Closure Versus Interior Operators in Categorical Topology

Fikreyohans Solomon Assfaw (fikreyohans@aims.ac.za)
ፍቅረዮሐንስ ሰለሞን አስፋው
African Institute for Mathematical Sciences (AIMS)

Supervised by: Professor David Holgate
University of the Western Cape, South Africa

22 May 2014

Submitted in partial fulfillment of a structured masters degree at AIMS South Africa

Abstract

In general topology, the notions of closure and interior are dual to each other. Indeed, a topology on a set X can equivalently be described by giving either the open sets or the closed sets. Categorical closure operators were introduced in 1987 (Dikranjan and Giuli, 1987) in order to study topology in general categories. More recently interior operators have been introduced and studied (Vorster, 2000). In this categorical setting these notions of interior and closure are no longer exactly dual to each other. This essay will begin with a survey of interior and closure operators in categories. It will then proceed to investigate the current research question of what is the link between interior and closure in a category.

Key Words: Galois connection, $(\mathbb{E}, \mathbb{M})$ factorization structures, subobject, closure operator, interior operator, and transformation operator.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Fikreyohans Solomon Assfaw, May 22 2014

Contents

1. Introduction

A category is a mathematical structure satisfying a number of abstractly defined axioms. Category theory is a relatively young branch of mathematics that arose from algebraic topology and designed to describe various structural concepts from different mathematical fields in a uniform way. It was introduced by Saunders Mac lane and Samuel Eillenberg in 1940 and first appeared in a paper in 1945 (Eilenberg and MacLane, 1945). Category theory provides a variety of notions which expand on the lattice theoretic concept of closure operator [cf. Freyd (1964), Kennison (1965)]. Both categorical and lattice theoretic views of closure operators play a vital role in theoretical computer science (Scott, 1982).

A categorical closure operator c assigns to every subobject $R \xrightarrow{r} X \in \mathbb{M}$ a subobject $c_X(r) : C_X[R] \rightarrow X \in \mathbb{M}$ such that it is extensive, monotone and compatible with taking images or equivalently, preimages, in the same way as the usual topological closure is compatible with continuous maps. Closure operators have played an important role in the development of categorical topology by introducing topological ideas in a topology-free environment. They have been used to characterize and extend topological notions like epimorphisms, separation, compactness, and connectedness to an arbitrary category [cf. Castellini (1986), Clementino et al. (1996),Clementino (2001)]. Besides, categorical closure operators provide a unified approach to many different mathematical notions that otherwise would appear rather unrelated. The formal theory of categorical closure operators was introduced by Dikranjan and Giuli (Dikranjan and Giuli, 1987). Thereafter, they have been studied by the same authors and other researchers [cf.Dikranjan et al. (1989), Castellini et al. (1994), Holgate (1998), Giuli and Tholen (2000), Castellini (2001), Holgate (2000), Brümmer et al. (2005), Castellini and Holgate (2010)]. It is well known that the associated closure and interior operators provide equivalent descriptions in general topology. But categorical interior operators have been ignored for some time. A notion of categorical interior operator was formally introduced by S.J.R. Vorster (Vorster, 2000). A categorical interior operator i maps each $R \xrightarrow{r} X \in \mathbb{M}$ to $i_X(r) : i_X[R] \rightarrow X \in \mathbb{M}$ such that it is contractive, monotone and only compatible with taking pre-images, in contrary to closure operator. In this categorical setting closure and interior operators are not dual. So, it makes sense to study the link between closure and interior operators. The following is a list of some of the papers that have contributed to the development of interior operators: Vorster (2000), Luna-Torres et al. (2010), Holgate and Šlapal (2010), Castellini and Ramos (2010), Holgate and Šlapal (2011), Castellini (2011), Castellini (2013), Razafindrakoto (2012), Castellini and Murcia (2013), and Razafindrakoto and Holgate (2014).

The aim of this research project is to study a link between closure and interior. Our approach to closure and interior in a category is via closure and interior operators. We first present the general theory of closure and interior operators. Chapter 2 provides the basic categorical concepts such as Galois connections, $(\mathbb{E}, \mathbb{M})$ factorization structure, and subobjects which are tools and appear repeatedly in the general theory of closure and interior operators. Chapter 3 deals briefly with the current notion of closure operators together with some basic important properties. The general theory of interior operators together with the main results are included in chapter 4. Finally, in chapter 5 we present three possible links between closure and interior operators.

Due to the restriction of pages we have only included some of the proofs of the needed basic results of categorical concepts in chapter 2. Throughout this work we assume that the reader is familiar with Topology, Algebra, Order and Lattice theory. For further references, we suggest the reader consult Munkres (1975), Fuchs (1970), Engelking (1989), Davey and Priestley (2002) and Grätzer (2003).

2. Basic Concepts of Category Theory

This chapter states some of the important categorical preliminaries required to deal with interior and closure operators. The relevant material can be found in Adamek et al. (2004), Dikranjan and Tholen (1995) and Castellini (2003). For further references the reader could consult Mac Lane (1998), Awodey (2010), Simmons (2011), Holgate and Razafindrakoto (2011) and Borceux (1994). We start by defining category.

2.1 Categories and Functors

2.1.1 Definition. A category **C** is an algebraic structure consisting of

1. a class $Ob(\mathbf{C})$ of objects: denoted by $A, B, C, \cdots$;
2. a class $Mor(\mathbf{C})$ of morphisms called arrows: denoted by $f, g, h, \cdots$;
3. for each arrow f, there are given objects $dom(f), cod(f)$, called the domain and codomain of f respectively. If $A = dom(f)$ and $B = cod(f)$ then f is described as a morphism from A to B, and is represented by $f : A \to B$ or $A \overset{f}{\longrightarrow} B$.The set of all morphisms from A to B is denoted by $Hom_{\mathbf{C}}(A, B)$.
4. a composition operator $\circ : Mor(\mathbf{C}) \times Mor(\mathbf{C}) \to Mor(\mathbf{C})$, assigning to each pair of morphisms $f : A \to B$ and $g : B \to C$ with $cod(f) = dom(g)$, a composite morphism $g \circ f : A \to C$ such that for morphisms $f : A \to B, g : B \to C$ and $h : C \to D$ we have always $h \circ (g \circ f) = (h \circ g) \circ f$. That is, composition is an associative operation.
5. an operator $1 : Ob(\mathbf{C}) \to Mor(\mathbf{C})$ assigning for each object A, an identity morphism $1_A : A \to A$ such that for every morphism $f : A \to B$ the equations $f \circ 1_A = f$ and $1_B \circ f = f$ hold. These laws are called unit laws.

2.1.2 Examples. 1. The category **Set** of Sets:

- the objects are sets, the arrows are functions from one set to another, the composition of arrows is the usual composition of functions. That is, for $f : A \to B$ and $g : B \to C$ we define $\forall a \in A \quad g \circ f : A \to C$ by $(g \circ f)(a) = g(f(a))$,
- the identity arrow on an object A is the identity map $1_A : A \to A$ defined by $\forall a \in A \quad 1_A(a) = a$ and the associative and unit laws clearly hold true.

2. The category **Grp** of groups:

- the objects are groups, the arrows are group homomorphisms, the composition of arrows is the usual composition of group homomorphism. Indeed, if $f : G \to H$ and $g : H \to N$ are group homomorphism then $\forall a, b \in G \quad (g \circ f)(ab) = g(f(ab)) = g(f(a)f(b)) = g(f(a))g(f(b)) = (g \circ f)(a)(g \circ f)(b)$, which is again a group homomorphism,
- the identity arrow on an object G is the identity map $1_G : G \to G$ defined by $\forall g \in G \quad 1_A(g) = g$. In fact 1_G is a group homomorphism since $\forall g, h \in G \quad 1_A(gh) = gh = 1_A(g)1_A(h)$. Also, the associative and unit laws clearly hold true.

3. The category **Top** of topological spaces: the objects are topological spaces, the arrows are continuous maps, the composition of arrows is the composition of continuous maps.

   Indeed, if $f : X \to Y$ and $g : Y \to Z$ are continuous maps then for any open set O in Z we obtain $g^{-1}(O)$ is open in Y since g is continuous. Also, since f is continuous we have $f^{-1}(g^{-1}(O)) = (g \circ f)^{-1}(O)$ is open in X. Therefore, $g \circ f$ is continuous.

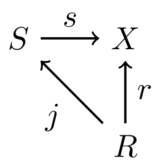
   - the identity arrow on an object X is the identity map $1_X : X \to X$ defined by $\forall x \in X \quad 1_X(x) = x$. In fact for any open set O in X we obtain $1_X^{-1}(O) = O$ is an open set in X. Hence 1_X is a continuous function and the associative and unit laws clearly hold true.

4. Let $(P, \leqslant)$ be a pre-ordered set viewed as a category with objects the elements of P and taking the unique morphism $p \to q$ defined by $p \to q \Leftrightarrow p \leqslant q$, then the reflexive and transitive properties on $\leqslant$ imply for arrows $f : p \to q$ and $g : q \to r$ in P we have $p \leqslant q$ and $q \leq r \Rightarrow p \leq r \Leftrightarrow g \circ f : p \to r$ exists . In fact there is no arrow from p to q when $p \nleqslant q$. Also since there is at most one arrow between two objects the composition is well defined and is associative and for each object p in P we have $p \leqslant p \Leftrightarrow p \to p$. Hence for each object p the identity arrow $1_p : p \to p$ exists. Also, since there is at most one arrow between two objects the unit laws hold for free. Therefore any pre-ordered set is a category.

5. Let $(M, \circ, 1_M)$ be a monoid with "$\circ$" as the associative binary operation defined on M and 1_M as the unit element of M then we have a single formal object, elements of M as arrows with 1_M as identity arrow and for arrows m and n in M we have the composition $m \circ n$ is always defined as they have the same domain and codomain. Moreover, since the binary operation $\circ$ is associative on M we have that composition is associative. For each arrow $m \in M$ we have $m \circ 1_M = m = 1_M \circ m$ as 1_M is the unit element. Hence a monoid is a category with just one object.

6. The slice Category $\mathbf{C}/X$ of a category $\mathbf{C}$ over an object $X \in \mathbf{C}$ has

   - Objects: $\mathbf{C}$-arrows whose codomain is X,
   - Arrows: an arrow from $r : R \to X$ to $s : S \to X$ is an arrow $j : R \to S \in \mathbf{C}$ such that $r = s \circ j$, as shown in the diagram below.



2.1.3 Remark. 1. Any category with at most one arrow between any two objects is a pre-order.

2. The arrows of a given category may not be functions. As an example take a category of finite sets A, B and C as objects, and an arrow $f : A \to B$ to be a matrix $f = (f_{ij})_{i<a, j<b}$ of natural numbers with $a = |A|$ and $b = |B|$, where $|C|$ is the number of elements in a set C. The composition of the arrows is the usual matrix multiplication, the identity arrows are the usual unit matrices.

2.1.4 Definition. Let $\mathbf{C}$ be a category with composition operation $\circ$ and identity arrow 1_A for each object A, then the dual (or opposite) category of $\mathbf{C}$, denoted by $\mathbf{C}^{op}$ is a category with $Ob(\mathbf{C}^{op}) = Ob(\mathbf{C})$, arrows $f : B \to A$ where $f : A \to B$ is an arrow in $\mathbf{C}$, $\forall f : A \to B, g : B \to C \in \mathbf{C}^{\ op}$ we have $g \circ^{op} f = f \circ g$ and $\forall A \in \mathbf{C}^{op}$, $1_A^{op} = 1_A$.

2.1.5 Remark. $(\mathbf{C}^{op})^{op} = \mathbf{C}$.

As stated in Adamek et al. (2004) the notion of category theory is well balanced in the sense that every concept has a dual concept. The "two for the price of one " principle holds. This allows us to be economical. More formally stated, we have:

2.1.6 Proposition. If a property p holds for all categories, then the property p^{op} holds for all categories, where p^{op} is the statement obtained by exchanging the domain and codomain of each morphism as well as exchanging the order of the composition in p.

Proof. Suppose a property p holds for all categories $\mathbf{C}$ and then since each opposite category is a category we have $\forall \mathbf{C}^{op} \quad p$ is true . This implies $\forall (\mathbf{C}^{op})^{op} = \mathbf{C} \quad p^{op}$ is true. □

This is called the duality principle. Categorical duality turns out to be a very powerful and far reaching phenomenon.

2.1.7 Examples.

1. Let $\mathbf{C} = (M, \circ, 1_M)$ be a monoid considered as category then $\mathbf{C}^{op} = (M, \circ^{op}, 1_M)$ where $m \circ^{op} n = n \circ m$.
2. Let $\mathbf{C} = (P, \leqslant)$ be a pre-ordered set considered as category then $\mathbf{C}^{op} = (P, \leqslant^{\mathsf{op}})$ where the new partial order relation $\leqslant^{\mathsf{op}}$ is defined by $p \leqslant^{\mathsf{op}} q \Leftrightarrow q \leqslant p$.

2.1.8 Definition. A functor F from a category $\mathbf{C}$ to a category $\mathbf{D}$, denoted by, $F : \mathbf{C} \to \mathbf{D}$ is a morphism that assigns to every object A in $\mathbf{C}$ an object $F(A)$ in $\mathbf{D}$ and to every morphism $f : A \to B$ in $\mathbf{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathbf{D}$ such that $\forall A \in \mathbf{C}, F(1_A) = 1_{F(A)}$ and $\forall f : A \to B$ and $g : B \to C$ in $\mathbf{C}, F(g \circ f) = F(g) \circ F(f)$. In other words, F is a homomorphism which preserves the algebraic structure of categories. As result, categories form a category, with functors as arrows.

2.1.9 Examples.

1. Let $\mathbf{C}$ be a category. Define $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ by $1_{\mathbf{C}}(f : A \to B) = f : A \to B$ then $1_{\mathbf{C}}(1_A) = 1_{\mathbf{C}}(1_A : A \to A) = 1_A : A \to A = 1_{1_{\mathbf{C}}(A)}$ and $1_{\mathbf{C}}(g \circ f) = g \circ f = 1_{\mathbf{C}}(g) \circ 1_{\mathbf{C}}(f)$. Hence $1_{\mathbf{C}}$ is a functor and we call it the identity functor.
2. Let $\mathbf{C}$ and $\mathbf{D}$ be two categories such that $X, Y \in \mathbf{C}$ and $Z \in \mathbf{D}$ then $\exists F_Z : \mathbf{C} \to \mathbf{D}$ given by $F_Z(f : X \to Y) = 1_Z : Z \to Z$, which is a constant functor.
3. Let $F : \mathbf{Top} \to \mathbf{Set}$ be defined by $F(f : X \to Y) = f : X \to Y$, by forgetting the topologies on the spaces and the continuity of f. F is called a forgetful functor.
4. Let $(P, \leqslant_1)$ and $(Q, \leqslant_2)$ be pre-ordered sets which are viewed as categories then an order preserving map $F : P \to Q$ is a functor. Note that an order preserving map F is a map such that $\forall p_1, p_2 \quad p_1 \leqslant_1 p_2 \Rightarrow F(p_1) \leqslant_2 F(p_2)$.
5. If $(M_1, \circ, 1_{M_1})$ and $(M_2, \odot, 1_{M_2})$ are monoids viewed as categories then any monoid homomorphism $F : M_1 \to M_2$ is a functor since by definition of a homomorphism $F(m \circ n) = F(m) \odot F(n)$.

2.2 Abstract Stuctures

2.2.1 Definition. Let $h : A \to B$ be a morphism in a category $\mathbf{C}$ then h is said to be

1. an isomorphism (or iso) if there exists a morphism $k : B \to A$ such that $h \circ k = 1_B$ and $k \circ h = 1_A$.
2. a monomorphism (or mono) if for any pair of morphisms $f, g : C \to A \in \mathbf{C}, h \circ f = h \circ g \Rightarrow f = g$.
3. a section (or split monomorphism) if there exists a morphism $k : B \to A$ such that $k \circ h = 1_A$.

2.2.2 Remark. The dual of iso, mono, and section are iso, epi and retraction respectively. Hence we can use 2 and 3 in the Definition 2.2.1 to define epi and retraction respectively.

In the category of **Set** monomorphisms, epimorphisms and isomorphisms are the injective, surjective and bijective functions respectively. In a group viewed as category every element is an isomorphism and in a poset viewed as category only identity arrows are isomorphisms.

2.2.3 Proposition. 1. Every section is a monomorphism and dually a retraction is an epimorphism.

2. The first factor of a monomorphism is a monomorphism.
3. Every iso is mono and epi.
4. A morphism which is both a monomorphism and a retraction is an isomorphism. Also, dually a morphism which is both an epimorphism and a section is an isomorphism.

Proof. 1. Let $s : A \to B$ be a section then $\exists r : B \to A$ such that $r \circ s = 1_A$. Thus for a pair of morphisms $u, v : C \to A$ such that $s \circ u = s \circ v$ we obtain $r \circ s \circ u = r \circ s \circ v \Rightarrow (r \circ s) \circ u = (r \circ s) \circ v$. As a result, $1_A \circ u = 1_A \circ v \Rightarrow u = v$. Therefore, s is monomorphism.

2. Let $A \xrightarrow{h} B$ be monomorphism such that $h = g \circ f$ then for any pairs of morphisms r and s such that $f \circ r = f \circ s$ we obtain $f \circ r = f \circ s \Rightarrow g \circ f \circ r = g \circ f \circ s$. Hence $h \circ r = h \circ s$ since $h = g \circ f$. This in turn implies $r = s$, since h is monomorphism . Therefore f is monomorphism
3. Follows from 1 since an iso is both a section and a retraction.
4. Suppose $A \xrightarrow{m} B$ is a monomorphism and a retraction then from the definition of retraction there exist a morphism $B \xrightarrow{n} A$ such that $m \circ n = 1_B$. Besides, from the definition of monomorphism we have $m \circ (n \circ m) = (m \circ n) \circ m = 1_B \circ m = m = m \circ 1_A$. Thus $n \circ m = 1_A$. Therefore we get a morphism $B \xrightarrow{n} A$ such that $m \circ n = 1_B$ and $n \circ m = 1_A$. Hence m is an isomorphism.

□

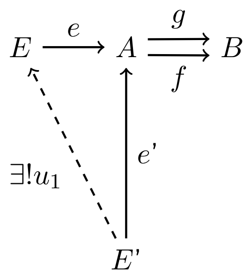
2.2.4 Definition. Let I be an index set then

1. a source is a family of morphisms $(f_i : A \to A_i)_{i \in I}$ in a category **C** having the same domain.
2. a monosource is a source $(f_i : A \to A_i)_{i \in I}$ such that $\forall i \in I (f_i \circ g = f_i \circ h \Rightarrow g = h)$.

