Lévy Processes and Lévy Kintchine Theorem

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Abstract

In this essay we discuss Lévy processes and look at some of the mathematical ideas behind them. The mathematical ideas we look at are measures, sigma-algebras and probability spaces. We then look at two important results which enable us to determine the probability measures and distributions of random variables. We then review the Lévy-Kintchine theorem which presents a formula which is used to characterize Lévy processes by their characteristic functions. Finally we consider properties and simulations of Lévy processes.

Contents

1. Basics

1.1 Introduction

Lévy processes were named after the French mathematician Paul Lévy. They are stochastic processes with stationary and independent increments. Lévy processes play a fundemental role in financial mathematics and other fields of science such as physics, engineering and actuarial science. In financial mathematics the main reason Lévy processes are important is that they can desribe the observed reality of financial markets in more accurate ways than models based on Brownian motion. The reason Lévy processes are accurate is that they can generalize Brownian motion to include jumps. In real life, stock prices move discontinously and occasionally with large jumps.

In this paper we will look at some important ideas pertaining to Lévy processes. In chapter two we discuss the mathematical ideas of *σ-algebras, measures*, as well as *probability spaces*. We then look at *stochastic processes* and discuss a few examples of such processes. The two important examples we will look at are *Brownian motion* and *Poisson processes*. The main reason we discuss these two examples is that they turn out to be Lévy processes. In the third chapter we discuss two important ideas, *characteristic functions* and *infinite divisibility*, and look at some examples of these ideas. The Lévy-Kintchine theorem, which enables us to characterize all Lévy processes by looking at their characteristic functions, is discussed in the fourth chapter. We also state some properties of Lévy process and perform some simple simulations.

1.2 Models Built on Brownian Motion

Brownian motion is a popularly-used stochastic process for modelling the fluctuation of stock prices. We define Brownian motion as follows.

Definition 1.1 *A stochastic process $(W_t)_{t\geq 0}$ is called a standard Brownian motion if:*

1. *$W_0 = 0$ (a.s.),*
2. *$W_t - W_s \sim N(0, t-s)$ implies stationary increment,*
3. *W_t is continuous in $t \geq 0$ (a.s.),*
4. *for each $n \geq 1$ and any times $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, the random variables $\{W_{t_r} - W_{t_{r-1}}\}$ for $r \geq 1$ are independent implies independent increment.*

We will discuss Brownian motion in further detail in Chapter 2. From the beginning of financial modelling, when the model of price S_t of an asset was proposed by Louis Bachelier at Paris Bourse to be

$$S_t = S_0 + \sigma W_t,$$

Brownian motion and financial modelling have been tied together.

The Black Scholes model arised from the multiplicative version of Bachelier's model, where the log-price $\ln S_t$ follows a Brownian motion with drift(a non-random linear component)

$$S_t = S_0 \exp[\mu t + \sigma W_t].$$

Then by applying Itô's formula the model is written in the so-called local form as a stochastic differential equation for S_t:

$$\frac{dS_t}{S_t} = \sigma dW_t + (\mu + \frac{\sigma^2}{2})dt.$$

The process S_t is called a *geometric Brownian motion*.

Black Scholes assume continuity in the sample paths, as this is one characteristic of Brownian motion. This assumption is what makes this model behave inappropriately across different time scales. Prices move in a discontinuous manner across different time scales, it is therefore important that their behaviour is simulated appropriately because it is at discontinuous points where most of the risk is concentrated. This has then led to the proposal of various models which allow for jumps to occur. Such models are built from various stochastic processes. We will study in this essay a class of stochastic processes called Lévy processes.

2. Preliminaries

2.1 Measures, σ-algebras and Probability Spaces

The notion of a measure can be generalized from familiar notions of length, area and volume to more abstract notions. Let us take for instance a set $\mathbb{D}$. Intuitively a measure μ on $\mathbb{D}$ associates to a certain subset $A \subset \mathbb{D}$ a positive number $\mu(A) \in [0, \infty]$. The subset A to which a number is associated is called a measurable set. We can naturally say that the empty set has measure 0: $\mu(\emptyset) = 0$. If we have two disjoint measurable sets A and B, then their union $A \cup B$ should be measurable. We can naturally define the measure of this union as $\mu(A \cup B) = \mu(A) + \mu(B)$; this is called the additivity property. This property can be extended to infinite sequences if $(A_n)_{n \in \mathbb{N}}$ is a sequence of disjoint measurable subsets, then

$$\mu(\bigcup_{n \geq 1} A_n) = \sum_{n \geq 1} \mu(A_n).$$

It is also possible for μ to be infinite, for example take $\mathbb{D} = \mathbb{R}$ then $\mu(\mathbb{R}) = \infty$. Since $A \subset \mathbb{D}$ we have that $A \bigcup A^c = \mathbb{D}$, where A^c is the complement of A. So if $\mu(\mathbb{D}) < \infty$ then $\mu(A^c) = \mu(\mathbb{D}) - \mu(A)$. It is therefore natural to require that any measurable set has its complement being also measurable. The above discussion motivates the following definiton.

Definition 2.1 *Let S be a non-empty set and $\mathcal{F}$ a collection of subsets of S. We call $\mathcal{F}$ a σ-***algebra** *if the following hold:*

1. $S \in \mathcal{F}$.
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.
3. *If $(A_n)_{n \in \mathbb{N}}$ is a sequence of subsets in $\mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.*

The pair $(S, \mathcal{F})$ is called a **measurable space**.

A **measure** on $(S, \mathcal{F})$ is a mapping $\mu : \mathcal{F} \to [0, \infty]$ that satisfies the following conditions

1. $\mu(\emptyset) = 0$.
2. $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ for every sequence $(A_n)_{n \in \mathbb{N}}$ of mutually disjoint sets in $\mathcal{F}$.

We then call the triplet $(S, \mathcal{F}, \mu)$ a **measure space**. We will now specialize the idea of a measure space to that of the *probability space.*

Definition 2.2 *A* **probability space** *is a measure space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{P}(\Omega) = 1$. A measurable set $A \in \mathcal{F}$, called an event, is a set of scenarios to which probability can be assigned. A probability* *measure assigns a number between 0 and 1 to each event and we say that this number is the probability that the event will occur. We have*

$$\begin{aligned} \mathbb{P} : \mathcal{F} &\to [0,1] \\ A &\mapsto \mathbb{P}(A). \end{aligned}$$

2.2 Stochastic Processes

Definition 2.3 *A family* $\{X_t : t \geq 0\}$ *of random variables on* $\mathbb{R}^d$ *with parameter* $t \in [0, \infty)$ *defined on a probability space* $(\Omega, \mathcal{F}, P)$ *is called a* **stochastic process.**

The time parameter t in the definition can be either discrete or continuous, but we will only deal with the continuous case in this paper. For each occurence of randomness $\omega \in \Omega$, we have a trajectory $X(\omega) : t \to X_t(\omega)$ which is a function of time and it is called the sample path of the process. We will assume that the sample paths are right continuous. However some of the processes we will encounter will have discontinuous sample paths, *e.g* Poisson processes, hence we need a space that will allow for discontinuous functions or functions with jumps. It is therefore natural to consider a class of *cadlag* functions which includes discontinuous functions.

Definition 2.4 *A function* $f : \mathbb{R}^+ \to \mathbb{R}^d$ *is* **cadlag** *if it is right continuous with left limits, i.e.*

$$f(t_-) = \lim_{s \to t, s < t} f(s),$$

$$f(t_+) = \lim_{s \to t, s > t} f(s),$$

both exist and $f(t) = f(t_+)$.

