Calculus of Variations and Optimal Control In Economic Models

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22 May 2014

Submitted in partial fulfillment of a structured masters degree at AIMS South Africa

Abstract

This project aims to investigate a master approach of the theory of calculus of variations and optimal control in the context of economic models.

The work environment is the focus of economic optimization models in the calculus of variations and optimal control, by performing research of various economic concepts and applying some mathematical results more relevant to the problems studied.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Andwilile Abrahamu George, 22 May 2014

Contents

1. Introduction

In this project we will consider two methods of analysing dynamic optimization in economic models. First, we start with calculus of variation which provides the mathematical theory to solve extremity functional problems for which a given functional has a stationary value either maximum or minimum, then we will state and prove the fundamental integral of Euler-Lagrange equation. Moreover, we will introduce the Noether theorem which describes a relationship between the invariance of the action integral with respect to given groups of transformations and some identities satisfied by the Euler-Lagrange equation.

Second, we will study the method of optimal control in Chapter Three which is an extension of calculus of variations and it is a mathematical optimization method for deriving optimal control theory. Optimal control is the pathway of the control variable to maximize the cost functional which satisfying ordinary differential equations. Moreover, we will see the method of the maximum principle which is the main tool for obtaining optimal strategies while classical variational theory could not handle some of the constraints involved in a realistic optimal control problem.

Furthermore, in Chapter Four we will study the application in economic models. Since most of economic models determined by a Lagrangian function and various constraints are very difficult to solve the equations of motions, then we introduce the Noether theorem and conservation laws which are useful to find the functions, which are constant along the temporal evolution of the system in order to know relevant aspects of the initial dynamic system. Likewise, in this Chapter, we will see a few classical examples in economic models that reflect the use of calculus of variation and optimal control.

2. Calculus of Variations

2.1 Introduction

The study of calculus of variations started in 1696 after John Bernoulli proposed the problem known as the Brachistochrone Problem to find a path along which a particle sliding under gravity takes a minimum time. However, Bernoulli himself was the first to provide a correct solution after many scientists submitted several solutions including famous scientists like Leibniz, Newton and L'Hospital.

The calculus of variation is the solution for optimization problems of functionals of one or more variables. The classical solutions to minimization (or maximization) problems in calculus of variations are prescribed by boundary-value problems which are associated with Euler-Lagrange's equations.

In this chapter, we will see the basic mathematical analysis for minimization (or maximization) functions known as the calculus of variations.

2.2 Optimality Conditions (Euler-Lagrange equations)

Definition 2.2.1. Consider the problem of the calculus of variations given by

$$J(x(t)) = \int_a^b F(t, x(t), \dot{x}(t))dt\,, \tag{2.2.1}$$

where $J(x(t))$ is called an objective functional or cost functional and F is the Lagrangian when $a, b \in \mathbb{R}$, $a < b$, $\dot{x}(t) = \frac{dx(t)}{dt}$ and

$$x(\cdot) \in C^2\left([a,b];\, \mathbb{R}^n\right);$$
$$F(\cdot,\cdot,\cdot) \in C^2\left([a,b] \times \mathbb{R}^n \times \mathbb{R}^n;\, \mathbb{R}\right).$$

2.2.2 The First Variation. As we know from calculus, that given a function, $f : \mathbb{R}^K \mapsto \mathbb{R}$ we can find its critical points. At these critical points the function attains its minimum or maximum values. This involved two steps. First, we use the closed interval test for find the points where the gradient of the function vanishes. Second, we use the second derivatives to classfy the critical points as local maxima, minima or saddle points.

Let S be a set of functions $x, y, \ldots$, then a functional J defined on S is a mapping $J : S \mapsto \mathbb{R}^1$ such that each $x \in S$ a real number in $J(x)$. Now assume that $S \subseteq N$ where N is a normed linear space.

Definition 2.2.3. Suppose x_s be a one-parameter family of elements in $S \subseteq N$, with $||x_s - x|| < \delta$ of the form

$$x_s = x + s\eta,$$

where $x \in S, \eta \in N, \delta, s > 0$. This implies that the first variation of J at x in the direction η is defined as:

$$\delta J(x, \eta) = \frac{d}{ds} J(x + s\eta)\bigg|_{s=0}, \tag{2.2.2}$$

where $\eta(t) = (\eta^1(t), \ldots, \eta^n(t)) \in C_n^2$ and $\eta(a) = \eta(b) = 0$.

Theorem 2.2.4. *If $x \in S$ is a relative minimum for the functional $J : S \mapsto \mathbb{R}^1$, then $\delta J(x, \eta) = 0$ for all $\eta \in N$ satisfying the assumptions of definition* (2.2.3)*, (Logan, 1977).*

Lemma 2.2.5. If $x(t)$ is a relative minimum of the functional J defined by definition (2.2.1), then

$$\int_a^b \left(\frac{\partial F}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\right)\right) \cdot \eta(t) dt = 0$$

for all $\eta(t) \in C_n^2$ with $\eta(a) = \eta(b) = 0$.

Proof. From the theorem (2.2.4) we are required to calculate $\delta J(x(t), \eta(t))$ and equating to zero. By referring the definition (2.2.3), we have

$$\begin{aligned}\delta J(x(t), \eta(t)) &= \frac{d}{ds} J(x(t) + s\eta(t))\bigg|_{s=0} \\ &= \frac{d}{ds}\left[\int_a^b F(t, x(t) + s\eta(t)), \dot{x}(t) + s\dot{\eta}(t))dt\right]_{s=0} \\ &= \int_a^b \frac{\partial}{\partial s}\left(F(t, x(t) + s\eta(t)), \dot{x}(t) + s\dot{\eta}(t))\right)\bigg|_{s=0} dt.\end{aligned}$$

We apply the chain rule and evaluating at $s = 0$, we have

$$\delta J(x(t), \eta(t)) = \int_a^b \left(\frac{\partial F(t, x(t), \dot{x}(t))}{\partial x(t)} \cdot \eta(t) + \frac{\partial F(t, x(t), \dot{x}(t))}{\partial \dot{x}(t)} \cdot \dot{\eta}(t)\right) dt. \tag{2.2.3}$$

Now, let us eliminate $\dot{\eta}(t)$ term in the integrand by integrating the last term of equation (2.2.3) by parts and have in mind that $\eta(b) = \eta(a) = 0$, by theorem (2.2.4), we have

$$\delta J = \int_a^b \left(\frac{\partial F}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\right)\right) \cdot \eta(t) dt = 0.$$

□

Lemma 2.2.6. If a continuous function $f(t) : [a, b] \mapsto \mathbb{R}$ is such that

$$\int_a^b f(t)\phi(t)dt = 0 \tag{2.2.4}$$

for every $\phi \in C^2[a, b]$ satisfying $\phi(a) = \phi(b) = 0$. Then $f(t) \equiv 0, \quad t \in [a, b]$.

Proof. See e.g (Logan, 1977), Lemma 1.2. □

Theorem 2.2.7. *(The necessary condition of Euler-Lagrange). Assume $x(t) \in S_n^2$ and $F(t, x(t), \dot{x}(t)) \in S_n^2$ satisfy the Lemma* (2.2.5)*. Then, $x(t)$ is a critical function of $J(x)$ if and only if*

$$\frac{\partial F}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\right) = 0 \quad (k = 1, \ldots, n) \quad for \quad a \leqslant t \leqslant b. \tag{2.2.5}$$

Proof. Lemma (2.2.5) holds for all $\eta(t) \in C_n^2$ with $\eta(a) = \eta(b) = 0$. Also, this holds when all of its components vanish except one, i.e., $\eta(t) = (0, \dots, 0, \eta^i(t), 0, \dots, 0)$, for some fixed i. Therefore,

$$\int_a^b \left(\frac{\partial F}{\partial x^i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}^i} \right) \eta^i(t) dt = 0$$

for all $\eta^i(t)$ (i fixed) of class $C^2(a, b)$ that vanishes at a and b.

This implies that the expression $\left(\frac{\partial F}{\partial x^i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}^i}\right)$ is continuous on $[a, b]$ and hence, by lemma (2.2.6), we have

$$\frac{\partial F}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^i}\right) = 0,$$

where i is arbitrary. □

2.3 Noether's Theorem

Noether's Theorem in the calculus of variations express a relationship between the invariance of the action integral with respect to given groups of transformations, and some identities satisfied by Euler-Lagrange expressions. Noether's theorem are divided into categories of two types such as the first theorem which deals with transformations depending on scalar parameters, and the second theorem which deals with transformations depending on functions. In this essay, we will concentrate on Noether's First Theorem.

2.3.1 Noether's First Theorem. Consider a Lagrange function $F : I \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^1$, which is twice continuously differentiable in each of its $2n + 1$ arguments, where $F \in C^2$ and $I \subseteq \mathbb{R}^1$ is an open interval of real numbers. Therefore, the variational integral or fundamental integral are given by

$$J(x) = \int_a^b F(t, x(t), \dot{x}(t))dt, \tag{2.3.1}$$

where $[a, b] \subset I$ and $x \in C_n^2[a, b]$ is the set of all vector functions $x(t) = (x^1(t), \dots, x^n(t)), \quad t \in [a, b]$.

Now consider the transformation which is given by

$$\bar{t} = \phi(t, x, s), \quad (s = s^1, \dots, s^z) \tag{2.3.2}$$

$$\bar{x}^k = \psi^k(t, x, s) \quad (k = 1, \dots, n). \tag{2.3.3}$$

In economic terms, ϕ can be considered as "subjective" time, while ψ represent "technical" or "taste" change.

