Stability Analysis of Selected Epidemic Models

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Abstract

In this Project our main interest is to study the stability of SIS, SIR, SIRS epidemic models. Lyapunov's indirect method uses linearization about an equilibrium point to determine local stability. If the eigenvalues of the Jacobian matrix are all negative or have negative real parts then the equilibrium point is locally asymptotically stable. However, in situations where the eigenvalues are difficult to compute, we used the Routh-Hurwitz conditions or a Corollary of Gershgorin's circle theorem.

Establishing global stability of the equilibria requires an ability to define an appropriate Lyapunov function. In the Project we have used a combination of quadratic, logarithmic and composite Lyapunov functions to establish global stability in various epidemiological models. Finally, we also investigated stability of a mathematical model for the spread of the malaria.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Belthasara ASSAN, 22 May 2014

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1. Introduction

One of the challenges in mathematical epidemiology is how to establish local and global stability of the equilibria. Local stability is establish by Lyapunov's indirect method, while global stability uses Lyuapunov functions.

Lyapunov's indirect method uses linearization about an equilibrium point to determine local stability. If the eigenvalues of the Jacobian matrix are all negative or have negative real parts, then the equilibrium point is locally asymptotically stable. However, finding the eigenvalues can sometimes be very daunting. In such situations either the Routh-Hurwitz conditions or a Corollary of Gershgorin's circle theorem can be used.

Establishing global stability of the equilibria requires an ability to define an appropriate Lyapunov function. In the Project we have successfully used a combination of quadratic, logarithmic and composite Lyapunov functions to establish global stability in various epidemiological models.

Keywords: Epidemic models, disease-free equilibrium, endemic equilibrium, Lyapunov's indirect, Local stability, Routh-Hurwitz criteria, Gershgorin's Circle theorem, global stability, Lyapunov functions, and Lyapunov's direct methods.

1.1 Stability of Nonlinear Dynamical Systems

1.1.1 Introduction. In this chapter we review the stability of critical points of nonlinear dynamical systems. We will then apply these properties to analyze the stability properties of the equilibrium points of typical epidemiological models.

We consider an **autonomous system** of differential equations of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1.1.1}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$, $\mathbf{f}(\mathbf{x}) = (f_1(x_1, x_2, \cdots, x_n), f_2(x_1, x_2, \cdots, x_n), \cdots, f_n(x_1, x_2, \cdots, x_n))^T$, and $\mathbf{f}$ does not depend explicitly on t.

1.1.2 Definition (Equilibrium Point)**.** A point $\mathbf{x}^* \in \mathbb{R}^n$ is called an **equilibrium point** of (1.1.1) if

$$\mathbf{f}(\mathbf{x}^*) = 0$$

Equilibrium points of dynamical systems represent *constant solutions* of the system and therefore give an indication of the long-term behaviour of the system Allen (2007).

1.1.3 Local stability.

1.1.4 Definition (Stability in the sense of Lyapunov)**.** An equilibrium point $\mathbf{x}^*$, of the first-order system in (1.1.1) is said to be **locally stable** provided that, if the initial values $\mathbf{x_0}$ is sufficiently close to $\mathbf{x}^*$ then the solution $\mathbf{x(t)}$ remains close to $\mathbf{x}^*$ for all $t \geq 0$.

More precisely, the equilibrium point $\mathbf{x}^*$ is **stable** if, for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\|\mathbf{x_0} - \mathbf{x}^*\| < \delta \Rightarrow \|\mathbf{x(t)} - \mathbf{x}^*\| < \epsilon.$$

The critical point $\mathbf{x} = \mathbf{x}^*$ is called **unstable** if it is not stable Benyah (2013).

Intuitively, we say an equilibrium point is **locally stable** if all solutions which start near $\mathbf{x}^*$ (meaning that all initial points in a neighbourhood of $\mathbf{x}^*$) remain close $\mathbf{x}^*$ for all time Benyah (2013).

Figures (1.1) and (1.2) below show respectively, a *stable* and an *unstable* equilibrium point in $\mathbb{R}^2$.

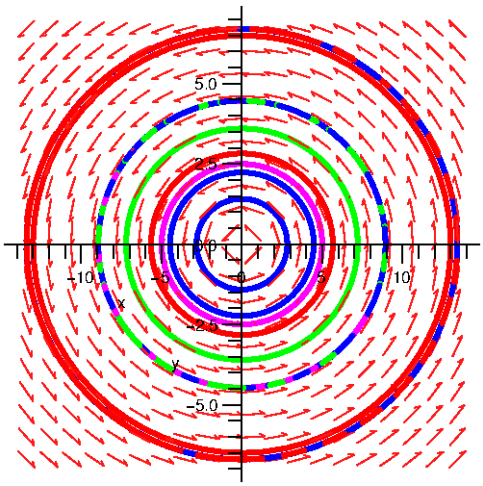

Figure 1.1: A stable point at $(0, 0)$ in $\mathbb{R}^2$

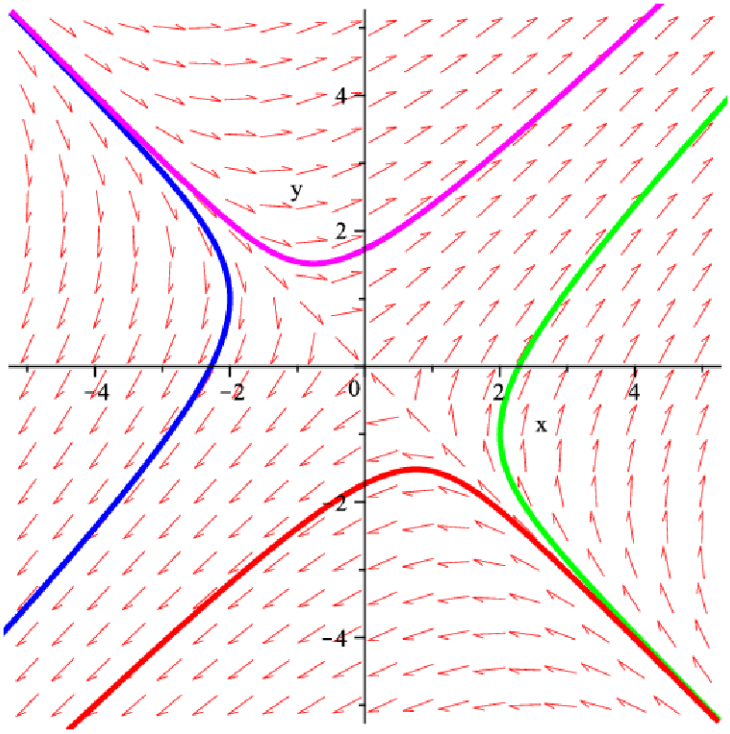

Figure 1.2: An unstable (saddle) point at $(0, 0)$ in $\mathbb{R}^2$

1.1.5 Remark. The stability criterion given above does not require that trajectories starting close to the $\mathbf{x}^*$ tend to $\mathbf{x}^*$ eventually Benyah (2013).

1.1.6 Definition (Asymptotic Stability)**.** An equilibrium point $\mathbf{x}^*$ is said to be **locally asymptotically stable** if $\mathbf{x}^*$ is locally stable and furthermore, all solutions starting near $\mathbf{x}^*$ tend towards $\mathbf{x}^*$ as $t \to \infty$.

That is, there exists $\delta > 0$ such that

$$\|\mathbf{x_0} - \mathbf{x}^*\| \quad < \quad \delta \quad \Rightarrow \quad \lim_{\mathbf{t}\to\infty} \mathbf{x}(\mathbf{t}) = \mathbf{x}^* \tag{1.1.2}$$

Figures (1.1), (1.3) and (1.2) below show respectively a *stable*, *asymptotically stable* and an *unstable* (saddle) point $(0,\, 0)$ in $\mathbb{R}^2$ Allen (2007).

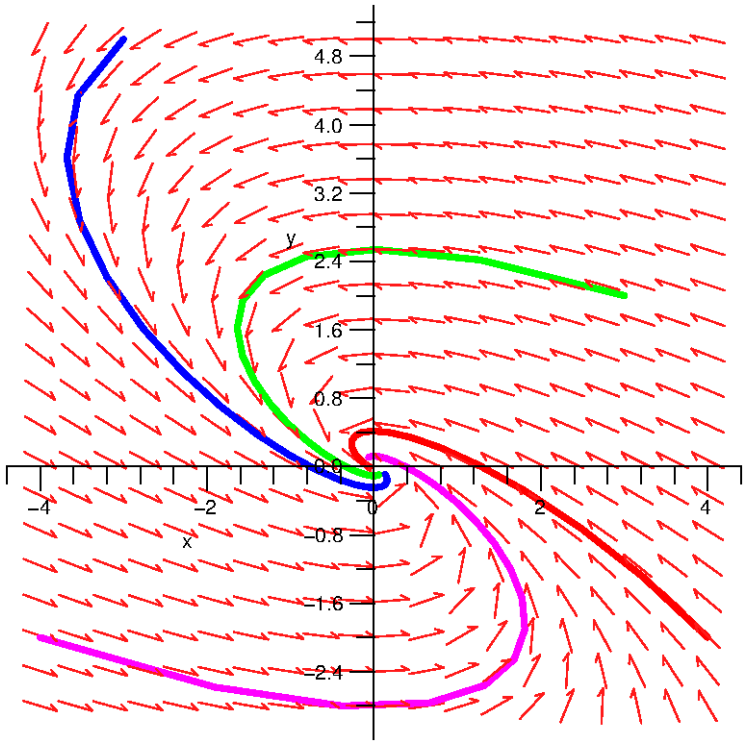

Figure 1.3: An asymptotically stable centre at $(0,0)$ in $\mathcal{R}^2$

For many nonlinear systems explicit solutions $\mathbf{x}(\mathbf{t})$, are rarely available. Without such solutions system behavior as required by (1.1.4) and (1.1.6) cannot be directly evaluated. Lyapunov (1892) recognized this difficulty and developed two methods (*indirect* and *direct* methods) for assessing (1.1.4) and (1.1.6). The *indirect method* involves linearizing $\mathbf{f}$ at $\mathbf{x}^*$ Allen (2007).