2.2.5 Remark. Dual concept: sink, episink.

2.2.6 Definition. Let $f, g : A \to B$ be parallel morphisms then a morphism $E \xrightarrow{e} A$ is said be an equalizer of f and g if $f \circ e = g \circ e$ and for any morphism $E' \xrightarrow{e'} A$ such that $f \circ e' = g \circ e'$, there is a unique morphism $E' \xrightarrow{u_1} E$ such that $e' = e \circ u_1$. The situation is given in the diagram below.

If every pair of morphisms $u, v : A \to B$ in category **C** has an equalizer then we say that **C** has equalizers.



2.2.7 Remark. Dual concept: coequalizer.

2.2.8 Proposition. Let $E \overset{e}{\longrightarrow} A$ be an equalizer of $f, g : A \to B$ then e is a monomorphism and unique up to isomorphism.

Proof. The proof can be found in Appendix A.1. □

2.2.9 Definition. A regular monomorphism is morphism which is an equalizer of some pair of morphisms.

2.2.10 Remark.

1. The dual of a regular monomorphism is regular epimorphism.
2. Regular monomorphisms are monomorphisms.

2.2.11 Proposition. A section is a regular monomorphism.

Proof. Let $s : A \to B$ be section then $\exists r : B \to A$ such that $r \circ s = 1_A$. Hence $(s \circ r) \circ s = s \circ (r \circ s) = s = 1_B \circ s$ and for any $e : E \to B$ such that $(s \circ r) \circ e = 1_B \circ e$ we have a unique morphism $r \circ e : E \to A$ with $s \circ r \circ e = e$. Therefore, s is an equalizer of $s \circ r$ and 1_B. □

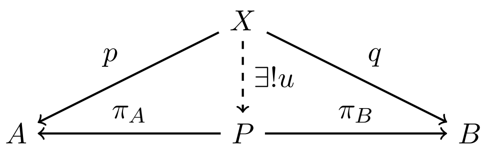
2.2.12 Definition. Let **C** be a category then we say that an object **0** is initial if $\forall A \in Ob(\mathbf{C}) \exists !$ morphism $\mathbf{0} \to A$ and an object **1** is terminal if $\forall A \in Ob(\mathbf{C}) \exists !$ morphism $A \to \mathbf{1}$. An object which is both initial and terminal is called zero or null.

2.2.13 Examples.

1. In the category of **Set, Pos,Top**, the object with the empty underlying set is initial and the object with the one element underlying set is terminal. Note: **Pos** is a category of posets and monotone maps.
2. In the category of groups **Grp** we have any trivial group $\{e : e$ is an identity of a group $G\}$ is a zero object.
3. In Posets $(P, \leqslant_P)$ considered as category we have the least element (if it exists) is an initial object and the greatest element (if it exists) is a terminal object.

2.2.14 Remark. Initial objects are unique up to isomorphism and by duality terminal objects are unique up to isomorphism.

2.2.15 Definition. The product of two objects A and B in a category **C** is an object P together with two projection morphisms $A \overset{\pi_A}{\leftarrow} P, P \overset{\pi_B}{\rightarrow} B$ such that for any object X and morphisms $p : X \to A$ and $q : X \to B$ $\exists !$ $X \overset{u}{\rightarrow} P$ such that the diagram below commutes.

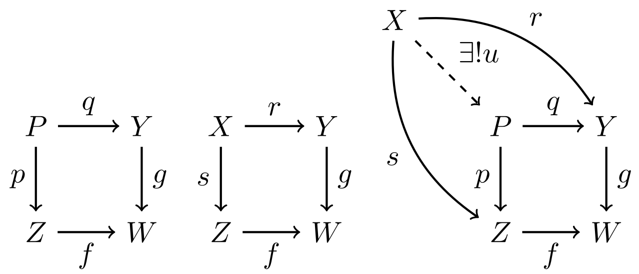


2.2.16 Proposition. Products are unique up to isomorphism.

Proof. The proof can be found in Appendix A.2. □

2.2.17 Examples. Let $(P, \leq_P)$ be a poset viewed as category then the product of p and q in P is given by the meet $p \wedge q$ of p and q.

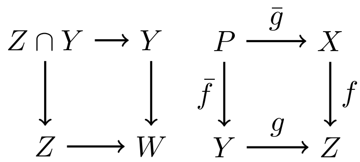
2.2.18 Definition. The pullbacks of morphisms $Z \overset{f}{\rightarrow} W, Y \overset{g}{\rightarrow} W$ having the same codomain is a commutative first below square such that for any commutative second below square $\exists ! u : X \to P$ for which the third diagram below commutes.



Moreover, we say that q is the pullback of f along g and p is the pullback of g along f. If g is mono then p is usually denoted by $f^{-1}(g)$ and is often called the inverse image of g along f.

If every pair of morphisms $f : Z \to W, g : Y \to W$ in **C** has a pullback then we say that **C** has pullbacks.

2.2.19 Examples. 1. Let $P(A)$ be power set of A and $(P(A), \subseteq)$ be a poset viewed as category then the left square below is a pullback in $(P(A), \subseteq)$.



2. Let $f : X \to Z$ and $g : Y \to Z$ be functions in **Set** such that $P = \{(x, y) \in X \times Y : f(x) = g(y)\}$ and $\bar{f}, \bar{g}$ are defined by $\bar{f}(x, y) = y$ and $\bar{g}(x, y) = x$ then the right square above is pullback.

2.2.20 Remark. 1. From the above two examples we can deduce that pullbacks can be viewed as a generalizations of both intersections and products.

2. Since pullbacks are given by the universal mapping property, they are unique up to isomorphism (essentially unique).

2.2.21 Proposition. 1. Monomorphisms and regular monomorphisms are stable under pullback (closed under the formation of pullbacks).

2. If two squares are pullbacks then the outer rectangle is also a pullback.

3. If the right square and the outer rectangle are pullbacks then the left square is also a pullback.

Proof. The proof can be found in Appendix A.3. □

2.3 Adjoints

Adjoint functor occur frequently in many branches of Mathematics, and the adjoint functor theorems have a surprising range of applications. In this context, Saunders MacLane, who is one of the founders of category theory, is remembered for his slogan, "Adjoint functors arise everywhere."

2.3.1 Definition. Let $F, G : \mathbf{C} \to \mathbf{D}$ be two functors between category **C** and **D** then a natural transformation δ from F to G, denoted by $\delta : F \Longrightarrow G$, is a map that assigns to each **C**-object A, a **D**-morphism $\delta_A : FA \to GA$ and for any **C**-morphism $f : A \to B$, the left square below commutes.

$$\begin{array}{ccc} FA & \xrightarrow{\delta_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\delta_B]{} & GB \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\eta_G} & G^{op} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{op}=f} \\ H & \xrightarrow[\eta_H]{} & H^{op} \end{array}$$

2.3.2 Examples. Every group is naturally isomorphic to its opposite group. To show this first let **Grp** be the category of groups and G^{op} is the opposite group of G, which has the same elements as G and its group operation is defined by $a \circ^{op} b = b \circ a$ where $\circ$ is the operation in G. Define $1_{\textbf{Grp}}$: **Grp** $\to$ **Grp** by $1_{\textbf{Grp}}(G \xrightarrow{f} H) = G \xrightarrow{f} H$, which is the identity functor. Also define op : **Grp** $\to$ **Grp** by $op(G \xrightarrow{f} H) = G^{op} \xrightarrow{f^{op}=f} H^{op}$. In fact, for a group homomorphism $G \xrightarrow{f} H$ we have $f^{op}(a \circ^{op} b) = f(a \circ^{op} b) = f(b \circ a) = f(b) \circ f(a) = f(a) \circ^{op} f(b) = f^{op}(a) \circ f^{op}(b)$. Hence, $G^{op} \xrightarrow{f^{op}=f} H^{op}$ is a group homomorphism. Thus op is also a functor on **Grp**. Next let G be any group and define $\eta_G : G \to G^{op}$ by $\eta_G(g) = g^{-1}$ which is shown by the right square above then $\eta_G(g \circ h) = (g \circ h)^{-1} = h^{-1} \circ g^{-1} = g^{-1} \circ^{op} h^{-1} = \eta_G(g) \circ^{op} \eta_G(h) \quad \forall g, h \in G$. Hence η_G is a group homomorphism. Also, for $g \in G$ we have $(\eta_G \circ \eta_G)(g) = \eta_G(\circ \eta_G(g)) = \eta_G(g^{-1} = (g^{-1})^{-1}) = g = 1_G(g) \Rightarrow \eta_G$ is its own inverse. Furthermore, let $f : G \to H$ be a group homomorphism then $(\eta_H \circ f)(g) = \eta_H(f(g)) = (f(g))^{-1} = f(g^{-1}) = f^{op}(g^{-1})) = f^{op}(\eta_G(g)) = (f^{op} \circ \eta_G)(g)) \quad \forall g \in G \Rightarrow \eta_H \circ f = f^{op} \circ \eta_G$. Hence the above diagram commutes.Therefore there is a isomorphism natural transformation $\eta : 1_{\textbf{Grp}} \Longrightarrow op$.

Next, we discuss a correspondence between posets, called Galois connection.

2.3.3 Definition. Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be pre-ordered sets viewed as categories. A pair of mappings (viewed as a functor) $F : P \to Q$ and $G : Q \to P$ such that $\forall x \in P, y \in Q \quad F(x) \leqslant_Q y \Leftrightarrow x \leqslant_P G(y)$ is called a Galois connection between P and Q. A Galois connection between P and Q is denoted by a pair (F, G). The map F is called a left Galois adjoint and G is the right Galois adjoint. If $x \in P$ and $y \in Q$ are such that $F(x) = y$ and $G(y) = x$, then x and y are said to be corresponding fixed points of the Galois connection.

2.3.4 Remark. The functions F and G in the Definition 2.3.3 are order preserving. Indeed, for $x, x' \in P$ such that $x \leq x'$ we have that $x' \leqslant G(F(x'))$, since $F(x') \leqslant F(x')$. Consequently $x \leqslant G(F(x'))$ and hence $F(x) \leqslant F(x')$. Similarly, we can show that G is an order preserving map.

2.3.5 Proposition. The composition of a Galois connection is also a Galois connection.

Proof. The proof can be found in Appendix A.4. □

2.3.6 Proposition. Let (F, G) be a Galois connection between $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ then F and G uniquely determine each other.

Proof. To prove the uniqueness of G, suppose (F, G') is also a Galois connection between $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ then for $y \in Q$ since $(F \circ G)(y) \leqslant_Q y$ we obtain $G(y) \leqslant_P G'(y)$. Also, since $(F \circ G')(y) \leqslant_Q y$ we obtain $G'(y) \leqslant_P G(y)$. Consequently $\forall y \in Q \quad G(y) = G'(y)$ and hence $G = G'$. Similarly, the uniqueness of F follows. □

2.3.7 Proposition. Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be pre-ordered sets and $F : P \to Q$ and $G : Q \to P$ be two maps then (F, G) is a Galois connection between P and Q if and only if

1. $\forall x \in P$, $\forall y \in Q \quad x \leqslant_P (G \circ F)(x)$ and $(F \circ G)(y) \leqslant_Q y$ and

2. F and G are both monotone.

Proof. $(\Rightarrow)$ Suppose (F, G) is a Galois connection between P and Q then $F(x) \leqslant_Q y \Leftrightarrow x \leqslant_P G(y)$ and hence by reflexive property of $\leqslant_P$ and $\leqslant_Q$ we obtain

$\forall x \in P \quad F(x) \leqslant_Q F(x) \Leftrightarrow x \leqslant_P (G \circ F)(x)$ and $\forall y \in Q \quad G(y) \leqslant_P G(y) \Leftrightarrow (F \circ G)(y) \leqslant_Q y$. Also, by the Remark 2.3.4 F and G are monotone.

$(\Leftarrow)$ Suppose 1 and 2 in the Proposition 2.3.7 hold true. Now, let $p \in P$ and $q \in Q$ such that $F(p) \leqslant_Q q$ then by monotonicity of G we have $F(p) \leqslant_Q q \Rightarrow (G \circ F)(p) \leqslant_P G(q)$. Hence $p \leqslant_P G(q)$, since $p \leqslant_P (G \circ F)(p)$. Also, for $p \leqslant_P G(q)$ we obtain $p \leqslant_P G(q) \Rightarrow F(p) \leqslant_Q (F \circ G)(q)$. As a result $F(p) \leqslant_Q q$, since $(F \circ G)(q) \leqslant_Q q$. Therefore, (F, G) is a Galois connection. □

2.3.8 Proposition. Let (F, G) be a Galois connection between partially ordered sets $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ then $F \circ G \circ F = F$ and $G \circ F \circ G = G$.

Proof. The proof can be found in Appendix A.5 . □

2.3.9 Proposition. Let (F, G) be a Galois connection between partially ordered sets $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ with $0, 0'$ and $1, 1'$ as the least and largest element of P and Q respectively then $F(0) = 0'$ and $G(1') = 1$.

Proof. The proof can be found in Appendix A.6. □

2.3.10 Proposition.

1. Let (F, G) be a Galois connection between partially ordered sets $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ then F and G preserves suprema and infima respectively. In particular, $\forall x, x' \in P \quad F(x \vee x') = F(x) \vee F(x')$ and $\forall y, y' \in Q \quad G(y \wedge y') = F(y) \wedge F(y')$.
2. Let P and Q be two partially ordered sets and assume that all suprema exist in P. Let $F : P \to Q$ be a function that preserves suprema then (F, G), where $\forall y \in Q \quad G(y) = sup\{x \in P : F(x) \leqslant_Q y\} = \bigvee\{x \in P : F(x) \leqslant_Q y\}$, is a Galois connection .
3. Let P and Q be two partially ordered sets and assume that all infima exist in Q. Let $G : Q \to P$ be a function that preserves infima then (F, G), where $\forall x \in P \quad F(x) = inf\{y \in Q : x \leqslant_P G(y)\} = \bigwedge\{y \in Q : x \leqslant_P G(y)\}$, is a Galois connection.

Proof. The proof can be found in Appendix A.7. □

2.3.11 Definition. Let **C** and **D** be two categories and $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be two functors then F is said to be a left adjiont to G if there exists a natural transformation $\eta : 1_{\mathbf{C}} \Rightarrow GF$ such that $\forall A \in Ob(\mathbf{C})$ and $\forall\ A \xrightarrow{f} GB$ where $B \in Ob(\mathbf{C})$ there exists a unique arrow $FA \xrightarrow{f^*} B$ such that $(Gf^*) \circ \eta_A = f$. The transformation $\eta : 1_{\mathbf{C}} \Rightarrow GF$ is called the unit of the adjunction.
2.3.12 Remark. Galois connections are a special case of adjoint functors when the functors are between two posets considered as categories.
2.3.13 Examples. Let $f : X \to Y$ be a function between two sets X and Y with respective power sets $(P(X), \subseteq)$ and $(P(Y), \subseteq)$ viewed as posets and hence as categories. Define maps $f(-) : P(X) \to P(Y)$ by $f(A) = \{f(a) : a \in A\}$ for $A \subseteq X$ and $f^{-1}(-) : P(Y) \to P(X)$ by $f^{-1}(B) = \{x \in X : f(x) \in B\}$ for $B \subseteq Y$. Then, $(f(-), f^{-1}(-))$ is a Galois connection and we call it image preimage Galois adjunction.Indeed, Suppose $f(A) \subseteq B$ then $a \in A \Rightarrow f(a) \in f(A) \subseteq B \Rightarrow f(a) \in B \Rightarrow a \in f^{-1}(B)$. Thus $\forall a(a \in A \Rightarrow a \in f^{-1}(B))$ and hence $A \subseteq f^{-1}(B)$. Conversely, suppose $A \subseteq f^{-1}(B)$ then $b \in f(A) \Rightarrow b = f(a)$ for some $a \in A$. But $a \in A \Rightarrow a \in f^{-1}(B) \Rightarrow f(a) \in B \Rightarrow b = f(a) \in B$. Thus $\forall b(b \in f(A) \Rightarrow b \in B)$ and hence $f(A) \subseteq B$.

2.4 $(\mathbb{E},\mathbb{M})$ Factorization and Subobjects

2.4.1 ($(\mathbb{E},\mathbb{M})$ Factorization). Let $f : X \to Y$ be a function in the category of **Sets** then f can be factored through its image. Indeed, let $e : X \to f(X) = \{f(x) : x \in X\}$ be defined by $\forall x \in X \quad e(x) = f(x)$, which is the codomain restriction of f and $m : f[X] \to Y$ be defined by $\forall x \in X \quad m(f(x)) = f(x)$, which is the inclusion map then $f = m \circ e$. From now onwards except stated otherwise, all the objects and morphisms are in a fixed arbitrary category **C**.

2.4.2 Definition. Let $\mathbb{E}$ and $\mathbb{M}$ be classes of morphisms in **C** such that

1. each of $\mathbb{E}$ and $\mathbb{M}$ is closed under compositions with isomorphisms in the following sense:
   - $[e \in \mathbb{E}$ and i isomorphism in **C** such that $i \circ e$ exists $] \Rightarrow i \circ e \in \mathbb{E}$ and
   - $[m \in \mathbb{M}$ and i isomorphism in **C** such that $m \circ i$ exists $] \Rightarrow m \circ i \in \mathbb{M}$,
2. **C** has $(\mathbb{E},\mathbb{M})$-factorizations (of morphisms); that is each morphism f in **C** has a factorization $f = m \circ e$, with $e \in \mathbb{E}$ and $m \in \mathbb{M}$, and
3. **C** has the unique $(\mathbb{E},\mathbb{M})$-diagonalization property; that is , for each commutative left below square with $e \in \mathbb{E}$ and $m \in \mathbb{M} \quad \exists!$ diagonal $d : Z \to Y$ such that $f = d \circ e$ and $g = m \circ d \Leftrightarrow$ the right diagram below commutes.

$$\begin{array}{ccc} X & \xrightarrow{e} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[m]{} & W \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{e} & Z \\ {\scriptstyle f}\downarrow & \overset{d}{\swarrow} & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[m]{} & W \end{array}$$

Then $(\mathbb{E},\mathbb{M})$ is called a factorization structure for morphisms in **C** and **C** is called $(\mathbb{E},\mathbb{M})$-structured.