Again consider an infinitesimal transformation of coordinates and field functions obtained by expanding equations (2.3.2) and (2.3.3) by Taylor series about $s = 0$ and we obtain

$$\bar{t} = t + \tau_r(t, x)s^r + 0(s), \tag{2.3.4}$$

$$\bar{x}^k = x^k + \xi_r^k(t, x)s^r + 0(s), \tag{2.3.5}$$

where τ_r and ξ_r^k are the infinitesimal generators of the transformation in (2.3.2) and (2.3.3) given by

$$\tau_r(t, x) = \frac{\partial \phi(t, x, 0)}{\partial s^r}, \quad \xi_r^k(t, x) = \frac{\partial \psi^k(t, x, 0)}{\partial s^r}.$$

Suppose $x : [a, b] \mapsto \mathbb{R}^n$ be a curve of class C_n^2 given by $x = x(t)$; then for sufficiently small s, the transformation

$$\bar{t} = \phi(t, x(t), s) \tag{2.3.6}$$

is invertible (Logan, 1977). The unique inverse function of (2.3.6) can be written as $t = T(\bar{t}, s)$, and by substituting it into the equation $\bar{x}^k = \psi(t, x(t), s)$, we have

$$\begin{aligned} \bar{x}^k &= \psi(T(\bar{t}, s), x(T(\bar{t}, s)), s) \\ &\equiv \bar{x}^k(\bar{t}) = \bar{x}^k(\phi(t, x(t), s)) = \psi^k(t, x(t), s). \end{aligned} \tag{2.3.7}$$

Remark 2.3.2. The geometrical interpretation of the transformation for the case of $n = 1$ and $z = 1$, where the curve $x = x(t)$ in the tx plane is mapped to a one-parameter family under the transformation $t \mapsto \bar{t} = \phi(t, x(t), s)$ and $x \mapsto \bar{x} = \psi(t, x(t), s)$ in the $\bar{t}\bar{x}$ plane, provided s is sufficiently small.

The two figures 2.1 below describe this transformation.

In the tx plane the functional is given by

$$J(x) = \int_a^b F(t, x(t), \dot{x}(t))dt.$$

And the functional for the $\bar{t}\bar{x}$ plane is given by

$$J(\bar{x}) = \int_{\bar{a}}^{\bar{b}} F(\bar{t}, \bar{x}(\bar{t}), \dot{\bar{x}}(\bar{t}))d\bar{t}.$$

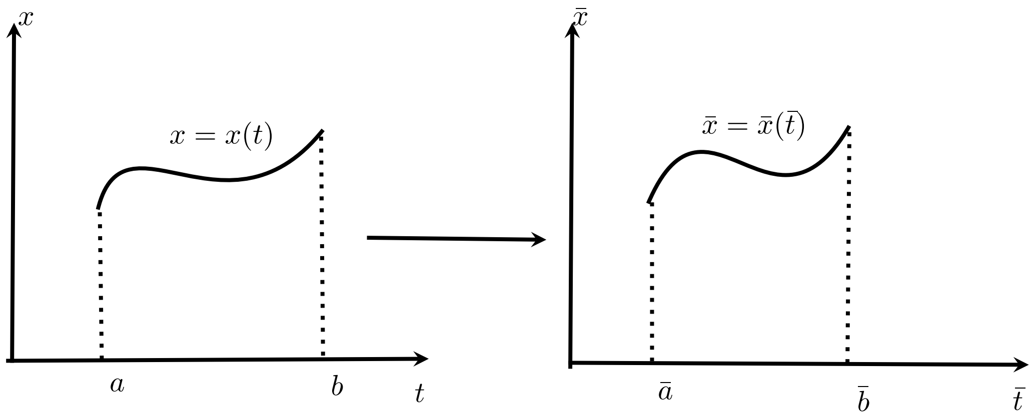

Figure 2.1: One parameter Transformation

J is invariant under the given transformation if $J(x) = J(\bar{x})$ or at least so up to the first order term in s.

2.3.3 Invariance Definition. The fundamental integral is divergence-invariant, or invariant up to a divergence term, if there exist z functions $\Phi_r : I \times \mathbb{R}^n \mapsto \mathbb{R}^1, \quad r = 1, \ldots, z$ of class C^1 such that

$$F\left(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}(\bar{t})}{d\bar{t}}\right)\frac{d\bar{t}}{dt} - F\left(t, x(t), \frac{dx(t)}{dt}\right) = s^r \frac{d\Phi_r}{dt}(t, x(t)) + 0(s). \tag{2.3.8}$$

Remark 2.3.4. The fundamental integral (2.3.1) is absolutely invariant if $\Phi_r \equiv 0$ and divergence invariant if Φ_r can be determined in the process of solving the system.

Example 2.3.5. Consider $F = T - U$, where T is the kinetic energy given by $T = \frac{1}{2}M_k\dot{x}^k$ and U is the potential energy $U(t, x)$. Assume that the potential energy is translation invariant, i.e., $U(\bar{t}, \bar{x}) = U(t, x)$, where $\bar{x}^k = x^k + s$, $\bar{t} = t$.

Differentiating the transformation factor by t, we have

$$\frac{d\bar{t}}{dt} = 1 \Rightarrow d\bar{t} = dt$$

and

$$\frac{d\bar{x}^k}{d\bar{t}} = \frac{d\bar{x}^k}{dt} \Big/ \frac{d\bar{t}}{dt} = \frac{dx^k}{dt}.$$

From Invariance definition of equation (2.3.8) and remark (2.3.4), we have

$$\begin{aligned}&\frac{1}{2}M_k\dot{\bar{x}}^k - U(t, \bar{x}) - \left(\frac{1}{2}M_k\dot{x}^k - U(t, x)\right)\\ &= \frac{1}{2}M_k\dot{\bar{x}}^k - \frac{1}{2}M_k\dot{x}^k - U(t, \bar{x}) + U(t, x) \equiv 0.\end{aligned}$$

Therefore, in this case F is absolutely invariant under translation.

Example 2.3.6. Suppose the functional $J(x)$ defined by

$$J(x) = \int_a^b \left(1 + \dot{x}(t)^2\right)^{\frac{1}{2}} dt$$

and the one-parameter group is invariant under rotation $\bar{t} = t - sx, \quad \bar{x} = x + st$.

Differentiate these rotation with respect to t, we have

$$\frac{d\bar{t}}{dt} = 1 - s\dot{x} \quad \text{and} \quad \frac{d\bar{x}}{dt} = \dot{x} + s, \quad \text{but} \quad \frac{d\bar{x}}{d\bar{t}} = \frac{d\bar{x}}{dt} \Big/ \frac{d\bar{t}}{dt} = \frac{\dot{x} + s}{1 - s\dot{x}}.$$

From Invariance definition of equation (2.3.8) and remark (2.3.4), we have

$$\begin{aligned}&\left(1 + \left(\frac{d\bar{x}}{d\bar{t}}\right)^2\right)^{\frac{1}{2}} \frac{d\bar{t}}{dt} - (1 + \dot{x}^2)^{\frac{1}{2}} = \left(1 + \left(\frac{\dot{x} + s}{1 - s\dot{x}}\right)^2\right)^{\frac{1}{2}} (1 - s\dot{x}) - \left(1 + \dot{x}^2\right)^{\frac{1}{2}}\\ &= \left((1 - s\dot{x})^2 + (\dot{x} + s)^2\right)^{\frac{1}{2}} - \left(1 + \dot{x}^2\right)^{\frac{1}{2}} = \left((1 + \dot{x}^2)(1 + s^2)\right)^{\frac{1}{2}} - \left(1 + \dot{x}^2\right)^{\frac{1}{2}}\\ &= (1 + \dot{x}^2)^{\frac{1}{2}}[(1 + s^2)^{\frac{1}{2}} - 1] \quad \text{we use binomial expansion to expand} \quad (1 + s^2)^{\frac{1}{2}}\\ &= (1 + \dot{x}^2)^{\frac{1}{2}}(1 + \frac{1}{2}s^2 - \frac{1}{8}s^4 + \frac{1}{16}s^6 - \cdots - 1)\\ &= (1 + \dot{x}^2)^{\frac{1}{2}}(\frac{1}{2}s^2 - \frac{1}{8}s^4 + \frac{1}{16}s^6 - \cdots) \quad \text{for} \quad s \to 0\\ &= 0_{(s)}.\end{aligned}$$

Therefore, the functional $J(x)$ is absolutely invariant under the rotation group.

2.3.7 The Fundamental Invariance Identities. The fundamental integral can be invariant either absolutely or up to an exact differential, under an $z-$parameter family of continuous transformations, the corresponding Lagrangian must satisfy certain conditions involving the Lagrangian, its derivatives, and the infinitesimal generators of the continuous transformations. On applying the Euler-Lagrange expressions, the resulting formulation is known as Noether's theorem.

Theorem 2.3.8. *(Necessary condition for the fundamental integral). If the variational integral* (2.3.1) *is divergence-invariant under the* $z-$*parameter transformations equations* (2.3.2) *and* (2.3.3)*, then the Lagrangian* $F(t, x, \dot{x})$ *and its derivatives satisfy the* $z-$*identities*

$$\frac{\partial F}{\partial t}\tau_r + \frac{\partial F}{\partial x^k}\cdot \xi_r^k + \frac{\partial F}{\partial \dot{x}^k}\cdot\left(\frac{d\xi_r^k}{dt} - \dot{x}^k\frac{d\tau_r}{dt}\right) + F\frac{d\tau_r}{dt} = \frac{d\Phi_r}{dt},$$

$$\textit{where} \quad (r = 1, \dots, z), \quad \tau_r = \frac{\partial \phi(t, x, 0)}{\partial s^r}, \quad \xi_r^k = \frac{\partial \psi^k(t, x, 0)}{\partial s^r}.$$

Proof. From invariant definition of equation (2.3.8), we know that

$$F\left(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}(\bar{t})}{d\bar{t}}\right)\frac{d\bar{t}}{dt} - F\left(t, x(t), \frac{dx(t)}{dt}\right) = s^r\frac{d\Phi_r}{dt}(t, x(t)) + 0(s), \tag{2.3.9}$$

then we introduce the infinitesimal transformation of coordinates (2.3.4) and (2.3.5) in equation (2.3.9) as follows:

$$F\left(t + s^r\tau_r + 0_{(s)}, x^k + s^r\xi_r^k + 0_{(s)}, \frac{\dot{x}^k + s^r\dot{\xi}_r^k + 0_{(s)}}{1 + s^r\dot{\tau}_r + 0_{(s)}}\right)\frac{d\bar{t}}{dt} - F\left(t, x(t), \dot{x}(t)\right) = s^r\frac{d\Phi}{dt}. \tag{2.3.10}$$

Differentiating both sides of equation (2.3.10) with respect to parameter s^r and substituting $s = 0$, we obtain

$$\frac{\partial F}{\partial t}\tau_r + \frac{\partial F}{\partial x^k}\xi_r^k + \frac{\partial F}{\partial \dot{x}^k}\left(\frac{d\xi_r^k}{dt} - \dot{x}^k\frac{d\tau_r}{dt}\right) + F\frac{d\tau_r}{dt} = \frac{d\Phi_r}{dt}. \tag{2.3.11}$$

Remark 2.3.9. Theorem (2.3.8) is useful to verify invariance and to compute the infinitesimal generators τ and ξ of the transformations.

