

Incompressibility: revisited

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Abstract

For the study of incompressibility, the assumption for taking the density as being constant is not appropriate. In order to investigate the conditions for incompressibility in the ocean, the rate of change of the density following the fluid motion has to be negligible compared to the divergence of the velocity. In this study, we make use of dimensional analysis. From that, the achievement of the research will lead us to require that the speed of sound must be very large compared to the characteristic velocity of the fluid and the diffusion of heat and salt should be low.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



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1. Introduction

Global warming is one of the most challenging environmental problems nowadays. In order to protect humanity and to deal with it, numerous communities in the world are studying it, including scientists in different areas of applied mathematics. For that, numerical models are performed to predict future events. Elaborating a correct model that fits well with the reality is very important to produce correct results from interpretations. Actually, applications in Computational Fluid Dynamics applied to the ocean often assume the density to be constant.

The continuity equation reads

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0,$$

where ρ is the density of the fluid, \mathbf{u} is the velocity of fluid motion and D/Dt is the material derivative. In fact, some ambiguities and wrong perceptions suppose the density to be constant to get the flow incompressible. Despite the variations of the density according to temperature and salinity, the material derivative of the density can be neglected for the ocean. In front of that, the goal of this study is to investigate when the material derivative of the density should be neglected, or in another words, to find conditions to make the term $D\rho/Dt$ negligible compared to $\nabla \cdot \mathbf{u}$.

This work is organized as follows. To begin with, some background in Fluid Dynamics will be introduced. Then, some balance equations will be provided in order to apply the dimensional analysis to explicitly address the conditions of incompressibility of flow in the ocean.

2. Governing equations of fluid motion

Fluid Dynamics is the study of fluid motion, and to be able to make a mathematical description of it, governing equations are derived from principles of physics such as the conservation of mass, the conservation of momentum and the conservation of energy. According to Muller (2006), for the ocean, giving an equation for motion of the whole ocean is not possible since it is in local thermodynamic equilibrium. Therefore taking an infinitesimal small fluid element and applying thermodynamic laws on it is necessary. However, a fluid element or fluid particle can be seen as a large ensemble of molecules from a microscopic point of view. In this work, the Cartesian coordinate system is used to localize the fluid element during its motion.

Since the importance of understanding the physical meaning of the terms involved in an equation cannot be underestimated, the following section will explain the use of time derivatives in fluid dynamics.

2.1 Time derivatives

Three types of time derivatives appear in equations in fluid dynamics. These are the total derivative, the material derivative or substantial derivative and the partial derivative. Their uses differ by the way of describing the fluid motion. In this study, a fluid element is seen from a macroscopic point of view in such a way that it can be mathematically considered as a point.

Furthermore, let us consider, for example, a scalar field $\phi(x, y, z, t)$ which is a property of the fluid element at the position (x, y, z) and at time t where x, y, z might be functions of t . That is to say, at a corresponding time and position (x, y, z, t) , $\phi(x, y, z, t)$ is the mean value of ϕ for all the molecules inside the fluid element. According to the continuum hypothesis in fluid mechanics, fluid properties are well-defined for an infinitesimal fluid element and differential calculus can be applied in the derivation of equations. For the description of fluid flow, there are two different approaches used, namely a Lagrangian approach (where the observer is moving with the fluid) and an Eulerian approach where the observer is stationary.

2.1.1 The total derivative . The derivative $\frac{d\phi}{dt}$ expresses the rate of change of ϕ between two very close points (x, y, z) and $(x + \Delta x, y + \Delta y, z + \Delta z)$ where the fluid element passes during a short time Δt tending to zero. Assuming that the property ϕ of a fluid element is differentiable, the Taylor expansion can be expressed as follows,

$$\begin{aligned} \phi(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) = & \phi(x, y, z, t) + \frac{\partial\phi(x, y, z, t)}{\partial t} \Delta t + \frac{\partial\phi(x, y, z, t)}{\partial x} \Delta x + \frac{\partial\phi(x, y, z, t)}{\partial y} \Delta y \\ & + \frac{\partial\phi(x, y, z, t)}{\partial z} \Delta z + \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta y^2) + \mathcal{O}(\Delta z^2) + \mathcal{O}(\Delta t^2). \end{aligned} \quad (2.1.1)$$

The derivative of a real-valued multivariable function ϕ with respect to time t is given by

$$\frac{d\phi}{dt} = \lim_{(\Delta x, \Delta y, \Delta z, \Delta t) \rightarrow (0, 0, 0, 0)} \frac{\phi(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) - \phi(x, y, z, t)}{\Delta t}. \quad (2.1.2)$$

Therefore, from equation (2.1.1) it follows that

$$\frac{d\phi}{dt} = \lim_{(\Delta x, \Delta y, \Delta z, \Delta t) \rightarrow (0, 0, 0, 0)} \frac{\partial \phi(x, y, z, t)}{\partial t} \frac{\Delta t}{\Delta t} + \frac{\partial \phi(x, y, z, t)}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial \phi(x, y, z, t)}{\partial y} \frac{\Delta y}{\Delta t} + \frac{\partial \phi(x, y, z, t)}{\partial z} \frac{\Delta z}{\Delta t}. \quad (2.1.3)$$

And then, as a result of equation (2.1.3),

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} \frac{dx}{dt} + \frac{\partial \phi}{\partial y} \frac{dy}{dt} + \frac{\partial \phi}{\partial z} \frac{dz}{dt}. \quad (2.1.4)$$

However, the total derivative of a function can be defined for any given path. Therefore, during the motion of the fluid element, the observer can follow it by taking a path different from the pathline of the fluid element, and the observer will keep following it at a velocity \mathbf{v} . Thus, the time rate of change of ϕ according to the observer will be a function of his position at any time.

Equation (2.1.4) can then be written as

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi, \quad (2.1.5)$$

where \mathbf{v} is the velocity of the observer following the fluid particle.

2.1.2 The material derivative. With the Lagrangian approach, the observer moves with the fluid particle on its pathline. The velocity of the observer is therefore equal to the velocity \mathbf{u} of the fluid particle at any time and position. The material derivative $\frac{D\phi}{Dt}$ determines the rate of change of the property ϕ of the fluid particle during its motion along its pathline, and therefore

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi. \quad (2.1.6)$$

The material and total derivatives are equivalent, i.e. $\frac{D}{Dt} \equiv \frac{d}{dt}$, only if the time rate of change following the pathline of the fluid element is considered.

2.1.3 The partial derivative . The partial derivative $\frac{\partial}{\partial t}$ is regularly used in the Eulerian method, i.e. when the observer is stationary. At a fixed point (x_0, y_0, z_0) in space, the rate of change of the property ϕ is expressed by $\frac{\partial \phi}{\partial t}$.