2.4.3 Examples. $(Iso(\mathbf{C}), Mor(\mathbf{C}))$ is trivial factorization structure for morphisms in **C**. To see this suppose $e \in \mathbb{E} = Iso(\mathbf{C}), m \in \mathbb{M} = Mor(\mathbf{C})$ and $i \in Iso(\mathbf{C})$ such that $i \circ e$ and $m \circ e$ exist then since Iso is closed under composition we have $i \circ e \in \varepsilon$. Also, since the composition $m \circ e$ exists we have $m \circ e \in \mathbb{M}$. In addition for any morphism $f : X \to Y$ we have $f = f \circ 1_X$ with $f \in \mathbb{M} = Mor(\mathbf{C}), 1_X \in \mathbb{E} = Iso(\mathbf{C})$ and hence factorization in **C**. Furthermore for any left above commutative square with $e \in \mathbb{E}$ and $m \in \mathbb{M}$, since e is an isomorphism there exists $u : Z \to X$ such that $e \circ u = 1_Z, u \circ e = 1_X$. Hence there exists $d : Z \to Y$ given by $d = f \circ u$ such that $f = d \circ e$ and $g = m \circ d$. Besides, e is isomorphism implies e is epimorphism and for if d' is such a diagonal we get $de = f = d'e \Rightarrow d = d'$. So, d is unique. Similarly we can show that $(Mor(\mathbf{C}), Iso(\mathbf{C}))$ is also trivial factorization structure for morphisms in any category **C**.

2.4.4 Lemma. Let **C** be $(\mathbb{E},\mathbb{M})$-structured, $e \in \mathbb{E}$ and $m \in \mathbb{M}$ such that the first below diagram commutes, then e is an isomorphism and $f \in \mathbb{M}$.

$$\begin{array}{ccc} \bullet & \xrightarrow{e} & \bullet \\ {\scriptstyle 1}\downarrow & \overset{d}{\swarrow} & \downarrow{\scriptstyle m} \\ \bullet & \xrightarrow[f]{} & \bullet \end{array} \qquad \begin{array}{ccc} \bullet & \xrightarrow{e} & \bullet \\ {\scriptstyle e}\downarrow & \overset{z}{\swarrow} & \downarrow{\scriptstyle m} \\ \bullet & \xrightarrow[m]{} & \bullet \end{array} \qquad \begin{array}{ccc} A_i & \xrightarrow{e_i} & B \\ {\scriptstyle r_i}\downarrow & \overset{d}{\swarrow} & \downarrow{\scriptstyle s} \\ M & \xrightarrow[m]{} & C \end{array}$$

Proof. Since $e = 1 \circ e$, $e \circ d \circ e = e \circ 1 = e$ and $m \circ e \circ d = f \circ d = m$ we have the second above diagram commutes for $z = 1$ and for $z = e \circ d$. Therefore, by the uniqueness of diagonalization property we have $1 = e \circ d$. Hence, e is an isomorphism since from the first diagram we get $d \circ e = 1$. Thus, since M is closed under composition with isomorphisms we obtain $f = m \circ e \in \mathbb{M}$. □

2.4.5 Proposition. If **C** is $(\mathbb{E}, \mathbb{M})$-structured then

1. $\mathbb{E} \cap \mathbb{M} = Iso(\mathbf{C})$,
2. each of $\mathbb{E}$ and $\mathbb{M}$ is closed under composition ,
3. $(\mathbb{E}, \mathbb{M})$ factorizations are unique up to isomorphism,
4. $n, n \circ m \in \mathbb{M} \Rightarrow m \in \mathbb{M}$,
5. n is monomorphism and $n \circ m \in \mathbb{M} \Rightarrow m \in \mathbb{M}$ and
6. $\mathbb{M}$ is stable under pullback

Proof. The proof can be found in Appendix (A.8). □

2.4.6 Definition. Let $\mathbb{E}$ be a conglomerate of sinks and $\mathbb{M}$ be a class of morphisms such that

1. each of $\mathbb{E}$ and $\mathbb{M}$ is closed under compositions with isomorphisms in the following sense:

- $[(A_i \overset{e_i}{\to} B)_{i \in I}$ is a sink in $\mathbb{E}$, and $B \overset{v}{\to} C$ is an isomorphism $] \Rightarrow [$ the sink $(A_i \overset{v \circ e_i}{\to} C)_{i \in I}$ is also in $\mathbb{E}]$
- $[m \in \mathbb{M}, i$ isomorphism in **C** such that $m \circ i$ exists $] \Rightarrow m \circ i \in \mathbb{M}$,

2. **C** has $(\mathbb{E}, \mathbb{M})$-factorizations (of sinks); that is, each sink s in **C** has a factorization $s = m \circ e$, with $e \in \mathbb{E}$ and $m \in \mathbb{M}$, and
3. **C** has the unique $(\mathbb{E}, \mathbb{M})$-diagonalization property; that is, if $B \overset{s}{\to} C$ and $M \overset{m}{\to} C$ are **C**-morphisms with $m \in \mathbb{M}$, and $e = (A_i \overset{e_i}{\to} B)_{i \in I}$ and $r = (A_i \overset{r_i}{\to} M)_{i \in I}$ are sinks in **C** with $e \in \mathbb{E}$, such that $m \circ r = s \circ e$, then $\exists!$ diagonal $d : B \to M$ such that for every $i \in I$ the third above diagram commutes. Then $(\mathbb{E}, \mathbb{M})$ is called a factorization structure (for sinks) on **C** and **C** is called an $(\mathbb{E}, \mathbb{M})$-category (for sinks).

2.4.7 Remark.

1. Any $(\mathbb{E}, \mathbb{M})$-category for sinks is also an $(\mathbb{E}, \mathbb{M})$-category for single morphisms, with $\mathbb{E}$ consisting of all morphisms (singleton sinks) in $\mathbb{E}$.
2. In order to develop the theory of closure and interior operators from now onwards throughout this project, except stated otherwise, we consider a category **C** and a fixed class $\mathbb{M}$ of **C**-monomorphisms that contains all isomorphisms and the assumption that **C** is $\mathbb{M}$-complete, that is, $\mathbb{M}$ is closed under composition and Pullbacks of $\mathbb{M}$-morphisms exist and belong to $\mathbb{M}$, and multiple pullbacks (possibly large) families of $\mathbb{M}$-morphisms with the same codomain exist and belong to $\mathbb{M}$. As a consequence, there is an $(\mathbb{E}, \mathbb{M})$ factorization system for sinks in **C** (cf. Adamek et al. (2004) for the dual results).

2.4.8 M-Subobjects. The notion of subobject provides the categorical formulations for structures such as subsets in set theory, subgroup in group theory and subspaces in topology. We define closure operators at each subsets of a certain set, subgroups of a given group and subspaces of a given topological space in the category of **Sets**, **Top** and **Grp** respectively. In arbitrary category we define closure operators on a suitable axiomatically defined notion of subobjects. Subobjects are described by special morphisms in **C** which may be thought of as inclusion maps.

2.4.9 Definition. The class of all $\mathbb{M}$-morphisms with codomain X is called the subobjects of X. Subobjects of X, denoted by $Sub_{\mathbf{C}}(X) = \{m \in \mathbb{M} : \text{codmain of } m \text{ is } X\}$, form a category. When there is no confusion we use $Sub(X)$ for $Sub_{\mathbf{C}}(X)$.

2.4.10 Remark.

1. In $Sub_{\mathbf{C}}(X)$ the objects are morphisms $M \overset{m}{\longrightarrow} X$ in **C** whose codomain is X and the arrow between the objects $M \overset{m}{\longrightarrow} X$ and $N \overset{n}{\longrightarrow} X$ is given by $M \overset{j}{\longrightarrow} N$ such that $m = n \circ j$. In fact since n is monic there is at most one j and hence $Sub_{\mathbf{C}}(X)$ is a pre-ordered category. Also, since m is monic we have j is also monic.
2. The subobjects are ordered by $m \leq n \Leftrightarrow \exists j(m = n \circ j)$ in $Sub_{\mathbf{C}}(X)$ is pre-order relation. Indeed, this relation is reflexive and transitive. Hence $Sub_{\mathbf{C}}(X)$ is a preordered class.
3. If $n \leq m$ and $m \leq n$ then they are isomorphic and we write $m \cong n$. We do not distinguish between isomorphic subobjects.Indeed, $m \leq n$ and $n \leq m \Rightarrow \exists i, j$ such that $m = n \circ j$ and $n = m \circ i$. Hence, $m \circ 1_M = m = n \circ j = m \circ i \circ j = m \circ (j \circ i) \Rightarrow j \circ i = 1_M$, since m is monic . Similarly, since n is monic we obtain $j \circ i = 1_N$.
4. $\cong$ is an equivalence relation. Hence $Sub_{\mathbf{C}}(X)$ modulo $\cong$ is a poset so that we can use all lattice theoretic concepts and we write $m = n$ for isomorphic subobjects. As the consequence of our assumptions on $\mathbb{M}$ we have $Sub_{\mathbf{C}}(X)$ is a complete lattice. We use 0_X and 1_X to denote the least and greatest element of $Sub_{\mathbf{C}}(X)$ respectively.
5. The joins and meets in $Sub_{\mathbf{C}}(X)$ are denoted by the usual symbols $\vee, \bigvee$ and $\wedge, \bigwedge$, respectively.

2.4.11 Definition. Let $(M_i \overset{m_i}{\to} X)_{i \in I}$ be a family of subobjects of an object X, indexed by class I, then we say that a subobject $M \overset{m}{\to} X$ is an intersection of the family $(M_i, m_i)_{i \in I}$, denoted by $\bigcap_{i \in I} m_i$, if

1. m factors through each m_i, that is, $\forall i \in I \quad \exists u_i : M \to M_i$ such that $m = m_i \circ u_i$ and
2. any morphism $f : Y \to X$ that factors through each m_i must also factor through m.

That is, m is infimum of m_i with respect to $\leq$. Intersections are unique up to isomorphism and can be interpreted as multiple pullbacks.

2.4.12 Theorem. *Let* **C** *be an* $(\mathbb{E}, \mathbb{M})$*-category (for sinks)then the following properties hold.*

1. *Every isomorphism is in both* $\mathbb{M}$ *and* $\mathbb{E}$*.*
2. *Every* $m \in \mathbb{M}$ *is a monomorphism.*
3. $\mathbb{M}$ *is closed under* $\mathbb{M}$*-relative first factors, that is,* $n, n \circ m \in \mathbb{M} \Rightarrow m \in \mathbb{M}$*.*
4. $\mathbb{M}$ *and* $\mathbb{E}$ *are closed under composition.*
5. *Pullbacks of* **C***-morphisms in* $\mathbb{M}$ *exist and belong to* $\mathbb{M}$*. That is, Inverse images of* $\mathbb{M}$*-subobjects along* **C***-morphism exist.*
6. *The* $\mathbb{M}$*- subobjects of every* **C***-object form a (possibly large) complete lattice superma are formed via* $(\mathbb{E}, \mathbb{M})$*-factorization and if* **C** *has arbitrary intersections, then infima are formed via intersections.*

Proof. The proof for the dual result can be found in Adamek et al. (2004). □

3. Closure Operators

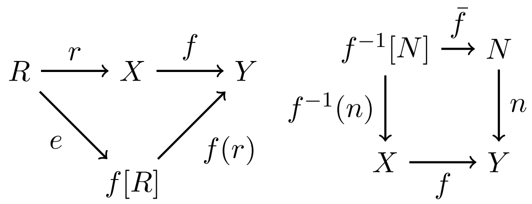
In this chapter we discuss closure operators in arbitrary category. The relevant material can be found in Dikranjan and Tholen (1995) and Castellini (2003). We begin by presenting a history of closure operators.

3.1 Introduction

Closure operators were first introduced in Analysis by Moore (1909) and Riesz (1909) and since then they have been defined, studied and intensively used in Logic (Hertz, 1922), (Tarski, 1929), Algebra (Birkhoff, 1937), (Pierce, 1972), Topology (Kuratowski, 1922), (Cech, 1937) and Lattice theory (Birkhoff, 1940). In 1987, Dikranjan and Giuli (Dikranjan and Giuli, 1987) introduced the concept of closure operators in an arbitrary category. The establishment of the categorical notion of closure operators has unified various important notions and has led to interesting examples and applications in diverse areas of Mathematics. The basic idea behind the theory of the closure operators is to provide a tool that allows us to extend to an arbitrary category **C**, without any topology, topological concepts such as closedness, denseness, separatedness, regularity, compactness and connectedness as is done with the classical Kuratowski operator in general topology. We define categorical closure operators for any category with a suitable axiomatically defined notion of subobject . Closure operators may be interpreted as factorization systems, every morphism $f : X \to Y \in \mathbf{C}$ can be factored through the closure of its image, $X \to c[f(X)] \to Y$.

3.2 Basic Properties of Closure Operators

3.2.1 Definition (Image/pre-image)**.** Let $r \in Sub(X)$ and $n \in Sub(Y)$ then we define the image of r under $f : X \to Y \in \mathbf{C}$ as the $\mathbb{M}$-component of the $(\mathbb{E}, \mathbb{M})$ factorization of $f \circ r$ such that the left diagram below commutes.



We define the pre-image of n under f, denoted by $f^{-1}(n)$, as the pullback of n along f which is shown by the right diagram above.

3.2.2 Remark. 1. We usually denote the d$f^{-1}(c_Y(g^{-1}(n))) \cong c_X(f^{-1}(g^{-1}(n))) \cong c_X((g\circ f)^{-1}(n))$omain of $n \in Sub(Y)$ by N. Since $\mathbb{M}$ is stable under pullback, we have $f^{-1}(n) \in \mathbb{M}$ and hence $f^{-1}(n) \in Sub(X)$.

2. We write $(e, f(r))$ to describe the $(\mathbb{E}, \mathbb{M})$ factorization of $f \circ r$ with e and $f(r)$ as the $\mathbb{E}$ and $\mathbb{M}$ component of the $(\mathbb{E}, \mathbb{M})$ factorization of $f \circ r$ respectively.

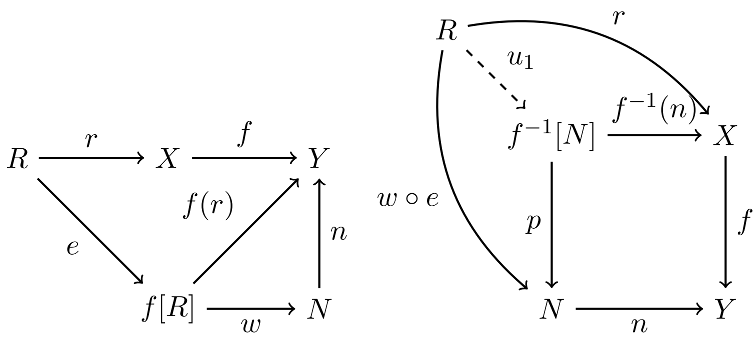
3.2.3 Proposition. Any morphism $f : X \to Y$ in **C** induces an image/preimage adjunction

$$Sub(X) \underset{f^{-1}(-)}{\overset{f(-)}{\rightleftarrows}} Sub(Y)$$

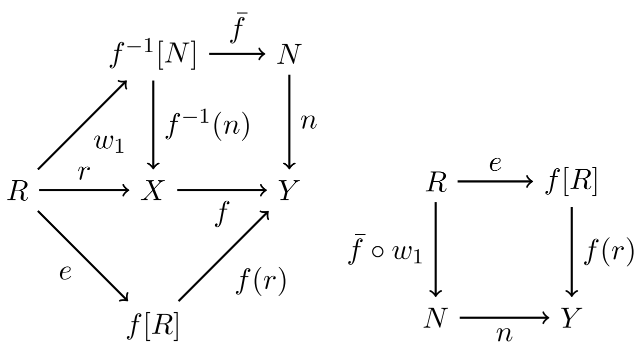
Proof. Let $r \in Sub(X)$ and $n \in Sub(Y)$. We now show $f(r) \leqslant n \Leftrightarrow r \leqslant f^{-1}(n)$.

($\Rightarrow$) suppose $f(r) \leqslant n$ then $\exists! w : f[R] \to N$ such that $f(r) = n \circ w$. Thus we obtain the commutative diagram below on the left. As consequence $f \circ r = n \circ w \circ e$ and hence the solid arrow diagram below on the right commutes. Thus, by the pullback property $\exists! u_1 : R \to f^{-1}[N]$ such that $r = f^{-1}(n) \circ u_1$ and $w \circ e = p \circ u_1$. Hence $r \leq f^{-1}(n)$.

$$\text{Therefore, } f(r) \leq n \Rightarrow r \leq f^{-1}(n). \tag{3.2.1}$$



($\Leftarrow$) suppose $r \leqslant f^{-1}(n)$ then $\exists! w_1 : R \to f^{-1}[N]$ such that $r = f^{-1}(n) \circ w_1$ and the left hand diagram commutes. Thus $f(r) \circ e = f \circ r = f \circ f^{-1}(n) \circ w_1 = n \circ \bar{f} \circ w_1$. As result we obtain the right below square with $e \in \mathbb{E}$ and $n \in \mathbb{M}$.



Therefore, $\exists! d : f[R] \to N$ such that $f(r) = n \circ d$ and $\bar{f} \circ w_1 = d \circ e$ and this in turn implies $f(r) \leq n$.

$$\text{Thus } r \leq f^{-1}(n) \Rightarrow f(r) \leq n. \tag{3.2.2}$$

Now, combining the inequalities in (3.2.1) and (3.2.2) we have $\forall r \in Sub(X)$, $n \in Sub(Y)$ $f(r) \leq n \Leftrightarrow r \leq f^{-1}(n)$. Hence the proof. $\square$

3.2.4 Remark. Since $Sub(X)$ and $Sub(Y)$ are posets by the Proposition 2.3.7 and A.7 we get

- $\forall r, s \in Sub(X) \quad r \leq s \Rightarrow f(r) \leq f(s)$;
- $\forall r \in Sub(X), \quad \forall n \in Sub(Y) \quad r \leq f^{-1}(f(r))$ and $f(f^{-1}(n)) \leq n$;

- $\forall r_i \in Sub(X)$, $\forall n_i \in Sub(Y)$ $\quad f(\bigvee_{i\in I)} r_i) = \bigvee_{i\in I} f(r_i)$ and $f^{-1}(\bigwedge_{i\in I} n_i) = \bigwedge_{i\in I} f^{-1}(n_i)$;
- Since 1_X is the largest element of $Sub(X)$ we get $f^{-1}(1_Y) \leq 1_X$ and $f(1_X) \leq 1_Y$. As a result $f^{-1}(1_Y) \leq 1_X$ and $1_X \leq f^{-1}(1_Y)$ and hence $f^{-1}(1_Y) \cong 1_X$. Similarly, for the least element 0_X of $Sub(X)$ we get $f(0_X) \cong 0_Y$.

3.2.5 Definition. A closure operator c on $\mathbf{C}$ with respect to $\mathbb{M}$ is a family of functions

$\{c_X : Sub(X) \to Sub(X) | X \in \mathbf{C}\}$ such that for all X in $\mathbf{C}$ we have:

(C_1) (Expansiveness) $\forall r \in Sub(X) \quad r \leqslant c_X(r)$;

(C_2) (Order Preservation) $\forall r, s \in Sub(X) \quad r \leqslant s \Rightarrow c_X(r) \leqslant c_X(s)$ and

(C_3) (Continuity) $\forall r \in Sub(X) \quad f : X \to Y \in \mathbf{C} \Rightarrow f(c_X(r)) \leqslant c_Y(f(r))$.