□

2.3.10 Noether's Theorem and Conservation laws. The classical theorem of Noether on invariant variational problems can be derived from the fundamental invariance identities.

Theorem 2.3.11. *(Noether's Theorem). If fundamental integral is invariant under transformations* (2.3.2) *and* (2.3.3)*, then there exist* z *identities of the form*

$$\frac{d}{dt}\left(\left(F - \dot{x}^k\frac{\partial F}{\partial \dot{x}^k}\right)\tau_r + \frac{\partial F}{\partial \dot{x}^k}\xi_r^k - \Phi_r\right) = 0, \quad (r = 1, \dots, z),$$

i.e.,

$$\left(F - \dot{x}^k\frac{\partial F}{\partial \dot{x}^k}\right)\tau_r + \frac{\partial F}{\partial \dot{x}^k}\xi_r^k - \Phi_r = \textit{constant} \tag{2.3.12}$$

along all the solutions of Euler-Lagrange's equations (2.2.5)*.*

Proof. We note that

$$\frac{\partial F}{\partial \dot{x}^k}\frac{d\xi_r^k}{dt} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\xi_r^k\right) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\right)\xi_r^k$$

and

$$\frac{\partial F}{\partial t}\tau_r = \left(\frac{dF}{dt} - \frac{\partial F}{\partial x^k}\dot{x}^k - \frac{\partial F}{\partial \dot{x}^k}\ddot{x}\right)\tau_r,$$

likewise,

$$\frac{\partial F}{\partial \dot{x}^k}\dot{x}^k\frac{d\tau_r}{dt} + \frac{\partial F}{\partial \dot{x}^k}\ddot{x}^k\tau_r = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\dot{x}^k\tau_r\right) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\right)\dot{x}^k\tau_r\,.$$

Substitute these three relations into (2.3.11), we get

$$\left[\frac{\partial F}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\right)\right]\xi_r^k - \left[\frac{\partial F}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\right)\right]\dot{x}^k\tau_r + \frac{d}{dt}\left[F\tau_r + \frac{\partial F}{\partial \dot{x}^k}\xi_r^k - \frac{\partial F}{\partial \dot{x}^k}\dot{x}^k\tau_r\right] = \frac{d\Phi_r}{dt}, \quad (r = 1, \ldots, z)\,.$$

But

$$\frac{\partial F}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\right) = 0,$$

which can be expressed as

$$\frac{d}{dt}\left[F\tau_r + \frac{\partial F}{\partial \dot{x}^k}\xi_r^k - \frac{\partial F}{\partial \dot{x}^k}\dot{x}^k\tau_r - \Phi_r\right] = 0.$$

Thus, can be written as follows:

$$\frac{d}{dt}\left(\left(F - \dot{x}^k\frac{\partial F}{\partial \dot{x}^k}\right)\tau_r + \frac{\partial F}{\partial \dot{x}^k}\xi_r^k - \Phi_r\right) = 0 \quad (r = 1, \ldots, z). \tag{2.3.13}$$

□

Remark 2.3.12. Equation (2.3.12) in economic and physics applications are interpreted as the conservation laws of the system.

3. Optimal Control

3.1 Introduction

Optimal control is the standard method for solving dynamic optimization problems for which the calculus of variations is not appropriate, for example consider those involving constraints on the derivatives of functions.

In optimal control problems we have two basic concepts such as the control system which is described by ordinary differential equations (3.1.2) and the cost function which assigns a cost value to each admissible control (3.1.1).

Now consider the simple optimal control problem

$$J(u) = \int_a^b F(t, x(t), u(t))dt, \tag{3.1.1}$$

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(a) = x_0, \quad x(b) = x_1, \tag{3.1.2}$$

where the Lagrangian $F : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and the velocity vector $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are assumed to be C^1 functions with respect to all the arguments. In agreement with the calculus of variations, we assume the admissible state trajectories to be piece-wise smooth, and the admissible control functions to be piece-wise constant with no restrictions on their values: $x(\cdot) \in PC^1([a, b]; \mathbb{R}^n)$, $u(\cdot) \in PC([a, b]; \mathbb{R}^m)$.

Remark 3.1.1. The fundamental problem of the calculus of variations,

$$I[q(\cdot)] = \int_a^b F\left(t, q(t), \dot{q}(t)\right) \longrightarrow \min\,, \tag{3.1.3}$$

is a particular case of problem (3.1.1) and (3.1.2): In that case $f(t, q, u) = u$.

3.2 The Maximum Principle

An alternative way for obtaining optimal strategies for systems of optimal control problem is the maximum principle. Classical variational theory could not easily handle some constraints involved in a realistic optimal control problem, so to solve this we use the maximum principle (Hawary and Christensen, 1979).

3.2.1 Basic Fixed-Endpoint Control Problem. Consider the Basic Fixed-Endpoint Control Problem of the optimal control problem with the following additional specifications $f = f(x, u)$ and $F = F(x, u)$ with no t-argument such as the control system and the running cost which are time-independent but f, f_x, F and F_x are continuous. The target set is $S = [a, \infty) \times \{x_1\}$ which is a free-time, fixed-endpoint problem and $K \equiv 0$, if the terminal cost is absent (Liberzon, 2012), section 3.3.2. In this special problem, the maximum principle takes the following definition and is followed by the theorem.

Definition 3.2.2. An admissible pair $(x(.), u(.))$ satisfying the control system $\dot{x}(t) = f(t, x(t), u(t))$ of optimal control problem (3.1.1) and (3.1.2) is called a process, where $t \in [a, b]$ (Frederico and Torres, 2007).

Theorem 3.2.3 (Maximum Principle for the Basic Fixed- Endpoint Control Problem)**.** *Suppose u^* : $[a, b] \mapsto U$ is an optimal control in the global sense and $x^* : [a, b] \mapsto \mathbb{R}^n$ is the corresponding optimal state trajectory. Then there exist a function $p^* : [a, b] \mapsto \mathbb{R}^n$ such that the following properties hold:*

1. *x^* and p^* satisfy the canonical equations*

$$\dot{x^*} = \frac{\partial H}{\partial p}(t, x^*, u^*, p^*), \tag{3.2.1}$$

$$\dot{p^*} = -\frac{\partial H}{\partial x}(t, x^*, u^*, p^*) \tag{3.2.2}$$

with the boundary conditions $x^*(a) = x_0$ and $x^*(b) = x_1$.

2. *For each fixed t, the function $u \mapsto H(t, x^*(t), u, p^*(t), p_0^*)$ has a global maximum at $u = u^*$, i.e., the inequality*

$$H((t, x^*(t), u^*(t), p^*(t)) \geq H((t, x^*(t), u, p^*(t)) \tag{3.2.3}$$

holds for all $t \in [a, b]$ and all $u \in U$.

3.

$$\frac{\partial H}{\partial u}((t, x^*(t), u^*(t), p^*(t)) = 0 \tag{3.2.4}$$

for all $t \in [a, b]$,

where the Hamiltonian $H : [a, b] \times \mathbb{R}^n \times U \times \mathbb{R}^n$ is defined as

$$H(t, x, u, p) = p \cdot f(t, x, u) - F(t, x, u). \tag{3.2.5}$$

Proof. The Maximum Principle is a necessary optimality condition which can be obtained from a general Lagrange multiplier theorem in spaces of infinite dimension. Let us introduce the Hamiltonian function

$$H(t, x, u, p) = p \cdot f(t, x, u) - F(t, x, u), \tag{3.2.6}$$

where p is the Lagrange multiplier or generalized momenta. The multiplier theorem insists that the optimal control problem is equivalent to the maximization of the augmented functional.

$$J[x(.), u(.), p(.)] = \int_a^b H\left(t, x(t), u(t), p(t)\right) - p(t) \cdot \dot{x}(t))\, dt. \tag{3.2.7}$$

Suppose these $(x^*(.), u^*(.), p^\cdot(.))$ solve the problem and consider arbitrary C^1 functions $\eta_1, \eta_3 : [a, b] \mapsto \mathbb{R}^n, \eta_1(.)$ vanishing at a and b $(\eta_1(.) \in C_0^1([a, b]))$ and arbitrary continuous $\eta_2 : [a, b] \mapsto \mathbb{R}^m$. And suppose s be a scalar. By the definition of the maximizer, we have

$$J\left[(x^* + s\eta_1)(.), (u^* + s\eta_2)(.), (p^* + s\eta_3)(.)\right] \leq J[x^*(.), u^*(.), p^*(.)]. \tag{3.2.8}$$

Differentiating (3.2.8) with respect to s and setting $s = 0$, we have

$$\frac{d}{ds} J\left[(x^* + s\eta_1)(.), (u^* + s\eta_2)(.), (p^* + s\eta_3)(.)\right]\Big|_{s=0} \tag{3.2.9}$$

$$= \int_a^b \left[\frac{\partial H}{\partial x} \cdot \eta_1(t) + \frac{\partial H}{\partial u} \cdot \eta_2(t) + \frac{\partial H}{\partial p} \cdot \eta_3(t) - \eta_3(t) \cdot \dot{x}^*(t) - p^*(t) \cdot \dot{\eta}_1(t)\right] dt = 0. \tag{3.2.10}$$

Let us integrate the term $p^*(t) \cdot \dot{\eta}_1(t)$, by parts, in order to eliminate $\dot{\eta}_1(t)$

$$\int_a^b \left[\frac{\partial H}{\partial x} \cdot \eta_1(t) + \frac{\partial H}{\partial u} \cdot \eta_2(t) + \frac{\partial H}{\partial p} \cdot \eta_3(t) - \eta_3(t) \cdot \dot{x}^*(t) + \dot{p}^*(t) \cdot \eta_1(t) \right] dt - p^*(t) \cdot \eta_1(t) \Big|_a^b$$

and we substitute the limit for $\eta_1(a) = \eta_1(b) = 0$, then we obtain

$$\int_a^b \left[\left(\frac{\partial H}{\partial x} + \dot{p}^*(t) \right) \cdot \eta_1(t) + \frac{\partial H}{\partial u} \cdot \eta_2(t) + \left(\frac{\partial H}{\partial p} - \dot{x}^*(t) \right) \cdot \eta_3(t) \right] dt = 0. \tag{3.2.11}$$

Since we know that 3.2.11 was obtained for any variation $\eta_1(.), \eta_2(.)$ and $\eta_3(.)$. Now choosing $\eta_1(t) = \eta_2(t) = 0$ and $\eta_3(.)$ arbitrary, we obtain the control system (3.2.1)

$$\dot{x}^*(t) = \frac{\partial H}{\partial p} (t, x^*(t), u^*(t), p^*(t)), \quad t \in [a, b]. \tag{3.2.12}$$

Again, with $\eta_1(.)$ arbitrary and $\eta_2(t) = \eta_3(t) = 0$, we obtain the adjoint system (3.2.2)

$$\dot{p}^*(t) = -\frac{\partial H}{\partial x} (t, x^*(t), u^*(t), p^*(t)), \quad t \in [a, b]. \tag{3.2.13}$$

Finally, with $\eta_2(.)$ arbitrary and $\eta_1(t) = \eta_3(t) = 0$, we obtain the stationary condition (3.2.4)

$$\frac{\partial H}{\partial u} (t, x^*(t), u^*(t), p^*(t)) = 0, \quad t \in [a, b]. \tag{3.2.14}$$

The proof is completed. □

Definition 3.2.4. Any triplet $(x(.), u(.), p(.))$ satisfying the conditions of theorem (3.2.3) is called maximum extremal (Frederico and Torres, 2007).