1.1.7 Local Stability: Lyapunov's Indirect Method.

1.1.8 Theorem (Lyapunovs indirect method)**.** *Let $\mathbf{x}^*$ be an equilibrium point of the system of differential equations*

$$\frac{\mathrm{d}\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(\mathbf{0}) = \mathbf{x_o}$$

where $\mathbf{f} : D \to \mathbb{R}^n$ is continuously differentiable and D is a neighbourhood of the $\mathbf{x}^*$. Let the Jacobian matrix $\mathbf{A}$ at $\mathbf{x}^*$ be:

$$Let \quad \mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}|_{\mathbf{x}=\mathbf{x}^*}$$

such that the linearized system is

$$\frac{\mathrm{d}\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}, \quad \mathbf{u} = \mathbf{x} - \mathbf{x}^*$$

Then

(a) $\mathbf{x}^*$ *is asymptotically stable if* $Re(\lambda_i(\mathbf{A})) < 0$ *for* $i = 1, \cdots, n$.

(b) $\mathbf{x}^*$ *is unstable if* $Re(\lambda_i(\mathbf{A})) > 0$ *for at least one* i Ramakrishnan (2003).

where $Re(\lambda_i(\mathbf{A}))$ designates the real part of the i-th eigenvalue of $\mathbf{A}$. Since $\mathbf{A}$ is only defined at $\mathbf{x}^*$, stability determined by the indirect method is restricted to small neighbourhoods of $\mathbf{x}^*$. For this reason, it is called **local stability** Allen (2007).

1.1.9 Local Stability of 2-dimensional Nonlinear Dynamical Systems. We illustrate Lyapunov's indirect method by considering the first-order autonomous system of differential equations in two variables x and y of the form

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned} \tag{1.1.3}$$

We recall that an **equilibrium point** (**steady state**, **fixed point** or **critical point**) (x^*, y^*) of (1.1.3) is a point satisfying $f(x^*, y^*) = g(x^*, y^*) = 0$.

We assume that the functions f and g are continuously differentiable in a neighbourhood of (x^*, y^*). Then the Taylor Formula for f and g about the critical point (x^*, y^*) gives

$$\begin{aligned} \frac{dx}{dt} &= f(x^*, y^*) + f_x(x^*, y^*)(x - x^*) + f_y(x^*, y^*)(y - y^*) + F(x - x^*, y - y^*) \\ &= f_x(x^*, y^*)(x - x^*) + f_y(x^*, y^*)(y - y^*) + F(x - x^*, y - y^*) \\ \frac{dy}{dt} &= g(x^*, y^*) + g_x(x^*, y^*)(x - x^*) + g_y(x^*, y^*)(y - y^*) + G(x - x^*, y - y^*) \\ &= g_x(x_*, y_*)(x - x_*) + g_y(x_*, y_*)(y - y^*) + G(x - x^*, y - y^*) \end{aligned} \tag{1.1.4}$$

since $f(x^*, y^*) = g(x^*, y^*) = 0$. $F(x - x^*, y - y^*)$ and $G(x - x^*, y - y^*)$ represent the higher-order terms.

Let $u = x - x^*, v = y - y^*$ so that $\frac{dx}{dt} = \frac{du}{dt}$, and $\frac{dy}{dt} = \frac{dv}{dt}$. Then, in matrix form the system in (1.1.4) becomes

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} F(u, v) \\ G(u, v) \end{bmatrix} \tag{1.1.5}$$

where $a = f_x(x^*, y^*)$, $b = f_y(x^*, y^*)$, $c = g_x(x^*, y^*)$, $d = g_y(x^*, y^*)$.

The matrix $\mathbf{J} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called the **Jacobian matrix** of the system (1.1.3) evaluated at the critical point (x^*, y^*). We assume that the Jacobian matrix is not singular at the critical point (x^*, y^*); such a critical point is said to be *"isolated"* From Theorem (1.1.8) the nonlinear system (1.1.3) is locally asymptotically stable if the eigenvalues of the Jacobian matrix J, are negative or have negative real parts if they are complex. For a 2 system, the eigenvalues are the roots of the characteristic equation can easily be solved. However, for an $n \times n$ system ($n \geq 3$) the solution of the characteristic equation is non-trivial. The Routh-Hurwitz conditions give necessary and sufficient conditions for all roots of the characteristics polynomial to have negative parts, thus implying local asymptotic stability Allen (2007).

1.1.10 The Routh-Hurwitz Conditions.

1.1.11 Theorem (Routh-Hurwitz Criteria)**.** Given the polynomial,

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n$$

where the coefficients a_i are real constants, $i = 1, ..., n$, define the n Hurwitz matrices using the coefficients a_i of the characteristics polynomial:

$$H_1 = (a_1), \quad H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}, \quad H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}, \quad H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

where $a_j = 0$ if $j > n$. All of the roots of the polynomial $P(\lambda)$ are negative or have negative real parts, if and only if the determinants of all Hurwitz matrices, $det(H_j) > 0,\ j = i, 2, \cdots, n$ are positive Allen (2007).

1.1.12 Example. For $n = 2$, the characteristic equation is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0$$

and the corresponding Hurwitz matrix is $H_2 = \begin{bmatrix} a_1 & 1 \\ 0 & a_2 \end{bmatrix},\ det(H_2) = a_1a_2 > 0.$

The Routh- Hurwitz criteria are $a_1 > 0,\ a_2 > 0.$

1.1.13 Example. for $n = 3$, the characteristic equation is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

and the corresponding Hurwitz matrix is $H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{bmatrix}, det(H_3) = a_3(a_1a_2 - a_3) > 0,$

The Routh-Hurwitz criteria implies that $a_3 > 0,\ a_1a_2 > a_3.$

1.1.14 Theorem. (Routh-Hurwitz conditions for $n = 2$): Let $\mathbf{A}$ be the coefficient matrix of the linear system

$$\frac{\mathrm{d}\mathbf{x}}{dt} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then critical point $\mathbf{x}^* = (0, 0)$

(a) is stable if $trace(\mathbf{A}) < 0$ and $det(\mathbf{A}) > 0$

(b) is unstable if either $trace(\mathbf{A}) > 0$ or $det(\mathbf{A}) > 0$

Proof. Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The eigenvalues of $\mathbf{A}$ are the roots of the characteristics polynomial

$$P(\lambda) = \lambda^2 - (a + d)\lambda - (bc - ad) = 0.$$

The roots are

$$\lambda_1 = \frac{(a+d)}{2} - \sqrt{\left(\frac{a+d}{2}\right)^2 - (ad - bc)}, \quad \lambda_2 = \frac{(a+d)}{2} + \sqrt{\left(\frac{a+d}{2}\right)^2 - (ad - bc)}$$

If $(a+d) < 0$ and $(ad-bc) > 0$ then, λ_1 and λ_2 are both negative if they are real, or have negative real parts if they are complex.

Note that $(a+d)$ is the trace of $\mathbf{A}$ and $(ad-bc)$ is the determinant of $\mathbf{A}$. □

The Routh-Hurwitz conditions, as simple as they look, can sometimes be very difficult to apply, if $n \geq 3$.

Another method for determining local stability is the following Corollary of Gershgorin's Circle Theorem.

1.1.15 Corollary of Gershgorin's Circle Theorem.
1.1.16 Corollary. Let A be an $n \times n$ matrix with real entries. If the diagonal elements of A satisfy

$$a_{ii} < -r_i, \quad \text{where} \quad r_i = \sum_{j=1, j\neq i}^{n} \mid a_{ij} \mid$$

for $i = 1, ..., n$, then the eigenvalues of A are negative or have negative real parts Allen (2007).

1.2 Global Stability: Lyapunov's direct method

1.2.1 Some Definitions and Theorems.
1.2.2 Definition. (Positive definite functions) A continuously differentiable function $V : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ is said to be

(a) **positive definite** in a region U of R^n that contains the origin if

(i) $V(0) = 0$ and

(ii) $V(\mathbf{x}) > 0$, for $\mathbf{x} \in U$ and $\mathbf{x} \neq 0$.

(b) **negative definite** in a region U of R^n that contains the origin if

(i) $V(0) = 0$ and

(ii) $V(\mathbf{x}) < 0$

(c) $V(\mathbf{x})$ is said to be

(i) **positive semi-definite** if $V(\mathbf{x}) \geq 0$ for $\mathbf{x} \in U$ and $\mathbf{x} \neq 0$.

(ii) **positive semi-definite** if $V(\mathbf{x}) \geq 0$ for $\mathbf{x} \in U$ and $\mathbf{x} \neq 0$.

Allen (2007)

1.2.3 The Concept of Lyapunov's Stability.
1.2.4 Theorem (Lyapunov Stability). Let $\mathbf{x} = \mathbf{0}$ be an equilibrium point for a system described by: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where $\mathbf{f} : U \to \mathbb{R}^n$ is a locally Lipschitz and $U \subset \mathbb{R}^n$ a domain that contains the origin. Let $V : U \to \mathbb{R}$ be a continuously differentiable, positive definite function in U.

(a) If $\dot{V}(\mathbf{x}) = [\frac{\partial V}{\partial \mathbf{x}}]^T \mathbf{f}(\mathbf{x}) \leq 0$, then $\mathbf{x} = \mathbf{0}$ is a stable equilibrium point.

(b) If $\dot{V}(\mathbf{x}) < 0$, then $\mathbf{x} = \mathbf{0}$ is an asymptotically stable equilibrium point.

In both cases above V is called a **Lyapunov function**. Moreover, if the conditions hold for all $\mathbf{x} \in \mathbb{R}^n$ and $\|\mathbf{x}\| \to \infty$ implies that $V(\mathbf{x}) \to \infty$, then $\mathbf{x} = \mathbf{0}$ is **asymptotically stable** in (a) and **globally asymptotically stable** in (b) Allen (2007).