3.2.6 Remark. 1. We sometimes ignore the subscript of c_X and write c for c_X when no confusion is likely and the domain of $c_X(r)$ is denoted by $c_X[R]$.

2. Lattice theorists usually define a closure operator c of a lattice L with least element 0 to be a function $c : L \to L$ satisfying the above conditions (C_1), (C_2) and condition (C_3) replaced by $c(c(r)) = c(r)$, which is called idempotence condition. From the categorical point of view, these systems of axioms turn out to be both insufficient and too restrictive. As pointed out in Dikranjan and Tholen (1995) they ignore the important continuity condition and a closure operator is available in $Sub(X)$ $\forall X \in$ $\mathbf{C}$. We can have a link between any two $\mathbb{M}$-subobjects via the continuity condition of a closure operators.

3. $\forall r, s \in Sub(X) \quad r \cong s \Rightarrow c_X(r) \cong c_X(s)$.

4. Condition (C_1) implies that for every closure operator c on $\mathbf{C}$ and $r \in Sub(X)$ there is a canonical factorization. That is, $\exists j_r$ such that $r = c_X(r) \circ j_r$

5. We denote the ordered conglomerate of all closure operators on $\mathbf{C}$ with respect to $\mathbb{M}$ by $CL(\mathbf{C}, \mathbb{M})$. $CL(\mathbf{C}, \mathbb{M})$ pre-ordered pointwise, writing $c \leqslant c'$ if for all $r \in \mathbb{M} \quad c(r) \leqslant c'(r)$.

3.2.7 Proposition. Consider the axiom $(C_4) \quad \forall f : X \to Y \in \mathbf{C}, r \in Sub(X)$ and $n \in Sub(Y) \quad f(r) \leqslant n \Rightarrow f(c_X(r)) \leqslant c_Y(n)$.Then $[(C_2)$ and $(C_3)] \Leftrightarrow (C_4)$.

Proof. $(\Rightarrow)$ Suppose $[(C_2)$ and $(C_3)]$ hold true and $\forall r \in Sub(X)$ and $n \in Sub(Y) \quad f(r) \leq n$.Then

$$\begin{aligned} f(r) \leqslant n &\Rightarrow c_Y(f(r)) \leqslant c_Y(n), \text{ by } (C_3) \\ &\Rightarrow f(c_X(r)) \leqslant c_Y(f(r)) \leqslant c_Y(n), \text{ by } (C_3) \\ &\Rightarrow f(c_X(r)) \leqslant c_Y(n) \text{ as we are in a lattice} \end{aligned}$$

Hence, the forward direction is true.

$(\Leftarrow)$ Suppose (C_4) holds. Then

- Let $r \leqslant s$ then for $1_X : X \to X$ we have $r \leqslant s \Rightarrow 1_X(r) \leqslant s \Rightarrow 1_X(c_X(r)) \leqslant c_X(s) \Rightarrow c_X(r) \leqslant c_X(s)$. Therefore, $r \leqslant s \Rightarrow c_X(r) \leqslant c_X(s)$ and hence (C_2).
- Let $r \in Sub(X)$ then $f(r) \in Sub(Y)$ and by reflexivity of $\leqslant$ we have $f(r) \leq f(r)$. By (C_4) for $n = f(r)$ we obtain $f(c_X(r)) \leq c_Y(f(r))$ and hence (C_3) . Therefore, the reverse direction is also true.

Hence the required result. □

3.2.8 Proposition. Consider the axiom (C_3') $\forall f : X \to Y \in \mathbf{C}$ and $n \in Sub(Y)$ $c_X(f^{-1}(n)) \leqslant f^{-1}(c_Y(n))$. In the presence of (C_2) we have $(C_3) \Leftrightarrow (C_3')$.

Proof. $(\Rightarrow)$ Suppose $\forall f : X \to Y \in \mathbf{C}$ and $r \in Sub(X)$ $f(c_X(r)) \leqslant c_Y(f(r))$. Then for $n \in Sub(Y)$ we have $f^{-1}(n) \in Sub(X)$. Hence by (C_3)

$$f(c_X(f^{-1}(n))) \leqslant c_Y(f(f^{-1}(n))) \tag{3.2.3}$$

Because of the adjunction between f and f^{-1} we have $f(f^{-1}(n)) \leqslant n$ and hence

$$c_Y(f(f^{-1}(n))) \leqslant c_Y(n) \tag{3.2.4}$$

Now, combining the equations (3.2.3) and (3.2.4) and the fact that (f, f^{-1}) is a Galois connection we obtain $f(c_X(f^{-1}(n))) \leqslant c_Y(n) \Rightarrow c_X(f^{-1}(n)) \leqslant f^{-1}(c_Y(n))$ and hence (C_3').

$(\Leftarrow)$ Suppose (C_3') $\forall f : X \to Y \in \mathbf{C}$ and $n \in Sub(Y)$ $c_X(f^{-1}(n)) \leqslant f^{-1}(c_Y(n))$. Then for $r \in Sub(X)$ we have $f(r) \in Sub(Y)$. Hence putting $n = f(r)$ in (C_3') we have $c_X(f^{-1}(f(r))) \leqslant f^{-1}(c_Y(f(r))) \Rightarrow c_X(r) \leqslant c_X(f^{-1}(f(r))) \leqslant f^{-1}(c_Y(f(r)))$, as $r \leqslant (f^{-1} \circ f)(r)$. Again $c_X(r) \leqslant c_X(f^{-1}(f(r))) \leqslant f^{-1}(c_Y(f(r))) \Rightarrow c_X(r) \leqslant f^{-1}(c_Y(f(r))) \Rightarrow f(c_X(r)) \leqslant c_Y(f(r))$, since (f, f^{-1}) is a Galois connection. Hence (C_3). □

3.2.9 Definition. A morphism $f : X \to Y \in \mathbf{C}$ which satisfies condition $(C_3)'$ is said to be c-continuous.

3.2.10 Proposition. The composition of c-continuous morphisms in $\mathbf{C}$ is also c-continuous.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ be c-continuous morphisms in $\mathbf{C}$. Let $n \in Sub(Z)$ then since g is c-continuous we have

$$c_Y(g^{-1}(n)) \leqslant g^{-1}(c_Z(n)) \tag{3.2.5}$$

Also, since $g^{-1}(n) \in Sub(Y)$, f is c-continuous and by the Proposition 3.2.3 we obtain

$$\begin{aligned} &c_X(f^{-1}(g^{-1}(n))) = c_X((g \circ f)^{-1}(n)) \leqslant f^{-1}(c_Y(g^{-1}(n))) \\ &\Rightarrow f(c_X((g \circ f)^{-1}(n))) \leqslant c_Y(g^{-1}(n)) \end{aligned} \tag{3.2.6}$$

Now, combining the inequalities (3.2.5) and (3.2.6) we get

$$f(c_X((g \circ f)^{-1}(n))) \leqslant g^{-1}(c_Z(n)) \Rightarrow c_X((g \circ f)^{-1}(n)) \leqslant f^{-1}(g^{-1}(c_Z(n))) \Rightarrow c_X((g \circ f)^{-1}(n)) \leqslant (g \circ f)^{-1}(c_Z(n)$$

Therefore, $\forall n \in Sub(Z)$ $c_X((g \circ f)^{-1}(n)) \leqslant (g \circ f)^{-1}(c_Z(n))$ and hence $g \circ f$ is c-continuous. □

3.2.11 Remark. The definition of c assumes of course that all morphisms are c-continuous.

3.3 Closed and Dense Subobjects

3.3.1 Definition. An $\mathbb{M}$-subobject $r \in Sub(X)$ such that

1. $r \cong c_X(r)$ is called c-closed (in X), that is, if $\exists$ an isomorphism $j_r : R \to c_X[R]$ such that $r = c_X(r) \circ j_r$,
2. $c_X(r) \cong 1_X$ is called c-dense in X, that is, if $\exists$ an isomorphism $j_{c_X(r)} : c_X[R] \to X$ such that $c_X(r) = 1_X \circ j_{c_X(r)}$.

$f^{-1}(c_Y(g^{-1}(n))) \cong c_X(f^{-1}(g^{-1}(n))) \cong c_X((g \circ f)^{-1}(n))$ From the continuity condition of closure operators we have a c-closedness and c-denseness are preserved by pre-images and images respectively.
3.3.2 Proposition. Let $f : X \to Y$ be a morphism then

1. n is c-closed in $Y \Rightarrow f^{-1}(n)$ is c-closed in X
2. r is c-dense in X and $f \in \mathbb{E} \Rightarrow f(r)$ is c-dense in Y.

Proof. 1. Suppose n is c-closed in Y. Then $n \cong c_Y(n) \Rightarrow c_Y(n) \leqslant n$. Now, $c_Y(n) \leqslant n \Rightarrow f^{-1}(c_Y(n)) \leqslant f^{-1}(n)$. But by (C_3) $c_X(f^{-1}(n)) \leqslant f^{-1}(c_Y(n))$ and hence $c_X(f^{-1}(n)) \leqslant f^{-1}(n)$. Thus by (C_1) $c_X(f^{-1}(n)) \cong f^{-1}(n)$ and hence $f^{-1}(n)$ is c-closed.

2. Suppose r is c-dense in X and $f \in \mathbb{E}$ then $c_X(r) \cong 1_X$ and $f(1_X) \cong 1_Y$. Hence by (C_3) we obtain $1_Y \cong f(1_X) \cong f(c_X(r)) \leq c_Y(f(r)) \leq 1_Y$.This implies $1_Y \cong c_Y(f(r))$ and hence $f(r)$ is c-dense in Y.

□

3.3.3 Remark. Let $\mathbb{M}_c$ be the class of c-closed $\mathbb{M}$-subobjects then by the Proposition 3.3.2 we can conclude that $\mathbb{M}_c$ is stable under pullback.
3.3.4 Corollary. Let n and k be monomorphisms such that $n \circ k$ is an c-closed $\mathbb{M}$-subobjects then k is a c-closed $\mathbb{M}$-subobject.

Proof. If $n \circ k \in \mathbb{M}$ with n monic, then we have a pullback diagram

$$\begin{array}{ccc} K & \xrightarrow{1_K} & K \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle n \circ k} \\ N & \xrightarrow[n]{} & Y \end{array}$$

Therefore, by the Proposition 4.2.2 we have $k = n^{-1}(n \circ k) \in \mathbb{M}$ is c-closed. □

3.3.5 Proposition. Let c be a closure operator on X with respect to $\mathbb{M}$ and $(R_i \xrightarrow{r_i} X)$ be a family of $\mathbb{M}$-subobjects of X then

1. $c_X(\bigwedge_{i \in I} r_i) \leqslant \bigwedge_{i \in I} c_X(r_i)$
2. $\bigvee_{i \in I} c_X(r_i) \leqslant c_X(\bigvee_{i \in I} r_i)$
3. $\forall i \in I, r_i$ is c-closed $\Rightarrow \bigwedge_{i \in I} r_i$ is c-closed
4. $\forall i \in I, r_i$ is c-dense $\Rightarrow \bigvee_{i \in I} r_i$ is c-dense

Proof. 1. Since $\forall i \in I$ $\bigwedge_{i \in I} r_i \leqslant r_i$ then by the order preservation condition of closure operators we have that $\forall i \in I$ $c_X(\bigwedge_{i \in I} r_i) \leqslant c_X(r_i)$. As result we obtain $c_X(\bigwedge_{i \in I} r_i) \leqslant \bigwedge_{i \in I} c_X(r_i)$, since $\bigwedge_{i \in I} c_X(r_i)$ is the greatest lower bound (meet) of $\{c_X(r_i) : i \in I\}$.

2. Similar to 1 in the Proposition 3.3.5.

3. From the expansiveness condition of closure operators we have that $\bigwedge_{i\in I} r_i \leqslant c_X(\bigwedge_{i\in I} r_i)$. Also, by 1 in the Proposition 3.3.5 we have that $c_X(\bigwedge_{i\in I} r_i) \leqslant \bigwedge_{i\in I} c_X(r_i) \cong \bigwedge_{i\in I} r_i$, since each r_i is c-closed. Hence $\bigwedge_{i\in I} r_i \leqslant c_X(\bigwedge_{i\in I} r_i) \leqslant \bigwedge_{i\in I} r_i$. Therefore, $\bigwedge_{i\in I} r_i \cong c_X(\bigwedge_{i\in I} r_i)$ and hence $\bigwedge_{i\in I} r_i$ is c-closed.
4. Since 1_X is the largest element of $Sub(X)$ we have that $c_X(\bigvee_{i\in I} r_i) \leqslant 1_X$. Also, since each r_i is c-dense in X we obtain $\forall i \in I \quad c_X(r_i) \cong 1_X$ and hence by 2 in the Proposition 3.3.5 $1_X \cong \bigvee_{i\in I}(1_X) \cong \bigvee_{i\in I}(c_X(r_i)) \leqslant c_X(\bigvee_{i\in I} r_i)$. Therefore, $c_X(\bigvee_{i\in I} r_i) \cong 1_X$ and hence $\bigvee_{i\in I} r_i$ is c-dense.

□

3.3.6 Definition. A morphism $f : X \to Y$ such that $\forall r \in Sub(X) \quad f(c_X(r)) \cong c_Y(f(r))$ is called c-closed (c-preserving). It is called c-open if $\forall n \in Sub(Y) \quad f^{-1}(c_Y(n)) \cong c_X(f^{-1})(n)$

3.3.7 Proposition. The composition of c-closed (c-open) morphisms is also c-closed (c-open).

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are c-open morphisms then $\forall n \in Sub(Z) \quad c_Y(g^{-1}(n)) \cong g^{-1}(c_Z(n))$. As a result by the c-openness of f we obtain $(g \circ f)^{-1}(c_Z(n)) \cong f^{-1}(g^{-1}(c_Z(n))) \cong f^{-1}(c_Y(g^{-1}(n))) \cong c_X(f^{-1}(g^{-1}(n))) \cong c_X((g \circ f)^{-1}(n))$. Hence $g \circ f$ is also c-open. Analogously, c-closed morphisms are closed under composition. □

3.4 Examples of Closure Operators

The following are a few examples of closure operators.

1. Let **Top** be the category of topological spaces and continuous functions with the (episink, embedding) factorization structure and let $r : R \to X$ be an embedding, without loss of generality we assume $R \in P(X)$.

   (a) Define $k_X : P(X) \to P(X)$ by $k_X(R) = \bigcap\{B \subseteq X : R \subseteq B \text{ and } B \text{ is a closed set}\}$ then

   - since $R \subseteq B \quad \forall B \in \{B \subseteq X : R \subseteq B \text{ and } B \text{ is a closed set}\}$ we have $R \subseteq \bigcap\{B \subseteq X : R \subseteq B \text{ and } B \text{ is a closed set}\} = k_X(R)$.
   - for $R \subseteq S$ since $k_X(S)$ is a closed set such that $R \subseteq S \subseteq k_X(S)$ and $k_X(R)$ is the smallest closed set containing R we have $k_X(R) \subseteq k_X(S)$
   - for a continuous function $f : X \to Y$ and $R \subseteq Y$ we have

   $$\begin{aligned} f^{-1}(k_Y(R)) &= f^{-1}\Big(\bigcap\{B \subseteq Y : R \subseteq B \text{ and } B \text{ is a closed set}\}\Big) \\ &= \bigcap\{f^{-1}(B) : B \text{ is closed set in } Y \text{ and } R \subseteq B(\text{ hence } f^{-1}(R) \subseteq f^{-1}(B))\} \\ &\supseteq \bigcap\{V \supseteq f^{-1}(R) : V \text{ is closed set in } X\} = k_X(f^{-1}(R)) \end{aligned}$$

   Note that continuous functions preserve closed sets under pre-image and $k_X(f^{-1}(R))$ is the smallest closed set containing $f^{-1}(R)$ and $\bigcap\{f^{-1}(B) : B \text{ is closed set in } Y \text{ and } R \subseteq B\}$ is a closed set containing $f^{-1}(R)$. Therefore, the intersection of all closed sets in X that are supersets of R, denoted by $k_X(R)$, is a closure operator on **Top**. In fact this operator is the usual topological closure operator. It is also known as the Kuratowoski closure operator.

(b) Define $q_X(R) = \bigcap\{F \supseteq R : F$ is clopen in $X\}$, which is the intersection of all closed and open (clopen) subsets of X which contain R then analogously to the proof in (a) we see that q_X is a closure operator on **Top**. It is called the quasicomponent closure operator.

(c) Let $\delta_X(R) = \{x \in X : \exists$ a sequence $\{x_n\}$ in R such that $\{x_n\}$ converges to x in $X\}$ then

- let $r \in R$ then there exists a constant sequence $\{r\}$ in R such that the sequence $\{r\}$ converges to r in X. Thus r is also in $\delta_X(R)$. Therefore as r is arbitrarily chosen we have $R \subseteq \delta_X(R)$ and hence the expansiveness condition of closure operators holds true,
- suppose $R \subseteq S$ then $x \in \delta_X(R)$ implies $\exists$ sequence $\{x_n\}$ in R such that $\{x_n\}$ converges to x in X. But since $R \subseteq S$ we have the sequence $\{x_n\}$ is also in S. Thus $x \in \delta_X(S)$ and hence $\delta_X(R) \subseteq \delta_X(S)$. Hence the order preservation condition of closure operators holds true.
- Let $f : X \to Y$ be a continuous function such that $y \in f(\delta_X(R))$ then $x \in \delta_X(R)$ such that $y = f(x)$. But $x \in \delta_X(R) \Rightarrow \exists$ a sequence $\{x_n\}$ in R such that $\{x_n\}$ converges to x. Hence by continuity of f we have $\{f(x_n)\}$ in $f(R)$ converges to $f(x) = y$ in $f(X)$. Therefore, $y \in \delta_Y(f(R))$ and hence $f(\delta_X(R)) \subseteq \delta_Y(f(R))$, which is the continuity condition of the closure operators. Therefore $\delta_X(R)$ is a closure operator. It is called the sequential closure of R in X.