3.2.5 Basic Variable-Endpoint Control Problem. The Basic variable endpoint control problem is similar to the basic fixed endpoint control problem except the target set of the form $S = [a, \infty) \times S_1$, where S_1 is a $k-$dimensional surface in $\mathbb{R}^n$ for some non-negative integer $k \leq n$. By constrained optimization, (Liberzon, 2012) the tangent space can be defined as a surface of equality constraints

$$S_1 = \{x \in \mathbb{R}^n : h_1(x) = h_2(x) = \cdots = h_{n-k}(x) = 0\},$$

where $h_i, \quad i = 1, \dots, n-k$ are C^1 functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Also we suppose that every $x \in S_1$ is a regular point. For case $k = n$, we obtain $S_1 = \mathbb{R}^n$ (which gives a free-time, free-endpoint problem). On the other hand, if $k = 0$, the surface S_1 decreases either to a single point like Basic fixed endpoint control problem, or to a set consisting of individual points.

Remark 3.2.6. The main difference between the maximum principle for a basic fixed endpoint control problem and the basic variable endpoint control problem is that it lies only on the boundary conditions for the system of canonical equations.

Theorem 3.2.7 (Maximum Principle for the Basic Variable Endpoint Control Problem). *Suppose u^* : $[a, b] \mapsto U$ is an optimal control in the global sense and $x^* : [a, b] \mapsto \mathbb{R}^n$ is the corresponding optimal state trajectory. Then there exist a function $p^* : [a, b] \mapsto \mathbb{R}^n$ and a constant $p_0^* \leq 0$ satisfying $p_0^*, p^* \neq (0, 0)$ for all $t \in [a, b]$ and having the following properties:*

1. *x^* and p^* satisfy the canonical equations in Theorem (3.2.3) with respect to the Hamiltonian defined in (3.2.5), with boundary conditions $x^*(a) = x_0$ and $x^*(b) \in S_1$.*

2. *$H((x^*(t), u^*(t), p^*(t), p_0^*) \geq H((x^*(t), u, p^*(t), p_0^*)$, holds for all $t \in [a, b]$ and all $u \in U$.*
3. *$H((x^*(t), u^*(t), p^*(t), p_0^*) = 0$, for all $t \in [a, b]$.*
4. *The vector $p^*(b)$ is perpendicular to the tangent space to S_1 at $x^*(b)$*

$$p^*(b) \cdot d = 0, \quad \forall d \in T_{x^*(b)} S_1. \tag{3.2.15}$$

The interpretation of this theorem

The necessary condition (3.2.15) is called the transversality condition. By constrained optimization, (Liberzon, 2012), we know that the tangent space can be written as

$$T_{x^*(b)} S_1 = \{d \in \mathbb{R}^n : \langle \nabla g_i(x^*(b)), d \rangle = 0, \quad \text{where} \quad i = 1, \ldots, n-k\}. \tag{3.2.16}$$

Also $p^*(b)$ from (3.2.15) can be written as a linear combination of the gradient vectors $\nabla g_i(x^*(b)$, where $i = 1, \ldots, n-k$.

In the case of $n = k$, this implies that $S_1 = \mathbb{R}^n$, the transversality condition decreases to $p^*(b) = 0$ since the tangent space is the entire $\mathbb{R}^n$. On the other hand, when $S_1 = \{x_1\}$, its tangent space is 0.

In general the basic fixed endpoint control problem and the basic variable endpoint control problem, we have considering n boundary conditions imposed on (x^*, p^*) at $t = a$ and n more at $t = b$. This situation gives the correct total number of boundary conditions to specify a solution of the $2n-$ dimensional system (3.2.1) and (3.2.2). In the basic fixed endpoint control problem, $x^*(b)$ was fixed and $p^*(b)$ was free while in the case of basic variable endpoint control problem, we have k degrees of freedom for $x^*(b) \in S_1$ but only $n-k$ degrees of freedom for $p^*(b)$ is perpendicular with $T_{x^*(b)} S_1$. This implies that the freer the state, the less free the co-state. Furthermore, each additional degree of freedom for $x^*(b)$ eliminates one degree of freedom for $p^*(b)$.

3.3 Noether's Theorem For Optimal Control

Emmy Noether established the main fundamental result to find conservation laws in the calculus of variations. Now we illustrate the standard argument used to derive Noether's theorem and conservation laws in the optimal control sense (Rocha and Torres, 2006).

Consider a one-parameter group of C^1 transformations of the form

$$\eta^s(t, x, u, p) = (\eta_t^s(t, x, u, p), \eta_x^s(t, x, u, p), \eta_u^s(t, x, u, p), \eta_p^s(t, x, u, p)), \tag{3.3.1}$$

where s indicates the independent parameter of the transformations. Setting the parameter value $s = 0$ correspond to the identity transformation

$$\eta^0(t, x, u, p) = (\eta_t^0(t, x, u, p), (\eta_x^0(t, x, u, p), (\eta_u^0(t, x, u, p), (\eta_p^0(t, x, u, p)) = (t, x, u, p). \tag{3.3.2}$$

Consider the infinitesimal generators associated with the group of transformations (3.3.1)

$$\begin{aligned} T(t, x, u, p) &= \frac{d}{ds} \eta_t^s(t, x, u, p) \bigg|_{s=0}, \quad X(t, x, u, p) = \frac{d}{ds} \eta_x^s(t, x, u, p) \bigg|_{s=0}, \\ U(t, x, u, p) &= \frac{d}{ds} \eta_u^s(t, x, u, p) \bigg|_{s=0}, \quad P(t, x, u, p) = \frac{d}{ds} \eta_p^s(t, x, u, p) \bigg|_{s=0}. \end{aligned} \tag{3.3.3}$$

Definition 3.3.1. The optimal control problem is said to be invariant under a one parameter group of C^1 transformations (3.3.1) if, and only if,

$$\frac{d}{ds}\left\{\left[H(\eta^s(t,x(t),u(t),p(t))) - \eta_p^s(t,x(t),u(t),p(t)) \cdot \frac{\frac{d\eta_x^s(t,x(t),u(t),p(t))}{dt}}{\frac{d\eta_t^s(t,x(t),u(t),p(t))}{dt}}\right] \cdot \frac{d\eta_t^s(t,x(t),u(t),p(t))}{dt}\right\}\bigg|_{s=0} = 0, \tag{3.3.4}$$

where H is the Hamiltonian (refer equation (3.2.5)).

The condition (3.3.4) is equivalent to

$$\frac{d}{ds}\left\{H(\eta^s(t,x(t),u(t),p(t)))\frac{d\eta_t^s(t,x(t),u(t),p(t))}{dt} - \eta_p^s(t,x(t),u(t),p(t))\frac{d\eta_x^s(t,x(t),u(t),p(t))}{dt}\right\}\bigg|_{s=0} = 0. \tag{3.3.5}$$

From the identity transformation (3.3.2) and the infinitesimal generators in (3.3.3), the equation (3.3.5) after differentiating with respect to s, we obtain

$$\frac{\partial H}{\partial t}T + H\frac{d}{dt}T + \frac{\partial H}{\partial x} \cdot X + \frac{\partial H}{\partial u} \cdot U + \frac{\partial H}{\partial p} \cdot P - P \cdot \dot{x}(t) - p(t) \cdot \frac{d}{dt}X = 0, \tag{3.3.6}$$

where all functions are evaluated at $(t, x(t), u(t), p(t))$ unless otherwise indicated. Along Maximum extremal $x(.), u(.), p(.)$ equalities, we have seen in (3.2.12), (3.2.13) and (3.2.14), respectively ,as

$$\dot{x^*}(t) = \frac{\partial H}{\partial p}, \quad \dot{p^*}(t) = -\frac{\partial H}{\partial x}, \quad \frac{\partial H}{\partial u} = 0. \tag{3.3.7}$$

Remark 3.3.2. If $x(.), u(.)$ are minimizers or a maximizers of optimal control, then there exists a non-zero pair $p_0, p(.)$ such that $(x(.), u(.), p_0, p(.))$ is a maximum extremal of optimal control. Furthermore, the function $H(t, x(t), u(t), p_0, p(t))$ is an absolutely continuous function of t and satisfies the equality (Torres, 2002)

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}. \tag{3.3.8}$$

Now by substituting (3.3.7) and (3.3.8) into (3.3.6), we get

$$T\frac{dH}{dt} - \dot{p}(t) \cdot X - p(t) \cdot \frac{dX}{dt} + H\frac{dT}{dt} = 0. \tag{3.3.9}$$

Then, (3.3.9) can be written as

$$\frac{d}{dt}(p(t) \cdot X - HT) = 0. \tag{3.3.10}$$

Hence, we have proved Noether's Theorem for optimal control problems.