The most commonly used Lyapunov candidate functions are:

(1) Quadratic functions: $V(x) = \frac{c_1}{2}(\mathbf{x}_1 - \mathbf{x}_1^*)^2 + \frac{c_2}{2}(\mathbf{x}_2 - \mathbf{x}_2^*)^2 + ... + \frac{c_n}{2}(\mathbf{x}_n - \mathbf{x}_n^*)^2$ where the equilibrium points are given by $\mathbf{x}_1^*, \mathbf{x}_2^*, ...$

(2) Logarithmic function: $V(x) = c_1(\mathbf{x}_1 - \mathbf{x}_1^* - \mathbf{x}_1^* ln(\frac{\mathbf{x}_1}{\mathbf{x}_1^*})) + c_2(\mathbf{x}_2 - \mathbf{x}_2^* - \mathbf{x}_2^* ln(\frac{\mathbf{x}_2}{\mathbf{x}_2^*})) + ... + c_n(\mathbf{x}_n - \mathbf{x}_n^* - \mathbf{x}_n^* ln(\frac{\mathbf{x}_n}{\mathbf{x}_n^*})) + ...$.

(3) Composite function: $V(x_1, x_2, ..., x_n) = \frac{c}{2}\left[\sum_{i=1}^{n}(\mathbf{x}_i - \mathbf{x}_i^*)\right]^2$ Leon (2009).

2. The SIS, SIR And SIRS Epidemic Model

2.1 SIS Epidemic Model

The SIS epidemic Model is used to study diseases for which infected individuals return to the susceptible class on recovery. because the disease confers no immunity against re-infection. Examples of such diseases include the common cold and sexually-transmitted diseases Accessed April 2014.

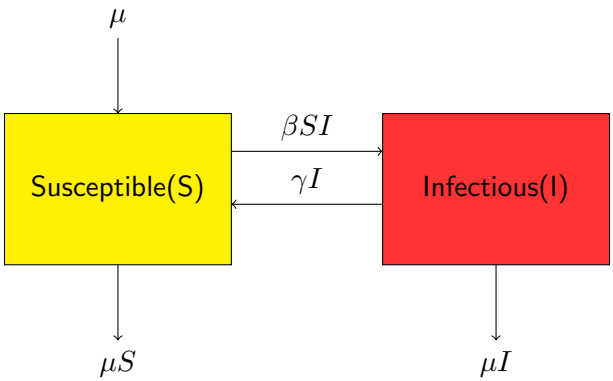

Figure 2.1: Compartment of the SIS model

2.1.1 An SIS epidermic model.

$$\begin{array}{ll} \frac{\mathrm{dS}}{\mathrm{dt}} & = \mu - \beta SI + \gamma I - \mu S, \qquad S(0) = S_o \\ \frac{\mathrm{dI}}{\mathrm{dt}} & = \beta SI - \gamma I - \mu I, \qquad I(0) = I_o \end{array} \tag{2.1.1}$$

All parameter considered here are positive constant. γ is the rate at which infectious individuals move back to the susceptible class. μ is the natural death rate assume to be equal to the birth rate; β, is the transmission rate.

2.1.2 Equilibrium Points, and Stability. The system has a unique disease free equilibrium point $E^o = (1, 0)$ and endemic steady state $E^* = \left(\frac{\gamma+\mu}{\beta}, \frac{\beta-(\gamma+\mu)}{\beta}\right)$

2.1.3 Basic Reproduction Number (2.1.1)**.**

$$\mathcal{R}_0 = \frac{\beta}{\gamma + \mu}$$

The threshold quantity $\mathcal{R}_0$ is often called the **basic reproduction number**. It is represents the average number of secondary infections that occur when one infectious individual is introduced into a completely susceptible population.

2.2 Stability Analysis

Let

$$\Omega = \{(S, I) \in \mathbb{R}^2_+ : S \geq 0, I \geq 0, S + I \leq 1\}$$

2.2.1 Theorem. *The disease free equilibrium* (E^o) *is locally asymptotically stable in* Ω *if* $\mathcal{R}_0 \leq 1$ *and unstable if* $\mathcal{R}_0 > 1$.

Proof. The Jacobian matrix of the system is given by

$$J = \begin{pmatrix} -\beta I - \mu & -\beta S + \gamma \\ \beta I & \beta S - \gamma - \mu \end{pmatrix}$$

Evaluating at the disease free equilibrium point gives

$$J_{(1,0)} = \begin{pmatrix} -\mu & -\beta + \gamma \\ 0 & \beta - \gamma - \mu \end{pmatrix} \text{ The eigenvalues are } \lambda_1 = -\mu, \quad \lambda_2 = \beta - \gamma - \mu.$$

Since $\lambda_1 < 0$, the disease free equilibrium point (E^0) of (2.1.1) is

1. locally asymptotically stable if $\lambda_2 < 0$. That is, if $\beta - \gamma - \mu < 0$, or $\frac{\beta}{\gamma+\mu} < 1$, and
2. unstable if $\lambda_2 > 0$. That is, if $\frac{\beta}{\gamma+\mu} > 1$.

From (1) and (2) we deduce that $\mathcal{R}_0 = \frac{\beta}{\gamma+\mu}$. □

2.2.2 Local stability of the endemic equilibrium. We express the endemic equilibrium in terms of $\mathcal{R}_0$ as $E^* = \left(\frac{1}{\mathcal{R}_0}, 1 - \frac{1}{\mathcal{R}_0}\right)$

2.2.3 Theorem. *The endemic equilibrium* E^* *is locally asymptotically stable in* Ω *if* $\mathcal{R}_0 > 1$ *and unstable if* $\mathcal{R}_0 \leq 1$.

Proof. Evaluating the Jacobian matrix at the endemic equilibrium gives

$$J_{(S^*,I^*)} = \begin{pmatrix} -\beta(1 - \frac{1}{R_o}) - \mu & -\mu \\ \beta(1 - \frac{1}{R_o}) & 0 \end{pmatrix}$$

$$tr(E^*) = -\beta(1 - \frac{1}{\mathcal{R}_0}) - \mu \quad \text{and} \quad det(E^*) = \beta\mu(1 - \frac{1}{\mathcal{R}_0})$$

The trace is negative and the determinant is positive if $\mathcal{R}_0 > 1$. Hence the endemic equilibrium point is locally asymptotically stable if $\mathcal{R}_0 > 1$ and unstable if $\mathcal{R}_0 \leq 1$. □

2.2.4 Global Stability of disease-free equilibrium.

2.2.5 Theorem. *The disease-free equilibrium* E^o *of* (2.1.1) *is globally asymptotically stable in* Ω *if* $\mathcal{R}_0 \leqslant 1$.

Proof. Define $V : \{(S, I) \in \Omega : S > 0\} \to \mathbb{R}$ by

$$V(S, I) = I.$$

Then V is $\mathbf{C}^1$ on the interior of Ω, E^o is global minimum of V on Ω, and $V(S^o, I^o) = 0$. The time derivative of V computed along solutions of (2.1.1) is

$$\begin{aligned} \dot{V} &= \frac{\mathrm{dI}}{dt}, \\ &= [\beta SI - (\gamma + \mu)I] \\ &= (\gamma + \mu)(\mathcal{R}_0 S - 1)I \\ &\leq 0, \quad if \mathcal{R}_0 \leq 1 \end{aligned}$$

Now $\dot{V}(S,I) = 0 \iff S = 1$, if $\mathcal{R}_0 < 1$. And $\dot{V}(S,I) = 0 \iff I = 0, \quad if \quad \mathcal{R}_0 = 1$ Hence, the disease-free equilibrium is globally asymptotically stable if $\mathcal{R}_0 \leq 1$. □

2.2.6 Global Stability of Endemic equilibrium.

2.2.7 Theorem. *If $\mathcal{R}_0 > 1$ then the unique endemic equilibrium E^* of the equations (2.1.1) is globally asymptotically stable on the interior of Ω*

Proof. Define $V : \{(S,I) \in \Omega : S, I > 0\} \to \mathbb{R}$ by

$$V = \frac{1}{2}[(S - S^*) + (I - I^*)]^2 + k(I - I^* - I^* ln \frac{I}{I^*})$$

Then,

$$\dot{V} = [(S - S^*) + (I - I^*)]\frac{\mathrm{d(S+I)}}{\mathrm{dt}} + k\frac{(I - I^*)}{I}\frac{\mathrm{dI}}{\mathrm{dt}}$$
$$\dot{V} = [(S - S^*) + (I - I^*)](\mu - \mu S - \mu I) + k\frac{(I - I^*)}{I}(\beta SI - \gamma I - \mu I)$$
$$\dot{V} = [(S - S^*) + (I - I^*)](-\mu[(S - S^*) + (I - I^*)]) + k\beta(I - I^*)(S - S^*)$$
$$\dot{V} = -\mu(S - S^*)^2 - \mu(I - I^*)^2 - 2\mu(S - S^*)(I - I^*) + k\beta(I - I^*)(S - S^*)$$
$$\dot{V} = -\mu(S - S^*)^2 - \mu(I - I^*)^2 + (K\beta - 2\mu)(S - S^*)(I - I^*) \qquad where \quad k = \frac{2\mu}{\beta}$$
$$\dot{V} = -\mu(S - S^*)^2 - \mu(I - I^*)^2$$

and $\dot{V} = 0 \iff S = S^*, I = I^*$

From theorem, this implies that for all initial conditions (S_o, I_o) solution converges to the endemic equilibrium (E^*) if $\mathcal{R}_0 > 1$. Hence (S^*, I^*) is globally asymptotically stable. □

2.3 SIR Epidemic Model

The SIR model is used to study the dynamics of diseases where recovered individuals acquire immunity against re-infection. Such diseases include *measles*, *chicken pox* or *rubella*.