(d) Define $\theta_X(R) = \{x \in X : R \cap \overline{N_x} \neq \emptyset \quad \forall$ neighborhoods N_x of $x \in X\}$ where $\overline{N_x}$ is the usual Kuratowski closure of the neighbourhood N_x then

- for $x \in R$ we have $x \in R \cap \overline{N_x} \quad \forall$ neighborhood N_x of $x \in X$, as $x \in \overline{N_x}$. Thus $R \cap \overline{N_x} \neq \emptyset \quad \forall$ neighborhoods N_x of $x \in X$. Therefore, $x \in \theta_X(R)$ and $R \subseteq \theta_X(R)$.
- for $R \subseteq S$ we have $R \cap \overline{N_x} \subseteq S \cap \overline{N_x}$ for any neighborhood N_x of $x \in X$. Therefore, $\theta_X(R) \subseteq \theta_X(S)$ and
- for a continuous function $f : X \to Y$ and $R \subseteq Y$ and since $\theta_X(R) = \{x \in X : R \cap \overline{N_x} \neq \emptyset \quad \forall$ neighborhoods N_x of $x \in X\} = \bigcap\{\overline{N} : R \subseteq N, N$ is open $\}$ we have

$$\begin{aligned} f^{-1}(\theta_Y(R)) &= f^{-1}(\bigcap\{\overline{N} : R \subseteq N, N \text{ is open }\}) \\ &= \bigcap\{f^{-1}(\overline{N}) : R \subseteq N, N \text{ is open }\}) \\ &\supseteq \bigcap\{\overline{f^{-1}(N)} : f^{-1}(R) \subseteq f^{-1}(N), f^{-1}(N) \text{ is open }\}) \\ &\supseteq \bigcap\{\overline{M} : f^{-1}(R) \subseteq M, M \text{ is open }\}) = \theta_X(f^{-1}(R)) \end{aligned}$$

Therefore, $\theta_X(R)$, is a closure operator on **Top**. It is called the θ closure operator.

2. Let **Grp** be the category of groups and homomorphisms with the (episink, monomorphism) factorization structure and let $r : R \to X$ be a subgroup, with out loss of generality we assume $R \leq X$ then the intersection of all normal subgroups of X containing R is a closure operator on **Grp**.This is known as the normal closure of R. Indeed, for $c_X(R) = \bigcap\{N \trianglelefteq X : R \subseteq N\}$
   - Since $R \leq X$ (R is subgroup of X), $R \subseteq c_X(R)$ and the intersection of normal subgroups is also normal we have $c_X(R) \trianglelefteq X$ and $R \leq c_X(R)$
   - for $R \leq S$ since $c_X(R)$ is the smallest normal subgroup containing R and $c_X(S)$ is a normal subgroup containing R as $R \leq S$ we have $c_X(R) \leq c_X(S)$.

- for a homomorphism $f : X \to Y$ and, $R \leq Y$ since the pre-image of a normal subgroup under a group homomorphism is also normal and $c_X(f^{-1}(R))$ is the smallest normal subgroup containing $f^{-1}(R)$ we have $f^{-1}(c_Y(R)) = f^{-1}(\bigcap\{N \trianglelefteq Y : R \subseteq N\}) = \bigcap\{f^{-1}(N) : R \subseteq N \text{ and } N \trianglelefteq Y\} \supseteq \bigcap\{P \trianglelefteq X : f^{-1}(R) \subseteq P\} = c_X(f^{-1}(R))$.

3.4.1 Definition. A closure operator is said to be

1. grounded if $\forall X \in \mathbf{C} \quad c_X(0_X) \cong 0_X$,
2. idempotent if $(\forall X \in \mathbf{C})(\forall r \in Sub(X)) \quad c_X(c_X(r)) \cong c_X(r)$,
3. additive if $\forall X \in \mathbf{C}$ and $r, s \in Sub(X)$ we have $c_X(r \vee s) \cong c_X(r) \vee c_X(s)$.

3.4.2 Examples. The Kuratowski closure operator of **Top** is additive, idempotent and grounded while θ-closure operator of **Top** isn't idempotent.

4. Interior Operators

This chapter presents interior operators in an arbitrary category and we refer to the recent research articles Vorster (2000), Castellini and Ramos (2010), Luna-Torres et al. (2010), Castellini (2011), and Castellini (2013).

4.1 Basic Properties of Interior Operators

The following definition of interior operator in an arbitrary category was introduced by S.J.R. Vorster (Vorster, 2000).

4.1.1 Definition. An interior operator i on **C** with respect to $\mathbb{M}$ is a family of functions $\{i_X : Sub(X) \to Sub(X) | X \in \mathbf{C}\}$ such that for all X in **C** we have:

(I_1) (Contractibility) $\forall r \in Sub(X) \quad i_X(r) \leqslant r$;

(I_2) (Order Preservation) $\forall r, s \in Sub(X) \quad r \leqslant s \Rightarrow i_X(r) \leqslant i_X(s)$ and

(I_3) (Continuity) $\forall n \in Sub(Y) \quad f : X \to Y \in \mathbf{C} \Rightarrow f^{-1}(i_Y(n)) \leqslant i_X(f^{-1}(n))$.

From the Contractibility condition we have that for every interior operator i on **C** and $\forall r \in Sub(X)$ there is a canonical factorization $i_X(r) = r \circ j_r$.

4.1.2 Remark.

1. We sometimes ignore the subscript of i_X and write i for i_X when no confusion is likely and the domain of $i_X(r)$ is denoted by $i_X[R]$.
2. We denote the ordered conglomerate of all interior operators on **C** with respect to $\mathbb{M}$ by $INT(\mathbf{C}, \mathbb{M})$. $INT(\mathbf{C}, \mathbb{M})$ pre-ordered pointwise, writing $i \leqslant i'$ if for all $r \in \mathbb{M} \quad i(r) \leqslant i'(r)$.

4.1.3 Definition. A morphism $f : X \to Y \in \mathbf{C}$ which satisfies condition (I_3) is said to be i-continuous.

4.1.4 Proposition. Consider the axiom (I_3') $\quad f : X \to Y \in \mathbf{C}$ such that $\forall r \in Sub(X), n \in Sub(Y) \quad f(r) \leqslant i_Y(n) \Rightarrow r \leqslant i_X(f^{-1}(n))$. Then we have $(I_3) \Leftrightarrow (I_3')$.

Proof. ($\Rightarrow$) Suppose (I_3) holds. Then for $f : X \to Y \in \mathbf{C}$ such that $\forall r \in Sub(X)$and $n \in Sub(Y) \quad f(r) \leqslant i_Y(n)$. Then by the Proposition 3.2.3 we have $f(r) \leqslant i_Y(n) \Rightarrow r \leqslant f^{-1}(i_Y(n)) \leqslant i_X(f^{-1}(n))$. Thus $f(r) \leqslant i_Y(n) \Rightarrow i_X(f^{-1}(n))$ and hence (I_3').

($\Leftarrow$) Suppose $[(I_3') \quad f : X \to Y \in \mathbf{C}$ such that $\forall r \in Sub(X), n \in Sub(Y) \quad f(r) \leqslant i_Y(n) \Rightarrow r \leqslant i_X(f^{-1}(n))]$ holds then by the Proposition 3.2.3 , for $r = f^{-1}(i_Y(n))$ we obtain $f(f^{-1}(i_Y(n))) \leqslant i_Y(n) \Rightarrow f^{-1}(i_Y(n)) \leqslant i_X(f^{-1}(n))$. Hence ($I_3$) holds true. Therefore, the result. □

4.1.5 Proposition. The composition of i-continuous morphisms in **C** is also i-continuous.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are i-continuous morphisms in **C**. Let $n \in Sub(Z)$ then since g is i-continuous we have

$$g^{-1}(i_Z(n)) \leqslant i_Y(g^{-1}(n)) \tag{4.1.1}$$

Also, since $g^{-1}(n) \in Sub(Y)$ and f is i-continuous by the Proposition 3.2.3 we obtain

$$\begin{aligned} &f^{-1}(i_Y(g^{-1}(n))) \leqslant i_X(f^{-1}(g^{-1}(n))) = i_X((g \circ f)^{-1}(n)) \\ &\Rightarrow i_Y(g^{-1}(n)) \leqslant f(i_X((g \circ f)^{-1}(n))) \end{aligned} \tag{4.1.2}$$

Now, combining the inequalities (4.1.1) and (4.1.2) we get

$$g^{-1}(i_Z(n)) \leqslant f(i_X((g\circ f)^{-1}(n))) \Rightarrow f^{-1}(g^{-1}(i_Z(n))) \leqslant i_X((g\circ f)^{-1}(n)) \Rightarrow (g\circ f)^{-1}(i_Z(n)) \leqslant i_X((g\circ f)^{-1}(n))$$

Therefore, $\forall n \in Sub(Z) \quad (g\circ f)^{-1}(i_Z(n)) \leqslant i_X((g\circ f)^{-1}(n))$ and hence $g\circ f$ is i-continuous. □

4.1.6 Remark. The definition of i assumes of course that all morphisms are i-continuous.

4.2 Open and Isolated $\mathbb{M}$-Subobjects

In this section we present the notion of open subobjects.

4.2.1 Definition. An $\mathbb{M}$-subobject $r \in Sub(X)$ such that $r \cong i_X(r)$ is called i-open (in X). That is, $\exists$ an isomorphism $j_r : i_X(R) \to R$ such that $i_X(r) = r \circ j_r$. Condition (I_3) tells us i-openness is preserved by pre-images.

4.2.2 Proposition. The pre-image of an i-open $\mathbb{M}$-subobject is also i-open.

Proof. Similar to the proof of 1 in the Proposition 3.3.2. Note that there is a symmetry between interior and closure when continuity in terms of pre-images is used. □

4.2.3 Remark. Let $\mathbb{M}_i$ be the class of i-open $\mathbb{M}$-subobjects then by the Proposition 4.2.2 we can conclude that $\mathbb{M}_i$ is stable under pullback. That is, pullbacks of i-open $\mathbb{M}$- subobjects are i-open.

4.2.4 Corollary. Let n and k be monomorphisms such that $n \circ k$ is an i-open $\mathbb{M}$-subobjects then k is an i-open $\mathbb{M}$-subobject.

Proof. The proof is essentially the same to the Corollary 3.3.4 because of the symmetry that we have between closure and interior. □

4.2.5 Definition. An $\mathbb{M}$-subobject $r \in Sub(X)$ such that $i_X(r) \cong 0_X$ is called i-isolated in X.

4.2.6 Proposition. Let i be an interior operator on X with respect to $\mathbb{M}$ and $(R_i \overset{r_i}{\to} X)$ be a family of $\mathbb{M}$-subobjects of X then

1. $i_X(\bigwedge\limits_{i\in I} r_i) \leqslant \bigwedge\limits_{i\in I} i_X(r_i) \leqslant \bigwedge\limits_{i\in I} r_i$.
2. $\bigvee\limits_{i\in I} i_X(r_i) \leqslant i_X(\bigvee\limits_{i\in I} r_i)$.
3. $\forall i \in I, r_i$ is i-open $\Rightarrow \bigvee\limits_{i\in I} r_i$ is i-open. That is, the supremum of a family of i-open $\mathbb{M}$-subobjects is i-open.
4. $\forall i \in I, r_i$ is i-isolated $\Rightarrow \bigwedge\limits_{i\in I} r_i$ is i-isolated. That is, the infimum of a family of i-isolated $\mathbb{M}$-subobjects is i-isolated.

Proof. Analogous to the proof of the Proposition 3.3.5. □

4.2.7 Definition. A morphism $f : X \to Y \in$ **C** is said to be i-open if $\forall r \in Sub(X) \quad f(i_X(r)) \leqslant i_Y(f(r))$. This definition is based on Castellini (2013).

4.2.8 Proposition. The composition of i-open morphisms is also an i-open morphism.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are i-open morphisms. Then from f is i-open we obtain $\forall r \in Sub(X) \quad f(i_X(r)) \leqslant i_Y(f(r))$. Also applying g to both sides of the inequality we have $g(f(i_X(r))) \leqslant g(i_Y(f(r)))$. But by i-openness of g we also get $g(i_Y(f(r))) \leqslant i_Z(g(f(r)))$. Thus $(g \circ f)(i_X(r)) = g(f(i_X(r))) \leqslant i_Z(g(f(r))) = i_Z((g \circ f)(r))$ and hence $g \circ f$ is i-open morphism. □

4.2.9 Proposition. Every i-open morphism assigns to each i-open $\mathbb{M}$-subobject an i-open $\mathbb{M}$-subobject. Moreover, if i is idempotent the converse is also true.

Proof. Suppose $f : X \to Y$ is any i-open morphism and r is i-open $\mathbb{M}$-subobject . From the fact that r is i-open we have $i_X(r) \cong r$. As a result $f(i_X(r)) \cong f(r)$. But since f is i-open morphism we obtain $f(i_X(r)) \leqslant i_Y(f(r))$. Hence $f(r) \leqslant i_Y(f(r))$. Consequently $f(r) \cong i_Y(f(r))$ and hence $f(r)$ is i-open. On the other hand, suppose f is assigns to each i-open $\mathbb{M}$-subobject an i-open $\mathbb{M}$-subobject and i is idempotent then $i_X(i_X(r)) \cong i_X(r)$ and hence $i_X(r)$ is i-open. As a result $f(i_X(r))$ is an i-open $\mathbb{M}$-subobject. That is, $i_Y(f(i_X(r))) \cong f(i_X(r))$. But by (I_1) and (I_2) we have $i_X(r) \leqslant r \Rightarrow f(i_X(r)) \leqslant f(r) \Rightarrow i_Y(f(i_X(r))) \leqslant i_Y(f(r))$. Thus, $f(i_X(r)) \cong i_Y(f(i_X(r))) \leqslant i_Y(f(r))$. Therefore, $f(i_X(r)) \leqslant i_Y(f(r))$ and hence f is i-open. □

4.2.10 Proposition. $f : X \to Y$ is an i-open morphism $\Leftrightarrow \forall n \in Sub(Y) \quad f^{-1}(i_Y(n)) \cong i_X(f^{-1}(n))$.

Proof. $(\Rightarrow)$ Suppose f is i-open morphism. Then by substituting $r = f^{-1}(n)$ in to the definition of i-openness of f we obtain $f(i_X(f^{-1}(n))) \leqslant i_Y(f(f^{-1}(n)))$. But since (f, f^{-1}) is a Galois connection then by (I_2) we get $i_Y(f(f^{-1}(n))) \leqslant i_Y(n)$ and hence $f(i_X(f^{-1}(n))) \leqslant i_Y(n)$. As a result $i_X(f^{-1}(n)) \leqslant f^{-1}(i_Y(n))$. By (I_3) we also have $f^{-1}(i_Y(n)) \leqslant i_X(f^{-1}(n))$. Therefore, $f^{-1}(i_Y(n)) \cong i_X(f^{-1}(n))$. $(\Leftarrow)$ Suppose $\forall n \in Sub(Y) \quad f^{-1}(i_Y(n)) \cong i_X(f^{-1}(n))$ then for $n = f(r)$ we obtain $f^{-1}(i_Y(f(r))) \cong i_X(f^{-1}(f(r)))$. But since (f, f^{-1}) is a Galois connection then by (I_2) we get $i_X(r) \leqslant i_X(f^{-1}(f(r))) \cong f^{-1}(i_Y(f(r)))$ and hence $i_X(r) \leqslant f^{-1}(i_Y(f(r)))$. As a result $f(i_X(r)) \leqslant i_Y(f(r))$ and hence f is i-open morphism. □

4.3 Examples of Interior Operators

The following are a few examples of interior operators.

1. Let **Top** be the category of topological spaces and continuous functions with the (episink, embedding) factorization structure and let $r : R \to X$ be an embedding such that $X \in$ **Top**. With out loss of generality we assume $R \in P(X)$.
   (a) Define $k^*_X(R) = \bigcup\{O \subseteq R : O \text{ is open set in } X\}$ then
      - since $R \supseteq O \quad \forall O \in \{O \subseteq R : O \text{ is open set in } X\}$ we have $R \supseteq k^*_X(R)$
      - for $R \subseteq S$ since $k^*_X(R)$ is an open set such that $k^*_X(R) \subseteq R \subseteq S$ and $k_X(S)$ is the largest open set contained S we have $k^*_X(R) \subseteq k^*_X(S)$ and
      - for a continuous function $f : X \to Y$ and $R \subseteq Y$ we have

$$\begin{aligned} f^{-1}(k^*_Y(R)) &= f^{-1}(\bigcup\{O \subseteq R : O \text{ is open set in } Y\}) \\ &= \bigcup\{f^{-1}(O) : O \text{ is open set in } Y \text{ and } O \subseteq R\} \\ &\subseteq \bigcup\{V \subseteq f^{-1}(R) : V \text{ is open set in } X\} = k^*_X(f^{-1}(R)). \end{aligned}$$

Note that the pre-image of continuous functions preserves open sets and $k^*_X(f^{-1}(R))$ is the largest open set contained in $f^{-1}(R)$. Therefore, the union of all open sets in X that are subsets of R, denoted by $k^*_X(R)$, is an interior operator on **Top**. This operator is called the Kuratowoski interior operator.

(b) Define $q^*_X(R) = \bigcup\{F \subseteq R : F \text{ is clopen in } X\}$, which is the union of all closed and open (clopen) subsets of X that are contained in R then analogously to the proof in (a) we see that q^*_X is an interior operator on **Top**. It is called the quasicomponent interior operator.

(c) Define $\theta^*_X(R) = \{x \in R : \exists \text{ a neighbourhood of } N_x \text{ of } x \in X \text{ such that } \overline{N_x} \subseteq R\}$, where $\overline{N_x}$ is the usual Kuratowski closure of the neighbourhood N_x then

- from the definition of $\theta^*_X(R)$ we have $\theta^*_X(R) \subseteq R$,
- for $R \subseteq S$ we have $\{x \in R : \exists \text{ a neighbourhood of } N_x \text{ of } x \in X \text{ such that } \overline{N_x} \subseteq R\} \subseteq \{x \in S : \exists \text{ a neighbourhood of } N_x \text{ of } x \in X \text{ such that } \overline{N_x} \subseteq S\}$, by transitivity property of subsets. Hence $\theta^*_X(R) \subseteq \theta^*_X(S)$ and
- for a continuous function $f : X \to Y$ and $R \subseteq Y$ we have $f^{-1}(\theta^*_Y(R)) = f^{-1}(\{r \in R : \exists \text{ a neighbourhood of } N_r \text{ of } r \in Y \text{ such that } \overline{N_r} \subseteq R)$ Now, let $x \in f^{-1}(\theta^*_Y(R))$ then $f(x) \in \theta^*_Y(R) = \{r \in R : \exists \text{ a neighbourhood of } N_r \text{ of } r \in Y \text{ such that } \overline{N_r} \subseteq R\}$. That is, $f(x) \in R, \exists$ a neighbourhood of $N_{f(x)}$ of $f(x) \in Y$ such that $\overline{N_{f(x)}} \subseteq R$. As result we get $f(x) \in \overline{N_{f(x)}} \subseteq R$ and hence $x \in f^{-1}(\overline{N_{f(x)}}) \subseteq f^{-1}(R)$. But since $f^{-1}(\overline{N_{f(x)}})$ is a closed set containing $f^{-1}(N_{f(x)})$ and $\overline{f^{-1}(N_{f(x)}}$ is the smallest closed set containing $f^{-1}(N_{f(x)})$ we obtain $x \in \overline{f^{-1}(N_{f(x)})} \subseteq f^{-1}(\overline{N_{f(x)}}) \subseteq f^{-1}(R) \Rightarrow x \in \overline{f^{-1}(N_{f(x)})} \subseteq f^{-1}(R)$. Thus there exists a neighbourhood $f^{-1}(N_{f(x)})$ of $x \in X$ such that $\overline{f^{-1}(N_{f(x)})} \subseteq f^{-1}(R)$. As consequence we have $x \in \theta^*_X(f^{-1}(R))$.Therefore, $\theta^*_X(R)$, is an interior operator on **Top**. It is called the θ^* interior operator.