The derivation of the governing equations for mass conservation below is an application of the correct use of the time derivatives in fluid dynamics.

2.2 The continuity equation in three dimensions

One of the fundamental physical principles of fluid dynamics is the conservation of mass. In the following paragraphs, constitutive equations for mass conservation are derived.

2.2.1 The Eulerian approach. One way of applying the Eulerian approach is by considering a finite control volume V where a constant amount of fluid passes through it. This control volume is fixed at a point in space. For the ocean, the mass conservation of fluids within this fixed control volume means that the mass flow rate of fluid entering into the control volume has to be the same as the mass flow

rate of fluid out of the control volume. This is because we are considering the parcel of seawater as a one-phase system.

Therefore for a volume with a boundary A ,

$$\left\{ \begin{array}{l} \text{mass flow out through} \\ \text{the control surface } A \end{array} \right\} = \left\{ \begin{array}{l} \text{time rate of decrease of mass} \\ \text{inside the control volume } V. \end{array} \right\} \quad (2.2.1)$$

According to [Munson et al. \(2013\)](#), the total mass of molecules inside the control volume is

$$\int_V \rho dV. \quad (2.2.2)$$

Consequently, the total rate of decrease is given by

$$-\frac{d}{dt} \int_V \rho dV. \quad (2.2.3)$$

On the other hand, the net rate of mass efflux across a small surface dA is ([Lifchitz, 1971](#))

$$\rho \mathbf{u} \cdot \mathbf{n} dA, \quad (2.2.4)$$

where \mathbf{u} and \mathbf{n} are, respectively, the velocity of the fluid element and the unitary vector which is normal to the surface. For the whole control surface, the net rate of mass efflux across the surface of the control volume is then

$$\oint_A \rho \mathbf{u} \cdot \mathbf{n} dA. \quad (2.2.5)$$

Therefore, by employing the expressions in (2.2.5) and (2.2.3) in the equality expressed in (2.2.1), the *integral form of the continuity equation for the Eulerian approach* is given by ([Munson et al., 2013](#); [Lifchitz, 1971](#))

$$\frac{\partial}{\partial t} \int_V \rho dV + \oint_A \rho \mathbf{u} \cdot \mathbf{n} dA = 0. \quad (2.2.6)$$

According to the divergence theorem,

$$\oint_A \rho \mathbf{u} \cdot \mathbf{n} dA = \int_V \nabla \cdot (\rho \mathbf{u}) dV. \quad (2.2.7)$$

And assuming that $\frac{\partial \rho}{\partial t}$ exists and is continuous on the domain, the theorem on the differentiation under the integral sign ([Marsden and Hoffman, 1993](#)) can be applied as follows

$$\frac{\partial}{\partial t} \left(\int_V \rho dV \right) = \int_V \frac{\partial \rho}{\partial t} dV. \quad (2.2.8)$$

Then by using (2.2.7) and (2.2.8) in (2.2.6), it follows that

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) dV = 0. \quad (2.2.9)$$

Since the control volume V was chosen arbitrary then the integrand vanishes, that is to say

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (2.2.10)$$

Consequently, *the differential form of the continuity equation for the Eulerian approach* is given by equation (2.2.10) ([Munson et al., 2013](#)).

2.2.2 The Lagrangian approach. This tags individual fluid particles in motion and observes the change in properties happening to these particles through time. First, in order to derive the integral form of the continuity equation for the Lagrangian approach, a moving finite control volume V which contains the tagged fluid particles is considered. As [Anderson \(2008\)](#) well explains, for the control volume observed, the mass of fluids inside remains constant but as the volume may change due to the motion, that implies a variation in density. So, for a small volume dV given and ρ its density, the mass of the fluid particles within the volume dV is

$$\rho dV. \quad (2.2.11)$$

Then the total mass of fluid particles in the control volume V is equal to

$$\int_V \rho dV. \quad (2.2.12)$$

According to [Anderson \(2008\)](#), as the mass is conserved and from the definition of the material derivative, the *integral form of the continuity equation for the Lagrangian approach* is given by

$$\frac{D}{Dt} \int_V \rho dV = 0. \quad (2.2.13)$$

Secondly, to derive the differential form of the continuity equation for the Lagrangian approach, the previous idea is again repeated but for an infinitesimal volume δV which contains an amount of fluid δm in motion. The mass density is denoted by

$$\rho = \frac{\delta m}{\delta V}. \quad (2.2.14)$$

The amount of fluid in δV is constant, thus

$$\frac{D(\delta m)}{Dt} = 0. \quad (2.2.15)$$

But from (2.2.14), $\delta m = \rho \delta V$ and then

$$\frac{D(\delta m)}{Dt} = \frac{D(\rho \delta V)}{Dt}. \quad (2.2.16)$$

Therefore, by using equation (2.2.16) in equation (2.2.15), it follows that

$$\rho \frac{D(\delta V)}{Dt} + \delta V \frac{D\rho}{Dt} = 0. \quad (2.2.17)$$

This can be simplified as

$$\frac{D\rho}{Dt} + \rho \left[\frac{1}{\delta V} \frac{D(\delta V)}{Dt} \right] = 0. \quad (2.2.18)$$

According to [Anderson \(2008\)](#),

$$\begin{aligned} \frac{D(\delta V)}{Dt} &= \int_{\delta A} \mathbf{u} \cdot \mathbf{dA}, \text{ as the volume of fluid crossing } \delta A \text{ is } \delta V = (\mathbf{u}\Delta t) \cdot \mathbf{dA}. \\ &= \int_{\delta V} \text{div } \mathbf{u} dV, \text{ by applying the divergence theorem.} \\ &= \delta V \text{div } \mathbf{u}, \text{ as } \delta V \text{ was considered to be very small and so } \text{div } \mathbf{u} \text{ is nearly constant in it.} \end{aligned} \quad (2.2.19)$$

Using (2.2.19) into (2.2.18), the *differential form of the continuity equation for the Lagrangian approach* is (Anderson, 2008)

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{u} = 0. \quad (2.2.20)$$

2.2.3 Remark. It is important to notice that these equations of mass conservation are mathematically equivalent. Although, when using the differential form of the mass conservation equation for the Eulerian approach, the differentiability of the density must be assumed. This makes the integral form of the continuity equation more practical.

2.3 Equation of state for seawater

As the ocean is in local thermodynamic equilibrium, the principles in thermodynamics can be well applied for the study of oceanic motions for each fluid particle considered. Then, any thermodynamic property of a fluid element in seawater can be expressed as a function of some intensive and independent thermodynamic variables. More precisely, according to the Gibbs' phase rule, the degree of freedom f' of this function depends on the number of the phases p' when the fluid is in equilibrium and the number of components c' of the fluid. Then, following ter Haar Wergeland (1966, p 82)

$$f' = 2 + c' - p'.$$

This rule is quite important in thermodynamics, and necessarily in the study of the ocean. The function of state which expresses the density in terms of other intensive and independent thermodynamic variables is the one that is most used. And it is called the equation of state.