According to Kermack and Mc. Kendrick, 1927 (Kermack and McKendric (1927))

- "The population is divided into disjoint classes that change with time:
  - a **Susceptible class:** individuals who can incur the disease but are not yet infective.
  - b **Infective class:** individuals who are transmitting the disease to others.
  - c **Removed class:** individuals who are removed form the susceptible-infective interaction by immunity or isolation
- The population has a constant size N
- The death and birth rate is denoted by μ
- expected length of time an infection remains infectious $\frac{1}{\mu}$
- The average number of contacts per infective per day which result in infection is denoted by β

- The average fraction of susceptible infected by the infective class is βSI
- Individuals recover from the infective class at a per capita constant rate γ" Esteva (Accessed April 2014).

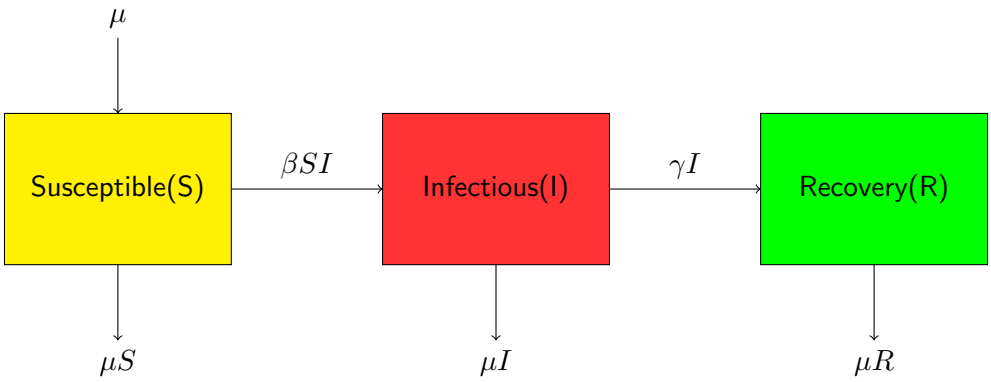

Figure 2.2: Compartment of the SIR model

The dynamics of an SIR epidemic model have the following form:

$$\begin{aligned} \frac{\mathrm{dS}}{\mathrm{dt}} &= \mu - \beta SI - \mu S, \\ \frac{\mathrm{dI}}{\mathrm{dt}} &= \beta SI - \gamma I - \mu I, \\ \frac{\mathrm{dR}}{\mathrm{dt}} &= \gamma I - \mu R \end{aligned} \tag{2.3.1}$$

From $S(t) + I(t) + R(t) = 1$, we have $R(t) = 1 - (S(t) + I(t))$. It is therefore enough to consider only the first two equations.

2.3.1 Equilibrium point. The system has a disease-free $E^o = (1, 0)$ and an endemic equilibrium point $E^* = \left(\frac{\gamma+\mu}{\beta}, \frac{\mu(\beta-(\gamma+\mu))}{\beta(\gamma+\mu)}\right)$

2.4 Stability Analysis

2.4.1 Basic Reproduction Number. The basic reproduction number is $R_0 = \frac{\beta}{\gamma+\mu}$. Expressing the endemic equilibrium in $\mathcal{R}_0$ gives $E^* = \left(\frac{1}{\mathcal{R}_0}, \frac{\mu(\mathcal{R}_0-1)}{\beta}\right)$

2.4.2 Local Stability Analysis.

$$\Omega = \{(S, I) \in \mathbb{R}^2_+ : S \geq 0, I \geq 0, \quad S + I \leq 1\}$$

The following theorems give conditions for the local stability of the disease-fee and endemic equilibria.
2.4.3 Theorem. *The disease-free equilibrium E^o of (2.3.1) is locally asymptotically stable in Ω if $\mathcal{R}_0 \leq 1$ and unstable if $\mathcal{R}_0 > 1$.*
2.4.4 Theorem. *The endemic equilibrium E^* of (2.3.1) is locally asymptotically stable in Ω if $\mathcal{R}_0 > 1$ and unstable $\mathcal{R}_0 < 1$.*

The proofs of the above theorems follow the same pattern as that of the SIS model given above.

2.4.5 Global Stability of disease-free equilibrium.

2.4.6 Theorem. *The disease-free equilibrium E^o of (2.3.1) is globally asymptotically stable in Ω if $\mathcal{R}_0 \leqslant 1$*

.

Proof. We define a composite Lyapunov candidate function $V : \{(S, I) \in \Omega : S > 0\} \to \mathbb{R}$ by $V(S, I) = \frac{1}{2}[(S - S^o) + I]^2 + kI$.

$$\begin{aligned}
\dot{V} &= [(S - S^o) + I]\frac{\mathrm{d(S + I)}}{dt} + k\frac{\mathrm{dI}}{dt}, \\
&= [(S - S^o) + I](\mu - \mu(S + I) - \gamma I) + k[\beta SI - (\gamma + \mu)I] \\
\dot{V} &= [(S - S^o) + I](\mu S^o - \mu(S + I) - \gamma I) + k[\beta S^o - (\gamma + \mu)]I \\
&= [(S - S^o) + I](-\mu(S - S^o) - (\mu + \gamma)I) + k[\beta S^o - (\gamma + \mu)]I \\
&= -\mu(S - S^o)^2 - (\mu + \gamma)I^2 - I(S - S^o)(\gamma + 2\mu) - k\beta\left[\frac{(\gamma + \mu)}{\beta} - S^0\right] I
\end{aligned}$$

Expressing it in the basic reproduction number:

$$= -\mu(S - S^o)^2 - (\mu + \gamma)I^2 - (k\beta - (\gamma + 2\mu))\frac{(\gamma + \mu)}{\beta}[\mathcal{R}_0 - 1]\, I \quad k = \frac{\gamma + 2\mu}{\beta}$$

$$\leqslant 0 \quad if \quad R_o \leqslant 1$$

Now$\dot{V}(S, I) < 0 \iff S^o = 1, \quad if \quad R_o < 1. \quad and \quad \dot{V}(S, I) = 0 \iff I = 0. \quad if \quad R_o = 1$

Since all the model parameters are positive and variables are non-negative, it follows that $\dot{V}(S, I) \leqslant 0$ for $\mathcal{R}_0 \leqslant 1$ with $\dot{V}(S, I) = 0$. Hence, V is a Lyapunov function on Ω. □

2.4.7 Global stability of the endemic equilibrium.

2.4.8 Theorem. *If $\mathcal{R}_0 > 1$ then the endemic equilibrium E^* of the model above is globally asymptotically stable in the interior of Ω.*

Proof. Define a logarithmic Lyapunov candidate function $V : \{(S, I) \in \Omega : S, I > 0\} \to \mathbb{R}$

$$V(S, I) = c_1[S - S^* - S^* ln\left(\frac{S}{S^*}\right)] + c_2[I - I^* - I^* ln\left(\frac{I}{I^*}\right)]$$

$$\text{for some} \quad c_1, c_2 > 0, \quad \text{differentiating} \quad V \quad gives$$

$$\dot{V} = c_1(S - S^*)(-\beta I - \mu + \frac{\mu}{S}) + c_2(I - I^*)(\beta S - (\gamma + \mu))$$

$$\dot{V} = c_1(S - S^*)(-\beta I + \beta I^* - \frac{\mu}{S^*} + \frac{\mu}{S}) + c_2(I - I^*)(\beta S - \beta S^*)$$

$$\dot{V} = \beta c_1(S - S^*)(-I + I^* - \mu\frac{1}{S^*} + \frac{1}{S}) + \beta c_2(I - I^*)(S - S^*)$$

$$\dot{V} = \beta c_1(S - S^*)(-I + I^* - \mu\frac{S - S^*}{SS^*}) + \beta c_2(I - I^*)(S - S^*)$$

$$\dot{V} = -\beta c_1(S - S^*)(I - I^*) + \beta c_2(I - I^*)(S - S^*) - \mu\frac{(S - S^*)^2}{SS^*}$$

$$\dot{V} = \beta(c_2 - c_1)(S - S^*)(I - I^*) - c_1\mu\frac{(S - S^*)^2}{SS^*} \quad Choosing \quad c_1 = c_2 = 1$$

$$\dot{V} = -\mu\frac{(S - S^*)^2}{SS^*} \leqslant 0 \quad and \quad \dot{V} = 0 \iff S = S^*$$

Hence the endemic equilibrium is globally asymptotically stable in the interior of Ω, when it exist. □

2.5 SIRS Model

The SIRS epidemic model is used to study the dynamics of diseases, where individuals who have recovered from the disease are not permanently immuned against re-infection. After a period of time they become susceptible and therefore can become infected again. Examples include the *common cold*, and *malaria*.