By interpreting the index t as a time, one must take into account the fact that events become less uncertain as more information on that event becomes available with time. One must then describe progressively how information on a particular event is revealed. This will be done by introducing the notion of a *filtration*. This will then allow us to define other important ideas such as *past information*, *predictability* and *adaptiveness* of the process.

Definition 2.5 *A* **filtration** *or* **information flow** *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *is an increasing family of* σ*-algebras* $(\mathcal{F}_t)_{t \in [0,\infty]}$ *:* $\mathcal{F}_s \subseteq \mathcal{F}_t$ *if* $s \leq t$.

$\mathcal{F}_t$ can be interpreted as the information available at time t, which increases with time. If $\mathcal{F}$ is the set of all possible events, then $\mathcal{F}_t \subseteq \mathcal{F}$. We will denote the filtration $(\mathcal{F}_t)_{t \in [0,\infty]}$ by the symbol $\mathbb{F}$. A probability space which is equipped with a filtration $\mathbb{F}$ is called a *filtered probability space* and it is denoted by $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. One can instinctively say that the probability of the occurence of a random event will change as more information is revealed with time. The *filtration* $\mathcal{F}$ describes the information flow which can be used to distinguish quantities which are known given the current information from quantities which are still regarded as random at a given time t. An observer can then decide whether an event $A \in \mathcal{F}$ has occured or not given the information $\mathcal{F}_t$. We call a random variable X an $\mathcal{F}_t$-**measurable random variable** if the value of X can be revealed at time t.

Definition 2.6 *For $i = 1, 2$, let $(S_i, \mathcal{F}_i)$ be measurable spaces. A mapping $f : S_1 \to S_2$ is said to be $(\mathcal{F}_1, \mathcal{F}_2)$-measurable if $f^{-1}(A) \in \mathcal{F}_1$ for all $A \in \mathcal{F}_2$.*

Definition 2.7 *A stochastic process $(X_t)_{t\in[0,\infty]}$ is said to be $\mathcal{F}_t$-**adapted** with respect to the information structure $(\mathcal{F}_t)_{t\in[0,\infty]}$ if for each $t \in [0, \infty]$, the value of X_t is revealed at time t : the random variable X_t is $\mathcal{F}_t$-measurable.*

Most of the events that will be dealt with happen at random times, where a random time is a positive random variable $T \geq 0$ representing the time at which some event is going to take place. Given an information flow $(\mathcal{F}_t)$, a natural question to ask is whether given the information $\mathcal{F}_t$ one can determine if an event has happened $(T \leq t)$ or not $(T > t)$. If this is true, then T is called a *stopping time*, which we define formally as follows:

Definition 2.8 *A random variable $T : \Omega \to [0, \infty]$ is a stopping time if the event $\{T \leq t\} \in \mathcal{F}_t$ for every $t \in [0, \infty]$.*

Two fundamental examples of stochastic processes are the *Poisson process* and the *Brownian motion*. We give a formal definition of these two stochastic processes.

2.2.1 Brownian Motion

In Chapter 1 we have defined Brownian motion in Definition 1.1. We will now elaborate on the meaning of the properties of Brownian motion. Property (1) says that Brownian motion starts at zero. Property (2) is called the *independent increments* property which implies that the process at the current time does not depend on the previous process. Property (3) says that the distribution of the increment $W_t - W_s$ depends only on $t - s$ and this property is called the *stationary increment property*. Lastly, Property (4) simply says that the sample paths of the Brownian motion are continuous.

Figure 2.1 illustrates a typical sample path of the *Brownian motion*.

2.2.2 Poisson Process

Definition 2.9 *(Poisson process) Let $(\tau_i)_{i\geq 1}$ be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^{n} \tau_i$. The process $(N_t, t \geq 0)$ defined by*

$$N_t = \sum_{n\geq 1} 1_{t\geq T_n}$$

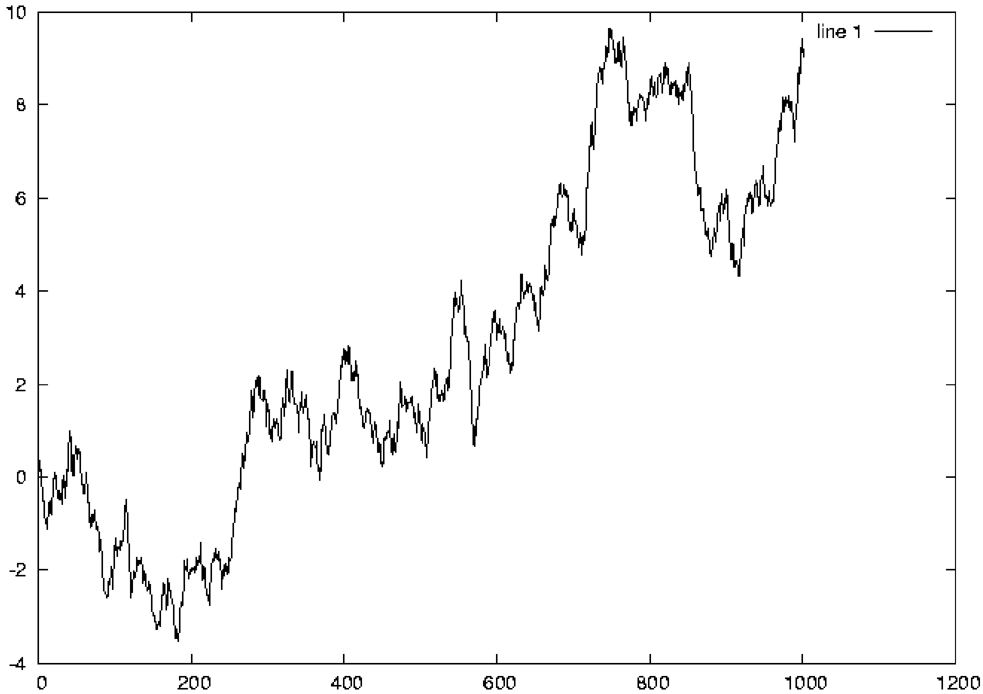


Figure 2.1: Sample path of Brownian motion

is called a Poisson process with intensity λ *on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$.

The process N_t obeys the following conditions:

- For any $t > 0$, $\{N_t\}_{t>0}$ is almost surely finite.
- For any $\omega \in \Omega$ the sample path $t \mapsto N_t$ is piecewise constant and increases by unit jumps.
- The sample paths $t \mapsto N_t$ are cadlag.
- For any $t > 0$, $N_{t-} = N_t$ with probability 1.
- For any $t > 0$, N_t follows a *Poisson distribution*.
- $\{N_t\}_{t \geq 0}$ has independent increments, i.e. for any $t_0 < t_1 < \cdots < t_n$, $N_{t_n} - N_{t_{n-1}}, \ldots, N_{t_1} - N_{t_0}$ are independent random variables.
- For any $0 < s < t$, $N_t - N_s$ has the same distribution as N_{t-s}, i.e the process has stationary increments.

The Poisson process N_t counts the number of random times $\{T_n, n \geq 1\}$ occuring in the interval $[0, t]$, where the random times T_n are partial sums of a sequence of i.i.d. exponential random variables. Poisson processes can be used to model different kinds of phenomena such as the arrival of customers in a shop or the arrival of telephone calls at a switch board of a particular institution.