For instance, from Eckart (1960), a thermodynamic variable, for the case of pure fluid, can be expressed as a function of two of the variables ρ (or v), p , T , η which are, respectively, the density (or specific volume), the pressure, the temperature and the specific entropy because $p' = 1$ and $c' = 1$.

Seawater differs from fresh water by its chemical components. As the chemical proportion of different salt ions in seawater is nearly constant, the concentration of salts or dissolved solids in seawater can be expressed as a function of one variable which is the salinity. It is defined as the amount of dissolved solids in one kilogram of seawater. It is often expressed in g/kg (or psu, or ppt).

According to Muller (2006), by applying the Gibbs' phase rule for the ocean, a thermodynamic variable can be written as a function of any three intensive and independent variables since the variable salinity S is introduced for seawater. Through this study, the equation of state will use the pressure p , the temperature T and the salinity S as thermodynamic variables.

Therefore, the equation of state for the density is

$$\rho = \rho(p, T, S). \quad (2.3.1)$$

Then, from (2.3.1) the total differential form is

$$d\rho = \left(\frac{\partial \rho}{\partial p} \right)_{T,S} dp + \left(\frac{\partial \rho}{\partial T} \right)_{S,p} dT + \left(\frac{\partial \rho}{\partial S} \right)_{T,p} dS. \quad (2.3.2)$$

By dividing both sides of equation (2.3.2) by dt , the time rate of change of the density defined in equation (2.3.1) is given by

$$\frac{d\rho}{dt} = \left(\frac{\partial\rho}{\partial p}\right)_{T,S} \frac{dp}{dt} + \left(\frac{\partial\rho}{\partial T}\right)_{S,p} \frac{dT}{dt} + \left(\frac{\partial\rho}{\partial S}\right)_{T,p} \frac{dS}{dt}. \quad (2.3.3)$$

By applying the Lagrangian approach, $\frac{D}{Dt} = \frac{d}{dt}$, equation (2.3.3) can be written as

$$\frac{D\rho}{Dt} = \left(\frac{\partial\rho}{\partial p}\right)_{T,S} \frac{Dp}{Dt} + \left(\frac{\partial\rho}{\partial T}\right)_{S,p} \frac{DT}{Dt} + \left(\frac{\partial\rho}{\partial S}\right)_{T,p} \frac{DS}{Dt}. \quad (2.3.4)$$

As our goal investigates when the material derivative of the density can be neglected, equation (2.3.4) will be very useful in the following section.

3. The material derivative of the density

The purpose of this work is to find meaningful conditions on the incompressibility assumption for the ocean case. According to [Olbers et al. \(2012, p 119\)](#), a fluid flow is incompressible when the differential form of the continuity equation (2.2.18) for the Lagrangian approach can be reduced to

$$\nabla \cdot \mathbf{u} = 0. \quad (3.0.1)$$

For that, it is necessary to search for the conditions on when, for a considered fluid element, the material derivative of the density is negligible compared to the divergence of the velocity. Nonetheless, a misunderstanding of the notion of incompressibility is revealed in some books.

3.1 Thermodynamic coefficients

In order to understand equation (2.3.4) on a physics level, it is necessary to introduce specific coefficients describing certain physical properties. According to [Muller \(2006\)](#), if the density ρ depends on the pressure p , the temperature T and the salinity S , then the following definitions can be introduced to simplify equation (2.3.4):

- the isothermal compressibility coefficient

$$\kappa = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial p} \right)_{T,S}, \quad (3.1.1)$$

- the thermal expansion coefficient

$$\alpha = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_{p,S}, \quad (3.1.2)$$

- the haline contraction coefficient

$$\beta = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial S} \right)_{p,T}. \quad (3.1.3)$$

Then, equation (2.3.4) can be rewritten in terms of these coefficients as follows

$$\frac{D\rho}{Dt} = \rho \left(\kappa \frac{Dp}{Dt} - \alpha \frac{DT}{Dt} + \beta \frac{DS}{Dt} \right). \quad (3.1.4)$$

Also, [Muller \(2006\)](#) considers some additional thermodynamic variables to describe the state of a fluid element, such as:

- the specific volume v [$m^3 kg^{-1}$] defined as the inverse of the density,
- the specific internal energy e [$m^2 s^{-2}$],
- the specific entropy η [$m^2 s^{-2} K^{-1}$],
- the specific enthalpy $h = e + pv$ [$m^2 s^{-2}$],

- the specific free enthalpy (or specific Gibbs function) $g = e + pv - T\eta$ [m^2s^{-2}],
- the specific heat at constant pressure is denoted by c_p ,

$$\text{by definition, } c_p = \left(\frac{\partial h}{\partial T} \right)_{p,S}; \text{ and since } h = g + T\eta, \text{ then } c_p = T \left(\frac{\partial \eta}{\partial T} \right)_{p,S}, \quad (3.1.5)$$

- the specific heat at constant specific volume denoted by c_v ,

$$\text{by definition, } c_v = \left(\frac{\partial e}{\partial T} \right)_{v,S}; \text{ but as } e = f + T\eta, \text{ then } c_v = T \left(\frac{\partial \eta}{\partial T} \right)_{v,S}, \quad (3.1.6)$$

- the speed of sound c .

Now, let us make a first step on finding the conditions when the material derivative of the density is negligible compared to the divergence of the velocity. In order to achieve that, evaluating some thermodynamic coefficients in equation (3.1.4) will be helpful.

The isothermal compressibility coefficient can be expressed as a function of another meaningful variables.

For that, $\left(\frac{\partial \rho}{\partial p} \right)_{T,S}$ will be rewritten in another form.

According to Muller (2006),

$$\left(\frac{\partial p}{\partial \rho} \right)_{\eta,S} = c^2. \quad (3.1.7)$$

In Munson et al. (2013), this expression is obtained by considering an infinitesimal control volume at rest, filled with fluid, which becomes in motion after applying a small pressure force on it. Supposing that an observer follows the control volume, the speed of sound can be derived in an adiabatic process by using the equations of conservation of mass and momentum.