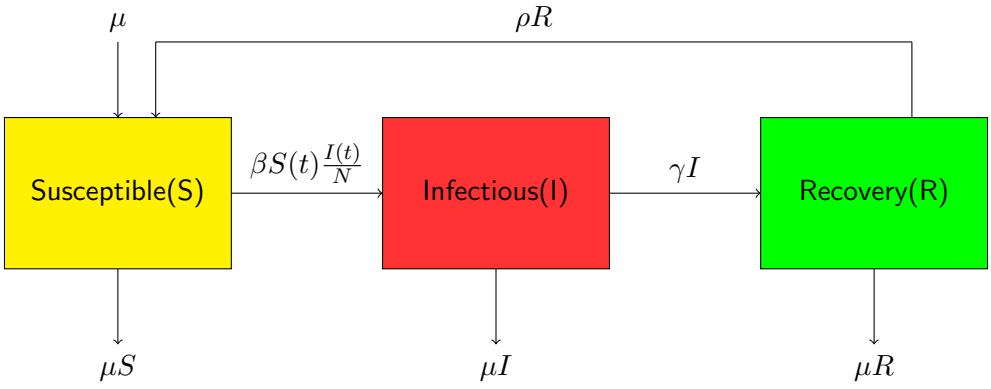

Figure 2.3: Compartment of the SIRS model

2.5.1 SIRS Model. The SIRS model using *standard incidence* is given by

$$\begin{aligned} \frac{dS}{dt} &= \mu N - \beta S(t)\frac{I(t)}{N} - \mu S(t) + \rho R(t) \\ \frac{dI}{dt} &= \beta S(t)\frac{I(t)}{N} - \mu I(t) - \gamma I(t) \\ \frac{dR}{dt} &= \gamma I(t) - \mu R(t) - \rho R(t) \end{aligned} \tag{2.5.1}$$

where $S(t) + I(t) + R(t) = N(t)$. Let $s = \frac{S(t)}{N}, \quad i = \frac{I(t)}{N}, \quad r = \frac{R(t)}{N}$, the system becomes

$$\begin{aligned}\dot{s} &= \mu - \beta si - \mu s + \rho r \\ \dot{i} &= \beta si - \mu i - \gamma i \\ \dot{r} &= \gamma i - \mu r - \rho r\end{aligned}$$

Then $r + i + s = 1 \Rightarrow r = 1 - i - s$. It is enough to consider only the first two equations.

$$\begin{aligned}\dot{s} &= \mu - \beta si - \mu s + \rho(1 - i - s) \\ \dot{i} &= \beta si - \mu i - \gamma i\end{aligned} \tag{2.5.2}$$

2.6 Stability Analysis

2.6.1 Equilibrium Point. The system has a disease-free equilibrium $E^0 = (1, 0)$ and an endemic equilibrium given by $E^* = \left(\frac{\mu+\gamma}{\beta}, \frac{(\mu+\rho)(\beta-(\mu+\gamma)}{\beta(\mu+\gamma+\rho)}\right)$

2.6.2 Basic Reproduction Number of (2.5.2). $\mathcal{R}_0 = \frac{\beta}{(\mu+\gamma)}$

Expressing the endemic equilibrium in terms of $\mathcal{R}_0$ gives, $E^* = \left(\frac{1}{\mathcal{R}_0}, \frac{\mu+\rho}{(\mu+\gamma+\rho)}\left(1 - \frac{1}{\mathcal{R}_0}\right)\right)$

2.6.3 Local Stability analysis of SIRS. The following theorem establishes conditions for the local stability of disease-free equilibrium. Let

$$\Omega = \{(S, I) \in \mathbb{R}^2_+ : S \geq 0, I \geq 0, \quad S + I \leq 1\}$$

2.6.4 Theorem. *The disease-free equilibrium E^o of (2.5.2) above is locally asymptotically stable in Ω if $\mathcal{R}_0 \leq 1$ and unstable if $\mathcal{R}_0 > 1$.*

Proof. □

The Jacobian matrix, evaluated at the disease-free equilibrium point is

$$J_{(1,0)} = \begin{pmatrix} -(\mu + \rho) & -(\beta + \rho) \\ 0 & \beta - (\mu + \gamma) \end{pmatrix}$$

The eigenvalues are $\lambda_1 = -(\mu + \rho), \ \lambda_2 = \beta - (\mu + \gamma)$. Since $\lambda_1 < 0$, the disease-free equilibrium is locally asymptotically stable if $\lambda_2 < 0$. That is, if $\beta - (\mu + \gamma) < 0$, or $\frac{\beta}{\mu+\gamma} = \mathcal{R}_0 < 1$.

2.6.5 Local Stability Analysis of the Endemic Equilibrium.
2.6.6 Theorem. *The endemic equilibrium E^* of (2.5.2) is locally asymptotically stable in Ω if $\mathcal{R}_0 > 1$ and unstable if $\mathcal{R}_0 \leq 1$.*

Proof. Evaluating the Jacobian matrix at the endemic equilibrium gives

$$J_{(E^*)} = \begin{pmatrix} -\beta\frac{\mu+\rho}{(\mu+\gamma+\rho)}\left(1-\frac{1}{\mathcal{R}_0}\right)-(\mu+\rho) & -(\frac{\beta}{\mathcal{R}_0}+\rho) \\ \beta\frac{\mu+\rho}{(\mu+\gamma+\rho)}\left(1-\frac{1}{\mathcal{R}_0}\right) & 0 \end{pmatrix}$$

$$tr_{E^*} = -\beta\frac{\mu+\rho}{(\mu+\gamma+\rho)}\left(1-\frac{1}{\mathcal{R}_0}\right)-(\mu+\rho)$$

$$det_{E^*} = \beta\frac{\mu+\rho}{(\mu+\gamma+\rho)}\left(1-\frac{1}{\mathcal{R}_0}\right)\cdot\left(\frac{\beta}{\mathcal{R}_0}+\rho\right)$$

The trace is negative and the determinant is positive if $\mathcal{R}_0 > 1$. Hence the equilibrium point locally asymptotically stable If $\mathcal{R}_0 > 1$. □

2.6.7 Global Stability of disease-free state.

2.6.8 Theorem. *The disease-free equilibrium E^o of (2.5.2) is globally asymptotically stable in Ω if $\mathcal{R}_0 \leqslant 1$.*

Proof. Define a composite Lyapunov function $V : \{(s,i) \in \Omega : s > 0\} \to \mathbb{R}$ by

$V(s,i) = \frac{1}{2}[(s-s^o)+(i-i^o)]^2 + ki.$

The time derivative of V computed along solutions of (2.5.2) is

$$\begin{aligned}
\dot{V} &= [(s-s^o)+i]\frac{\mathrm{d(s+i)}}{dt} + k\frac{\mathrm{di}}{dt}, \\
&= [(s-s^o)+i](\mu - s(\mu+\rho) - i(\mu+\gamma+\rho)) + k[\beta si - (\gamma+\mu)i] \\
\dot{V} &= [(s-s^o)+i](s^o(\mu+\rho) - s(\mu+\rho) - i(\mu+\gamma+\rho)) + k[(\beta s^o - (\gamma+\mu))]i \\
&= [(s-s^o)+i](-(\mu+\rho)(s^o-s) - i(\mu+\gamma+\rho)) - k\left[(\gamma+\mu) - \beta s^o\right] i \\
&= -(\mu+\rho)(s-s^o)^2 - i(\mu+\gamma+\rho)(s-s^o) - i(\mu+\rho)(s-s^o) - (\mu+\gamma+\rho)i^2 - k\beta\left[\frac{\gamma+\mu}{\beta} - s^o\right] i \\
&= -(\mu+\rho)(s-s^o)^2 - (\mu+\gamma+\rho)i^2 - ((2\mu+\gamma+2\rho)+k\beta)\left(\frac{(\gamma+\mu)}{\beta}\right)[1-\mathcal{R}_0)]\, i
\end{aligned}$$

$$\dot{V} = -(\mu+\rho)(s-s^o)^2 - (\mu+\gamma+\rho)i^2 - ((2\mu+\gamma+2\rho)-k\beta)\frac{(\gamma+\mu)}{\beta}[\mathcal{R}_0-1]\, i \quad \text{where} \quad k = \frac{2\mu+\gamma+2\rho}{\beta}$$

$\dot{V} \leqslant 0$ if $\mathcal{R}_0 \leqslant 1$

Now $\dot{V}(s,i) = 0 \iff s = 1, \quad if \quad \mathcal{R}_0 < 1.$ And $\dot{V}(s,i) = 0 \iff i = 0, \quad if \quad \mathcal{R}_0 = 1$

Since all the model parameters are positive and variables are non-negative, it follows that $\dot{V}(s,i) \leqslant 0$ for $\mathcal{R}_0 \leqslant 1$ with $\dot{V}(s,i) = 0$. □

2.6.9 Global Stability of Endemic Equilibrium.

2.6.10 Theorem. *If $\mathcal{R}_0 > 1$ then the unique endemic equilibrium E^* of the equations (2.5.2) is globally asymptotically stable on the interior of Ω*

Proof. Define $V : \{(s,i) \in \Omega : s,i > 0\} \to \mathbb{R}$ by

$V = \frac{1}{2}[(s-s^*)+(i-i^*)]^2 + k(i-i^*-i^* ln\frac{i}{i^*})$

The time derivative of V computed along solution of (2.5.2) is

$$\dot{V} = [(s-s^*)+(i-i^*)]\frac{\mathrm{d(s+i)}}{\mathrm{dt}} + k\frac{(i-i^*)}{i}\frac{\mathrm{di}}{\mathrm{dt}}$$

$$\dot{V} = [(s-s^*)+(i-i^*)](\mu - s(\mu+\rho) - (\mu+\gamma+\rho)i) + k\frac{(i-i^*)}{i}(\beta si - \gamma i - \mu i)$$

$$\dot{V} = [(s-s^*)+(i-i^*)](s^*(\mu+\rho) + i^*(\mu+\gamma+\rho) - s(\mu+\rho) - i(\mu+\gamma+\rho)) + k\beta(i-i^*)(s-s^*)$$

$$\dot{V} = -(\mu+\rho)(s-s^*)^2 - (\mu+\gamma+\rho)(i-i^*)^2 + (k\beta - (2\mu+2\rho+\gamma))(s-s^*)(i-i^*) \quad and \quad k = \frac{2\mu+2\rho+\gamma}{\beta}$$

$$\dot{V} = -(\mu+\rho)(s-s^*)^2 - (\mu+\gamma+\rho)(i-i^*)^2$$

and

$$\dot{V} = 0 \Longleftrightarrow s = s^*, i = i^*$$

Hence (s^*, i^*) is globally asymptotically stable. From theorem(0.1) implies that for all initial conditions(s_o, i_o) solution converges to the endemic equilibrium (E^*), if $\mathcal{R}_0 > 1$. □

3. Stability Analysis of a Vector-Host Model for Malaria Transmission

In this section we use some of the techniques developed in the earlier chapters to perform stability analysis of an SIS model for a vector-host model for malaria transmission.

3.1 A Vector-Host Model

Vector-host models are used to describe the spread of a disease between and within two populations. Example of such diseases are malaria, dengue fever and West Nile virus. These diseases are contracted by humans from sources such as mosquitoes, birds, rodents other than from human to human.