Theorem 3.3.3 (Noether's Theorem)**.** *If the optimal control problem is invariant under* (3.3.1)*, in the sense of definition* (3.3.1)*, then*

$$p(t) \cdot X(t,x(t),u(t),p(t)) - H(t,x(t),u(t),p(t))T(t,x(t),u(t),p(t)) = \textit{const} \tag{3.3.11}$$

along all the minimizers $x(.), u(.)$ of optimal control which are Maximum extremal.

4. Applications In Economics

4.1 Introduction

Both the Calculus of Variation and Optimal Control Problem are important methods in optimization problems. Now a days most companies find ways for maximizing their profits in a very complex environment. There are many factors involved in a company making a profit, these factors are always changing (dynamic). These dynamic factors can be inventory level, production rate, production cost, consumption, inventory holding cost, etc, and they all depend on time.

We also have external factors which can influence profit such as market conditions and the nature of the customers. But these cannot be controlled by the company. Economic models are created to assist companies to better predict future condition and decision-making. Creating, analysing and solving these complicated models will become more important as our society becomes more complex and more global.

4.2 Application I

In this application we will use Noether's Theorem and conservation laws to apply to economic models such as the Ramsey Growth model.

4.2.1 The Ramsey Growth Model. These model are described with the linear transformation function between consumption and capital accumulation such as

$$c = c(\dot{k}, k) = f(k) - nk - \dot{k}, \quad n \geq 0, \tag{4.2.1}$$

where c is the consumption (per capita C/L), $f(k) = Y/L$ the neoclassical (aggregate) production function (per capita) with $f'(k) > 0, f''(k) < 0, k = K/L$ is capita labour ratio, n is the sum of the constant population growth rate and the depreciation rate, and $\dot{k} = \frac{dk}{dt}$ the rate of capita accumulation (Webots). The main characteristics of this model is that the production possibility boundary is linear. Since the model is linear, then

$$\frac{\partial^2 c}{\partial \dot{k}^2} \equiv 0. \tag{4.2.2}$$

The objective of society is to maximize the social welfare functional

$$\begin{aligned} \text{max} \quad & J(k) = \int_0^T e^{-\rho t} U(c) dt, \\ \text{subject to} \quad & \dot{k} = f(k) - c - nk, \quad k(0) = k_0, \end{aligned} \tag{4.2.3}$$

where U is a utility function with properties $U' > 0, \quad U'' < 0$, and $\rho \geq 0$ is the fixed discount rate. The terminal time T can be finite or infinite but most in cases we consider T as infinite. Substitute (4.2.1) into (4.2.3) we get the standard calculus of variation. Then consider the Lagrangian

$$F(t, k(t), \dot{k}(t)) = e^{-\rho t} U[h(k(t)) - \dot{k}(t)], \tag{4.2.4}$$

$$\text{where} \quad h(k) = f(k) - nk. \tag{4.2.5}$$

Now let us consider the family of transformations below under which (4.2.3) is dynamically invariant

$$\bar{t} = t + s\tau(t,k), \tag{4.2.6}$$
$$\bar{k} = k + s\xi(t,k). \tag{4.2.7}$$

Recall the invariance identity (2.3.11), first we suppose that the fundamental integral (2.3.1) is absolutely invariant such that $\frac{d\Phi}{dt} = 0$, then we calculate the equation (4.2.4) as follows:

$$\frac{\partial F}{\partial t} = -\rho e^{-\rho t}U, \quad \frac{\partial F}{\partial k} = e^{-\rho t}U'h' \quad \text{and} \quad \frac{\partial F}{\partial \dot{k}} = -e^{-\rho t}U'. \tag{4.2.8}$$

Then substitute (4.2.8) into the invariance identity (2.3.11), we get

$$U\left(\frac{d\tau}{dt} - \rho\tau\right) + U'\left(h'\xi - \frac{d\xi}{dt} + \dot{k}\frac{d\tau}{dt}\right) = 0. \tag{4.2.9}$$

This is called the fundamental invariance equation for the simple Ramsey models. The problem (4.2.3) can be invariant under any arbitrary form of U and conservation laws hold for any utility function, then (4.2.9) means that

$$\frac{d\tau}{dt} - \rho\tau = 0, \tag{4.2.10}$$
$$h'\xi - \frac{d\xi}{dt} + \dot{k}\frac{d\tau}{dt} = 0. \tag{4.2.11}$$

By introducing total and partial derivatives, we have $\frac{d\tau}{dt} = \frac{\partial\tau}{\partial t} + \dot{k}\frac{\partial\tau}{\partial k}$ and $\frac{d\tau}{dt} = \frac{\partial\xi}{\partial t} + \dot{k}\frac{\partial\xi}{\partial k}$, substitute these into (4.2.10) and (4.2.11), we obtain

$$\frac{\partial\tau}{\partial t} + \dot{k}\frac{\partial\tau}{\partial k} - \rho\tau = 0, \tag{4.2.12}$$
$$h'\xi - \left(\frac{\partial\xi}{\partial t} + \dot{k}\frac{\partial\xi}{\partial k}\right) + \dot{k}\left(\frac{\partial\tau}{\partial t} + \dot{k}\frac{\partial\tau}{\partial k}\right) = 0. \tag{4.2.13}$$

Let us rearrange the coefficients of the power of $\dot{k}$ as follows:

$$\left(\frac{\partial\tau}{\partial t} - \rho\tau\right)\dot{k}^0 + \frac{\partial\tau}{\partial k}\dot{k}^1 = 0, \tag{4.2.14}$$
$$\left(h'\xi - \frac{\partial\xi}{\partial t}\right)\dot{k}^0 + \left(\frac{\partial\tau}{\partial t} - \frac{\partial\xi}{\partial k}\right)\dot{k}^1 + \frac{\partial\tau}{\partial k}\dot{k}^2 = 0. \tag{4.2.15}$$

Putting these coefficients equal to zero, then we obtain the invariance conditions

$$\frac{\partial\tau}{\partial t} - \rho\tau = 0, \tag{4.2.16}$$
$$\frac{\partial\tau}{\partial k} = 0, \tag{4.2.17}$$
$$h'\xi - \frac{\partial\xi}{\partial t} = 0, \tag{4.2.18}$$
$$\frac{\partial\tau}{\partial t} - \frac{\partial\xi}{\partial k} = 0. \tag{4.2.19}$$

Since $\frac{\partial\tau}{\partial k} = 0$ this implies that $\tau = \tau(t)$, then from (4.2.16), we have

$$\tau = ae^{\rho t}. \tag{4.2.20}$$

Taking partial derivative of equation (4.2.20) with respect to t, we have $\frac{\partial\tau}{\partial t} = a\rho e^{\rho t}$. From equation (4.2.18), we get

$$\xi = A(k)e^{h'(k)t}. \tag{4.2.21}$$

Since $\frac{\partial\tau}{\partial t} = \frac{\partial\xi}{\partial k}$ and $\tau = ae^{\rho t}$, we can show that $\frac{\partial^2\xi}{\partial k^2} \equiv 0$. Differentiating (4.2.21) with respect to k twice and using the fact that $\frac{\partial^2\xi}{\partial k^2} \equiv 0$, we have

$$t^2 h''(k)^2 A(k) + t^1[h''(k)A'(k) + (h''(k)A(k))'] + t^0 A''(k) = 0. \tag{4.2.22}$$

Setting the coefficients of power of t equal to zero, we get

$$A''(k) = 0, \tag{4.2.23}$$
$$h''(k)^2 A(k) = 0, \tag{4.2.24}$$
$$h''(k)A'(k) + (h''(k)A(k))' = 0. \tag{4.2.25}$$

Now we can consider two cases depending upon $h''(k) \neq 0$ or $h''(k) = 0$.

case 1. $h''(k) \neq 0$, this implies that $f''(k) \neq 0$ from (4.2.5) such that $f(k) \neq \alpha k + \beta$, where α and β are constants.. Therefore, in equation (4.2.24) $A(k) = 0$, so from (4.2.21) it imply that $\xi = 0$ (which means no conservation law with respect to $\bar{k}$). Again, from (4.2.19) it implies that $\frac{\partial\tau}{\partial t} = 0$, since $\xi = 0$. Then substitute this into (4.2.16), we get $\rho\tau = 0$, which means

$$\rho \neq 0, \quad \tau = 0. \tag{4.2.26}$$
$$\text{Hence} \quad \xi = 0, \quad \tau = 0 \quad \text{No conservation law.} \tag{4.2.27}$$
$$\rho = 0 \quad \text{(zero discount rate)}, \quad \tau = 1 \quad \text{(time translation)}. \tag{4.2.28}$$
$$\text{Hence} \quad \xi = 0, \quad \tau = \text{const} = 1 \quad \text{time translation.} \tag{4.2.29}$$

Substituting (4.2.29) into Noether's Theorem (2.3.11) and knowing that $\frac{d\Phi}{dt} = 0$, we have

$$\Omega = F\tau - \frac{\partial F}{\partial \dot{k}}\dot{k}\tau \quad \text{or we can write this by referring to (4.2.8) as}$$
$$\Omega = U + U'\dot{k} = \text{Hamiltonian=const}(\tau = 1). \tag{4.2.30}$$

This is a general production for $f(k) \neq \alpha k + \beta$.

case 2. $h''(k) = 0$, this implies that $f''(k) = 0$ such that $f(k) = \alpha k + \beta, \alpha > 0$ or $h(k) = (\alpha - n)k + \beta$. But from (4.2.23) $A''(k) = 0$ this implies that $A(k) = bk + d$ is linear. Substituting this into (4.2.21), we obtain

$$\xi = (bk + d)e^{(\alpha - n)t}, \quad h' = (\alpha - n). \tag{4.2.31}$$

Then, equation (4.2.31) can be verified in equation (4.2.18).

Now let us find the relationship of ξ and τ in (4.2.19). Differentiate (4.2.31) with respect to k and referring (4.2.19), we get

$$\frac{\partial\xi}{\partial k} = be^{(\alpha - n)t} = \rho a e^{\rho t} = \frac{\partial\tau}{\partial t}. \tag{4.2.32}$$

Then, by comparing the coefficient and its exponential, we get $b = \rho a$ and $\rho = \alpha - n$ or $\alpha = \rho + n$. These give the final form of the generator of the group as follows:

$$\tau = ae^{\rho t}, \tag{4.2.33}$$
$$\xi = (a\rho k + d)e^{\rho t}. \tag{4.2.34}$$

From $h(k) = (\alpha - n)k + \beta$, we have $h(k) = \rho k + \beta = f(k) - nk = \alpha k + \beta - nk$ or $f(k) = h(k) + nk = \rho k + \beta + nk = (\rho + n)k + \beta$, where a, β and d are arbitrary constants.