2. Let H be a subgroup of $G \in$**Grp**. Define $i_G(H) = \bigvee\{N \leq H : N \trianglelefteq G\}$, which is the subgroup generated by all the normal subgroups in G contained in H , then for a group homomorphism $f : G_1 \to G_2$ and K subgroup of G_2 we obtain $i_{G_2}(K) \trianglelefteq G_2$ as the subgroup generated by the family of normal subgroups is normal. Also, since the inverse image of a normal subgroup is normal we get $f^{-1}(i_{G_2}(K))$ is a normal subgroup contained in $f^{-1}(K)$. Hence $f^{-1}(i_{G_2}(K)) \leq i_{G_1}(f^{-1}(K))$. We can easily verify the other two conditions of interior operators. Therefore, $i_G(H)$ is an interior operator in **Grp**.

4.3.1 Definition. An interior operator is said to be

1. grounded(or standard) if $\forall X \in \mathbf{C} \quad i_X(1_X) \cong 1_X$.
2. idempotent if $\forall X \in \mathbf{C}$ and $r \in Sub(X)$ we have $i_X(r)$ is i-open, that is, $i_X(i_X(r)) \cong i_X(r)$ and
3. additive if $\forall X \in \mathbf{C}$ and $r, s \in Sub(X)$ we have $i_X(r \wedge s) \cong i_X(r) \wedge i_X(s)$

4.3.2 Examples. 1. The Kuratowoski interior operator is an idempotent interior operator.

2. The quasicomponent interior operator is idempotent. Indeed, by (I_1) we have $q^*_X(q^*_X(R)) \subseteq q^*_X(R)$. Also, let F be a clopen subset of X which occurs in the construction of $q^*_X(R)$ then $F \subseteq R$ and hence $F \subseteq q^*_X(R)$. Thus $q^*_X(R) \subseteq q^*_X(q^*_X(R))$. Therefore, $q^*_X(R) = q^*_X(q^*_X(R))$
3. The θ^* interior operator is not idempotent. Indeed, let $X = \{1, 2, 3\}$ and $\tau_X = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$ then $\theta^*_X(\{1, 3\}) = \{1\}$ and $\theta^*_X = (\{1\}) = \emptyset$. Note that $\overline{\{1\}} = \{1, 3\} \not\subseteq \{1\}$.

5. Link Between Interior and Closure Operators

This chapter investigates a number of approaches to the categorical link between interior and closure operators. We refer to Vorster (2000), Holgate and Šlapal (2010), Holgate and Šlapal (2011), Fiadeiro (1999) and Dikranjan and Tholen (2014).

5.1 Introduction

One of the oldest and most fruitful notions in Mathematics is duality (Atiyah, 2007). Duality is a means of translating concepts, theorems or mathematical structures in to respective concepts, theorems or structures in a one to one correspondence. Duality is a very powerful and useful principle. Its advantage is when we prove a theorem we expect its dual theorem and can often prove it using the dual argument. In General Topology interior and closure operators are dual to each other. That is, the complement of the closure is the interior of the complement and the complement of the interior is the closure of the complement. Indeed, for a topological space X such that $A \subseteq X$ we have

$$\overline{A} = \bigcap\{C : C \text{ closed and } A \subseteq C\}$$
$$\Rightarrow X \setminus \overline{A} = X \setminus \bigcap\{C : C \text{ closed and } A \subseteq C\} = \bigcup\{X \setminus C : C \text{ closed and } A \subseteq C\}$$
$$\Rightarrow X \setminus \overline{A} = \bigcup\{X \setminus C : X \setminus C \text{ open and } X \setminus C \subseteq X \setminus A\} = \bigcup\{O : O \text{ open and } O \subseteq X \setminus A\} = (X \setminus A)^{\circ}$$
$$\Rightarrow \overline{A} = X \setminus (X \setminus A)^{\circ}. \tag{5.1.1}$$

Also, replacing A by $X \setminus A$ in the equation (5.1.1) we obtain $\overline{X \setminus A} = X \setminus (X \setminus (X \setminus A))^{\circ}$. This implies $\overline{X \setminus A} = X \setminus A^{\circ}$. Hence $A^{\circ} = X \setminus \overline{X \setminus A}$.

5.2 Link Via Categorical Transformation Operators

The following definition of categorical transformation operator was introduced by Vorster (2000).

5.2.1 Definition. A categorical transformation operator or simply a transformation operator t on **C** with respect to $\mathbb{M}$ is a family of functions $\{t_X : Sub(X) \to Sub(X) | X \in \mathbf{C}\}$ such that for all X in **C** we have:

(T_1) (Involution) $\forall r \in Sub(X) \quad (t_X \circ t_X)(r) = r$;

(T_2) (Antimonotonocity) $\forall r, s \in Sub(X) \quad r \leq s \Rightarrow t_X(s) \leq t_X(r)$ and

(T_3) (Bicontinuity) $\forall n \in Sub(Y) \quad f : X \to Y \in \mathbf{C} \Rightarrow f^{-1}(t_Y(n)) \cong t_X(f^{-1}(n))$.

5.2.2 Remark. $\forall X \in \mathbf{C} \quad t_X$ is one to one map . Indeed, let $r, s \in Sub(X)$ such that $t_X(r) = t_X(s)$ then $t_X(t_X(r)) = t_X(t_X(s))$. Hence by the involution condition we have $r = s$.

5.2.3 Examples. The following are few examples of transformation operators.

1. Let **Set** be the category of all sets and functions with the (episink, inclusion) factorization structure and let $r : R \to X$ be an inclusion such that $X \in \mathbf{Set}$. Define $t_X(R) = X \setminus R = \{x \in X : x \notin R\}$ then

   - $t_X(t_X(R)) = X \setminus t_X(R) = X \setminus (X \setminus R) = R$. Therefore, $\forall R \subseteq X \quad t_X(t_X(R)) = R$,

- for $S, R \subseteq X$ such that $R \subseteq S$ we have $(X \setminus S) \subseteq (X \setminus R)$. As a result, $t_X(S) \subseteq t_X(R)$. Hence $R \subseteq S \Rightarrow t_X(S) \subseteq t_X(R)$ and
- for a function $f : X \to Y$ in **Set** and $R \subseteq Y$ we have $x \in f^{-1}(t_Y(R)) \Leftrightarrow f(x) \in t_Y(R) = Y \setminus R \Leftrightarrow f(x) \in Y \setminus R \Leftrightarrow x \in f^{-1}(Y \setminus R) \Leftrightarrow x \in X \setminus f^{-1}(R) = c_X(f^{-1}(R))$. Consequently $f^{-1}(t_Y(R)) \subseteq c_X(f^{-1}(R))$ and $c_X(f^{-1}(R)) \subseteq f^{-1}(t_Y(R))$. Therefore, $f^{-1}(t_Y(R)) = c_X(f^{-1}(R))$.

Hence t_X is a transformation operator on **Set** with respect to $\mathbb{M}$.

2. Let **Top** be the category of topological spaces and continuous functions and $\mathbb{M}$ be the class of all embeddings of subspaces then the map $t_X : Sub(X) \to Sub(X)$ that assigns to an embedding $r : R \to X$ the embedding $r' : R' \to X$, where $R' = X \setminus R$ with the usual subspace topology is a transformation operator on **Top** with respect to $\mathbb{M}$.
3. Let **Inn** be the category of finite dimensional inner product spaces and linear inner product preserving maps and $\mathbb{M}$ be the class of all embeddings of subspaces of X in to X. Define $t_X(R) = R^{\perp}$, where $R^{\perp}$ is the orthogonal complement of a subspace R of X then since for the constant map $f : \Re \to \Re$ defined by $f(x) = 0 \quad \forall x \in \Re$ we obtain $f^{-1}(\Re^{\perp}) = f^{-1}(0) = \Re$ and $(f^{-1}(\Re))^{\perp} = \Re^{\perp} = 0$. Therefore, $f^{-1}(t_{\Re}(\Re)) \nsubseteq t_{\Re}(f^{-1}(\Re))$ and hence t_X is not a transformation operator.
4. The category **Grp** of groups and homomorphism does not have a transformation operator [see Vorster (2000)].

5.2.4 Theorem. *Let* $c = \{c_X : Sub(X) \to Sub(X) | X \in \mathbf{C}\}$ *and* $t = \{t_X : Sub(X) \to Sub(X) | X \in \mathbf{C}\}$ *be closure and transformation operators on* **C** *with respect to* $\mathbb{M}$ *then the family of maps* $\{i_X : Sub(X) \to Sub(X) | X \in \mathbf{C}\}$ *given by* $\forall r \in Sub(X) \quad i_X(r) = (t_X \circ c_X \circ t_X)(r)$ *is an interior operator* **C** *with respect to* $\mathbb{M}$.

Proof. 1. Let $r \in Sub(X)$ then by the expansiveness condition of closure operators we obtain $t_X(r) \leq c_X(t_X(r))$. Hence by the antimonotonicity condition of transformation operators we get $t_X(c_X(t_X(r))) \leq t_X(t_X(r)) \Leftrightarrow (t_X \circ c_X \circ t_X)(r) \leq (t_X \circ t_X)(r))$. This in turn implies $i_X(r) \leq r$, since $i_X = t_X \circ c_X \circ t_X$. Hence the contractibility condition of interior operators holds true.

2. Suppose r and s in **C** such that $r \leq s$. Then by the antimonotonicity condition of transformation operators and order preservation of closure operators we obtain

$$
\begin{aligned}
& t_X(s) \leq t_X(r) \\
& \Rightarrow c_X(t_X(s)) \leq c_X(t_X(r)) \\
& \Rightarrow t_X(c_X(t_X(r))) \leq t_X(c_X(t_X(s))) \\
& \Rightarrow (t_X \circ c_X \circ t_X)(r) \leq t_X \circ c_{Also,X} \circ t_X(s) \\
& \Rightarrow i_X(r) \leq i_X(s) \\
& \therefore r \leq s \Rightarrow i_X(r) \leq i_X(s)
\end{aligned}
$$

Hence the order preservation of interior operators holds true.

3. Suppose $f : X \to Y \in \mathbf{C}$ and $n \in Sub(Y)$. Then by continuity condition of the closure operators

we obtain $\forall n \in Sub(Y)$

$$
\begin{aligned}
& c_X(f^{-1}(n)) \leq f^{-1}(c_Y(n)) \\
& \Rightarrow c_X(f^{-1}(t_Also, Y(n))) \leq f^{-1}(c_Y(t_Y(n))) \\
& \Rightarrow c_X(t_X(f^{-1}(n))) \cong c_X(f^{-1}(t_Y(n))) \leq f^{-1}(c_Y(t_Y(n))), \text{ by } (T_3) \\
& \Rightarrow t_X(f^{-1}(c_Y(t_Y(n)))) \leq t_X(c_X(t_X(f^{-1}(n)))), \text{ by } (T_2) \\
& \Rightarrow f^{-1}(t_Y(c_Y(t_Y(n)))) \cong t_X(f^{-1}(c_Y(t_Y(n)))) \leq t_X(c_X(t_X(f^{-1}(n)))), \text{ by } (T_3) \\
& \Rightarrow f^{-1}((t_Y \circ c_Y \circ t_Y)(n)) \leq (t_X \circ c_X \circ t_X)(f^{-1}(n)) \\
& \Rightarrow f^{-1}(i_Y(n)) \leq i_X(f^{-1}(n)) \\
& \therefore \forall n \in Sub(Y) \quad f^{-1}(i_Y(n)) \leq i_X(f^{-1}(n))
\end{aligned}
$$

Hence the continuity condition of interior operators holds true.

□

5.2.5 Theorem. *Let $i = \{i_X : Sub(X) \to Sub(X)|X \in \mathbf{C}\}$ and $t = \{t_X : Sub(X) \to Sub(X)|X \in \mathbf{C}\}$ be interior and transformation operators on **C** with respect to $\mathbb{M}$ then the family of maps $\{c_X : Sub(X) \to Sub(X)|X \in \mathbf{C}\}$ given by $\forall r \in Sub(X) \quad c_X(r) = (t_X \circ i_X \circ t_X)(r)$ is an interior operator **C** with respect to $\mathbb{M}$.*

Proof. Similar to the proof of the Theorem 5.2.4. □

5.2.6 Corollary. Categorical transformation operators establish a one to one correspondence between the collections of all closure and interior operators on **C** with respect to $\mathbb{M}$.

Proof. It is an immediate consequence of Theorem 5.2.4 and 5.2.5 □

5.2.7 Remark. The above corollary tells us there is a link between closure and interior operators via a transformation operators whenever it exists. That is, for categories that admit a transformation operators there is a bijective correspondence between interior and closure operators.

5.3 Link Via Closed ↔ Open notion

5.3.1 From interior to closure. We define closed subobjects with respect to interior operators based on the topological notion as in Holgate and Šlapal (2010).

5.3.2 Definition. $r \in Sub(X)$ is said to be

1. pseudocomplemented if there exists $r^* \in Sub(X)$ such that $s \leqslant r^* \Leftrightarrow s \wedge r = 0_X$. Such r^* is called the pseudocomplement of r.
2. complemented if there exists $\bar{r} \in Sub(X)$ such that $r \vee \bar{r} = 1_X$ and $r \wedge \bar{r} = 0_X$. Such $\bar{r}$ is called the complement of r.

5.3.3 Definition. Let $i \in INT(\mathbf{C}, \mathbb{M})$, $X \in \mathbf{C}$ and $r \in Sub(X)$ then we say that r is:

1. i^1-closed if $\forall s \in Sub(X) \quad i_X(r \vee s) \leqslant r \vee i_X(s)$.
2. i^2-closed if $\forall s \in Sub(X) \quad r \vee s = 1_X \Rightarrow r \vee i_X(s) = 1_X$.
3. i^3-closed if r is pseudocomplemented and $r^* = i_X(r^*)$.

5.3.4 Definition. A morphism $f : X \to Y$ reflects 0 if $f^{-1}(0_Y) = 0_X$(equivalently $f(r) = 0_Y \Leftrightarrow r = 0_X$).

5.3.5 Remark. If we assume $\mathbb{E}$ is stable under pullback along $\mathbb{M}$-morphisms then Frobenius reciprocity law holding. That is, $\forall f : X \to Y, r \in Sub(X), s \in Sub(Y) \quad f(r \wedge f^{-1}(s)) = f(r) \wedge s$.

5.3.6 Proposition. Let $\mathbb{E}$ be closed under pullback along $\mathbb{M}$-morphisms. A morphism $f : X \to Y$ reflects $0 \Leftrightarrow f^{-1}(-)$ preserves pseudocomplements .

Proof. ($\Rightarrow$) Suppose f reflects 0 with n^* as the pseudocomplement of $n \in Sub(Y)$ then $r \leqslant f^{-1}(n^*) \Leftrightarrow f(r) \leqslant n^* \Leftrightarrow f(r) \wedge n = 0_Y \Leftrightarrow f(r \wedge f^{-1}(n)) = 0_Y \Leftrightarrow r \wedge f^{-1}(n) = 0_X$. Hence $f^{-1}(n^*) = f^{-1}(n)^*$.

($\Leftarrow$) Suppose $f^{-1}(-)$ preserves pseudocomplements then $f^{-1}(0_Y) = f^{-1}(1_Y^*) = f^{-1}(1_Y)^* = 1_X^* = 0_X$. □

5.3.7 Proposition. Let $i \in INT(\mathbf{C}, \mathbb{M})$, and $R \xrightarrow{r} X \in \mathbb{M}$

1. i is grounded $\Rightarrow [r$ is i^1-closed $\Rightarrow r$ is i^2-closed $]$.
2. $Sub(X)$ is a Boolean algebra $\Rightarrow [r$ is i^2-closed $\Rightarrow r$ is i^3-closed $]$.
3. i is additive and $Sub(X)$ is a Boolean algebra $\Rightarrow [r$ is i^3-closed $\Rightarrow r$ is i^1-closed $]$

Proof.
1. Let i be grounded such that r is i^1-closed then $1_X = i_X(1_X) = i_X(r \vee s) \leqslant r \vee i_X(s)$. Hence $r \vee i_X(s) = 1_X$.
2. Let $Sub(X)$ be a Boolean algebra such that r is i^2-closed then $\bar{r} = r^*$ and $r \vee r^* = 1_X$. Hence $r \vee i_X(r^*) = 1_X \Rightarrow r^* \leqslant i_X(r^*)$. Therefore $r^* = i_X(r^*)$.
3. Let i be additive and $Sub(X)$ be Boolean algebra such that r is i^3-closed then $\bar{r} = r^*$. Also, $i_X(r \vee s) \wedge \bar{r} = i_X(r \vee s) \wedge i_X(\bar{r}) = i_X((r \vee s) \wedge \bar{r}) = i_X((r \wedge \bar{r}) \vee (s \wedge \bar{r})) = i_X(0_X \vee (s \wedge \bar{r})) = i_X(s \wedge \bar{r}) = i_X(s) \wedge i_X(\bar{r}) = i_X(s) \wedge \bar{r}$. As a result, $i_X(r \vee s) = i_X(r \vee s) \wedge 1_X = i_X(r \vee s) \wedge (r \vee \bar{r}) = [i_X(r \vee s) \wedge r] \vee [i_X(r \vee s) \wedge \bar{r}] \leqslant r \vee [i_X(r \vee s) \wedge \bar{r}] = r \vee (i_X(s) \wedge \bar{r}) = (r \vee i_X(s)) \wedge (r \vee \bar{r}) = (r \vee i_X(s)) \wedge 1_X = r \vee i_X(s)$.