3.2 The Model

We use an SIS epidemic model for the host population. Let S_h and I_h be respectively, susceptible and infected for humans. An SI model is used for vector population, with S_v and I_v respectively, the susceptible and infected vectors. It is assumed that infected vectors remain infected for life, and that the infection is not harmful to them. See Figure (3.1) below. Susceptible hosts (S_h) become infectious hosts (I_h) at a rate $\alpha S_h \frac{I_v}{N_v}$ through being bitten by infected vector (I_v). Similarly, susceptible vectors (S_v) bite infected humans at a rate $\gamma S_v \frac{I_h}{N_h}$ and move to infected class that is vector (I_h).

3.2.1 Compartment of the SIS model. From Figure (3.1), we derive the following system of differential equations:

3.2.2 Vector-Host Model for Malaria.

$$\begin{aligned} \frac{\mathrm{dS_h}}{\mathrm{dt}} &= \mu_h N_h - \mu_h S_h - \alpha S_h \frac{I_v}{N_v} + \alpha_h I_h \\ \frac{\mathrm{dI_h}}{\mathrm{dt}} &= \alpha S_h \frac{I_v}{N_v} - \mu_h I_h - \alpha_h I_h \\ \frac{\mathrm{dS_v}}{\mathrm{dt}} &= \mu_v N_v - \mu_v S_v - \gamma S_v \frac{I_h}{N_h} \\ \frac{\mathrm{dI_v}}{\mathrm{dt}} &= \gamma S_v \frac{I_h}{N_h} - \mu_v I_v \end{aligned} \tag{3.2.1}$$

Then $N_h = S_h + I_h$ and $N_v = S_v + I_v$ is a constant.

We let

$$s_h = \frac{S_h}{N_h}, \quad i_h = \frac{I_h}{N_h}, \quad s_v = \frac{S_v}{N_v}, \quad i_v = \frac{I_v}{N_v}$$

We obtain the following equivalent system of equation

$$\begin{aligned} \dot{s_h} &= \mu_h - \mu_h s_h - \alpha s_h i_v + \alpha_h i_h \\ \dot{i_h} &= \alpha s_h i_v - \mu_h i_h - \alpha_h i_h \\ \dot{s_v} &= \mu_v - \mu_v s_v - \gamma s_v i_h \\ \dot{i_v} &= \gamma s_v i_h - \mu_v i_v \end{aligned} \tag{3.2.2}$$

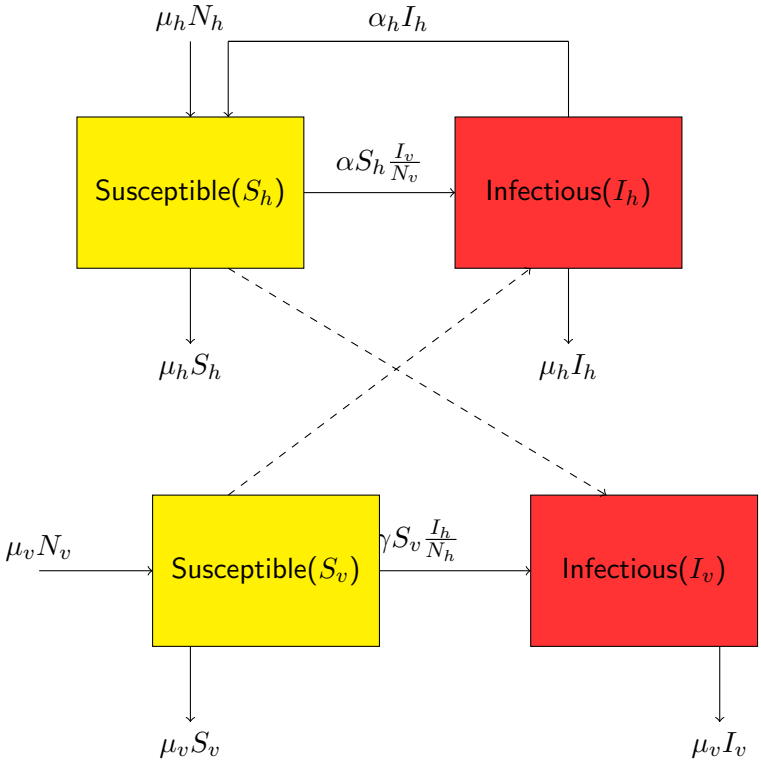

Figure 3.1: Compartment of the SIS-SI model

Parameter	Meaning
μ_h	birth and death rate of humans
μ_v	birth and death rate of vectors
α	human infection rate
γ	mosquito infection rate
α_h	human recovery rate

Figure 3.2: Parameters for the model

3.2.3 Equilibrium point. The system has a unique disease free equilibrium and an endemic equilibrium given respectively, by

$$\begin{aligned} E^o =& (S_h^o, I_h^o, S_v^o, I_v^o) = (N_h, 0, N_v, 0) \\ E^* =& (S_h^*, I_h^*, S_v^*, I_v^*) \\ =& \left(\frac{N_h(\mu_h + \alpha_h)(\gamma + \mu_v)}{(\alpha + \alpha_h + \mu_h)\gamma}, \frac{N_h(-\mu_h\mu_v - \mu_v\alpha_h + \alpha\gamma)}{\gamma(\alpha + \alpha_h + \mu_h)}, \frac{\mu_v N_v(\alpha + \alpha_h + \mu_h)}{\alpha(\gamma + \mu_v)}, \frac{N_v(-\mu_h\mu_v - \mu_v\alpha_h + \alpha\gamma)}{\alpha(\gamma + \mu_v)} \right) \end{aligned}$$

3.2.4 Stability analysis. Let

$$\Omega = \{(s_h, i_h, s_v, i_v) \in \mathbb{R}_+^4 : (s_h, i_h, s_v, i_v) \geq 0 \quad s_h + i_h \leq 1, \quad s_v + i_v \leq 1\}$$

3.2.5 The Basic Reproductive number. A threshold quantity $\mathcal{R}_1$, for the system is given by

$$\mathcal{R}_1 = \frac{\alpha\gamma}{\mu_v(\mu_h + \alpha_h)}$$

Therefore the basic reproductive number is

$$\mathcal{R}_0 = \sqrt{\mathcal{R}_1} = \sqrt{\frac{\alpha\gamma}{\mu_v(\mu_h + \alpha_h)}}$$

The threshold parameter $\mathcal{R}_0$ can be defined as square root of the "product of number of humans one mosquito infects during its infectious lifetime and number of mosquitoes one human infects during the duration of the infectious period" Azu-Tungmah (Accessed April 2014).

3.2.6 Local stability analysis of the disease-free equilibrium.

3.2.7 Theorem. *If $R_0 \leq 1$, then the disease free equilibrium point is locally asymptotically stable in Ω. And if $\mathcal{R}_0 > 1$, unstable.*

Proof. The Jacobian matrix is

$$J_{(S_h,I_h,S_v,I_v)} = \begin{pmatrix} -\frac{\alpha I_v}{N_v} - \mu_h & \alpha_h & 0 & -\frac{\alpha S_h}{N_v} \\ \frac{\alpha I_v}{N_v} & -\mu_h - \alpha_h & 0 & \frac{\alpha S_h}{N_v} \\ 0 & -\frac{\gamma S_V}{N_h} & -\frac{\gamma I_h}{N_h} - \mu_v & 0 \\ 0 & \frac{\gamma S_V}{N_h} & \frac{\gamma I_h}{N_h} & -\mu_v \end{pmatrix}$$

Evaluating it at the equilibrium point:

$$J_{E^o} = \begin{pmatrix} -\mu_h & \alpha_h & 0 & -\frac{\alpha N_h}{N_v} \\ 0 & -\mu_h - \alpha_h & 0 & \frac{\alpha N_h}{N_v} \\ 0 & -\frac{\gamma N_V}{N_h} & -\mu_v & 0 \\ 0 & \frac{\gamma N_V}{N_h} & 0 & -\mu_v \end{pmatrix}$$

From the Jacobian matrix we obtain the eigenvalues as follows:

$$\begin{aligned} \lambda_1 &= -\mu_h, \\ \lambda_2 &= -\mu_v, \\ \lambda_3 &= -\frac{1}{2}\mu_h - \frac{1}{2}\mu_v - \frac{1}{2}\alpha_h + \frac{1}{2}\sqrt{\mu_h^2 - 2\mu_h\mu_v + 2\mu_h\alpha_h + \mu_v^2 - 2\mu_v\alpha_h + \alpha_h^2 + 4\alpha\gamma} \\ \lambda_4 &= -\frac{1}{2}\mu_h - \frac{1}{2}\mu_v - \frac{1}{2}\alpha_h - \frac{1}{2}\sqrt{\mu_h^2 - 2\mu_h\mu_v + 2\mu_h\alpha_h + \mu_v^2 - 2\mu_v\alpha_h + \alpha_h^2 + 4\alpha\gamma} \end{aligned}$$

λ_3 can be rewritten as

$$\lambda_3 = -\frac{1}{2}\mu_h - \frac{1}{2}\mu_v - \frac{1}{2}\alpha_h + \sqrt{\left(\frac{\mu_h + \mu_h + \alpha_h}{2}\right)^2 - (\mu_h\mu_v + \mu_v\alpha_h - \alpha\gamma)}$$

λ_3 is only negative if and only if $(\mu_h\mu_v + \mu_v\alpha_h - \alpha\gamma) > 0$

Therefore the disease free equilibrium is stable provided $(\mu_h\mu_v + \mu_v\alpha_h - \alpha\gamma) > 0$. That is, if $\frac{\alpha\gamma}{\mu_v(\mu_h+\alpha_h)} = \mathcal{R}_1 \leq 1$. □

3.2.8 Local analysis of endemic equation. We now evaluate the Jacobian at the disease endemic equilibrium.

3.2.9 Theorem. *The endemic equilibrium of* (3.2.1) *is locally asymptotically stable in* Ω *if* $\mathcal{R}_0 > 1$, *and unstable if* $\mathcal{R}_0 \leq 1$.