By applying the chain rule,

$$\left(\frac{\partial p}{\partial \rho} \right)_{\eta,S} = \left(\frac{\partial p}{\partial T} \right)_{\eta,S} \left(\frac{\partial T}{\partial v} \right)_{\eta,S} \left(\frac{\partial v}{\partial \rho} \right)_{\eta,S}. \quad (3.1.8)$$

And by considering the cycle property for partial derivatives in Moran (2006) and the second expressions of c_p and c_v in equations (3.1.5) and (3.1.6),

$$\left(\frac{\partial p}{\partial T} \right)_{\eta,S} = \frac{-1}{\left(\frac{\partial T}{\partial \eta} \right)_{p,S} \left(\frac{\partial \eta}{\partial p} \right)_{T,S}} = - \left(\frac{\partial \eta}{\partial T} \right)_{p,S} \left(\frac{\partial p}{\partial \eta} \right)_{T,S} = - \frac{c_p}{T} \left(\frac{\partial p}{\partial \eta} \right)_{T,S}, \quad (3.1.9)$$

and

$$\left(\frac{\partial T}{\partial v} \right)_{\eta,S} = \frac{-1}{\left(\frac{\partial v}{\partial \eta} \right)_{p,S} \left(\frac{\partial \eta}{\partial T} \right)_{v,S}} = - \left(\frac{\partial \eta}{\partial v} \right)_{T,S} \left(\frac{\partial T}{\partial \eta} \right)_{v,S} = - \frac{T}{c_v} \left(\frac{\partial \eta}{\partial v} \right)_{T,S}. \quad (3.1.10)$$

Therefore by using equations (3.1.9) and (3.1.10) in equation (3.1.8),

$$\begin{aligned}
 \left(\frac{\partial p}{\partial \rho}\right)_{\eta,S} &= \frac{c_p}{c_v} \left(\frac{\partial p}{\partial \eta}\right)_{T,S} \left(\frac{\partial \eta}{\partial v}\right)_{T,S} \left(\frac{\partial v}{\partial \rho}\right)_{\eta,S}, \\
 &= \frac{c_p}{c_v} \left(\frac{\partial p}{\partial v}\right)_{T,S} \left(\frac{\partial v}{\partial \rho}\right)_{\eta,S}, \\
 &= \frac{c_p}{c_v} \left(\frac{\partial p}{\partial v}\right)_{T,S} \left(\frac{\partial \rho}{\partial v}\right)_{T,S} \left(\frac{\partial p}{\partial \rho}\right)_{\eta,S}, \\
 &= \frac{c_p}{c_v} \left(\frac{\partial p}{\partial v}\right)_{T,S},
 \end{aligned} \tag{3.1.11}$$

as the specific volume can be directly defined as a function of the density, then

$$\left(\frac{\partial \rho}{\partial v}\right)_{T,S} \left(\frac{\partial v}{\partial \rho}\right)_{\eta,S} = \frac{d\rho}{dv} \frac{dv}{d\rho} = 1.$$

By considering the equation (3.1.7), equation (3.1.11) is equivalent to

$$\left(\frac{\partial \rho}{\partial p}\right)_{T,S} = \frac{1}{c^2} \frac{c_p}{c_v}. \tag{3.1.12}$$

Muller (2006) characterizes the specific heat as the amount of heat that is required to increase the temperature of a unit mass of seawater. For c_v , the volume is held constant and for c_p the volume is allowed to expand at a constant pressure. Now, let us consider two cylinders filled with the same amount of freshwater and salt. One of the cylinders is provided with a fixed piston to hold the volume of the solution constant. Let us apply the same amount of heat to the two cylinders. According to Moran (2006, p 48), the first law of thermodynamics can be expressed as follows ,

$$\Delta e = Q + W, \tag{3.1.13}$$

where Q and W are, respectively, the (external and internal) heat and the work (done by the system or on the system).

The salinity is a thermodynamic variable and should also be taken into consideration. Then, the chemical energy obtained from the interaction between salt ions and water molecules has to be added to equation (3.1.13). According to Muller (2006, p 12), if the internal energy is $e = e(v, \eta, S)$ and the process is reversible, then the first law of thermodynamics is written as follows

$$de = -p dv + T d\eta + \Delta\mu dS, \tag{3.1.14}$$

where $\Delta\mu$ is the chemical potential difference. In this case, the chemical energy is expressed as a function of chemical potential difference. Assuming that the heat from the intermolecular friction is negligible, equation (3.1.14) may be explained as follows for the constant volume case and the constant pressure case. Recall that the same amount of heat is applied to the two identical solutions of seawater. The goal here is to compare the change in internal energy for the two cases to compare c_v and c_p .

- For the constant volume case, no work is done by the system, then $-p dv = 0$ in contrast to the constant pressure case which expresses the work done by the system by $-p dv < 0$.

- Furthermore, if the volume of the system is hold to be a constant, then the kinetic energy is higher compared to the case of having a constant pressure. This implies that the temperature of the solution for the constant volume case will increase easily compared to the other case. Also, since particles received more energy to vibrate for the constant volume case, then the entropy is higher in this situation compared to the constant pressure case.
- The chemical energy in the two cases is studied here.

Following Muller (2006, p 26), the chemical potential difference is defined as

$$\Delta\mu = \mu_s - \mu_w,$$

where μ_s and μ_w are the chemical potentials of salt and water characterizing the energy required to vary the concentrations of water c_w and salt c_s . Remember that the amount of particles is conserved for both cases. The chemical potential difference is related to the probability of the salt ions to interact with the water molecules. So the increase of this probability means increasing the chemical energy of the system. When the volume is allowed to expand, the distance between particles of the solution gets larger and larger and so the probability of interaction between salt ions and water molecules will tend to be low. But when the volume is kept constant, the salt ions and water molecules are more likely to interact compared to the constant pressure case. Thus, the chemical energy for the constant volume case is higher compared to the one obtained from the pressure constant case.

Then, after applying the same amount of heat to the two systems, the change in internal energy for the constant volume case is greater than the change in the case of constant pressure. Knowing that the internal energy is the total energy of the system, usually equal to the combination of the kinetic energy (which is related to the temperature of the system) and the potential energy, the potential energy can be neglected in our case as we consider a small amount of fluid in our approach. Then, when the internal energy increases, the temperature increases, meaning that more amount of heat is needed to augment the temperature of a unit mass of seawater for the constant pressure case compared to the constant volume. That implies c_p is greater than c_v . Then, $\frac{c_p}{c_v} > 1$. Also, since a heat quantity cannot tend to infinity in the concept of thermodynamics, in particular c_p does not tend to infinity, the ratio of specific heats $\frac{c_p}{c_v}$ is always a finite value that does not tend to zero.

According to Muller (2006, p 21),

$$c_v = c_p - \frac{T\alpha^2 v}{\kappa}.$$

Also, this implies

$$\frac{c_p}{c_v} > 1.$$

Following Olbers et al. (2012, p 24), the specific heats at constant pressure and at constant volume differ only about 1% in value from experimental results. Then, we can consider in the later section that the ratio of specific heats is approximately equal to 1.