□

5.3.8 Remark.
1. Let $F \subseteq \mathbb{M}$ be closed under pullback and $r : R \to X \in \mathbb{M}$. Define $c_X(r) = \bigwedge\{r' \in F : r \leqslant r'\}$ then $c_X(r)$ is an idempotent closure operator (Castellini and Holgate, 2003). Indeed, since $r \leqslant r' \quad \forall r' \in \{r' \in F : r \leqslant r'\}$ we obtain $r \leqslant \bigwedge\{r' \in F : r \leqslant r'\} \Rightarrow r \leqslant c_X(r)$. Also, for $r, s \in \mathbb{M}$ such that $r \leqslant s \Rightarrow s \in \{r' \in F : r \leqslant r'\}$ and hence $c_X(r) = \bigwedge\{r' \in F : r \leqslant r'\} \leqslant s \leqslant \bigwedge\{s' \in F : s \leqslant s'\} = c_X(s)$. And, for $f : X \to Y \in \mathbf{C}$ such that $n : N \to Y \in \mathbb{M}$ we get $f^{-1}(c_Y(n)) = f^{-1}(\bigwedge\{n' \in F : n \leqslant n'\}) = \bigwedge\{f^{-1}(n') : n \leqslant n'\} = \bigwedge\{f^{-1}(n') \in F : f^{-1}(n) \leqslant f^{-1}(n')\} \geqslant \bigwedge\{\bar{n} : f^{-1}(n) \leqslant \bar{n}\} = c_X(f^{-1}(n))$. In fact $c_X(r) \leqslant r' \quad \forall r' \in \{r' \in F : c_X(r) \leqslant r'\}$. Thus $c_X(r) \leqslant \bigwedge\{r' \in F : c_X(r) \leqslant r'\} = c_X(c_X(r))$. Also, since $c_X(r) \leqslant c_X(r)$ we have that $c_X(c_X(r)) = \bigwedge\{r' \in F : c_X(r) \leqslant r'\} \leqslant c_X(r)$. Therefore, $c_X(c_X(r)) = c_X(r)$.
2. Let $F = \mathbb{M}$ be the class of closed $\mathbb{M}$-subobjects then by the Remark 3.3.3 F is stable under pullback and hence by 1 in the Remark 5.3.8 we get $c(r) = \bigwedge\{r' \geqslant r : r'$ closed $\}$ is an idempotent closure operator.
3. If the class of i^1-closed (or i^2-closed or i^3-closed) is stable under pullback then it induces a respective idempotent closure operator.

5.3.9 Proposition. If $i \in INT(\mathbf{C}, \mathbb{M})$ and $(\mathbb{E}, \mathbb{M})$ is stable under pullback such that $f : X \to Y \in$ **C** reflects 0 then the class of i^3-closed stable under pullback .

Proof. Suppose $n \in Sub(Y)$ is i^3-closed then by the Proposition 5.3.6 since $f^{-1}(-)$ preserves pseudo-complements we obtain $f^{-1}(n)^* = f^{-1}(n^*) = f^{-1}(i_Y(n^*)) \leqslant i_X(f^{-1}(n^*)) = i_X(f^{-1}(n)^*)$.

□

5.3.10 From closure to Interior. We define open subobjects with respect to closure operators based on the topological notion as in Holgate and Šlapal (2010).

5.3.11 Definition. Let $c \in CL(\mathbf{C}, \mathbb{M})$, $X \in \mathbf{C}$ and $r \in Sub(X)$. We say that r is:

1. c^1-open if $\forall s \in Sub(X) \quad r \wedge c_X(s) \leqslant c_X(r \wedge s)$.
2. c^2-open if $\forall s \in Sub(X) \quad r \wedge s = 0_X \Rightarrow r \wedge c_X(s) = 0_X$.
3. c^3-open if r is pseudocomplemented and $r^* = c_X(r^*)$.

5.3.12 Proposition. Let $c \in INT(\mathbf{C}, \mathbb{M})$, and $R \overset{r}{\to} X \in \mathbb{M}$.

1. c is grounded $\Rightarrow [r$ is c^1-open $\Rightarrow r$ is c^2-open $]$.
2. $Sub(X)$ is a Boolean algebra $\Rightarrow [r$ is c^2-open $\Rightarrow r$ is c^3-open $]$.
3. c is additive and $Sub(X)$ is a Boolean algebra $\Rightarrow [r$ is c^3-open $\Rightarrow r$ is c^1-open $]$

Proof. Analogous to the proof of the Proposition 5.3.7. □

5.3.13 Remark. By assuming the joins commute with pre-image, which is always true in the category **Set** of sets we get the following.

1. Let $F \subseteq \mathbb{M}$ be closed under pullback and $r : R \to X \in \mathbb{M}$. Define $i_X(r) = \bigvee\{r' \in F : r' \leqslant r\}$ then $i_X(r)$ is an idempotent interior operator (Holgate and Šlapal, 2010). Indeed, since $r' \leqslant r \quad \forall r' \in \{r' \in F : r \leqslant r'\}$ we obtain $\bigvee\{r' \in F : r' \leqslant r\} \leqslant r \Rightarrow i_X(r) \leqslant r$. Also, for $r, s \in \mathbb{M}$ such that $r \leqslant s \Rightarrow r \in \{s' \in F : s' \leqslant s\}$ and hence $i_X(r) = \bigvee\{r' \in F : r' \leqslant r\} \leqslant r \leqslant \bigvee\{s' \in F : s' \leqslant s\} = i_X(s)$. And, for $f : X \to Y \in \mathbf{C}$ such that $n : N \to Y \in \mathbb{M}$ we get $f^{-1}(c_Y(n)) = f^{-1}(\bigvee\{n' \in F : n' \leqslant n\}) = \bigvee\{f^{-1}(n') : n' \leqslant n\} = \bigvee\{f^{-1}(n') \in F : f^{-1}(n') \leqslant f^{-1}(n)\} \leqslant \bigvee\{\bar{n} \in F : \bar{n} \leqslant f^{-1}(n)\} = i_X(f^{-1}(n))$. Also, since $i_X(r) \leqslant i_X(r)$ we have that $i_X(r) \leqslant i_X(i_X(r)) = \bigvee\{r' \in F : i_X(r) \leqslant r'\}$. Therefore, $i_X(i_X(r)) = i_X(r)$.
2. Let $F = \mathbb{M}$ be the class of open $\mathbb{M}$-subobjects then by the Remark 4.2.3 F is stable under pullback and hence by 1 in the Remark 5.3.13 we get $i(r) = \bigvee\{s \leqslant r : s$ open $\}$ is an idempotent interior operator.
3. If the class of c^1-open (or c^2-open or c^3-open) is stable under pullback then it induces a respective idempotent interior operator.

5.3.14 Proposition. Let $(\mathbb{E}, \mathbb{M})$ be stable and morphisms in **C** reflect 0 then c^3-open morphisms are closed under pullback for each closure operator c.

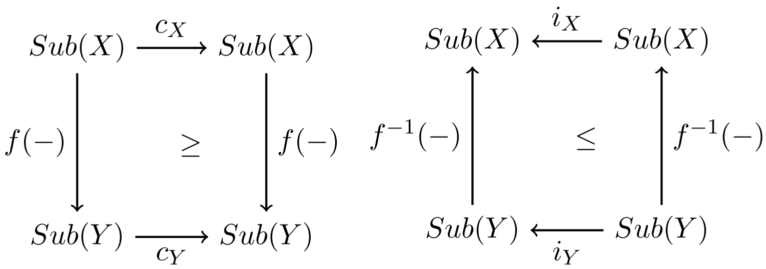
Proof. Similar to the proof of the Proposition 5.3.9. □

5.3.15 Remark. From the Remark 5.3.8 and 5.3.13 we can conclude that the closure c operator induced by interior operator i via the notion of closed is always idempotent. Also, i induced by the closure operator c via the notion of open may not be interior operator as it needs joins commute with preimage.

5.4 Link Via Lax Natural Transformation

5.4.1 Definition. An order enriched category is a category **C** where each hom-set is a pre-order and such that $[f \leq \bar{f}$ in $Hom_{\mathbf{C}}(A,B)$ and $g \leq \bar{g}$ in $Hom_{\mathbf{C}}(B,C)] \Rightarrow g \circ f \leq \bar{g} \circ \bar{f}$.
5.4.2 Definition. Let $F, G : \mathbf{C} \to \mathbf{D}$ be two functors where **D** is an ordered enriched category with order $\leq$. A transformation $\delta : F \Longrightarrow G$ is lax natural if for every arrow $f : A \to B \in \mathbf{C}, \delta_B \circ F(f) \leq G(f) \circ \delta_A$. It is op-lax natural if for every arrow $f : A \to B \in \mathbf{C}, \delta_B \circ F(f) \geq G(f) \circ \delta_A$.
5.4.3 Proposition. A closure operator c on **C** with respect to $\mathbb{M}$ is an op-lax natural transformation $(c_X : Sub(X) \to Sub(X))_{X \in \mathbf{C}}$ with $1_{Sub} \leq c$.

Proof. From the expansiveness condition of closure operators $r \leq c(r)$, which is in $Sub(X)$, we obtain $1_{Sub}(r) \leq c(r)$. Hence $1_{Sub} \leq c$. Also, from the order preservation property of closure operators $r \leq s \Rightarrow c(r) \leq c(s)$, which is in $Sub(X)$, we get c is a functor on poset $Sub(X)$. In fact $c(r \to s) = c(r) \to c(s)$ and from the continuity condition of closure operators $f(c_X(r)) \leq c_Y(f(r))$, which links $Sub(X)$ and $Sub(Y)$, we obtain the op-laxness of c and the situation is given by the below left diagram. □



5.4.4 Remark.

A interior operator i on **C** with respect to $\mathbb{M}$ is a lax natural transformation $(i_X : Sub(X) \to Sub(X))_{X \in \mathbf{C}}$ with $i \leq 1_{Sub}$. In fact conditions I_1, I_2 and I_3 of interior operators describe $1 \leq 1_{Sub}$, $i : Sub(X) \to Sub(X)$ a functor and laxness of i (which is shown in the right above diagram) respectively. From the two diagrams we can observe that the continuity condition of the two operators does not give us an equivalent description.

As you can see a link between closure and interior operators via lax natural transformation does not give us a bijective correspondence. As stated in the Remark 5.3.15 a link via closed and notions have some restrictions. But it is possible to have a one to one correspondence between closure and interior operators for categories satisfying some sort of complementation condition obtained via a transformation operator. Also, in the Proposition 3.2.8 the continuity condition of closure operators can be described in terms of images and inverse images while continuity condition of interior operators can not be described in terms of images. Indeed, for the set of real numbers with the usual Euclidean topology with $f(x) = x^2+1, R = [-2, 2]$ and i be the interior operator induced by the topology then $f(i(R)) = f((-2, 2)) = [1, 5)$ and $i(f(R)) = i(f[-2, 2]) = i[1, 5] = (1, 5)$.Thus $f(i(R)) \nsubseteq i(f(R))$. And, if we consider $A = \{0, 1\}$, the two element indiscrete topological space, $B = \{1\}$, the singleton topological space, $r : R = \{0\} \to A$ as inclusion map and $g : A \to B$ as the only possible function then $i(g(R)) = B, g(i(R)) = \emptyset$ and hence $i(g(R)) \nsubseteq g(i(R))$. Consequently the two operators can not be dual to each other. A current research work of D.Dikranjan and W.Tholen (Dikranjan and Tholen, 2014) illustrates departing from a suitable categorical description of closure operators one can obtain dual closure operators. As it is observed in this paper the dual closure operators are not equivalent to interior operators as the dual operators act on quotients rather than subobjects.

6. Conclusion

The aim of this research project was to investigate the link between interior and closure operators. We gave an introduction to categorical concepts that are essential in the study of closure and interior operators. We have presented a review on the general theory of categorical closure and interior operators together with a number of examples, taken from topology and algebra. A categorical closure operator is compatible with taking images or equivalently, pre-images, like the usual closure of a topological space is compatible with continuous maps. On the other hand, categorical interior operators are only compatible with taking pre-images. Contrary to the closure operator case the continuity condition of interior operator can not be expressed in terms of direct images. This shows that the theory of the two operators are not the symmetric counterpart of each other.

Three possible links between closure and interior operators have been investigated. The link via categorical transformation gives us a way to move from one operator to the other in sense that one induces the other. Set theoretic complementation for subobjects in every topological category over set provides a transformation operator. However, the category of groups **Grp** does not have these operators. This indicates that there are categories in which closure and interior are not necessarily equivalent.

The link via lax transformation illustrates well the similarity between the two operators via the first two conditions. The expansiveness and order preservation condition of a closure operator are dual to the respective contractibility and monotone condition of an interior operator. It also shows us that the continuity conditions of the two operators does not provide equivalent descriptions. However, it does give a clear categorical frame-work from which we can investigate the similarity more thoroughly.

The link via closed and open notions definitely does not give a good interior-closure correspondence; it is not possible to construct interior operators from closure operators via the notion of open in categories where the joins fail to commute with preimages. Also, we can not get non idempotent closure operators from interior operators via the notion of closed. Furthermore, the link does not give an adjunction between closure and interior operators.

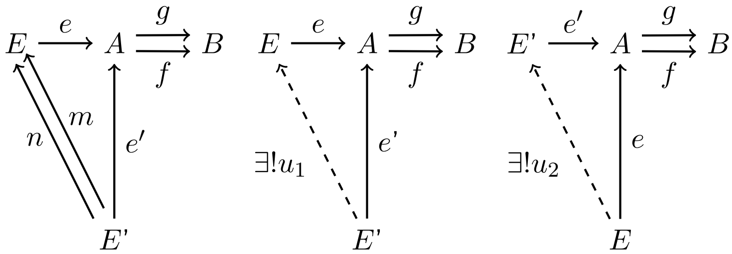
The theory of categorical closure operators is sufficiently deep. They have been studied for a relatively long period of time. However, the notion of categorical interior operators have been recently introduced and studied. Its theory still developing. As we have pointed out there are obvious categorical links between the two operators. These links need to be better understood and studied. The recent work of D.Dikranjan and W.Tholen as well as research on neighbourhood operators needs to be considered in further research.

Appendix A. Proofs of Selected Propositions

In this appendix we present proofs of selected propositions that are stated in chapter 2.

A.1 Proof of Proposition 2.2.8

Proof. 1. Suppose we have morphisms $m, n : E' \to E$ such that $e \circ m = e \circ n$ then since e is an equalizer of f and g we have for $e' = e \circ m = e \circ n$ the following: $f \circ e = g \circ e \Rightarrow f \circ e \circ m = g \circ e \circ n \Rightarrow f \circ e' = g \circ e'$. Thus $\exists! u_1 : E' \to E$ such that $e' = e \circ u_1$ and this in turn implies $m = u_1 = n$. Therefore, e is monomorphism.



2. Suppose $e : E \to A$ and $e' : E' \to A$ are two equalizers of f and g then from the definition of e as an equalizer of f and g we have a unique morphism,

$$u_1 : E' \to E \text{ such that } e' = e \circ u_1 \tag{A.1.1}$$

Also, since e' is an equalizer of f and g we have a unique morphism

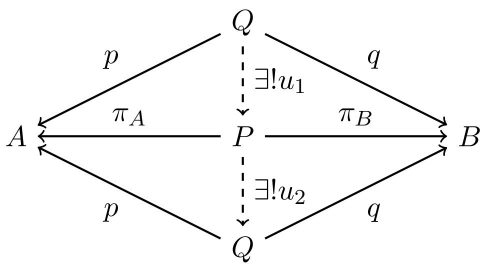
$$u_2 : E \to E' \text{ such that } e = e' \circ u_2 \tag{A.1.2}$$

Now , combining the equations (A.1.1) and (A.1.2) we obtain $e \circ u_1 \circ u_2 = e' u_2 = e = e \circ 1_E \Rightarrow u_1 \circ u_2 = 1_E$, since e is monomorphism. Similarly, $u_2 \circ u_1 = 1'_E$. Therefore, e and e' are isomorphic and hence the proof.

□

A.2 Proof of Proposition 2.2.16

Proof. Suppose an object P with arrows $\pi_A : P \to A, \pi_B : P \to B$ and an object Q with arrows $p : Q \to A, q : Q \to B$ are products of objects A and B as shown in the right diagram below. Then since P is a product we have $\exists u_1 : Q \to P$ such that $p = \pi_A \circ u_1$ and $q = \pi_B \circ u_1$. Also, since Q is a product we obtain $\exists u_2 : P \to Q$ such that $\pi_A = p \circ u_2$ and $\pi_B = q \circ u_2$. As consequence we get $p \circ u_2 \circ u_1 = \pi_A \circ u_1 = p = p \circ 1_Q$ and $q \circ u_2 \circ u_1 = \pi_B \circ u_1 = q = q \circ 1_Q$. Hence $u_2 \circ u_1 = 1_Q$, by the uniqueness condition of u_1 and u_2. Again, $\pi_A \circ u_1 \circ u_2 = p \circ u_2 = \pi_A = \pi_A \circ 1_P$ and $\pi_B \circ u_1 \circ u_2 = q \circ u_2 = \pi_B = \pi_B \circ 1_P$. Hence $u_1 \circ u_2 = 1_P$, by the uniqueness condition of u_1 and u_2. Therefore there exist an isomorphism $u_1 : Q \to P$ and hence $Q \cong P$. This completes the proof.

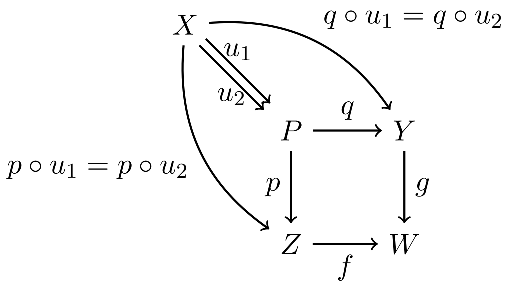


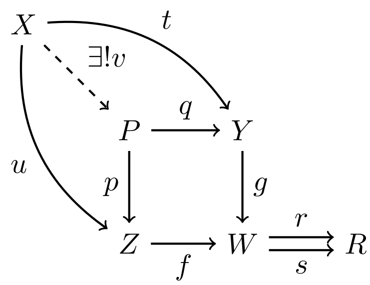
∎

A.3 Proof of Proposition 2.2.21

Proof. 1. Suppose f is a monomorphism in the below pullback square of f such that

(i) $q \circ u_1 = q \circ u_2$ for morphisms $u_1, u_2 : X \to P$ then by the commutativity of the pullback square we obtain $f \circ p \circ u_2 = g \circ q \circ u_1 = g \circ q \circ u_2 = f \circ p \circ u_2$. This implies $p \circ u_2 = p \circ u_1$ as f is monomorphism. Hence the outer below square commutes. Consequently by the universal mapping property (UMP) of pullbacks we obtain $u_1 = u_2$. Therefore, q is monomorphism.





(ii) in the above right diagram f is an equalizer of r and s. Then $r \circ g \circ q = r \circ f \circ p = s \circ f \circ p = s \circ g \circ q$. Also, suppose for a morphism $t : X \to Y$ we have $r \circ g \circ t = s \circ g \circ t$. Then since f is an equalizer of r and s we obtain $\exists! u : X \to Z$ such that $g \circ t = f \circ u$. Hence the above right outer square commutes. Therefore, by the UMP of pullbacks $\exists v : X \to P$ such that $t = qv$. Thus q is an equalizer of $r \circ g$ and $s \circ g$.