Form (4.2.4), $F = e^{-\rho t}U(\rho k + \beta - \dot{k})$ and refer Noether's Theorem (2.3.11) and have in mind that $\frac{d\Phi}{dt} = 0$ in the case of absolutely invariant, we have

$$\begin{aligned}
\Omega &= F\tau + \frac{\partial F}{\partial \dot{k}}(\xi - \dot{k}\tau) = \left(F - \dot{k}\frac{\partial F}{\partial \dot{k}}\right)\tau + \frac{\partial F}{\partial \dot{k}}\xi \\
&= -\text{Hamiltonian}.\tau + \frac{\partial F}{\partial \dot{k}}\xi = \text{const} \\
&= e^{-\rho t}Uae^{\rho t} - e^{-\rho t}U'[(a\rho k + d)e^{\rho t} - \dot{k}ae^{\rho t}] \\
&= aU - U'[a(h(k) - \dot{k}) + d - a\beta], \quad \text{where} \quad (h(k) = \rho k + \beta), \\
\text{or} \quad \Omega &= a[U(c) - U'(c)c] + (a\beta - d)U'(c) = \text{const}, \quad \text{where} \quad (h(k) - \dot{k} = c).
\end{aligned} \tag{4.2.35}$$

This is a linear technology for $f(k) = (\rho + n)k + \beta$.

We have already shown the conservation laws under the definition of absolute invariance. Now let us consider the conservation laws under the more general definition of divergence invariant (definition (2.3.8)). Consider the invariance of Theorem (2.3.8) as follows:

$$\frac{\partial F}{\partial t}\tau + \frac{\partial F}{\partial k}\xi + \frac{\partial F}{\partial \dot{k}}\left(\frac{d\xi}{dt} - \dot{k}\frac{d\tau}{dt}\right) + F\frac{d\tau}{dt} = \frac{d\Phi}{dt},$$

where $\Phi = \Phi(t, k)$ and $F = e^{-\rho t}U(h(k) - \dot{k})$. Substituting (4.2.8) into the equation above and have in mind that $\frac{d\xi}{dt} = \frac{\partial \xi}{\partial t} + \dot{k}\frac{\partial \xi}{\partial k}$, $\frac{d\tau}{dt} = \frac{\partial \tau}{\partial t} + \dot{k}\frac{\partial \tau}{\partial k}$ and $\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \dot{k}\frac{\partial \Phi}{\partial k}$, also we substitute these into the above equation and by differentiating partially twice with respect to k, we get

$$U'''\left\{\left(h'\xi - \frac{\partial \xi}{\partial t}\right) - \left(\frac{\partial \xi}{\partial k} - \frac{\partial \tau}{\partial t}\right)\dot{k} + \frac{\partial \tau}{\partial k}\dot{k}^2\right\} + U''\left\{\left(\frac{\partial \tau}{\partial t} - \rho\tau\right) + 2\left(\frac{\partial \xi}{\partial k} - \frac{\partial \tau}{\partial t}\right) - 3\frac{\partial \tau}{\partial k}\dot{k}\right\} = 0.$$

Setting the coefficients of the powers of k to zero, then the conservation laws may hold for any arbitrary utility function and we have

$$h'\xi - \frac{\partial \xi}{\partial t} = 0, \quad \frac{\partial \tau}{\partial k} = 0, \quad \frac{\partial \xi}{\partial k} - \frac{\partial \tau}{\partial t} = 0, \quad \frac{\partial \tau}{\partial t} - \rho\tau + 2\left(\frac{\partial \xi}{\partial k} - \frac{\partial \tau}{\partial t}\right) = 0.$$

This implies that

$$\frac{\partial \tau}{\partial t} - \rho\tau = 0. \quad \text{Since,} \quad \frac{\partial \xi}{\partial k} - \frac{\partial \tau}{\partial t} = 0, \tag{4.2.36}$$
$$\frac{\partial \tau}{\partial k} = 0, \tag{4.2.37}$$
$$h'\xi - \frac{\partial \xi}{\partial t} = 0, \tag{4.2.38}$$
$$\frac{\partial \xi}{\partial k} - \frac{\partial \tau}{\partial t} = 0. \tag{4.2.39}$$

This is exactly the same as in the case of absolute invariance obtained in equations (4.2.16)–(4.2.19). Therefore, there exist no additional conservation laws even if we allow for general definition of invariance.

4.2.2 Economic Interpretation. We consider a economic interpretation of Noether's Theorem (2.3.11), when it is applied to a typical welfare maximization problem with $n-$state variables.

If a welfare function is dynamically invariant under an z-parameter family of transformations resulting from technical change or taste change, then the following quantities are constant along any optimal path for the entire planning period (Webots).

(Hamilt.) $\begin{pmatrix}\text{Infinitesimal trans}\\ \text{formations of time}\end{pmatrix}+\sum_{i=1}^{n}\begin{pmatrix}\text{implicit}\\ \text{price of} \quad k_i\end{pmatrix}\begin{pmatrix}\text{infinitesimal transformations of}\\ k_i \quad \text{due to technical change}\end{pmatrix}+\begin{pmatrix}\text{null}\\ \text{term}\end{pmatrix}=$ $\begin{pmatrix}\text{measure of welfare per}\\ \text{infinitesimal change of time}\end{pmatrix}+\sum_{i=1}^{n}\begin{pmatrix}\text{value (effect)of}\\ \text{technical change}\end{pmatrix}+\begin{pmatrix}\text{null}\\ \text{term}\end{pmatrix}=\text{Const.}$

From the previous discussion we see that conservation laws for the Ramsey Growth models depend on either zero a discount rate or not, and either technology is linear or not. For example if production technology is of the general neoclassical type, $f(k)>0, f'(k)>0, f''(k)<0, k>0$. Then we have two cases:

Case 1. If the discount rate is positive $p>0$ and the production technology is not linear, then there exist no conservation law for an arbitrary form of utility function. On the other hand when technology is linear with the marginal product equal to the exogenously determined rate of $\rho+n$, there exists an invariance identity $U(c)=$ constant, i,e., $c=$constant for every period.

Case 2. If the discount rate is zero $(\rho=0)$ and the production technology is not linear, then the Lagrangian does not contain t explicit, therefore the value of the Hamiltonian is constant. The constancy of the Hamiltonian is what we call the conservation law which operates in the system when discount rate $\rho=0$. In economics we can interpret the constancy of the Hamiltonian (4.2.30) or the invariance under time-translation (4.2.28) to be the sum of the value of consumption $U(c)$ and the value investment $U'\dot{k}$ during which the measure of welfare in terms of consumptions remains constant. Then we conclude that the value of income measured in terms of consumption U' does not change over the planning period. On the other hand when technology is linear. In this case the conservation law (4.2.35) does not hold for all linear technology. This indicate that it holds only for a special type of linear technology with the marginal product of capital identically equal to $\rho+n$, the sum of the discount rate and the population growth rate. But we have two parameters a and d in the infinitesimal transformation (4.2.33) and (4.2.34), so we have two quantities which are constant by Noether's theorem. Thus we have

$$\tau_1=ae^{\rho t},\quad d=0 \quad \text{and} \quad \tau_2=0,\quad a=0, \tag{4.2.40}$$
$$\xi_1=a\rho k e^{\rho t},\quad d=0,\quad \text{and} \quad \xi_2=de^{\rho t},\quad a=0. \tag{4.2.41}$$

By substituting these cases into (4.2.35), we obtain

$$\Omega_1=a[U(c)-U'(c)c+\beta U'(c)=\text{constant}, \tag{4.2.42}$$
$$\Omega_2=-dU'(c)=\text{constant}. \tag{4.2.43}$$

From (4.2.43) we conclude that $U'(c)=$ const or $c=$ const, in economics this means that the optimally controlled per capita consumption must remain unchanged during the planning horizon. From (4.2.42), if $c=$ const, then Ω_1 which is the function of c must also be constant.

Theorem 4.2.3 (Conservation Laws for Simple Model of the Ramsey Growth)**.** *1. If the discount rate ρ is zero, there exists only one conservation law that the value of welfare measured in terms of consumption and investment remains constant i.e., $H=U(c)+U'\dot{k}=$ const.*

2. If the discount rate ρ is positive, there exists no conservation laws unless the production technology is a special linear type $f(k)=\alpha k+\beta=(\rho+n)k+\beta, \quad \alpha=\rho+n.$

4.3 Application II

The focus of this application is on the models of optimal economic growth. As with most economic applications of optimal control theory, we provide a complete economic interpretation of the necessary and sufficient conditions of the optimal economic growth model.

In this section we will discuss the two simple models of economic growth or capital accumulation. The first model we consider is a finite horizon fixed end-point model with stationary population. And the second model we consider is the exponentially growing population in the infinite horizon by introducing a new factor labor (Sethi and Thompson, 2000).

4.3.1 An Optimal Capital Accumulation Model. The aim of the problem is that of maximizing the present value of utility of consumption for society and also accumulating a specified capital stock by the end of the horizon.

Now consider a one-sector economy in which the stock of capital is denoted by $K(t)$. Let $Y(K)$ be the output rate of the economy, $C(t)$ be the amount of output allocated to consumption, $I(t)$ be the amount invested i.e., $I(t) = Y[K(t)] - C(t)$, and σ be the constant rate of depreciation of capital. Also let $U(C)$ be society utility of consumption, where we assume $U'(0) = \infty, \quad U'(C) > 0$ and $U''(C) < 0$, for $C > 0$, ρ denotes the social discount rate and T denotes the finite horizon. Suppose $Y(0) = 0, \quad Y(K) > 0, \quad Y'(K) > 0$ and $Y''(K) < 0$, for $K > 0$.