3.2 Evaluating the coefficients

3.2.1 Isothermal compressibility coefficient. From the analysis above, κ tends to zero if the speed of sound is infinitely large. In seawater, the speed of sound is a variable and can be expressed as a function

of p , T and S . A low value of the speed of sound is, for instance equal to 1445 m/s for $S = 34.85$ psu and $T = 0^\circ\text{C}$, and according to [Pickard and Emery \(1982, p 23\)](#) the speed of sound increases when temperature increases. As the speed of sound is high compared to $\frac{c_p}{c_v}$, the isothermal compressibility coefficient κ can be neglected.

Then, if the isothermal compressibility coefficient is nearly equal to zero, it implies that the flow is incompressible. And it follows that the speed of sound tends to infinity. But as κ expresses only the variation of the density with change in pressure, it follows only that the density may change with temperature and salinity. Thus, the density is not a fortiori constant.

3.2.2 Why are α and β non-zero?.

- Why is the thermal expansion coefficient α non-zero? By decomposing the expression of α in equation(3.1.2),

$$\begin{aligned}\alpha &= -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial v} \right)_{p,S} \left(\frac{\partial v}{\partial T} \right)_{p,S}, \\ &= \rho \left(\frac{\partial v}{\partial T} \right)_{p,S}.\end{aligned}\quad (3.2.1)$$

As the density cannot be null, and also, by varying the temperature the specific volume has to change if the mass is kept constant, therefore α cannot tend to zero.

- Why is the haline coefficient β non-zero? If m is the mass of the fluid element, let us denote m_w and m_s , respectively, the mass of freshwater and the mass of dissolved elements of the fluid such that $m = m_w + m_s$. In the expression of β , the temperature and the pressure are held constant, and the salinity varies. According to ([Olbers et al., 2012, p 13](#)), the variation of density with salinity happens when a change in the amount of dissolved solids or freshwater is made. That means,

$$\beta = \left(\frac{\partial \rho}{\partial S} \right)_{p,T} = \left(\frac{\partial \rho}{\partial m_s} \right)_{p,T} \left(\frac{\partial m_s}{\partial S} \right)_{p,T} + \left(\frac{\partial \rho}{\partial m_w} \right)_{p,T} \left(\frac{\partial m_w}{\partial S} \right)_{p,T}.\quad (3.2.2)$$

For a change in salinity, at least one of the term on the right hand side of equation (3.2.2) cannot vanish, and therefore, β is non-zero.

Balance equations will be introduced in the following section to express the material derivatives of the salinity and the temperature in equation (2.3.4) as a function of other thermodynamic variables.

3.3 Balance equations

3.3.1 Conservation of salt. To obtain an expression of the variation of salinity with respect to time, a similar expression is used for the conservation of salt, the continuity equation can be employed by substituting the density ρ in equation (2.2.18) with the salt density which is equal to ρS and by evaluating the behaviour of salt ions in the volume through its motion. First, suppose that the volume is at rest or moving slowly, then we have predominantly the diffusion of salt in water. During the diffusion, the salt ions spread from a region at high concentration in salt into a region with a low concentration in salt. From that, the salinity decreases through this process. That defines the salt flux \mathbf{J}_S from the Fick's law of diffusion in three dimensions ([Banks, 1994, p 327](#)):

$$\mathbf{J}_S = -k_S \nabla S, \quad (3.3.1)$$

where k_S is the diffusion coefficient or the diffusivity for salt. Secondly, when the volume moves faster than what we described previously, the movement of salt ions within the volume, which is caused by the flow, has to be considered. This is described by the convective term, which is equal to

$$(\rho S) \nabla \cdot \mathbf{u}. \quad (3.3.2)$$

Considering the diffusion and the convection, the conservation of salt is

$$\frac{D(\rho S)}{Dt} + (\rho S) \nabla \cdot \mathbf{u} + \nabla \cdot \mathbf{J}_S = 0. \quad (3.3.3)$$

After developing equation (3.3.3) and making some simplification using the conservation of mass in equation (2.2.18), equation (3.3.3) becomes (Olbers et al., 2012, p 30)

$$\begin{aligned} \rho \frac{DS}{Dt} &= -\mathbf{J}_S \\ &= \nabla \cdot (k_S \nabla S). \end{aligned} \quad (3.3.4)$$

But in some cases, sources and sinks of salt are considered. This is because of the change in salinity due to evaporation, addition of water through the rain or ice melt, or addition of dissolved solids transported by rivers. LeBlond and Mysak (1978, p 9) expresses the conservation of salt as follows

$$\frac{DS}{Dt} = \nabla \cdot (k_S \nabla S) + Q_S, \quad (3.3.5)$$

where Q_S covers all sources and sinks of salt.

A more general relation can be derived for the variation of the salinity. This can be done by considering the effect of pressure and temperature. For instance, if the temperature is high, then the salt ions receive a high energy to vibrate so that the volume will expand. Consequently, the salt flux decreases. Also, if the volume goes towards a deeper region, the pressure is high and the temperature might be low. In that case, the salt ions just receive a less amount of energy to move and then the flux increases. Following Olbers et al. (2012, p 56-57), the salt flux is expressed as follows

$$\mathbf{J}_S = -\rho k_S \left(\nabla S + \frac{\partial \mu / \partial p}{\partial \mu / \partial S} \nabla p \right) - \rho k_{ST} \nabla T, \quad (3.3.6)$$

where μ is the variation of the Gibbs function with change in salinity at constant pressure and temperature, and k_{ST} is the diffusivity for salt with change in temperature. An approximative value of k_S is equal to $1.2 \times 10^{-9} \text{m}^2 \text{s}^{-1}$ and also from experiments k_{ST} can often be neglected.

Alternatively, from Phillips (1966, p 11), the conservation of dissolved solids is more precisely developed as

$$\rho \frac{DS}{Dt} + \nabla \cdot \mathbf{s} = 0, \quad (3.3.7)$$

where \mathbf{s} is the flux density of salt (mass of salt per unit area per unit time) and

$$\mathbf{s} = -\rho k_S \left(\nabla S + \left(\frac{k_{ST}}{T} \right) \nabla T + \left(\frac{k_{SP}}{p} \right) \nabla p \right), \quad (3.3.8)$$

where k_{ST} and k_{SP} correspond to the molecular diffusion of salt from the temperature gradient and pressure gradient, respectively. In seawater at 20°C, $k_{SP} \approx 1.3 \times 10^{-5} \text{cm}^2 \text{sec}^{-1}$. Phillips (1966, p 11) also notes that k_{SP} is negligible in the ocean.