2. Diagram chase.

3. Diagram chase.

∎

A.4 Proof of Proposition 2.3.5

Proof. Let

$$P \underset{G}{\overset{F}{\rightleftarrows}} Q \quad , \quad Q \underset{K}{\overset{H}{\rightleftarrows}} R$$

be two Galois connections.Then for $x \in P, z \in R$ such that $(H \circ F)(x) \leqslant_R z$ we obtain

$$(H \circ F)(x) = H(F(x)) \leqslant_R z \Leftrightarrow F(x) \leqslant_Q K(z) \Leftrightarrow x \leqslant_P G(K(z)) = (G \circ K)(z)$$

Thus $(H \circ F)(x) \leqslant_R z \Leftrightarrow x \leqslant_P (G \circ K)(z)$. Therefore, $(H \circ F, G \circ K)$ is a Galois connection. □

A.5 Proof of Proposition 2.3.8

Proof. To show that $G \circ F \circ G = G$, let $y \in Q$ then by the Proposition 2.3.7 we have that $G(y) \leqslant_P (G \circ F)(G(y)) = (G \circ F \circ G)(y)$. Also, since $\forall y \in Q \quad (F \circ G)(y) \leqslant_Q y$ then by the order preservation of G , $G((F \circ G)(y)) = (G \circ F)(G(y)) \leqslant_P G(y)$. Therefore, $\forall y \in Q \quad (G \circ F \circ G)(y) = G(y)$ and hence $G \circ F \circ G = G$. In in the same way, the proof of $F \circ G \circ F = F$ follows. □

A.6 Proof of Proposition 2.3.9

Proof. Since $0' \leqslant F(0)$ and $0 \leqslant G(0') \Leftrightarrow F(0) \leqslant 0'$ we have $F(0) = 0$. Analogously, since $G(1') \leqslant 1$ and $F(1) \leqslant 1' \Leftrightarrow 1 \leqslant G(1')$ we have $G(1') = 1$. □

A.7 Proof of Proposition 2.3.10

Proof.

1. Suppose (F, G) be a Galois connection between P and Q. Let $A \subseteq P$ and $s = sup(A)$ then $\forall a \in A \quad a \leqslant_P s$ and $[\forall a \in A \quad a \leqslant_P u \Rightarrow s \leqslant_P u]$. These in turn imply $\forall a \in A \quad F(a) \leqslant_P F(s)$, since F and G are monotone. As a result, $F(s)$ is an upper bound for $F(A)$. Also, if q is an upper bound for $\{F(a) : a \in A\}$ then $\forall a \in A \quad F(a) \leqslant_Q q$. Thus $\forall a \in A \quad a \leqslant_P G(q)$ and hence $s \leqslant_P G(q)$ as $s = sup(A)$. Consequently $F(s) \leqslant_Q q$. Thus $F(s)$ is the least upper bound among the upper bounds of $F(A)$. Hence $F(s) = Sup(F(A)) \Rightarrow F(sup(A)) = Sup(F(A))$ for all subsets A of P. Dually the other proof follows.

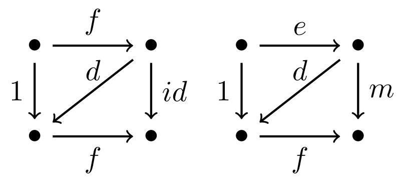
2. Let $x, x' \in P$ such that $x \leqslant_P x'$ then since F preserves suprema we have $x \leqslant x' \Leftrightarrow x \vee x' = x' \Leftrightarrow F(x \vee x') = F(x') \Leftrightarrow F(x) \vee F(x') \Leftrightarrow F(x') \Leftrightarrow F(x) \leqslant F(x')$. Hence F is monotone. To show that G is order preserving let $y, y' \in Q$ such that $y \leqslant y'$. Then $\{x \in P : F(x) \leqslant_Q y\} \subseteq \{x \in P : F(x) \leqslant_Q y'\}$. As a result $G(y) = \bigvee\{x \in P : F(x) \leqslant_Q y\} \subseteq \bigvee\{x \in P : F(x) \leqslant_Q y'\} = G(y')$. Hence G is order preserving. Also, for $x' \in P$ then by the definition of G we get $G((F(x'))) = \bigvee\{x \in P : F(x) \leqslant_Q F(x')\} \geqslant x'$ as $F(x') \leqslant F(x')$ and hence $x' \in \{x \in P : F(x) \leqslant_Q F(x')\}$. Thus $\forall x' \in A \quad x' \leqslant (G \circ F)(x')$. Besides, for $y \in Q$ we obtain $(F \circ G)(y) = F(G(y)) = F(\bigvee\{x \in P : F(x) \leqslant_Q y\}) = \bigvee\{F(x) \in Q : F(x) \leqslant_Q y\} \leqslant y$, since F preserves suprema. Therefore, by the Proposition 2.3.7 we get (F, G) is a Galois connection.

3. Similar to the proof of 2 in the Proposition A.7.

□

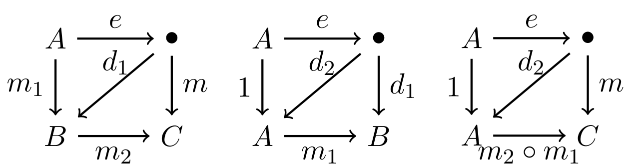
A.8 Proof of Proposition 2.4.5

Proof. 1. ($\Rightarrow$) suppose $f \in \mathbb{E} \cap \mathbb{M}$ then by diagonalization property we have $\exists! d$ such that the left below diagram commutes. Therefore, $d \circ f = f \circ d = 1 \Rightarrow f \in Iso(\mathbf{C})$



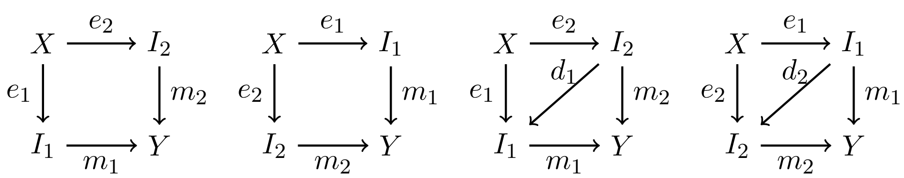
($\Leftarrow$) Suppose $f \in Iso(\mathbf{C})$. Then by $(\mathbb{E}, \mathbb{M})$-factorization of f we have $f = m \circ e$ with $m \in \mathbb{M}$ and $e \in \mathbb{E}$. Thus for $d = f^{-1} \circ m$ the right above diagram commutes. Therefore, by the Lemma 2.4.4 , $f \in \mathbb{M}$. By duality $f \in \mathbb{E}$.

2. Suppose $m_1 : A \to B$ and $m_2 : B \to C$ are morphisms in $\mathbb{M}$. If $m_2 \circ m_1 = m \circ e$ is an $(\mathbb{E}, \mathbb{M})$-factorization, then $\exists d_1$ and d_2 such that the diagrams



commute. Therefore, by the Lemma 2.4.4 we obtain $m_2 \circ m_1 \in \mathbb{M}$. Also, by duality we have $\mathbb{E}$ closed under composition.

3. Suppose (e_1, m_1) and (e_2, m_2) be two $(\mathbb{E}, \mathbb{M})$ factorization of $f : X \to Y$ then we have the first two commutative diagrams shown below.

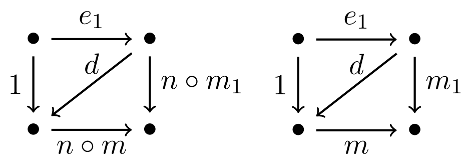


As result, the diagonalization property implies $\exists! d_1 : I_2 \to I_1$ and $d_2 : I_1 \to I_2$ such that the last two commutative diagrams shown above. Hence $m_2 = m_1 \circ d_1, e_1 = d_1 \circ e_2, m_1 = m_2 \circ d_2$ and $e_2 = d_2 \circ e_1$. Consequently $m_2 \circ 1_{I_2} = m_2 = m_1 \circ d_1 = m_2 \circ d_2 \circ d_1$ and hence $1_{I_2} = d_2 \circ d_1$ as they are unique. Similarly we obtain $1_{I_1} = d_1 \circ d_2$. Thus there exists a unique isomorphism d_1 such that the third above diagram commutes. Therefore $I_1 \cong I_2$.

4. Let (e_1, m_1) be the $(\mathbb{E}, \mathbb{M})$ factorization of m. That is, $m = m_1 \circ e_1$ with $m_1 \in \mathbb{M}$ and $e_1 \in \mathbb{E}$. Then by 2 in the Proposition 2.4.5 we obtain $n \circ m = (n \circ m) \circ 1 = (n \circ m_1) \circ e_1$ with $n \circ m, n \circ m_1 \in \mathbb{M}$ and $1, e_1 \in \mathbb{E}$. This implies $(1, n \circ m)$ and $(e_1, n \circ m_1)$ are the $(\mathbb{E}, \mathbb{M})$ factorization of $n \circ m$. As a result by 3 in the Proposition 2.4.5 there is an isomorphism d_1 such that $d_1 \circ e_1 = 1$. Thus $e_1 = d_1^{-1}$ and hence e_1 isomorphism. Therefore, $m = m_1 \circ e_1 \in \mathbb{M}$ as $\mathbb{M}$ is closed under composition with isomorphism.

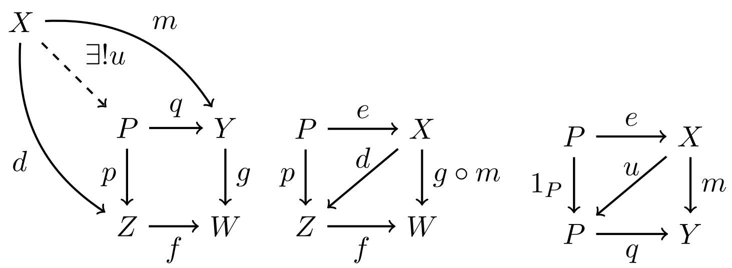
5. Let (e_1, m_1) be the $(\mathbb{E}, \mathbb{M})$ factorization of m. That is, $m = m_1 \circ e_1$ with $m_1 \in \mathbb{M}$ and $e_1 \in \mathbb{E}$. Then by 2 in the Proposition 2.4.5 we obtain $n \circ m = (n \circ m) \circ 1 = (n \circ m_1) \circ e_1$ with

$n \circ m, n \circ m_1 \in \mathbb{M}$ and $1, e_1 \in \mathbb{E}$. Hence by the unique diagonalization property $\exists! d$ such that the left below diagram commutes.



Consequently the fact that n is monomorphism implies $n \circ m \circ 1 = n \circ m_1 \circ e_1 \Rightarrow m \circ 1 = m_1 \circ e_1$ and hence the right above diagram also commutes. Therefore, by the Lemma 2.4.4 we get $m \in \mathbb{M}$.

6. Let the first solid inner square below is a pullback of $f \in \mathbb{M}$ and (e, m) be the $(\mathbb{E}, \mathbb{M})$ factorization of q then $f \circ p = g \circ q = (g \circ m) \circ e$. Hence $\exists! d : X \to Z$ such that the second below square commutes. Thus $f \circ d = g \circ m$ and hence the first outer solid square commutes. So by the pullback property $\exists! u : X \to P$ such that the first below diagram commutes. In particular we get $d = p \circ u$ and $m = q \circ u$. As consequence, $p \circ u \circ e = de = p$ and $q \circ u \circ e = m \circ e = q$ and hence $u \circ e = 1_P$.Thus the third diagram below commutes and hence by the Lemma 2.4.4, $q \in \mathbb{M}$.



□

Acknowledgements

First and foremost I would like to express my sincerest gratitude to my supervisor Professor David Holgate for his knowledge, advise, guidance, patience, leadership, commitment and encouragement throughout the research project. He taught and showed me the beauty and power of category theory. His recommendations, constructive comments and corrections have been invaluable. I feel lucky to have him as my supervisor. With out his support I could not have been able to finalize this project.

My appreciation goes to Professor Neil Turok, founder of AIMS, Professor Bary Green, Director of AIMS, Professor Jeff Sanders, Academic Director of AIMS, for their leadership, advice, encouragement, motivation, follow up and giving me the opportunity to study at AIMS. I would like to use this opportunity to thank all the AIMS family, lectures, tutors for their commitment. My special thanks goes to my tutor Chama Bensmail for her encouragement and assistance. I would also like to thank Jan Groenewald, the IT manager and computing lecturer, Frances Aron, the English and communication lecturer, for their continuous support and follow up. My thanks go to Professor Walter Tholen for sharing with me his paper.

Finally, I would like to thank my wife, Freweyni Tekle, for her encouragement, patience, and sacrifice. I am also grateful to every 2013/14 AIMS students in particular Monde, Amos and Yahya for their help, encouragement, sharing of their experiences. This work is dedicated to my wife. God bless my parents.

References

J. Adamek, H. Herrlich, and G. E. Strecker. *Abstract and Concrete Categories The Joy of Cats*. John Wiley and Sons, Inc,, 2004.

S. M. F. Atiyah. Duality in mathematics and physics. *Conferències FME*, page 69, 2007.

S. Awodey. *Category Theory*. Oxford University Press, 2010.

G. Birkhoff. The meaning of completeness. *Ann. Math.*, 38:57–60, 1937.

G. Birkhoff. *Lattice Theory*. American Mathematical Society, New York, 1940.

F. Borceux. *Handbook of Categorical Algebra: Category Theory*, volume 1. Cambridge University Press, 1994.

G. Brümmer, E. Giuli, and D. Holgate. Splitting closure operators. *Acta Mathematica Hungarica*, 106 (1-2):1–16, 2005.

G. Castellini. Closure operators, monomorphisms and epimorphisms in categories of groups. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 27(2):151–167, 1986.

G. Castellini. Categorical closure operators. In *Categorical Perspectives*, pages 109–150. Springer, 2001.

G. Castellini. *Categorical closure operators*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 2003.

G. Castellini. Interior operators in a category: Idempotency and heredity. *Topology and its Applications*, 158(17):2332–2339, 2011.

G. Castellini. Interior operators, open morphisms and the preservation property. *Applied Categorical Structures*, page Published online November, 2013.

G. Castellini and D. Holgate. Closure operator constructions depending on one parameter. *Quaestiones Mathematicae*, 26(3):289–305, 2003.

G. Castellini and D. Holgate. A categorical approach to absolute closure. *Acta Mathematica Hungarica*, 126(4):295–314, 2010.

G. Castellini and E. Murcia. Interior operators and topological separation. *Topology and its Applications*, 160(12):1476–1485, 2013.

G. Castellini and J. Ramos. Interior operators and topological connectedness. *Quaestiones Mathematicae*, 33(3):290–304, 2010.

G. Castellini, J. Koslowski, and G. E. Strecker. Regular closure operators. *Applied Categorical Structures*, 2(3):219–244, 1994.

E. Cech. On bicompact spaces. *Annals of Mathematics*, pages 823–844, 1937.

M. M. Clementino. On connectedness via closure operators. *Applied Categorical Structures*, 9(6): 539–556, 2001.

M. M. Clementino, E. Giuli, and W. Tholen. Topology in a category: compactness. *Portugaliae Mathematica*, 53(4):397–434, 1996.

B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 2002.

D. Dikranjan and E. Giuli. Closure operators i. *Topology and its Applications*, 27(2):129–143, 1987.

D. Dikranjan and W. Tholen. *Categorical structure of closure operators*, volume 346 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1995. With applications to topology, algebra and discrete mathematics.

D. Dikranjan and W. Tholen. Dual closure operators and their applications. *preprint*, 2014.

D. Dikranjan, E. Giuli, and W. Tholen. Closure operators ii. In *Proceedings of the Conference in Categorical Topology (Prague, 1988) World Scientific*, pages 297–335, 1989.

S. Eilenberg and S. MacLane. General theory of natural equivalences. *Transactions of the American Mathematical Society*, 58(2):231–294, 1945.

R. Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

J. L. Fiadeiro, editor. *Recent trends in algebraic development techniques*, volume 1589 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, 1999. papers from the 13th International Workshop on Algebraic Development Techniques (WADT'98) held in Lisbon, April 2–4,1998.

P. Freyd. *Abelian categories. An introduction to the theory of functors*. Harper's Series in Modern Mathematics. Harper & Row, Publishers, New York, 1964.

L. Fuchs. *Infinite abelian groups. Vol. I*. Pure and Applied Mathematics, Vol. 36. Academic Press, New York-London, 1970.

E. Giuli and W. Tholen. Openness with respect to a closure operator. *Applied Categorical Structures*, 8(3):487–502, 2000.

G. Grätzer. *General Lattice Theory*. Springer, 2003.

P. Hertz. Über Axiomensysteme für beliebige Satzsysteme. *Math. Ann.*, 87, 1922.

D. Holgate. Linking the closure and orthogonality properties of perfect morphisms in a category. *Commentationes Mathematicae Universitatis Carolinae*, 39(3):587–608, 1998.

D. Holgate. Compactification and closure. *Quaestiones Mathematicae*, 23(4):529–545, 2000.

D. Holgate and A. Razafindrakoto. Introduction to category theory, 2011. Lecture Notes.

D. Holgate and J. Šlapal. Closure, interior and neighborhood in a category. *Preprint*, 2010.

D. Holgate and J. Šlapal. Categorical neighborhood operators. *Topology and its Applications*, 158(17): 2356–2365, 2011.

J. F. Kennison. Reflective functors in general topology and elsewhere. *Transactions of the American Mathematical Society*, pages 303–315, 1965.

K. Kuratowski. Sur t'operation a de i'analysis situs. *Fund. Math.*, (3):182–199, 1922.

J. Luna-Torres et al. Interior operators and topological categories. *arXiv preprint arXiv:1010.4460*, 2010.

S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

E. H. Moore. On a form of general analysis with applications to linear differential equations and integral equations. In *Atti del IV Congresso Internazionale dei Matematici (Roma, 6-11 Aprile 1909)*, volume 2, pages 98–114, 1909.

J. R. Munkres. *Topology: a first course*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975.

R. Pierce. Closure spaces with application to ring theory. *Lecture Notes in Math.*, 246:516–616, 1972.

A. Razafindrakoto. On coarse and fine neighbourhood operators. *Topology and its Applications*, 159 (13):3067–3079, 2012.

A. Razafindrakoto and D. Holgate. Interior and neighbourhood. *Topology and its Applications*, 168: 144–152, 2014.

F. Riesz. Stetigkeitsbegriff and abstrakte mengenteh. *Atti del IV Congr. Int. di Mat., II (Roma)*, pages 18–24, 1909.

D. S. Scott. Domains for denotational semantics. In *Automata, languages and programming*, pages 577–610. Springer, 1982.

H. Simmons. *An Introduction to Category Theory*. Cambridge University Press, 2011.

A. Tarski. Remarques sur les notions fondamentales de is mdthodologie des mathematiques. *Ann. Soc. Math. Polon*, (7):270–272, 1929.

S. R. Vorster. Interior operators in general categories. *Quaestiones Mathematicae*, 23(4):405–416, 2000.