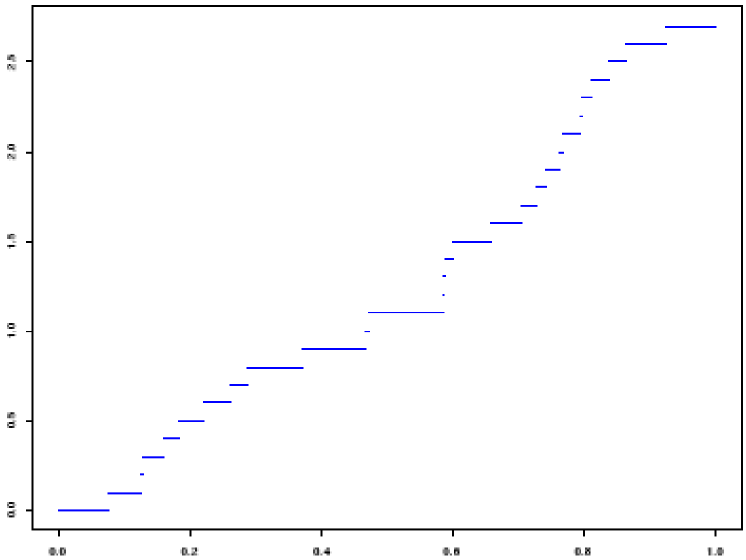


Figure 2.2: Sample path of a Poisson process

Figure 2.2 shows a typical behaviour of the *Poisson process*.

We now give a brief discussion on the *Gamma distribution* which we will use to prove an important property of the Poisson process.

Definition 2.10 *(Gamma distribution) If a variable* $x > 0$ *has a* Gamma distribution *denoted by* $\Gamma(\alpha, \beta)$ *where* α *is the shape parameter and* β *is the inverse scale parameter, then the probability density function of* x *is:*

$$f(x; \alpha, \beta) = x^{\alpha-1} \frac{\beta^{\alpha} e^{-\beta x}}{\Gamma(\alpha)}. \tag{2.1}$$

An exponential random variable X $\sim$*Exp*(λ) has a distribution $f(x) = \lambda e^{-\lambda x}$, $x > 0$. We observe that *Exp*$(\lambda)\sim \Gamma(1, \lambda)$. The *Gamma distribution* has the following property concerning summations.

Property 2.11 *If* X_i *has a* $\Gamma(\alpha_i, \beta)$ *distribution for* $i = 1, 2, \ldots, N$ *and the random variable* $Y = \sum_{n=1}^{N} X_i$*, then* Y *has the Gamma distribution*

$$Y \sim \Gamma\left(\sum_{i=1}^{N} \alpha_i, \beta\right).$$

From this property we deduce the following lemma:

Lemma 2.12 *If $T_n = \sum_{i=1}^{n} \tau_i$ where τ_i's are i.i.d.* Exp(λ) *random variables for some $\lambda > 0$, then T_n has the following probability density function:*

$$f(x) = x^{n-1}\frac{\lambda^n e^{-\lambda x}}{(n-1)!}, \quad x > 0.$$

Proof: $T_n = \tau_1 + \tau_2 + \cdots + \tau_n$. *Using the fact that* Exp(λ)$= \Gamma(1, \lambda)$ *and Property 2.11 it follows that $T_n \sim \Gamma(n, \lambda)$. This implies that T_n has the following probability density function:*

$$f(x) = x^{n-1}\frac{\lambda^n e^{-\lambda x}}{(n-1)!} \quad x > 0.$$

The following theorem states an important property of the distribution of a Poisson process:

Theorem 2.13 *Let $(N_t)_{t\geq 0}$ be a Poisson process, for any $t > 0$, the process $(N_t)_{t\geq 0}$ follows a Poisson distribution with parameter λt :*

$$\forall n \in \mathbb{N}, \mathbb{P}(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}. \tag{2.2}$$

Proof: *We have*

$$\begin{aligned}
\mathbb{P}(N_t = n) &= \mathbb{P}(N_t \text{ has exactly n jumps}) \\
&= \mathbb{P}(T_n \leq t, T_{n+1} > t) \\
&= \mathbb{E}[\mathbb{P}(T_n \leq t, T_{n+1} > t | T_n = s)] \\
&= \mathbb{E}[\mathbb{P}(T_{n+1} - T_n > t - s | T_n = s)] \\
&= \int_0^t \mathbb{P}(T_{n+1} - T_n > t - s) d\mu_{T_n(s)} \\
&= \int_0^t e^{-\lambda(t-s)} \lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} ds \\
&= \frac{e^{-\lambda t}\lambda^n}{(n-1)!} \int_0^t s^{n-1} ds \\
&= \frac{e^{-\lambda t}\lambda^n}{(n-1)!} \frac{t^n}{n} \\
&= \frac{e^{-\lambda t}(\lambda t)^n}{n!}.
\end{aligned} \tag{2.3}$$

From this theorem we can immediately deduce the following corollary:

Corollary 2.14 *Let $(N_t)_{t\geq 0}$ be a Poisson process. Then the expectation is given by*

$$\mathbb{E}[N_t] = \lambda t. \tag{2.4}$$

Proof:

$$
\begin{aligned}
\mathbb{E}[N_t] &= \sum_{n=0}^{\infty} n\mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} n\frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= \sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{(n-1)!} \\
&= \lambda t \sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \\
&= \lambda t e^{-\lambda t} e^{\lambda t} \\
&= \lambda t.
\end{aligned}
$$

2.2.3 Compensated Poisson Process

Using the Poisson process we can define another process called the *compensated Poisson process*.

Definition 2.15 *Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter λ. The process $(\bar{N}_t)_{t\geq 0}$ defined by:*

$$\bar{N}_t = N_t - \lambda t, \quad t \geq 0 \tag{2.5}$$

is called a **compensated Poisson process**.

Figure 2.3 figure shows the sample paths of the compensated Poisson process.

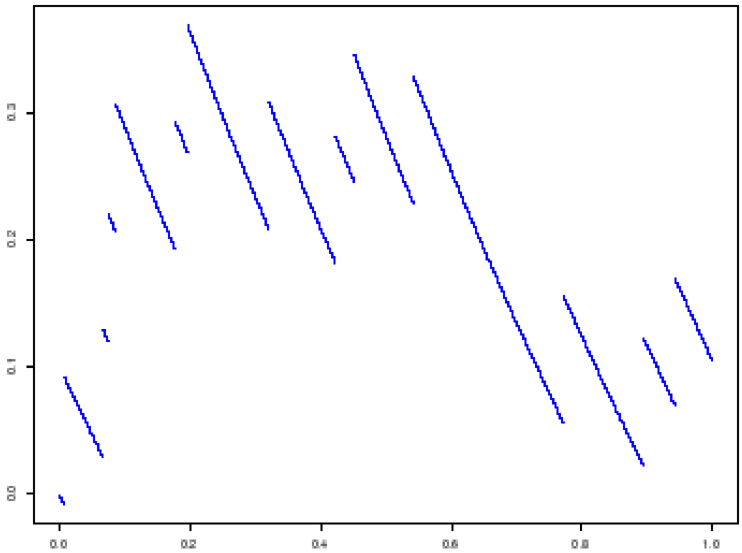


Figure 2.3: An example sample path of a compensated Poisson process

3. Lévy Processes, Characteristic Functions and Infinite Divisibility

3.1 Lévy Processes

Lévy processes are essentially stochastic processes with stationary and independent increments. This is a class of stochastic processes which encompasses the processes which are continuously random as well as processes with jumps.