In [E.Gill \(1995, p 70\)](#), the effects of pressure and temperature can be neglected in the flux density of salt, and also ρk_S is a function of state of the fluid which has a small variation regardless of its thermodynamic variables. Then

$$\frac{DS}{Dt} = k_S \nabla^2 S, \quad (3.3.9)$$

where k_S depends on the state of the fluid. The variation of salinity by following the fluid motion on the path of the particle can simply be expressed by equation (3.3.9) as the remaining terms in the other equations can be ignored for their small typical values. In equation (3.3.9), the diffusivity of salt in water $k_S \approx 1.5 \times 10^{-9} \text{m}^2 \text{s}^{-1}$ at 25°C according to [E.Gill \(1995, p 68\)](#).

3.3.2 Variation in temperature. The goal here is to give an expression of the material derivative of the temperature. According to [Pedlosky \(1987, p 11\)](#), the first law of thermodynamics is expressed as

$$\rho \frac{De}{Dt} = -p\rho \frac{D}{Dt} \left(\frac{1}{\rho} \right) + k \nabla^2 T + \chi + \rho Q, \quad (3.3.10)$$

where k is the thermal conductivity, Q is the rate of heat addition per unit mass by internal heat sources, and χ is the addition of heat due to the viscous dissipation. The fundamental thermodynamic relation gives a relation for the specific entropy ([Pedlosky, 1987, p 12](#))

$$T \frac{D\eta}{Dt} = \frac{De}{Dt} + p \frac{D}{Dt} \frac{1}{\rho}. \quad (3.3.11)$$

Using the expression of $\frac{De}{Dt}$ from equation (3.3.10) to substitute it in equation (3.3.11) and neglecting χ in the change in internal energy for the ocean case, equation (3.3.11) can be rewritten as

$$T \frac{D\eta}{Dt} = \frac{k}{\rho} \nabla^2 T + Q. \quad (3.3.12)$$

For the case of pure water, the specific entropy will not depend on salinity, meaning that $\eta = \eta(p, T)$. Thus, by using the thermodynamic relation

$$\left(\frac{\partial \eta}{\partial p} \right)_T = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial T} \right)_p, \quad (3.3.13)$$

equation (3.3.12) is expressed in the following ([Pedlosky, 1987, p 12](#))

$$c_p \frac{DT}{Dt} - \frac{T\alpha}{\rho} \frac{Dp}{Dt} = \frac{k}{\rho} \nabla^2 T + Q. \quad (3.3.14)$$

For a pure liquid where the effects of pressure on the density are very small, according to [Pedlosky \(1987, p 13\)](#) the equation of state can be simplified as

$$\rho = \rho_0 (1 - \alpha(T - T_0)). \quad (3.3.15)$$

In that case equation (3.3.14) becomes

$$\frac{DT}{Dt} = \frac{k}{\rho c_p} \nabla^2 T + \frac{Q}{c_p}, \quad (3.3.16)$$

where $\frac{k}{\rho c_p}$ is the coefficient of thermal diffusivity.

According to LeBlond and Mysak (1978, p 9), the conservation of internal energy is expressed by

$$\frac{D}{Dt}(\rho c_v T) = \nabla \cdot (k_T \nabla T) + Q_T, \quad (3.3.17)$$

where k_T is the thermal conductivity and Q_T denotes all sources and sinks of heat.

A more general balance equation for the variation in temperature can be expressed. In Olbers et al. (2012, p 58), we have

$$\rho c_p \frac{DT}{Dt} = -\nabla \cdot \mathbf{J}_H + \alpha T \frac{Dp}{Dt} + Q_T, \quad (3.3.18)$$

where Q_T is the temperature source and \mathbf{J}_H is the flux of internal energy¹ described as

$$\mathbf{J}_H = \mathbf{J}_{\text{rad}} + \mathbf{J}_{\text{chem}} + \mathbf{J}_T,$$

where \mathbf{J}_{rad} is the solar radiation flux, \mathbf{J}_{chem} is the chemical energy flux and \mathbf{J}_T denotes the molecular heat flux which is expressed as follows (Olbers et al., 2012, p 56)

$$\mathbf{J}_T = -\rho c_p k_T \nabla T - \rho k_{TS} \left(\nabla S + \frac{\partial \mu / \partial p}{\partial \mu / \partial S} \right).$$

Phillips (1966, p 12) shows also a precise expression of the time rate of temperature. The temperature is considered as a function of the density, the specific entropy and the salinity. From that, the material derivative of temperature is

$$\frac{DT}{Dt} = -(\alpha T c^2 / c_p) \nabla \cdot \mathbf{u} + (\rho c_v)^{-1} (Q - \nabla \cdot (k \nabla T)), \quad (3.3.19)$$

where Q denotes the heat due to the friction and other irreversible processes, and k is the thermal conductivity. Using equation (3.3.18) and neglecting the effects of solar radiation under some conditions, the effect of pressure and the intermolecular friction in the ocean, E.Gill (1995) simplifies equation (3.3.18) as follows

$$\rho c_p \frac{DT}{Dt} - \alpha T \frac{Dp}{Dt} = 0. \quad (3.3.20)$$

Notice that in (Muller, 2006, p 15), an expression of the adiabatic temperature gradient (or lapse rate) is given by

$$\Gamma = \frac{\alpha v T}{c_p}. \quad (3.3.21)$$

Using the expression (3.3.21) in equation (3.3.20), it follows that

$$\frac{DT}{Dt} - \Gamma \frac{Dp}{Dt} = 0. \quad (3.3.22)$$

But this needs a further investigation.

In the later sections, balance equations will be used to express the material derivative of the salinity and the temperature in the equation of state (2.3.4) to find the conditions of incompressibility.

¹ \mathbf{J}_H is also the flux of enthalpy according to the definition of the enthalpy.

3.4 Dimensionless numbers

In this section, some dimensionless numbers will be introduced for the next step on the dimensional analysis. The dimensionless numbers applicable to this study are the Reynolds number, the Prandtl number, the Peclet number, the Froude number, the Mach number and the Strouhal number.

3.4.1 Reynolds number. The Reynolds number is the ratio of inertial forces to viscous forces acting on a fluid. Using this definition, the expression of the Reynolds number Re becomes (Batchelor, 2000, p 215)

$$Re = \frac{uL\rho}{\mu}, \quad (3.4.1)$$

where u is the mean velocity of the fluid, L is the characteristic length scale of the flow and μ is the viscosity of the fluid. It gives as a result a description of the flow regimes, including laminar and turbulent flow. For instance, a laminar flow occurs at a lower Reynolds number where viscous forces are dominant, while, a turbulent flow occurs at a higher Reynolds number where the inertial forces are dominant.