The capital stock equation is

$$\dot{K} = Y(K) - C - \sigma K, \quad K(0) = K_0. \tag{4.3.1}$$

Consider the optimization problem of the calculus of variations

$$\max \text{J} = \int_0^T e^{-\rho t} U[C(t)]dt = \int_0^T e^{-\rho t} U[Y(K) - \sigma K - \dot{K}]dt.$$

Then, we convert the calculus of variations problem into optimal control theory, the equation of the investment flow $\dot{K}$ is the equation of the motion in the optimal control. Therefore, the optimal control problem is formulated as:

$$\max \quad J = \int_0^T e^{-\rho t} U[C(t)]dt, \tag{4.3.2}$$

$$\text{subject to} \quad \dot{K} = Y(K) - C - \sigma K, \quad K(T) = K_T. \tag{4.3.3}$$

Now we introduce the maximum principle to figure out the form of the solution and check it according to the maximum principle.

From (4.3.2) we form the current value Hamiltonian as

$$H = U(C) + \lambda[Y(K) - C - \sigma K]. \tag{4.3.4}$$

The current adjoint equation is

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial K} = (\rho + \sigma)\lambda - \lambda\frac{\partial Y}{\partial K}, \quad \lambda(T) = \alpha, \tag{4.3.5}$$

where α is any constant.

The optimal control is obtained by differentiating equation (4.3.4) with respect to C,

$$\frac{\partial H}{\partial C} = U'(C) - \lambda = 0. \tag{4.3.6}$$

Since $U'(0) = \infty$, this implies that the solution of this condition always gives $C(t) > 0$.

The economic interpretation of the Hamiltonian of this model consists of two parts, the first part gives the utility of current consumption and the second part gives the net investment evaluated by price λ (4.3.6) and reflects the marginal utility of consumption.

For the system to be economically optimal, the solution must satisfy the following three conditions.

1. The static efficiency condition (4.3.6) which maximizes the value of the Hamiltonian at each instant of time, provided that λ is accepted.
2. The dynamic efficiency condition (4.3.5) which forces the price λ of capital to change over time in case the capital stock always yield a net rate of return which is equal to the social discount rate ρ

$$d\lambda + \frac{\partial H}{\partial K} dt = \rho \lambda dt.$$

3. The long run foresight condition which establishes the terminal price $\lambda(T)$ of capital in such a way that exactly the terminal capital stock K_T is obtained at T.

Equations (4.3.3), (4.3.5) and (4.3.6) form a two-point boundary value problem which can be solved numerically or can be solved by the phase diagram method. But we will discuss this in the next section to solve the problem using the phase diagram method.

4.3.2 A One Sector Model With a Growing Labor Force. From the previous model, we introduce a new factor labor which is growing exponentially at a fixed rate $r > 0$. For introducing a per capita variable we will consider the infinite horizon version of this new model.

Let $N(t)$ be the amount of labor at time t. And we know that it is growing exponentially at rate r, then we have

$$N(t) = N(0)e^{rt}, \quad \text{where} \quad N(0) \quad \text{is constant.}$$

Let $Y(K, N)$ be the production function and we assume it to be concave and homogeneous of degree one in K and N. Let us define $k = K/N$. The per capita production function $f(k)$ is defined as

$$f(k) = \frac{Y(K, N)}{N} = Y(\frac{K}{N}, 1) = Y(k, 1). \tag{4.3.7}$$

From $k = \frac{K}{N} \Rightarrow K = kN$ differentiating this equation, we get

$$\dot{K} = k\dot{N} + N\dot{k} = krN + N\dot{k}.$$

Substitute $\dot{K}$ into equation (4.3.1) and let per capita consumption $c = \frac{C}{N}$, we have

$$\begin{aligned} \dot{k} &= f(k) - c - (r + \sigma)k, \quad \text{Let} \quad \alpha = r + \sigma, \\ \dot{k} &= f(k) - c - \alpha k, \quad k(0) = k_0. \end{aligned} \tag{4.3.8}$$

Suppose $u(c)$ be the utility of per capita consumption of c, where u must satisfy

$$u'(c) > 0 \quad \forall c > 0, \quad u''(c) < 0, \quad \lim_{c \mapsto 0} u'(c) = +\infty, \quad \lim_{c \mapsto \infty} u'(c) = 0. \tag{4.3.9}$$

The objective of the problem is to

$$\max \quad J = \int_0^\infty e^{-\rho t} u(c) dt, \tag{4.3.10}$$

$$\text{subject to} \quad \dot{k} = f(k) - c - \alpha k, \quad k(0) = k_0. \tag{4.3.11}$$

Now we use the maximum principle. From (4.3.10) and (4.3.11) we form the current Hamiltonian as

$$H = u(c) + \lambda[f(k) - c - \alpha k]. \tag{4.3.12}$$

The current adjoint equation is

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial k} = (\rho + \alpha)\lambda - f'(k)\lambda. \tag{4.3.13}$$

The optimal control is obtained by differentiating equation (4.3.12) with respect to c,

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0 \Rightarrow u'(c) = \lambda. \tag{4.3.14}$$

Since $H_{cc} = u''(c) < 0$, then we use the implicit function theorem to solve the equation $H_c = u'(c) - \lambda = 0$, in principle, for c as a locally C^1 function of λ. Suppose $c = \eta(\lambda)$, then let us use the implicit function theorem to find the derivative of $\eta(\lambda)$ with respect to λ,

$$\frac{d\eta(\lambda)}{d\lambda} = \eta'(\lambda) = \left.\frac{-H_{c\lambda}}{H_{cc}}\right|_{c=\eta(\lambda)} = \frac{1}{u''(\eta(\lambda))} < 0. \tag{4.3.15}$$

Assume $c = \eta(\lambda)$ is the solution of (4.3.14). We know that $u'(c) > 0$ because of (4.3.9) and f(k) is concave from the assumptions on $Y(K, N)$.

Equations (4.3.8), (4.3.13) and (4.3.14) form a complete autonomous system as follows:

$$\dot{\lambda} = (\rho + \alpha)\lambda - f'(k)\lambda, \tag{4.3.16}$$

$$\dot{k} = f(k) - \eta(\lambda) - \alpha k. \tag{4.3.17}$$

We see that this dynamic system is a function of only (k, λ) and their time derivatives as c have been substituted out of the system through the maximum principle (Caputo, 2005).

Now we need to find the steady state solution (k^*, λ^*) of the necessary and sufficient conditions (4.3.16) and (4.3.17). The steady state solution is obtained by setting $\dot{\lambda} = 0$ and $\dot{k} = 0$ in equations (4.3.16) and (4.3.17) and we get

$$\dot{\lambda} = 0 \Rightarrow (\rho + \alpha) - f'(k)\lambda = 0, \tag{4.3.18}$$

$$\dot{k} = 0 \Rightarrow f(k) - \eta(\lambda) - \alpha k = 0. \tag{4.3.19}$$

We obtain two simultaneously equations, then we solve these two equations for k and λ in terms of the parameters (ρ, α).

To find the local stability of the steady state (k^*, λ^*), first we compute the Jacobian matrix of the dynamical systems (4.3.16) and (4.3.17) with respect to (k, λ), and then we evaluate it at the steady state (k^*, λ^*).

Consider the Jacobian matrix

$$J\left(\lambda^{*}, k^{*}\right)) = \begin{bmatrix} \frac{\partial \dot{\lambda}}{\partial \lambda} & \frac{\partial \dot{\lambda}}{\partial k} \\ \frac{\partial \dot{k}}{\partial \lambda} & \frac{\partial \dot{k}}{\partial k} \end{bmatrix} = \begin{bmatrix} (\rho+\alpha) - f(k^{*}) & -\lambda f''(k^{*}) \\ -\eta'(\lambda^{*}) & f'(k^{*}) - \alpha \end{bmatrix}.$$

From (4.3.18) we have $\rho + \alpha - f'(k^{*}) = 0 \Rightarrow \rho = f'(k^{*}) - \alpha$ substituted into Jacobian matrix above, we get

$$J\left(\lambda^{*}, k^{*}\right) = \begin{bmatrix} 0 & -\lambda^{*} f''(k^{*}) \\ -\eta'(\lambda^{*}) & \rho \end{bmatrix}.$$

Since $\eta'(\lambda^{*}) < 0$ from (4.3.15), $f''(k^{*}) < 0$ and $\lambda^{*} > 0$,

then the trace of J is

$$\text{tr } J(k^{*}, \lambda^{*}) = 0 + \rho = \rho > 0,$$

the determinant of J

$$\begin{aligned} \det J &= 0 - \eta'(\lambda^{*})\lambda^{*} f''(k^{*}) \\ &= -\eta'(\lambda^{*})\lambda^{*} f''(k^{*}) < 0, \end{aligned}$$

$$\text{eigenvalues} = \frac{\text{tr}J \pm \sqrt{\text{tr}^2 J - 4\det J}}{2}.$$

Since $\text{tr}J > 0$ and $\det J < 0$, then eigenvalues are both real and are different signs such that one is positive while the other is negative. Thus, it implies that the steady state (k^{*}, λ^{*}) of the equations (4.3.16) and (4.3.17) is an unstable saddle point with two trajectories in the $k\lambda$ phase plane converging to it as $t \mapsto +\infty$.

Now let us construct the phase diagram, first we consider the $\dot{\lambda} = 0$ isocline which is defined as

$$\dot{\lambda} = [(\rho + \alpha) - f'(k)]\lambda = 0. \tag{4.3.20}$$

Since $\lambda > 0$, then $\dot{\lambda} = 0$ isocline which is the same when written $\dot{\lambda} = (\rho + \alpha) - f'(k) = 0$ and which indicates that $\dot{\lambda}$ is independent of λ. Therefore, the solution of equation (4.3.20) is the steady state solution for the per capita capital stock, such that $k = k^{*} > 0$. So in $k\lambda$ phase plane diagram, $\dot{\lambda} = 0$ isocline is a vertical line positioned at the steady state per capita capital stock level.

Similarly, consider $\dot{k} = 0$ isocline, which is defined as

$$\dot{k} = f(k) - \eta(\lambda) - \alpha k = 0. \tag{4.3.21}$$

Suppose that $\lambda = \Lambda(k)$ be the solution of equation (4.3.21), then the slope of the $\dot{k} = 0$ isocline in a neighborhood of the steady state solution is given by the implicit function theorem of equation (4.3.21). From this we have

$$\left.\frac{\partial \lambda}{\partial k}\right|_{\dot{\lambda}=\dot{k}=0} = \Lambda_k(k)|_{\dot{\lambda}=\dot{k}=0} = -\left.\frac{\partial \dot{k}}{\partial k} \Big/ \frac{\partial \dot{k}}{\partial \lambda}\right|_{\dot{\lambda}=\dot{k}=0} = \frac{f'(k^{*}) - \alpha}{\eta'(\lambda^{*})} = \frac{\rho}{\eta'(\lambda^{*})} < 0;$$

this calculation shows that $\dot{k} = 0$ isocline is negatively sloped in the $k\lambda$-phase plane.