Definition 3.1 *A stochastic process* $(X_t)_{t\geq 0}$ *on* $\mathbb{R}^d$ *is* stochastically continuous *or* continuous in probability *if, for every* $t \geq 0$ *and* $\epsilon > 0$

$$\lim_{s\to t} \mathbb{P}[|X_s - X_t| > \epsilon] = 0.$$

Definition 3.2 *A stochastic process* $(X_t)_{t\geq 0}$ *on* $\mathbb{R}^d$ *is a* **Lévy Process** *if it satisfies the following conditions:*

1. $X_0 = 0$ *a.s.*
2. *For any choice of* $n \geq 1$ *and* $0 \leq t_0 < t_1 \cdots < t_n$ *the random variables*

   $$X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$$

   are independent.
3. *The distribution of* $X_{s+t} - X_s$ *does not depend on* s.
4. X_t *is stochastically continuous.*

Two important examples of Lévy processes are Brownian motion and Poisson processes as discussed in the previous chapter.

Below we give another important example of a Lévy process called the Lévy jump-diffusion.

Example 3.3 *(***Lévy jump-diffusion process***)*
Let:

$$L_t = bt + \sigma W_t + \sum_{k=1}^{N_t} J_k - t\lambda\mathcal{K}. \tag{3.1}$$

where $b \in \mathbb{R}$ *is called the drift term,* $\sigma \in \mathbb{R}^+$, $W = (W_t)_{t\geq 0}$ is a standard Brownian Motion, $N = (N_t)_{t\geq 0}$ *is a Poisson process with parameter* λ *(so that* $\mathbb{E}[N_t] = \lambda t$*),* $J = (J_k)_{k\geq 1}$ is an independent and identically-distributed sequence of random variables with probability distribution F, and $\mathbb{E}[J] = \mathcal{K} < \infty$. F describes the distribution of the jump size. The process L_t is defined by the sum of the Brownian motion with drift and the compensated Poisson process.

Figure 3.3 shows a typical sample path of Lévy jump-diffusion.

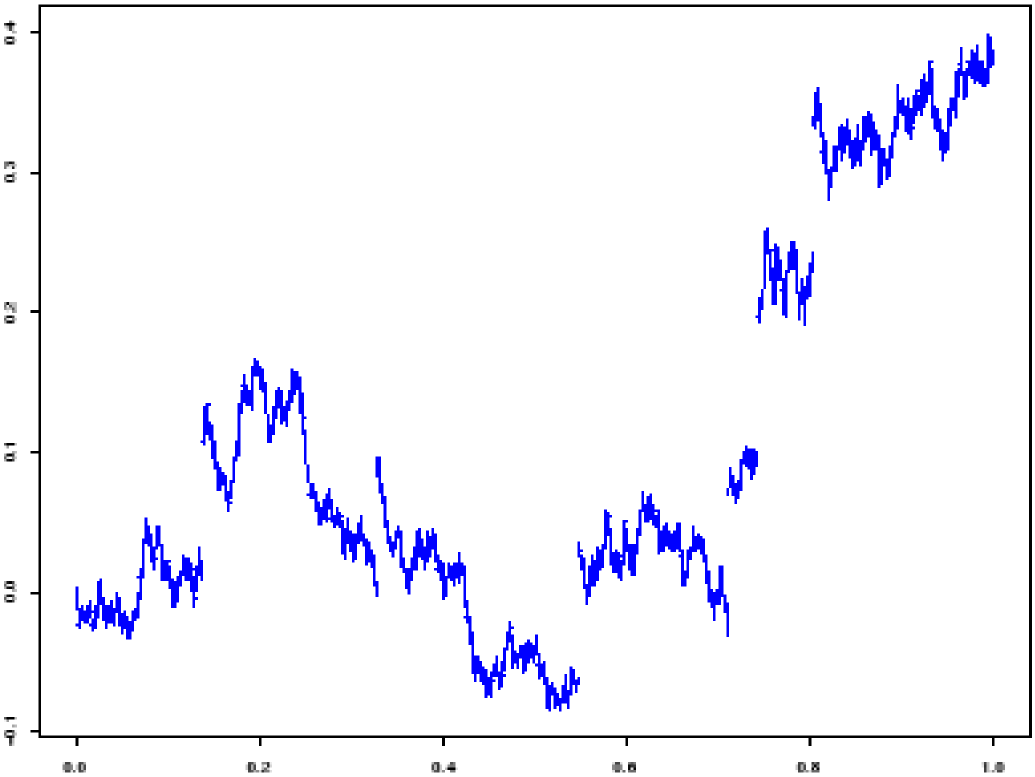


Figure 3.1: Sample path of a Lévy jump-diffusion process

3.2 Characteristic Functions

The *characteristic function* of a random variable is the Fourier transform of its distribution. The concept of characteristic functions is useful for studying random variables because many probabilistic properties of random variables correspond to analytical properties of their characteristic functions.

Definition 3.4 *A moment generating function of a random variable X is the function M_X defined by*

$$M_X(t) = \mathbb{E}[e^{tX}]$$

for all $t \in \mathbb{R}$ for which this expectation exists.

Example 3.5 *If X has a normal distribution with mean 0 and variance 1, i.e. $X \sim N(0,1)$,* *then*

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\ &= e^{-\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx \\ &= e^{-\frac{1}{2}t^2} \end{aligned}$$

Similarly, if $X \sim N(\mu, \sigma^2)$ i.e. X has mean μ and variance σ^2, then

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tu} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(u-\mu)^2) du \\ &= e^{\mu t} \int_{-\infty}^{\infty} e^{x\sigma t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx, \text{ by substitution } x = \frac{u-\mu}{\sigma} \qquad (3.2) \\ &= \exp(\mu t + \frac{1}{2}\sigma^2 t^2). \end{aligned}$$

Definition 3.6 *(Characteristic function) The characteristic function of an $\mathbb{R}^d$-valued random variable X is the function $\Phi : \mathbb{R}^d \to \mathbb{C}$ defined by*

$$\Phi_X(z) = \mathbb{E}[e^{i\langle z, X\rangle}] = \int_{\mathbb{R}^d} e^{i\langle z, x\rangle} \mu_X dx. \qquad (3.3)$$

where μ_X is the distribution of X and $\langle x, y\rangle$ denotes the Euclidean scalar product.

The following results give a connection between moment generating functions and characteristic functions. If the moment generating function M_X of X is finite in a non-trivial neighbourhood of the origion, the characteristic function of X may be found by substituting $s = it$ in the formula for $M_X(s)$:

$$\Phi_X(t) = M_X(it), \qquad (3.4)$$

for $t \in \mathbb{R}$.

The following theorem states some basic properties of characteristic functions. The first property concerns the characteristic function of the sum of two independent random variables.

Theorem 3.7 *If Φ_X is a characteristic function, then:*

1. *If X and Y are independent random variables then*

$$\Phi_{X+Y}(z) = \Phi_X(z)\Phi_Y(z). \qquad (3.5)$$

2. $\Phi_X(0) = 1$.