Proof. Evaluating the Jacobian matrix at the endemic equilibrium point gives

$$J_{E^*} = \begin{pmatrix} -\frac{-\mu_h\mu_v-\mu_v\alpha_h+\alpha\gamma}{\alpha+\mu_v} - \mu_h & \alpha_h & 0 & -\frac{\alpha N_h(\mu_h+\alpha_h)(\alpha+\mu_v)}{(\alpha+\alpha_h+\mu_h)\gamma N_v} \\ \frac{-\mu_h\mu_v-\mu_v\alpha_h+\alpha\gamma}{\alpha+\mu_v} & -(\mu+\alpha_h) & 0 & \frac{\alpha N_h(\mu_h+\alpha_h)(\alpha+\mu_v)}{(\alpha+\alpha_h+\mu_h)\gamma N_v} \\ 0 & -\frac{\alpha\mu_v N_v(\alpha+\alpha_h+\mu_h)}{\alpha_h(\alpha+\mu_v)N_h} & -\frac{-\mu_h\mu_v-\mu_v\alpha_h+\alpha+\gamma}{\alpha+\alpha_h+\mu_h} - \mu_v & 0 \\ 0 & \frac{\alpha\mu_v N_v(\alpha+\alpha_h+\mu_h)}{\alpha_h(\alpha+\mu_v)N_h} & \frac{-\mu_h\mu_v-\mu_v\alpha_h+\alpha+\gamma}{\alpha+\alpha_h+\mu_h} & -\mu_v \end{pmatrix}$$

$$\begin{aligned} \lambda_1 &= -\mu_h, \\ \lambda_2 &= -\mu_v, \\ \lambda_3 &= \frac{1}{2}\frac{-A - B\sqrt{\left(\frac{A}{B}\right)^2 + 4\mu_h\mu_v + 4\mu_v\alpha_h - 4\alpha\gamma}}{B} \\ \lambda_4 &= \frac{1}{2}\frac{-A + B\sqrt{\left(\frac{A}{B}\right)^2 + 4\mu_h\mu_v + 4\mu_v\alpha_h - 4\alpha\gamma}}{B} \end{aligned}$$

Where

$$\begin{aligned} A &= \mu_h^2\gamma + 2\mu_h\alpha_h\gamma + 2\mu_h\alpha\gamma + \mu_v^2\alpha + 2\alpha\gamma\mu_v + \alpha_h^2\gamma + 2\alpha_h\alpha\gamma + \alpha^2\gamma + \alpha\gamma^2 \\ B &= (\alpha+\mu_v)(\alpha+\alpha_h+\mu_h) \end{aligned}$$

λ_3 is negative and λ_4 will be negative provided $4\mu_h\mu_v + 4\mu_v\alpha_h - 4\alpha\gamma < 0$

$$\begin{aligned} \mathcal{R}_1 &= \frac{\alpha\gamma}{\mu_v(\mu_h+\alpha_h)} > 1 \\ &\Leftrightarrow \mathcal{R}_1 > 1 \end{aligned}$$

□

3.2.10 Global Stability of disease-free equilibrium.

3.2.11 Theorem. *The disease-free equilibrium* E^o *of the malaria model is globally asymptotically stable in* Ω *if* $R_0 \leqslant 1$.

Proof. Define $V : \{(s_h, i_h, s_v, i_v) \in \Omega : s_h, s_v > 0\} \to \mathbb{R}$ by

$$V(s_h, i_h, s_v, i_v) = (s_h - s_h^o - s_h^o log\frac{s_h}{s_h^o}) + i_h + k(s_v - s_v^o - s_v^o log\frac{s_v}{s_v^o}) + i_v$$

The time derivative of V computed along solutions of (3.2.2) is

$$\dot{V} = \left(\frac{s_h - s_h^o}{s_h}\right)\frac{\mathrm{ds_h}}{dt} + \frac{\mathrm{di_h}}{dt} + k\left(\frac{s_v - s_v^o}{s_v}\right)\frac{\mathrm{ds_v}}{dt} + \frac{\mathrm{di_v}}{dt}$$

$$\dot{V} = \left(\frac{s_h - s_h^o}{s_h}\right)[\mu_h - \mu_h s_h - \alpha s_h i_v + \alpha_h i_h] + [\alpha s_h i_v - \mu_h i_h - \alpha_h i_h]$$
$$+ k\left(\frac{s_v - s_v^o}{s_v}\right)[\mu_v - \mu_v s_v - \gamma s_v i_h] + [\gamma s_v i_h - \mu_v i_v]$$

From $i_h = 1 - s_h, s_h^o = 1$ and $s_v^o = 1$, we have

$$\dot{V} = \left(\frac{s_h - s_h^o}{s_h}\right)[-\mu_h(s_h - s_h^o) - \alpha_h(s_h - s_h^o)] + k\frac{(s_v - s_v^o)}{s_v}[-(s_v - s_v^o)]$$
$$- \left(\frac{(\mu_h + \alpha_h)}{\gamma s_v^o} - \frac{\alpha s_h^o}{\mu_v}\right) i_v$$

$$\dot{V} = -(\mu_h + \alpha_h)\frac{(s_h - s_h^o)^2}{s_h} - k\frac{(s_v - s_v^o)^2}{s_v} - \frac{(\mu_h + \alpha_h)}{\gamma s_v^o}\left(1 - \frac{\alpha s_h^o}{\mu_v}\frac{\gamma s_v^o}{(\mu_h + \alpha_h)}\right) i_v$$

since $\quad s_h^o = s_v^o = 1$

$$\dot{V} = -(\mu_h + \alpha_h)\frac{(s_h - s_h^o)^2}{s_h} - k\frac{(s_v - s_v^o)^2}{s_v} - \frac{(\mu_h + \alpha_h)}{\gamma s_v^o}(1 - \mathcal{R}_0)\, i_v$$

Taking $k = \frac{(\mu_h + \alpha_h)}{\gamma s_v^o}$

$$\dot{V} = -(\mu_h + \alpha_h)\frac{(s_h - s_h^0)^2}{s_h} - \frac{(\mu_h + \alpha_h)}{\gamma s_v^o}\frac{(s_v - s_v^0)^2}{s_v} - \frac{(\mu_h + \alpha_h)}{\gamma s_v^o}(1 - \mathcal{R}_0)\, i_v \leq 0.$$

Now $\quad \dot{V} \leq 0 \quad iff \quad \mathcal{R}_0 \leq 1$

That is $\quad \dot{V}(s_h, i_h, s_v, i_v) = 0 \Leftrightarrow s_h = s_h^o, \quad s_v = s_v^o \quad if \quad \mathcal{R}_0 < 1.$

And $\quad \dot{V}(s_h, i_h, s_v, i_v) = 0 \Leftrightarrow i_v = i_h = 0, \quad if \quad \mathcal{R}_0 = 1$

Hence $(s_h^o, i_h^o, s_v^o, i_v^o)$ is globally asymptotically stable. □

3.2.12 Global Stability of endemic equilibrium.

3.2.13 Theorem. *The endemic equilibrium E^* of (3.2.2) is globally asymptotically stable in Ω if $\mathcal{R}_0 > 1$.*

Proof. Define $V : \{(s_h, i_h, s_v, i_v) \in \Omega : s_h, i_h, s_v, i_v > 0\} \to \mathbb{R}$ by

$$V(s_h, i_h, s_v, i_v) = c_1(s_h - s_h^* - s_h^* log\frac{s_h}{s_h*}) + c_2(i_h - i_h^* - i_h^* log\frac{i_h}{i_h*})$$
$$+ c_3(s_v - s_v^* - s_v^* log\frac{s_v}{s_v*}) + c_4(i_v - i_v^* - i_v^* log\frac{i_v}{i_v*})$$

Where $c_1 = c_2 = \gamma s_v^* i_h^*$ and $c_3 = c_4 = \alpha s_h^* i_h^*$. Then V is $\mathbf{C}^1$ on the interior of Ω, E^* is global minimum of V on Ω, and $V(s_h, i_h, s_v, i_v) = 0$. The time derivative of V computed along solutions of (3.2.2) is

$$\dot{V} = \gamma s_v^* i_h^* \left(1 - \frac{s_h^*}{s_h}\right) \frac{\mathrm{ds_h}}{dt} + \gamma s_v^* i_h^* \left(1 - \frac{i_h^*}{i_h}\right) \frac{\mathrm{di_h}}{dt}$$
$$+ \alpha s_h^* i_v^* \left(1 - \frac{s_v^*}{s_v}\right) \frac{\mathrm{ds_v}}{dt} + \alpha s_h^* i_v^* \left(1 - \frac{i_v^*}{i_v}\right) \frac{\mathrm{di_v}}{dt}$$

$$\dot{V} = \gamma s_v^* i_h^* \left(1 - \frac{s_h^*}{s_h}\right) [\mu_h - \mu_h s_h - \alpha s_h i_v + \alpha_h i_h] + \gamma s_v^* i_h^* \left(1 - \frac{i_h^*}{i_h}\right) [\alpha s_h i_v - \mu_h i_h - \alpha_h i_h]$$
$$+ \alpha s_h^* i_v^* \left(1 - \frac{s_v^*}{s_v}\right) [\mu_v - \mu_v s_v - \gamma s_v i_h] + \alpha s_h^* i_v^* \left(1 - \frac{i_v^*}{i_v}\right) [\gamma s_v i_h - \mu_v i_v]$$