Now let us determine the vector fields for $\dot{\lambda} = 0$ and $\dot{k} = 0$ isocline in the neighborhood of the steady state. Differentiate equations (4.3.16) and (4.3.17) both with respect to λ and k and evaluate the result at the steady state solution (k^*, λ^*), we have

$$\left.\frac{\partial \dot{\lambda}}{\partial \lambda}\right|_{\dot{\lambda}=\dot{k}=0} = \rho + \alpha - f'(k)\big|_{\dot{\lambda}=\dot{k}=0} = \rho + \alpha - f'(k^*) \equiv 0, \tag{4.3.22}$$

$$\left.\frac{\partial \dot{\lambda}}{\partial k}\right|_{\dot{\lambda}=\dot{k}=0} = -\lambda f''(k)\big|_{\dot{\lambda}=\dot{k}=0} = -\lambda f''(k^*) > 0 \quad \text{since} \quad f''(k^*) < 0, \tag{4.3.23}$$

$$\left.\frac{\partial \dot{k}}{\partial \lambda}\right|_{\dot{\lambda}=\dot{k}=0} = -\eta'(\lambda)\big|_{\dot{\lambda}=\dot{k}=0} = -\eta'(\lambda^*) > 0 \quad \text{since} \quad \eta'(\lambda^*) < 0, \tag{4.3.24}$$

$$\left.\frac{\partial \dot{k}}{\partial k}\right|_{\dot{\lambda}=\dot{k}=0} = f'(k) - \alpha\big|_{\dot{\lambda}=\dot{k}=0} = f'(k^*) - \alpha = \rho > 0. \tag{4.3.25}$$

Consider the vector fields for $\dot{\lambda} = 0$.

From equation (4.3.22) a neighborhood of the steady state (k^*, λ^*), an increase in λ does not affect or change $\dot{\lambda}$ from zero, but in equation (4.3.23) shows that a neighborhood of the steady state (k^*, λ^*) an increase in k increases $\dot{\lambda}$ from zero to any positive number. This means that the direction of arrow points to the right of the $\dot{\lambda} = 0$ isocline, therefore $\dot{\lambda} > 0$. Likewise, $\dot{\lambda} < 0$, the direction of arrow points to the left of the $\dot{\lambda} = 0$ isocline, (see Figure 4.1 below).

The vector fields for $\dot{k} = 0$.

From equation (4.3.24) where there is a neighborhood of the steady state (k^*, λ^*), an increase in λ increases $\dot{k}$ from zero to a positive value, this implies that all points above the $\dot{k} = 0$ isocline in a neighborhood of the steady state (k^*, λ^*) have $k-$increasing over time likewise applying for points below $\dot{k} = 0$ isocline. Also equation (4.3.25) shows that in a neighborhood of the steady state, an increase in k increases $\dot{k}$ from zero to a positive value, therefore all points to the right of $\dot{k} = 0$ isocline in a neighborhood of the steady state k are increasing overtime (this is the same as the conclusion of equation (4.3.24)).

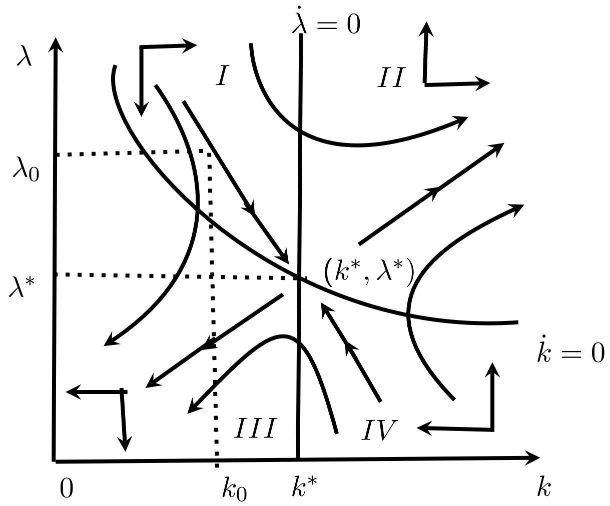

Figure 4.1: Phase diagram for optimal growth model

The point (k^*, λ^*) is the stationary equilibrium for optimal long-run.

So let us find whether there is a path satisfying the maximum principle which converges to the equilibrium (k^*, λ^*). But keep in mind that a path cannot start in regions II and III because the directions of the arrows point away from the stationary equilibrium (k^*, λ^*) which means it diverges from the equilibrium.

Now consider for case of $k_0 < k^*$, this implies that the value of λ_0 must be chosen in order to ensure (k_0, λ_0) is in region I, similarly for case $k_0 > k^*$, the point (k_0, λ_0) must be chosen to be in region IV.

Case I: $k_0 < k^*$.

We need to show that there exists a unique λ_0 associated with the given k_0. The point (k_0, λ_0) is shown by a dotted line in Figure 4.1 above. The function $k(t)$ is an increasing function of t as shown by the right horizontal direction arrow in region I. From equations (4.3.16) and (4.3.17), we have

$$\frac{d(ln\lambda)}{dk} = \frac{1}{\lambda}\frac{d\lambda}{dt}/\frac{dk}{dt} = \frac{f'(k) - (\rho + \alpha)}{\eta(\lambda) + \alpha k - f(k)}. \tag{4.3.26}$$

For $k < k^*$, the last expression of equation (4.3.26) is negative in region I and since $\eta(\lambda)$ decreases with λ, therefore it implies that $d(ln\lambda)/dk$ increasing with λ (Sethi and Thompson, 2000).

Suppose that $\lambda_1(k)$ and $\lambda_2(k)$ are two paths leading to (k^*, λ^*) and are such that the selected initial values satisfy $\lambda_1(k_0) > \lambda_2(k_0)$ but $d(lnk)/dk$ increases with λ. Therefore we have

$$\frac{dln[\lambda_1(k)/\lambda_2(k)]}{dk} = \frac{dln\lambda_1(k)}{dk} - \frac{dln\lambda_2(k)}{dk} > 0, \tag{4.3.27}$$
$$\text{where} \quad \lambda_1(k) > \lambda_2(k).$$

The inequality (4.3.27) is satisfied at k_0, and by (4.3.26), $\lambda_1(k)/\lambda_2(k)$ increases at k_0, this implies that the inequality also holds at $k_0 + \varepsilon$ where $\varepsilon > 0$ is small, then we replace k_0 by $k_0 + \varepsilon$ and we apply the same procedure and we conclude that the ratio $\lambda_1(k)/\lambda_2(k)$ increases as k increases. For this case, $\lambda_1(k)$ and $\lambda_2(k)$ cannot both converge to λ^* as $k \to k^*$.

The remaining task is to show that for $k_0 < k^*$, there exists λ_0 such that the trajectory converges to (k^*, λ^*), keep in mind that some trajectory start values of the adjoint variable in region I and the resulting trajectory (k, λ) enters regions II where it diverges, similarly for others trajectory from region I and enters region III and diverges. By continuity, there exists a starting value λ_0 such that the resulting trajectory (k, λ) converges to (k^*, λ^*).

Case II: $k_0 > k^*$.

This case follows the same procedure as $k_0 < k^*$, therefore the argument holds even for this case $k_0 > k^*$.

4.3.3 Economic interpretation of the Figure 4.1. An optimal growth path cannot pass into region II which is above the $\dot{k} = 0$ and right of the $\dot{\lambda} = 0$, then the product λk in this region would end up increasing too fast to satisfy the transversality condition when $t \to \infty$, therefore, in this region, people would postpone consuming forever. But in region III, the optimal growth path also cannot move in this region in which consumption is rising $(\dot{\lambda} < 0)$ and capital falling, because this exhaust, the capital stock in finite time. In order to avoid this we must choose the given initial value k_0 and the initial value λ_0 so as to put the economy on this trajectory in the saddle path or stable path. This is because the saddle path converges asymptotically to a stationary equilibrium, such that the growth will sustainable in the long run.

5. Conclusion

In this project, we have seen that the calculus of variations problem is a special case of the general control problem because we can convert calculus of variations into a control problem and we use the maximum principle to solve it.

In Chapter Two we have studied that under the hypothesis that a maximum (or minimum) solution exists, the solution must satisfy the Euler-Lagrange equation and the derivatives of a function at an extreme value vanishes provided that derivative exists. Note that not all functions have extreme value, for example $f(x) = x^3$ has a critical point at $x = 0$, but $f(x)$ has no extreme values.

We used the Noether Theorem and conservation laws in the Ramsey growth model to show that the conservation laws under the definition of absolute invariance are just the same as the definition of divergence invariant, therefore, in this case, we conclude that there exist no additional conservation laws even if we allow for a more general definition of invariance.

We constructed a phase diagram and used it to investigate the properties of a dynamic system in Figure 4.1 in which we have studied that an increase in the social discount rate, decrease the steady state stock of per capita capital stock k as the result, the steady state per capita output falls. But with a smaller per capita capital stock, society values it more at the margin, hence the steady state current value shadow price of the per capita capital stock rises. Therefore, we concluded that both the per capita capital stock and the per capita consumption rate are decreased with an increasing in the social discount rate ρ.

Acknowledgements

First, I would like to thank the Almighty God for giving me healthy life for the whole period of studying at AIMS. Secondly, I am very grateful to my supervisor Gastão S. F. Frederico who supervised me and proposed this essay topic, so I thank for his contribution and guidance and I will still very grateful to him forever. Thank you my tutor Hilaire M. Mobele for this support and guidance on this project.

I would like to express my gratitude to AIMS for giving me this golden opportunity to do a masters. Special thanks to the founder of AIMS, Neil Turok, Prof Barry Green, Prof Jeff Sanders, Jan, Frances Aron, A'eeda, Igsaan, Lynne and all AIMS staffs who always ensured our well-being and comfort while at AIMS. Also I would like to thank the Lecturers, Tutors and Students for maximum cooperation.

Last, I would like to thank my parents (Abrahamu Mwaiswaga and Christina Sege) and siblings for their moral support, prayers and encouragements.

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