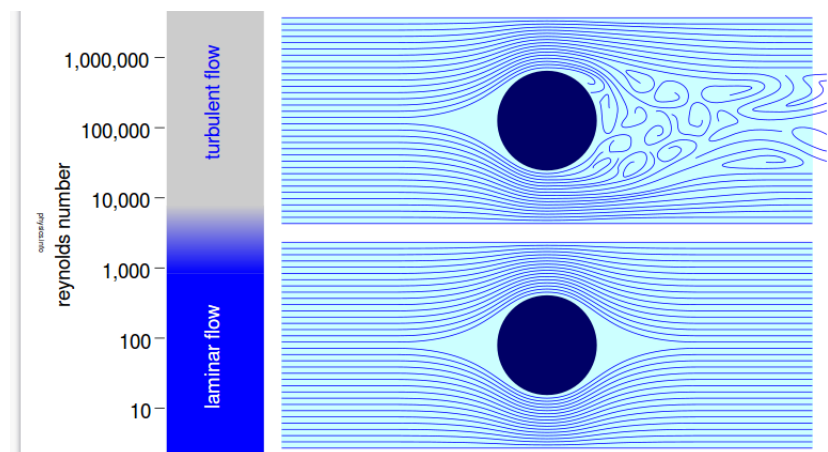


Figure 3.1: ²

In Figure 3.1, for a low Reynolds number (bottom panel) the path of the flow is continuous and regular because of the dominance of the viscous forces, on the other hand, for a high Reynolds number (top panel) the effect of the inertial forces causes the irregularity of the flow.

3.4.2 Prandtl number. Consider a situation where heat is transferred to the fluid, then the conduction and convection occur within the fluid. Recall that, conduction is a transfer of heat per unit length and it is defined for both solids and fluids. On the other hand, the convection happens only for fluids. The Prandtl number measures which process is faster, the conduction or the convection. It is often denoted as Pr and is therefore expressed by

$$Pr = \frac{\nu}{\alpha}, \quad (3.4.2)$$

where ν is the kinematic viscosity (SI units : m^2/s) and α (SI units : m^2/s) is the thermal diffusivity. The kinematic viscosity, here, determines the mobility of the molecules within the fluid for the convection

²<http://physics.info/turbulence/>

process. An alternative definition of the Prandtl number is

$$Pr = \frac{c_p \mu}{k}, \quad (3.4.3)$$

where μ is the dynamic viscosity (SI units : Pa s = (N s)/m²), k is the thermal conductivity (SI units : W/(m K)) and c_p is the specific heat at constant pressure (SI units : J/(kg K)).

3.4.3 Peclet number. According to Bird et al. (2007, p 268), the Peclet number is a non-dimensionless quantity describing the process dominating a transport of matter or thermodynamic quantity. The transport of any quantity in fluid occurs through the advection and the diffusion owing to a suitable gradient. The Peclet number Pe is given by

$$Pe = \frac{\text{advective transport rate}}{\text{diffusive transport rate}}. \quad (3.4.4)$$

In the case of heat transfer,

$$Pe = \frac{uL\rho c_p}{k} = RePr, \quad (3.4.5)$$

where k is the diffusivity coefficient of heat.

In the case of mass transfer,

$$Pe = \frac{uL}{k} = RePr, \quad (3.4.6)$$

where k is the diffusivity coefficient of the matter.

3.4.4 Froude number. The expression of the Froude number is given by

$$Fr = \frac{u}{v_w}, \quad (3.4.7)$$

where u is the characteristic velocity and v_w is the characteristic water wave propagation velocity. Also, the Froude number is the ratio of the inertial forces to the gravitational forces. In Bird et al. (2007, p 98),

$$Fr = \frac{v_w^2}{gl}, \quad (3.4.8)$$

where v_w is the velocity of water, g the acceleration due to gravity and l is the hydraulic depth. When the Froude number is high, the inertial forces are dominant and then the flow speed is high also. On the other hand, when it is low the gravitational forces dominate which makes the flow slow.

3.4.5 Mach number. The Mach number is a dimensionless number which determines the compressibility of the flow. It is often denoted by M or Ma and is expressed by (Batchelor, 2000, p 168)

$$M = \frac{u}{c}, \quad (3.4.9)$$

where u is the speed of a fluid and c is the speed of sound within the fluid.

3.4.6 The Strouhal number. Following Batchelor (2000, p 216), the Strouhal number describes the oscillations of the fluid flow. This number is often denoted by St or Sr and is expressed as follows

$$St = \frac{wL}{u}, \quad (3.4.10)$$

where w is the frequency of the oscillations in average. When the three variables w , L and u are independent, then the Strouhal number characterizes well the flow of the fluid as the Reynolds number. Moreover, the Strouhal number is a function of the Reynolds number.

In the following section, these dimensionless numbers will be useful to determine meaningful conditions of the incompressible fluid flow in the case of the ocean.

3.5 Dimensional analysis

3.5.1 Order of magnitude analysis and assumptions. Let us rewrite the mass conservation equation (2.2.20) by substituting equations (3.1.12), (3.3.14) and (3.3.9) as follows:

$$\begin{aligned} \nabla \cdot \mathbf{u} &= -\frac{1}{\rho} \frac{D\rho}{Dt}, \\ &= -\frac{1}{\rho} \left\{ \left(\frac{\partial \rho}{\partial p} \right)_{T,S} \frac{Dp}{Dt} + \left(\frac{\partial \rho}{\partial T} \right)_{S,p} \frac{DT}{Dt} + \left(\frac{\partial \rho}{\partial S} \right)_{T,p} \frac{DS}{Dt} \right\}, \\ &= -\frac{1}{\rho} \left\{ \frac{c_p}{c_v} \frac{1}{c^2} \left(\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p \right) + \frac{\rho \alpha}{c_p} \left(\frac{T \alpha}{\rho} \frac{Dp}{Dt} + \frac{k}{\rho} \nabla^2 T + Q \right) - \beta k_S \nabla^2 S \right\}. \end{aligned} \quad (3.5.1)$$

Let us consider a system and denote the characteristic velocity by U , the length scale by L , the mass by m and let us take an additional notation T to represent the time.

Then, the order of magnitude of the left hand side of (3.5.1) is given as follows

$$\nabla \cdot \mathbf{u} \approx O\left(\frac{U}{L}\right). \quad (3.5.2)$$

In the following subsections, the order of magnitude of each term on the right hand side of equation (3.5.1) will be compared to the one in the left hand side given in (3.5.2). As we want the material derivative of the density to be negligible compared to the divergence of the velocity of the fluid, that implies the order of magnitude of each term in the right hand side of equation (3.5.1) has to be very small compared to the order of magnitude of the divergence of the velocity of the fluid.