3. $\Phi_{aX+b}(z) = e^{i\langle z, b\rangle}\Phi_X(az)$, *for constants $a, b \in \mathbb{C}$*

Proof:

1. We have that:

$$\begin{aligned}\Phi_{X+Y}(z) &= \mathbb{E}[e^{i\langle z,X+Y\rangle}]\\ &= \mathbb{E}[e^{i\langle z,X\rangle}e^{i\langle z,Y\rangle}]\\ &= \mathbb{E}[e^{i\langle z,X\rangle}]\mathbb{E}[e^{i\langle z,Y\rangle}] \text{ by independence}\\ &= \Phi_X(z)\Phi_Y(z).\end{aligned}$$

2. This is trivial.

3. Using the linearity of the expectation:

$$\begin{aligned}\Phi_{aX+b}(z) &= \mathbb{E}[e^{i\langle z,aX+b\rangle}]\\ &= \mathbb{E}[e^{i\langle z,aX\rangle}e^{i\langle z,b\rangle}]\\ &= e^{i\langle z,b\rangle}\mathbb{E}[e^{i\langle za,X\rangle}]\\ &= e^{i\langle z,b\rangle}\Phi_X(az).\end{aligned}$$

In the following examples we find the characteristic functions of certain random variables.

Example 3.8 *If X has a normal distribution with mean μ and variance σ^2, then the moment generating function of X is*

$$M_X(s) = \exp(\mu s + \frac{1}{2}\sigma^2 t^2)$$

as found in Example 3.5. Now using Equation 3.4 and substituting $s = it$ we get

$$\Phi_X(t) = \exp(i\mu t - \frac{1}{2}\sigma^2 t^2)$$

which is the characteristic function of a random variable $X \sim N(\mu, \sigma^2)$.

Example 3.9 *If a random variable X has an exponential distribution with parameter λ, then its characteristic function is given by:*

$$\begin{aligned}\Phi_X(z) &= \mathbb{E}[e^{i\langle z,X\rangle}]\\ &= \int_0^\infty e^{izx}\lambda e^{-\lambda x}dx\\ &= \lambda\int_0^\infty e^{(iz-\lambda)x}dx\\ &= \frac{\lambda}{iz-\lambda}[e^{(iz-\lambda)x}]_0^\infty\\ &= \frac{\lambda}{iz-\lambda}\left[\lim_{x\to\infty} e^{(iz-\lambda)x} - 1\right]\end{aligned}$$

Using the result from complex analysis that absolute convergence implies convergence, we have

$$|e^{(iz-\lambda)x}| = |e^{izx}e^{-\lambda x}| = |e^{izx}||e^{-\lambda x}| = e^{-\lambda x}.$$

We then have that $\lim_{x\to\infty} e^{-\lambda x} = 0$ which yields

$$\begin{aligned}\Phi_X(z) &= \frac{\lambda}{iz-\lambda}[0-1] \\ &= \frac{\lambda}{\lambda - iz}.\end{aligned}$$

Example 3.10 In this example we find the characteristic function of a *univariate Levy jump-diffusion process* L_t which was defined in Example 3.3. Let*:*

$$L_t = bt + \sigma W_t + \sum_{k=1}^{N_t} J_k - t\lambda\mathcal{K}.$$

The characteristic function *of* L_t *is:*

$$\begin{aligned}\Phi_{L_t}(z) &= \mathbb{E}[e^{i\langle z, L_t\rangle}] \\ &= \mathbb{E}[\exp i\langle z, bt + \sigma W_t + \sum_{k=1}^{N_t} J_k - t\lambda\mathcal{K}\rangle] \\ &= \mathbb{E}[\exp(izbt + iz\sigma W_t + iz\sum_{k=1}^{N_t} J_k - izt\lambda\mathcal{K})] \\ &= \mathbb{E}[\exp(izbt)]\mathbb{E}[\exp(iz\sigma W_t)\exp(iz\sum_{k=1}^{N_t} J_k - izt\lambda\mathcal{K})]\end{aligned}$$

since the Brownian motion is independent of the jumps. We then get that

$$\Phi_{L_t}(z) = \exp(izbt)\mathbb{E}[\exp(iz\sigma W_t)]\mathbb{E}[\exp(iz\sum_{k=1}^{N_t} J_k - izt\lambda\mathcal{K})]. \tag{3.6}$$

We also have that:

$$W_t \sim N(0,\sigma^2 t) \Rightarrow \mathbb{E}[e^{izW_t}] = e^{-\frac{1}{2}\sigma^2 z^2 t}$$

$$N_t \sim \textit{Poisson}(\lambda t) \Rightarrow \mathbb{E}[e^{iz\sum_{k=1}^{N_t} J_k}] = e^{\lambda t(\mathbb{E}[e^{izJ}-1])}$$

If we put these expectations in Equation 3.6 we get:

$$\begin{aligned}\Phi_{L_t}(z) &= \exp(izbt)\exp\left[-\frac{1}{2}z^2\sigma^2 t\right]\exp[\lambda t(\mathbb{E}[e^{izJ}-1] - iz\mathbb{E}[J])] \\ &= \exp(izbt)\exp\left[-\frac{1}{2}z^2\sigma^2 t\right]\exp[\lambda t(\mathbb{E}[e^{izJ}-1-izJ])].\end{aligned}$$

Because the distribution of J is F, we have

$$= \exp(izbt)\exp[-\frac{1}{2}z^2\sigma^2 t]\exp[\lambda t\int_{\mathbb{R}}(e^{izx}-1-izx)F(dx)].$$

We take t out as a common factor,

$$\mathbb{E}[e^{i\langle z,L_t\rangle}] = \exp[t(izb - \frac{z^2\sigma^2}{2} + \lambda\int_{\mathbb{R}}(e^{izx}-1-izx)F(dx)]$$

which we can write as:

$$\Phi_{L_t}(z) = e^{t\psi(z)}, \tag{3.7}$$

where $\psi(z) = izb - \dfrac{z^2\sigma^2}{2} + \lambda\int_{\mathbb{R}}(e^{izx}-1-izx)F(dx)$
is called the *characteristic exponent.* From Equation 3.7 we then have that:

$$\Phi_{L_1}(z) = e^{\psi(z)}$$

$$\Rightarrow \Phi_{L_t}(z) = (\Phi_{L_1}(z))^t$$

We notice that the distribution of L_1 tells us the distribution of the entire process L_t. This distribution belongs to a class of *infinitely divisible distributions* which is the subject of our next discussion.

3.3 Infinite Divisibility

Definition 3.11 *The collection of all* Borel sets *on* $\mathbb{R}^d$*, denoted by* $\mathcal{B}(\mathbb{R}^d)$*, is called the* Borel σ-algebra. *It is the* σ*-algebra generated by the open sets in* $\mathbb{R}^d$*, that is the smallest* σ*-algebra that contains all open sets in* $\mathbb{R}^d$*. The elements of* $\mathcal{B}(\mathbb{R}^d)$ *are called* Borel sets.

Next we define an important notion called *convolution* which will be useful in finding the distribution of the sum of two independent random variables.

Definition 3.12 *Let* $\mathcal{M}_1(\mathbb{R}^d)$ *denote the set of all Borel probability measures on* $\mathbb{R}^d$*.The* **convolution** μ *of two distributions* μ_1 *and* μ_2 *on* $\mathbb{R}^d$*, denoted by* $\mu = \mu_1 * \mu_2$*, is a distribution defined by:*

$$\mu(A) = (\mu_1 * \mu_2)(A) = \int_{\mathbb{R}^d}\mu_1(A-x)\mu_2(dx),$$

for each $\mu_i, i = 1, 2$ *and each* $A \in \mathcal{B}(\mathbb{R}^d)$*, where we note that* $A - x = \{y - x, y \in A\}$.

