtained by measuring only the depth of the water at the gates. Moreover, we have also pointed out the condition of exponential stability of the solutions of the linearised system. Which are the control-tuning parameters, $|k_i| < |\lambda_i/\lambda_{n+i}|$ for each pool.

5. Conclusion

In this paper, we have discussed the sufficient condition for the exponential stability analysis of a linearised systems of hyperbolic conservation laws of the form

$$\partial_t \xi + \Lambda \partial_x \xi = 0, \tag{5.0.1}$$

with linear boundary conditions

$$\begin{pmatrix} \xi^+(0,t) \\ \xi^-(L,t) \end{pmatrix} = K \begin{pmatrix} \xi^+(L,t) \\ \xi^-(0,t) \end{pmatrix}. \tag{5.0.2}$$

The basic property is that the system (5.0.1), with linear boundary conditions (5.0.2), is exponentially stable if the sufficient condition $\rho(K) < 1$ is satisfied.

From the result of the application to open-channel, we have seen that the control actions are obtained only by measuring the depth at the gates.

We have also indicated the condition for exponential stability of the linearized system of Saint-Venant equations. We used Lyapunov stability analysis to show that the equilibrium solution is exponentially stable if the control-tuning parameters, $|k_i| < |\lambda_i / \lambda_{n+i}|$ $(i = 1, \dots, n)$ for each pool. But, the non-linear stability analysis is left for further study.

AppendixA.

A.1 Matrix Norm

A.1.1 Definition (Singular Value Decomposition)**.** (Gentle, 2007) Let A be an $n \times m$ real matrix. Then the matrix A can be factored as

$$A = UDV^T, \tag{A.1.1}$$

where U is an $n \times n$ orthogonal matrix, V is an $m \times m$ orthogonal matrix and D is an $n \times m$ diagonal matrix with entries,

$$\sigma_1 \geq \sigma_2 \geq \dots \sigma_p > 0, \quad p = \min\{n, m\},$$

called singular values of A. The factorisation (A.1.1) is called the singular value decomposition.

A.1.2 Definition (L_2 Matrix Norms)**.** (Gentle, 2007) The matrix norms with L_2 vectors norm is defined as

$$\|A\|_2 = \underset{\|x\|_2=1}{\|Ax\|_2}.$$

Moreover, the matrix 2-norm is also defined as

$$\|A\|_2 = \sqrt{\sigma_{\max}},$$

where $\sigma_{\max}$ is the largest singular value of A.

A.2 Quadratic Forms and Definiteness

A.2.1 Definition (Quadratic Form)**.** (Abadir and Magnus, 2005) A function $F : \mathbb{R}^2 \to \mathbb{R}$ is a quadratic form if it can be expressed as

$$F = a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2.$$

This can also be expressed in matrix form,

$$F(x) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x^T Ax,$$

where $x = (x_1 \quad x_2)^T$ and A is unique and symmetric.

In general, the quadratic form in $\mathbb{R}^n$ is

$$F(x) = \sum_{i,j=1}^{n} a_{i,j}x_ix_j = x^T Ax,$$

where $x = (x_1, \dots, x_n)^T$ and A is unique.

A.2.2 Definition (Definiteness)**.** (Abadir and Magnus, 2005) Let A be an $n \times n$ symmetric matrix. Then A is

1. Positive definite if $x^T Ax > 0 \quad \forall x \neq 0$ in $\mathbb{R}^n$.
2. Negative definite if $x^T Ax < 0 \quad \forall x \neq 0$ in $\mathbb{R}^n$.

A.2.3 Definition (Principal minors and leading principal minors)**.** (Abadir and Magnus, 2005) Let A be an $n \times n$ matrix. A $k \times k$ sub-matrix of A obtained by deleting any $n - k$ columns and the same $n - k$ rows from A is called a kth-order principal sub-matrix of A. The determinant of a $k \times k$ principal sub-matrix is called a kth-order principal minor of A. The leading principal minor is the determinant of the leading principal sub-matrix obtained by deleting the last $n - k$ rows and columns of an $n \times n$ matrix A.

A.2.4 Definition (Testing for Definiteness)**.** (Abadir and Magnus, 2005) A matrix:

1. A is positive definite if, and only if, all of its n-leading principal minors are strictly positive.
2. A is negative definite if, and only if, all of its n-leading principal minors alternate in sign, where $|A_1| < 0$, $|A_2| > 0$, $|A_3| < 0$, etc.

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