3.5.2 Change in pressure. The term $\partial p / \partial t$ represents a change in the pressure field. This term may generate pulses of pressure (with frequency w) that translate across the fluid generating pulses of density. Since p is a scalar quantity depending on time and position,

$$\partial p / \partial t = \lim_{\Delta t \rightarrow 0} \frac{\Delta p}{\Delta t},$$

but as Δt represents the time needed for having a change in pressure Δp , this implies that

$$\Delta t \propto \frac{1}{w} \text{ and so } \frac{\Delta p}{\Delta t} \propto \Delta p w.$$

Then, the evaluation of the first term in the right hand side of equation (3.5.1) with the divergence of the velocity of the fluid is given by

$$\frac{1}{\rho c^2} \frac{\partial p}{\partial t} \approx \frac{O\left(\frac{w \Delta p}{\rho c^2}\right)}{O\left(\frac{U}{L}\right)}. \quad (3.5.3)$$

But in terms of order of magnitude,

$$\Delta p \approx O\left(\frac{m}{LT^2}\right), \quad \rho \approx O\left(\frac{L^3}{m}\right).$$

Therefore, it follows that the order of magnitude in (3.5.3) is equal to

$$\begin{aligned} O\left(\frac{\frac{L^3}{m} \frac{1}{c^2} w \frac{m}{LT^2}}{\frac{U}{L}}\right) &= O\left(\frac{\frac{L^2}{T^2} \frac{1}{c^2} w}{\frac{U}{L}}\right), \\ &= O\left(\frac{ULw}{c^2}\right), \\ &= St \cdot M^2, \end{aligned}$$

where

$$St = \frac{wL}{U} \text{ and } M = \frac{U}{c},$$

where St is the Strouhal number and M is the Mach number. The first condition for neglecting $D\rho/Dt$ is that:

$$St \cdot M \ll 1.$$

This implies that the Strouhal number and the Mach are both very small. In that case, from the definitions of Strouhal number and Mach number,

$$w \ll \frac{U}{L} \text{ and } U \ll c.$$

So, the characteristic time of pulsation $t_0 \approx 1/w$ must be much longer than the characteristic time than a fluid particle resides in the system $t_r \approx L/U$. In addition, low velocities compared with the speed of sound also makes the Mach number small collaborating to the accuracy of the flow being considered as incompressible. Typical values for the Strouhal and the Mach numbers are, respectively, of the order 10^{-3} and 10^{-6} .

3.5.3 Compressibility of large scale flows. The second term on the right hand side of (3.5.1) represents the transport of pressure by the fluid. Since

$$\nabla p \approx O\left(\frac{mg}{L^3}\right),$$

it follows that

$$\frac{\mathbf{u} \cdot \nabla p}{\rho c^2} \approx \frac{O\left(\frac{gU}{c^2}\right)}{O\left(\frac{U}{L}\right)} = \frac{gL}{c^2} = \frac{M^2}{Fr}, \quad (3.5.4)$$

where

$$Fr = \frac{U^2}{gL}, \quad (3.5.5)$$

is the Froude number and M is the Mach number. Therefore, to neglect $D\rho/Dt$,

$$\frac{M^2}{Fr} \ll 1, \quad (3.5.6)$$

which is the same as stating that

$$\frac{gL}{c^2} \ll 1. \quad (3.5.7)$$

The length c^2/g is called the scale height and it is the characteristic scale in the system over which there are appreciable in pressure. For (3.5.7) to be of order of unity, $L \approx c^2/g$, which implies that for scales larger than the scale height, the gradient of pressure causes hydrostatically large gradients of density. The larger the system, the less accurate the incompressibility assumption. A typical value for the scale height is approximately equal to 200 km while the maximum depth of an ocean is approximately equal to 11 km (Pacific ocean). Also, a typical value for the Froude number is of the order 10^{-9} .

3.5.4 Compressibility caused by the temperature gradient. The fourth term on the right hand side of (3.5.1) represents an irreversible source of temperature (entropy) due to heat transfer. Scaling of this term leads to the following:

$$\frac{\alpha k \frac{\nabla^2 T}{c_p \rho}}{\nabla \cdot \mathbf{u}} \approx \frac{O\left(\frac{\alpha k \Delta T}{c_p \rho L^2}\right)}{O\left(\frac{U}{L}\right)},$$

with

$$Re = \frac{\rho U L}{\mu} \text{ and } Pr = \frac{\mu c_p}{k},$$

where Re is the Reynolds number and Pr is the Prandtl number. For neglecting $D\rho/Dt$

$$\frac{\alpha \Delta T}{Re Pr} \ll 1. \quad (3.5.8)$$

Therefore, the critical temperature difference should be

$$\Delta T \sim \frac{Re Pr}{\alpha}, \quad (3.5.9)$$

which implies that the flow must be fast to get a large Reynolds number, the substance should not transfer heat very well for having a high Prandtl number and the temperature difference must not be very large. Typical values for the Prandtl number for the heat transfer process and for the Reynolds number are, respectively, in the range of 10^{15} and 10^3 .

Terms that still need to be scaled are the third term which will mostly also become negligible for small Mach numbers, the term containing the sources and sinks of heat (Q), and the terms for the transport of salinity. It is expected that the Peclet number will be important in this scaling.

3.5.5 Compressibility caused by the salinity gradient. The last term in the right hand side of equation (3.5.1) represents the transport in salinity. Recall that

$$\beta \approx O\left(\frac{\rho}{S}\right) \text{ and } \nabla^2 S \approx O\left(\frac{S}{L^2}\right). \quad (3.5.10)$$

Using the result from (3.5.10), scaling of the term gives us the following result:

$$\frac{\beta k_S \nabla^2 S}{\rho \nabla \cdot \mathbf{u}} \approx \frac{O\left(\frac{k_S}{L^2}\right)}{O\left(\frac{U}{L}\right)} = O\left(\frac{k_S}{UL}\right) = \frac{1}{Pe}.$$

For neglecting $D\rho/Dt$, the following condition must be held

$$\frac{1}{Pe} \ll 1,$$

which implies that the diffusion term is low compared to the convection term in terms of salt transport with the flow. A typical value for the Peclet number is of the order 10^{12} .

4. Conclusions

The flow is incompressible if the isothermal compressibility coefficient is equal to zero. But again, the density might not be constant since it varies with temperature and salinity in the ocean.

In this essay, we studied the flow incompressibility for the ocean case when the material derivative of the density is negligible compared to the divergence of the velocity of the flow.

Through this study, the conditions for the incompressibility have been revisited. The speed of sound has to be very large compared to the characteristic velocity of the fluid, also low diffusions of heat and dissolved solids are required.

Remark that some of scaling may only be applicable to the ocean and might need further investigation.

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