Proposition 3.13 *Suppose* μ_1, μ_2 *are probability measures,* $\mu = \mu_1 * \mu_2$ *and* f *a bounded Borel function. Then*

$$\int_{\mathbb{R}^d} f(y)(\mu_1 * \mu_2)(dy) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x+y)\mu_1(dy)\mu_2(dx). \tag{3.8}$$

Definition 3.14 *Let A be a non-empty set, the indicator function denoted by* $\mathbb{I}_A$ *is defined as follows:*

$$\mathbb{I}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{3.9}$$

Theorem 3.15 *Let* A *be a Borel set,* X_1 *and* X_2 *be independent random variables with distributions* μ_1 *and* μ_2 *respectively, then the variable* $X = X_1 + X_2$ *has the distribution*

$$\mu_X = \mu_1 * \mu_2.$$

Proof: *Let*

$$\begin{aligned} \mathbb{P}(X_1 + X_2 \in A) &= \mathbb{E}(\mathbb{I}_A(X_1 + X_2)) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{I}_A(x+y)p(dx,dy) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{I}_A(x+y)\mu_1(dx)\mu_2(dy) \end{aligned}$$

because of independence of X_1 *and* X_2*. By Proposition 3.13 we have that*

$$\begin{aligned} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{I}_A(x+y)\mu_1(dy)\mu_2(dx) &= \int_{\mathbb{R}^d} \mathbb{I}_A(\mu_1 * \mu_2)(dz) \\ &= \int_A \mu_1 * \mu_2(dz) \\ &= \mu_1 * \mu_2(A). \end{aligned}$$

$\mathbb{P}(X_1 + X_2 \in A) = \mu_1 * \mu_2(A)$ *for all* $A \subset \mathbb{R}^d$ *implies that* $\mu_1 * \mu_2$ *is the distribution of the random variable* $X_1 + X_2$*.*

Theorem 3.16 *Let* μ_1, μ_2 *be distributions on* $\mathbb{R}^d$ *with characteristic functions* Φ_{μ_1} *and* Φ_{μ_2} *respectively. If* $\mu = \mu_1 * \mu_2$*, then the characteristic function* Φ_μ *of the distribution* μ *is:*

$$\Phi_\mu = \Phi_{\mu_1}\Phi_{\mu_2} \tag{3.10}$$

Proof: *Let f_1, f_2, f be density functions of μ_1, μ_2 and μ respectively, $x, y \in \mathbb{R}^d$ then*

$$\begin{aligned}
\Phi_\mu(t) &= \int_{\mathbb{R}^d} e^{itx} d\mu(x) \\
&= \int_{\mathbb{R}^d} e^{itx} f(x)dx \\
&= \int_{\mathbb{R}^d} e^{itx} (f_1 * f_2)(x)dx \\
(f_1 * f_2)(x) &= \int_{\mathbb{R}^d} f_1(x-y)f_2(y)dy \\
\Phi_\mu(t) &= \int_{\mathbb{R}^d} e^{itx} \int_{\mathbb{R}^d} f_1(x-y)f_2(y)dydx
\end{aligned}$$

Making the substitution $x = u + y$,

$$\begin{aligned}
\Phi_\mu(t) &= \int_{\mathbb{R}^d} e^{it(u+y)} \int_{\mathbb{R}^d} f_1(u)f_2(y)dydu \\
&= \int_{\mathbb{R}^d} e^{itu} f_1(u)du \int_{\mathbb{R}^d} e^{ity} f_2(y)dy \\
&= \int_{\mathbb{R}^d} e^{itu} d\mu_1(u) \int_{\mathbb{R}^d} e^{ity} d\mu_2(y) \\
&= \Phi_{\mu_1}(t)\Phi_{\mu_2}(t).
\end{aligned}$$

In the definitions that follow we define the Gausian distribution and the compound Poisson distribution. We will use the results obtained in Theorem 3.16 to find the characteristic function of the convolution of the two distributions. We will use this characteristic function when proving the Lévy-Kintchine theorem in the next chapter.

Definition 3.17 μ_1 *is called a* Gaussian distribution *on* $\mathbb{R}^d$ *if:*

$$\Phi_{\mu_1}(z) = \exp(-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle) \tag{3.11}$$

where $\gamma, z \in \mathbb{R}^d$ and A is a nonnegative-definite symmetric $d \times d$ matrix.

Definition 3.18 μ_2 *is a* compound Poisson distribution *if for some $\lambda > 0$ and measure F on $\mathbb{R}^d$ with $F(\{0\}) = 0$, $z \in \mathbb{R}^d$,*

$$\Phi_{\mu_2}(z) = \exp[\lambda \int_{\mathbb{R}^d} (e^{izx} - 1)F(dx).] \tag{3.12}$$

If we take the convolution of μ_1 and μ_2 in the Definitions 3.17 and 3.18 and use Theorem 3.16 we get that:

$$\mu = \mu_1 * \mu_2$$

which implies that

$$\Phi_\mu(z) = \Phi_{\mu_1}(z)\Phi_{\mu_2}(z)$$

$$\Phi_\mu(z) = \exp[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d}(e^{izx} - 1)\lambda F(dx)] \tag{3.13}$$

We now define infinite divisibility.

Definition 3.19 *Let X be a random variable taking values in $\mathbb{R}^d$ with distribution μ_X. We say that X is* infinitely divisible *if for all $n \in \mathbb{N}$ there exist independent and identically distributed random variables $X_1^{(1/n)}, \ldots, X_n^{(1/n)}$ such that*

$$X \stackrel{d}{=} X_1^{(1/n)} + \cdots + X_n^{(1/n)},$$

where $\stackrel{d}{=}$ denotes equality in distribution.

Equivalently we can say that:
The distribution μ_X of a random variable X is *infinitely divisible* if for all $n \in \mathbb{N}$ there exists another distribution $\mu_{X^{(1/n)}}$ of a random variable $X^{(1/n)}$ such that

$$\mu_X = \underbrace{\mu_{X^{(1/n)}} * \cdots * \mu_{X^{(1/n)}}}_{\text{n times}}. \tag{3.14}$$

We can also use the characteristic function of a random variable to characterise the infinite divisibility.

Definition 3.20 *The distribution of a random variable X is* infinitely divisible *if for all $n \in \mathbb{N}$, there exists a random variable $X^{(1/n)}$, such that*

$$\Phi_X(z) = (\Phi_{X^{(1/n)}}(z))^n$$

The following are examples of infinitely divisible distributions. We will show that the examples are infinitely divisible using Definition 3.20.

1. (Normal distribution), Let $X \sim N(\mu, \sigma^2)$. The characteristic function of the variable X is

given by

$$\begin{aligned}\Phi_X(z) &= \mathbb{E}[e^{i\langle z,x\rangle}]\\ &= \int_{\mathbb{R}^d} e^{i\langle z,x\rangle} d\mu_X(x)\\ &= \exp[iz\mu - \frac{1}{2}z^2\sigma^2]\\ &= \exp[izn\frac{\mu}{n} - \frac{1}{2}z^2 n\frac{\sigma^2}{n}]\\ &= \exp[n(iz\frac{\mu}{n} - \frac{1}{2}z^2\frac{\sigma^2}{n})]\\ &= (\exp[iz\frac{\mu}{n} - \frac{1}{2}z^2\frac{\sigma^2}{n}])^n\\ &= (\Phi_{X^{(1/n)}}(z))^n,\end{aligned}$$

i.e. $X^{(1/n)} \sim N(\frac{\mu}{n}, \frac{\sigma^2}{n})$.