Using $i_h = N_h - s_h, i_h^* = N_h^* - s_h^*,$

$$\dot{V} = \gamma s_v^* i_h^* \left(1 - \frac{s_h^*}{s_h}\right) [\alpha s_h^* {i_v}^* + (\alpha_h + \mu_h) s_h^* - \alpha s_h i_v - (\alpha_h + \mu_h) s_h] + \gamma s_v^* i_h^* \left(1 - \frac{i_h^*}{i_h}\right) [\alpha s_h i_v - \alpha s_h^* {i_v}^*]$$
$$+ \alpha s_h^* i_v^* \left(1 - \frac{s_v^*}{s_v}\right) [\gamma s_v^* i_h^* + \mu_v s_v^* - \mu_v s_v - \gamma s_v i_h] + \alpha s_h^* i_v^* \left(1 - \frac{i_v^*}{i_v}\right) [\gamma s_v i_h - \gamma s_v^* i_h^*]$$
$$\dot{V} = \gamma s_v^* i_h^* \left(1 - \frac{s_h^*}{s_h}\right) [\alpha s_h^* {i_v}^* - \alpha s_h i_v - (\alpha_h + \mu_h)(s_h - s_h^*)] + \gamma s_v^* i_h^* \left(1 - \frac{i_h^*}{i_h}\right) [\alpha s_h i_v - \alpha s_h^* {i_v}^*]$$
$$+ \alpha s_h^* i_v^* \left(1 - \frac{s_v^*}{s_v}\right) [\gamma s_v^* i_h^* - \gamma s_v i_h - \mu_v (s_v - s_v^*)] + \alpha s_h^* i_v^* \left(1 - \frac{i_v^*}{i_v}\right) [\gamma s_v i_h - \gamma s_v^* i_h^*]$$

$$\dot{V} = -(\alpha_h + \mu_h) \gamma s_v^* i_h^* \frac{(s_h - s_h^*)^2}{s_h} - \mu_v \alpha s_h^* i_v^* \frac{(s_v - s_v^*)^2}{s_v} - \gamma \alpha s_h^* i_h^* s_v^* i_v^* \left(\frac{s_h^*}{s_h} + \frac{s_v^*}{s_v} + \frac{s_h i_h^* i_v}{s_h^* i_h i_v^*} + \frac{s_v i_v^* i_h}{s_v^* i_v i_h^*} - 4\right) \leq 0$$

$\dot{V} = 0$ if and only if $s_h = s_h^*, i_h = i_h^*, s_v = s_v^*,$ and $i_v = i_v^*$.

Hence E^* is globally asymptotically stable in the interior of Ω, when it exist. □

3.3 Numerical Simulation

3.3.1 Estimation of Parameter. The parameters in (3.2.1) were estimated using clinical malaria data and demographics statistics of Ghana.

Those that were not available were obtained from malaria endemic countries which have similar environmental conditions published by researchers Azu-Tungmah (Accessed April 2014). The total population for Ghana in 2009 was 23837000 according to World Malaria Report 2010 and the population growth rate per year is also 1.855% by 2011 CIA World Factbook and other sources. Life expectancy at birth in 2010 is 64 years according to UNICEF Ghana Fact Sheet,malaria., 2012 Azu-Tungmah (Accessed April 2014). The probability of transmission of malaria infection from infectious humans to susceptible mosquitoes is estimated to be 0.42 and we estimate that it will take $(5 - 8)$days for human to recover, with effective treatment we expect human to recover in about 5days and for endemic in about 8days.

3.3.2 Population data for mosquito. According to Ghana Living Standards Survey Report, the estimated number of households in Ghana is 5.5 million with a higher proportion in the rural areas to be 3.1 million than in the urban areas which is 2.4 million. The quarterly data for Anopheles funestus and Anopheles gambiae mosquitoes in a region of Western Kenya was provided in Gimnig (2003).

In Chitnis (2005) he used the data provided by Gimnig (2003) and made an estimate of 2 Anopheles gambiae and 0.8 Anopheles funestus mosquitoes per house in high malaria transmission areas;

based on that Azu-Tungmah (Accessed April 2014) also estimate that there are 10 female Anopheles mosquitoes in each house in Ghana. He also use an estimate of 0.40 bites on humans per mosquito per day in Ghana.

The estimation of biting includes both, the dependence on the mosquito's gonotrophic cycle and the dependence on the mosquito's anthropophilic rate . The rate of transmission of infection from an infectious mosquito to a susceptible human is estimated to be 0.0655. We also assumed that life expectancy of an adult anopheles mosquito is assumed to be $25 - 30$ days.

The table 3.3 below shows the estimated parameters and their sources for (3.2.1). All rates are given per day.

symbol	Values	Source
α	0.070000	Azu-Tungmah (Accessed April 2014)
γ	0.420000	Azu-Tungmah (Accessed April 2014)
μ_v	$\frac{1}{30} = 0.03333$	Estimated
μ_h	$\frac{1}{(64\times 365.25)} = 0.00004$	UNICEF Ghana Fact Sheet,malaria.
α_h	$\frac{1}{5} - \frac{1}{8} = 0.20000 - 0.12500$	Estimated

Figure 3.3: Estimation of Parameters for the Simulation

Based on our theoretical findings, we can predict realistic conditions under which the disease will either persist or go extinct. To illustrate this point, we will now provide graphical representations of our results for both the cases in which $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$.

The figures below shows that over the time span of 30 and 90 days, when $\mathcal{R}_0 < 1$ the solutions converge to disease-free equilibrium point, $(i_h, i_v) = (0, 0)$ and the disease goes extinct. This indicates that, with the appropriate intervention we can eliminate infected mosquitoes from malaria endemic areas. In fact there are many places where mosquitoes abound but there is no malaria. Conversely, we note from the figure below that over the same time span, when $\mathcal{R}_0 > 1$ the solutions converge to the disease endemic equilibrium point $(i_h, i_v) = (i_h^*, i_v^*)$ and the disease will persist.

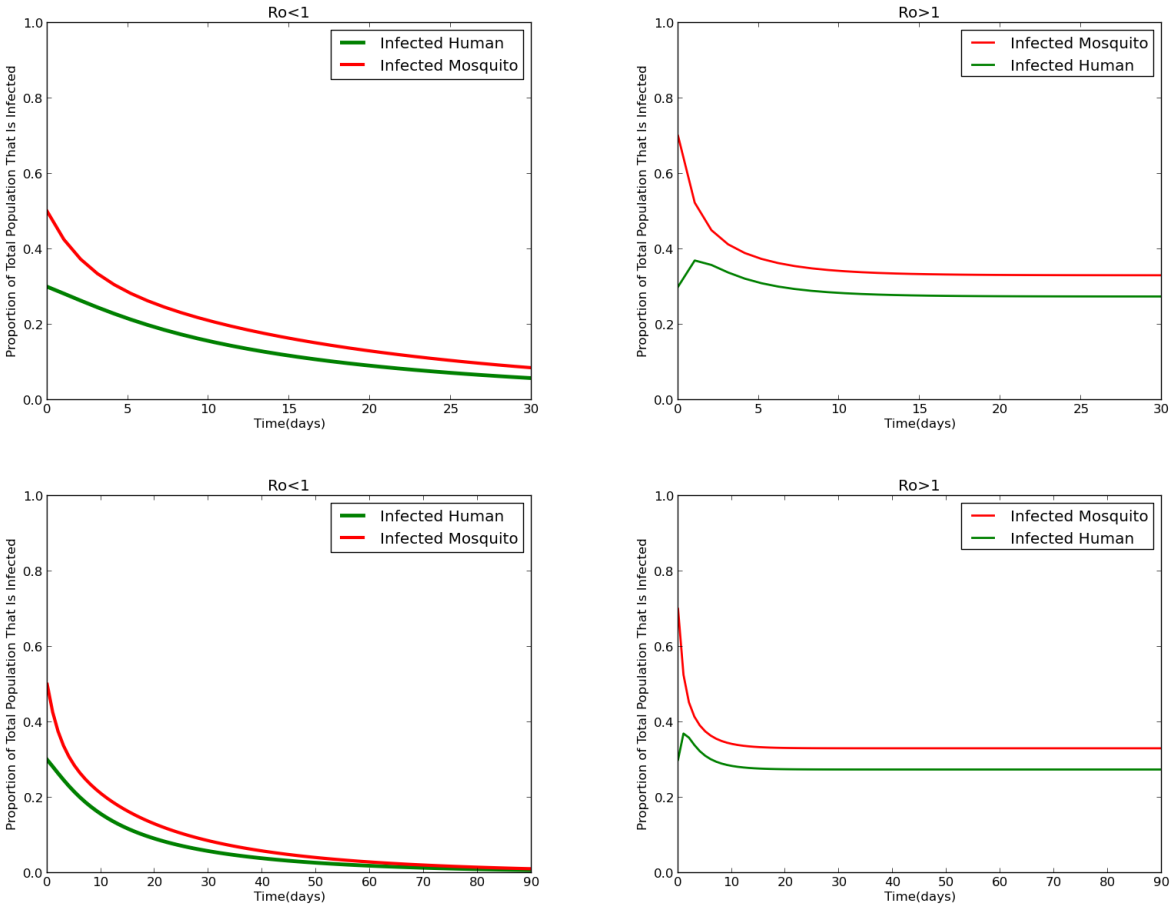

Figure 3.4: Analysis of the spread of Malaria Disease

4. Conclusion

In this Project our main interest is to study the stability of SIS, SIR, SIRS epidemic models. Lyapunov's indirect method uses linearization about an equilibrium point to determine local stability. If the eigenvalues of the Jacobian matrix are all negative or have negative real parts, then the equilibrium point is locally asymptotically stable. However, finding the eigenvalues can sometimes be very daunting. In such situations either the Routh-Hurwitz conditions or a Corollary of Gershgorin's circle theorem can be used. Establishing global stability of the equilibria requires an ability to define an appropriate Lyapunov function. In the Project we have used a combination of quadratic, logarithmic and composite Lyapunov functions to establish global stability in various epidemiological models.

Finally, the Project investigated stability of a mathematical model for the spread of the malaria. Simulations of the model show that, with the appropriate treatment measures infected mosquitoes in a closed environment can be eliminated thus reducing the vicious circle of the malaria epidemic.

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