2. (Poisson distribution), Let $X \sim$ Poisson(λ) be a random variable, then

$$\begin{aligned}\Phi_X(z) &= \mathbb{E}(e^{i\langle z,x\rangle})\\ &= \int_{\mathbb{R}^d} e^{i\langle z,x\rangle} d\mu_X(x)\\ &= \exp[\lambda(e^{iz} - 1)]\\ &= \exp[n\frac{\lambda}{n}(e^{iz} - 1)]\\ &= (\exp[\frac{\lambda}{n}(e^{iz} - 1)])^n\\ &= (\Phi_{X^{(1/n)}}(z))^n,\end{aligned}$$

i.e. $X^{(1/n)} \sim$ Poisson($\frac{\lambda}{n}$).

Thus the *Normal distribution* and *Poisson distribution* are infinitely divisible.

4. Lévy-Kintchine Theorem

4.1 Lévy-Kintchine Formula

In this chapter we will look at the Lévy-Kintchine theorem. The theorem presents a useful formula which enables us to characterise the infinite divisibility of Lévy processes by their characteristic function. This emphasizes the strength of the connection between Lévy processes and infinite divisibility. We will start by listing some important theorems and definitions that will be useful in the proof of the main theorem.

Theorem 4.1 *(Lévy continuity) If $(\phi_n)_{n\in\mathbb{N}}$ is a sequence of characteristic functions and there exists a function $\psi : \mathbb{R}^d \to \mathbb{C}$ such that for all $u \in \mathbb{R}^d$, $\phi_n(u) \to \psi(u)$ as $n \to \infty$ and ψ is continuous at zero, then ψ is the characteristic function of a probability distribution.*

Proof: See [CK98]

Definition 4.2 *(Lévy measure) Let ν be a Borel measure defined on $\mathbb{R}^d\backslash\{0\} = \{x \in \mathbb{R}^d, x \neq 0\}$. ν is a Levy measure if*

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d\backslash\{0\}} (|y|^2 \wedge 1)\nu(dy) < \infty, \tag{4.1}$$

where $(|y|^2 \wedge 1) = \min\{|y|^2, 1\}$.

The Lévy measure describes the expected number of jumps of a certain height in a time interval of unit length 1. If ν is a finite measure, i.e $\nu(\mathbb{R}) = \int_{\mathbb{R}} \nu(dy) = \lambda < \infty$, then $F(dy) := \dfrac{\nu(dy)}{\lambda}$ is a probability measure. Thus λ is the expected number of jumps and $F(dy)$ the distribution of the jump size y. If $\nu(\mathbb{R}) = \infty$, then an infinite number of jumps is expected. Properties such as *activity* and *variation* about the path of a Lévy process can be derived from its Lévy measure.

Theorem 4.3 *A Borel probability measure μ is infinitely divisible if there exists a vector $b \in \mathbb{R}^d$, a positive-definite symmetric $d \times d$ matrix A and a Lévy measure ν on $\mathbb{R}^d \setminus \{0\}$ such that, for all $u \in \mathbb{R}^d$,*

$$\Phi_\mu(u) = \exp\left\{e^{i\langle b,u\rangle} - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}^d\backslash\{0\}} [e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle \mathbb{I}_{\hat{B}}(y)]\nu(dy)\right\} \tag{4.2}$$

where $\hat{B} = B_1(0) = \{y \in \mathbb{R}^d : |y| < 1\}$ and $|y|$ denotes the Euclidean norm of y $\in \mathbb{R}^d$.

Converse: Any mapping of the form of Equation (4.2) is a characteristic function of an infinitely divisible probability measure on $\mathbb{R}^d$.

Proof: We only state the proof of the converse.This implies that we must show that the right-hand side of Equation (4.2) is a characteristic function. Let $(\alpha(n), n \in \mathbb{N})$ be a sequence in $\mathbb{R}^d$ that is monotonically decreasing to zero.
Define for all $u \in \mathbb{R}^d, n \in \mathbb{N}$

$$\Phi_n(u) = \exp\left\{ i\langle b - \int_{[-\alpha(n),\alpha(n)]^c \bigcap \hat{B}} y\nu(dy), u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{[-\alpha(n),\alpha(n)]^c} (e^{i\langle u,y\rangle} - 1)\nu(dy)\right\}. \tag{4.3}$$

Each Φ_n represents the convolution of the normal distribution with an independent compound Poisson distribution as in Equation (3.13) and it is therefore the characteristic function of a probability measure μ_n. We notice that

$$\Phi_\mu(u) = \lim_{n\to\infty} \Phi_n(u).$$

Using the Lévy continuity Theorem 4.1 and by showing that $\Phi_\mu(u)$ is continuous at zero, it follows that Φ_μ is a characteristic function. To show this we let

$$\psi_\mu(u) = \int_{\mathbb{R}^d\setminus\{0\}} [e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle \mathbb{I}_{\hat{B}}(y)]\nu(dy)$$

and we must show that ψ_μ is continuous at zero. Since ν is a Lévy measure we have

$$\psi_\mu(u) = \int_{\hat{B}} [e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle]\nu(dy) + \int_{\hat{B}^c} (e^{i\langle u,y\rangle} - 1)\nu(dy).$$

Using the identity

$$e^{iu} = \sum_{k=0}^{n-1} \frac{(iu)^k}{k!} + \theta\frac{|u|^n}{n!},$$

where $\theta \in \mathbb{C}$ satisfying $|\theta| < 1$, we have for $n = 2$ that

$$e^{i\langle u,y\rangle} = 1 + i\langle u, y\rangle + \theta\frac{|\langle u, y\rangle|^2}{2}$$

$$\Rightarrow e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle = \theta\frac{|\langle u, y\rangle|^2}{2}$$

$$\Rightarrow |e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle| \leq \theta\frac{|u|^2|y|^2}{2}$$

by the Cauchy-Schwartz inequality. This implies that

$$\int_{\hat{B}} |e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle|\nu(dy) < \frac{|u|^2}{2}\int_{\hat{B}} |y|^2\nu(dy) \to 0 \textit{ as } u \to 0,$$

since $\int_{\hat{B}} |y|^2\nu(dy) < \infty$ as ν is a Lévy measure.

Also

$$\int_{\hat{B}^c} |e^{i\langle u,y\rangle} - 1|\nu(dy) = 0 \textit{ if } u = 0$$

since $\nu(\{0\}) = 0$ (see Definition 4.2).
This shows that $\psi_\mu(u)$ is continuous at zero which in turn implies that $\Phi_\mu(u)$ is continuous at zero. By the Levy-continuity theorem Φ_μ is a characteristic function of an infinitely divisible random variable.

4.2 Properties of Lévy Process

From Theorem 4.3 we call the triplet (b, A, ν) the **characteristic** or **Lévy triplet**. From the characteristic triplet we get the following properties about the path of a Lévy process.

Proposition 4.4 *Let L be a Lévy process with triplet (b, A, ν). Then*

1. *If $\nu(\mathbb{R}) < \infty$ then almost all paths of L have a finite number of jumps on every compact interval. The Lévy process is said to be of* finite activity *in that case.*
2. *If $\nu(\mathbb{R}) = \infty$ then almost all paths of L have an infinite number of jumps on every compact interval. In this case the Lévy process has* infinite activity*.*

Proof: *See Theorem 21.3 in Sato (1999).*

Lévy measure determines whether a Lévy process has finite variation or not.

Proposition 4.5 *Let L be a Lévy process with triplet (b, A, ν)*

1. *If $A = 0$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$ then almost all paths of L have finite variation.*
2. *If $A \neq 0$ or $\int_{|x|\leq 1} |x|\nu(dx) = \infty$ then almost all paths of L have finite variation.*

Lévy measure can also determine the existence of moments of the Lévy process.

Proposition 4.6 *Let L be a Lévy process on $\mathbb{R}^d$ with characteristic triplet (b, A, ν), and $\hat{B}$ be an open ball as defined in Theorem 4.3. Then*

1. $\mathbb{E}[L(t)] = t\mathbb{E}[L(1)]$ *if and only if*
$$\int_{\mathbb{R}^d} |y|\mathbb{I}_{\hat{B}(y)}\nu(dy) < \infty,$$
and in this case
$$\mathbb{E}[L(1)] = bt + \int_{\mathbb{R}^d} |y|\mathbb{I}_{\hat{B}(y)}dy.$$
2. $\mathbb{E}[L(1)^2] < \infty$ *if and only if*
$$\int_{\mathbb{R}^d} |y|^2\mathbb{I}_{\hat{B}(y)}\nu(dy) < \infty,$$
and then
$$\mathbb{E}[L(1)^2] = \int_{\mathbb{R}^d} |y|^2\nu(dy) + A.$$

4.3 Simulating Lévy Processes

The following algorithm describes how to simulate Lévy jump-diffusion on a fixed-time grid. Recall that Levy jump-diffusion is the sum of a Brownian motion and a compensated Poisson process.

- Algorithm for Brownian motion with drift μ.
  The formula is given as $dX_t = \mu dt + \sigma dB_t$
  Let
  $$X_{t+\triangle t} = X_t + \mu \triangle t + \sigma \triangle B_t,$$
  where $B_t \sim N(0, \triangle t)$, i.e $\triangle B_t$ is a Gaussian random variable with mean 0 and variance $\triangle t$.

- Algorithm for compound Poisson process.
  Initialize $k = 0$

  Repeat *while* $\sum_{i=1}^{k} T_i < T$
  Set $k = k + 1$.
  Simulate $T_k \sim \exp(\lambda)$.
  Simulate Y_k from distribution $\mu = \nu/\lambda$.
  The trajectory is given by
  $$X_{t_j} = bt_i + X_{t+\triangle t} + \sum_{i=1}^{N_t} Y_i, \text{ where } N_t = sup\{k : \sum_{i=1}^{k} T_i \leq t\}.$$

The following code written in octave implements the Lévy jump-diffusion process.

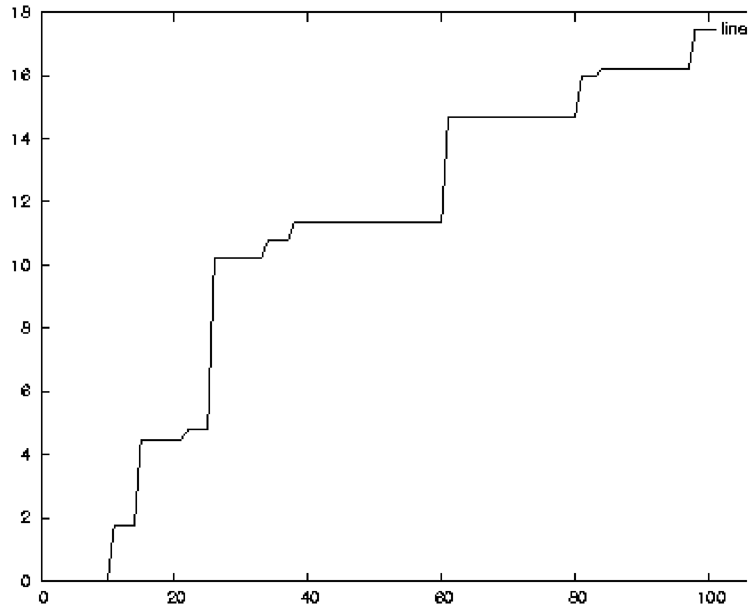
```
function D=jumpdiff(lam,mu,var,t)
X=0;
dt = 0.05;
N=length(t);
B=zeros(1,N);
for j=1:N
dBt=randn(1)*sqrt(dt);
X = X+mu*dt+var*dBt;
B(j)=X;
end
B; returns a vector holding information about variables which follow Brownian motion


TT = 0;
k=0;
N = length(t);
while sum(TT(1:k)) < t(N)
```

```
k=k+1;
Tk= (-1*log(rand())/log(exp(1)))/lam; simulating Tk which exponentially distributed
if k > 1
TT=[TT,Tk];
else
TT=Tk;
end
yk=(-1*log(rand())/log(exp(1)))/(lam*lam) simulating yk from distribution nu/lambda
if k > 1
y=[y,yk];
else
y=yk;
end
end
X=zeros(1,N);
w=0;
for j=1:N
while sum(TT(1:w))¡=t(j)
w=w+1;
end
n=w-1;
X(j)=sum(y(1:n));summing n number of events occuring at time t
end
X;    returns a vector which holds information about compound poisson process
D=zeros(1,N);
for k=1:N
D(k)=X(k)+B(k);
end
D;    vector for plotting data which give a jump-diffusion process
```

The following plots are obtained from the above simulation.

1. For the simulation Brownian motion see Figure 2.1.
2. Figure 4.1 shows the simulation of Lévy jump-diffusion process . The top figure is for the compound Poisson process. The bottom figure is for the Lévy jump-diffusion process. The vertical line represents a jump.



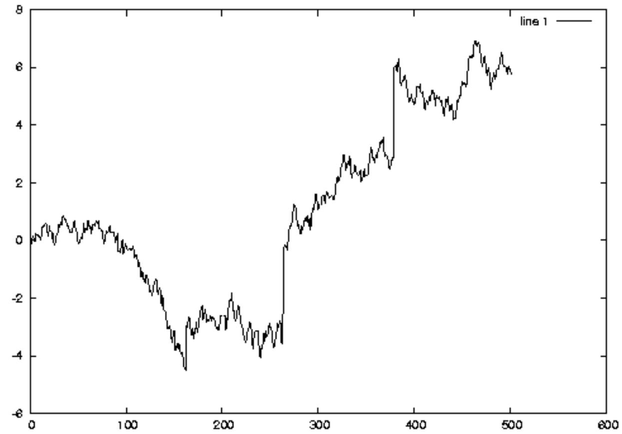


Figure 4.1: The simulation of sample path of compound Poisson process(top) and Lévy jump-diffusion process(bottom)

4.4 Summary

We considered the notion of probability space which is a space where stochastic processes are defined. Some stochastic processes are continuous and others are discontinuous, hence we need a space which accomodates both processes and for this we use a class called cadlag processes. The idea of filtration provides a way of revealing more information about a particular event as time progresses. Lévy processes are stochastic processes with stationary and independent increments. Two important examples of Lévy processes are Brownian motions and Poisson processes. Characteristic functions and infinite divisibility play an important role in describing the probability measure of a Lévy process. This is shown in the Lévy-Kintchine theorem. This theorem presents a Lévy-Kintchine formula which is used to characterize the measure of an infinitely-divisible random variable by its characteristic function. The properties of a Lévy process are derived from its Lévy measure. The measure determines whether the Lévy process has finite variation or not. The activity property tells us whether the path of a Lévy process has finite or infinite number of jumps. We finally considered an algorithm for implementing the Lévy jump-diffusion process and gave some code written in octave which implements the algorithm.

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