Introduction to Stochastic Portfolio Theory

Thibaut Zafack Takadong (thibaut@aims.ac.za)
African Institute for Mathematical Sciences (AIMS)

Supervised by Dr. Raouf Ghomrasni
University of the Witwatersrand

June 6, 2007

Abstract

In this work we give a first insight into *Stochastic Portfolio Theory*. Stochastic Portfolio Theory is a descriptive approach to Financial modelling born from the study of stock prices *log-return*. While general Financial modelling is based on the study of stock prices *arithmetic-return* and is normative. Standard assumptions and restrictions are seen from another perspective, become uncertain and sometimes false or not needed.

Résumé

Les Mathématiques Financières ont pour but d'analyser le comportement de portfolios et de stocks dans un marché financier. Ici, nous vous introduisons à la *Thèorie de Portfolio Stochastique* dont *Robert Fernholz* est le pionnier. Elle est basée sur le modéle logarithmique de prix de stocks plutôt que sur le modéle arithmique qui est utilisé par toutes les autres méthodologies. Cette nouvelle approche revèle des aspects jusqu'alors cachés du comportement de marchés financiers. Elle nous présente ce comportement sous un nouveau jour, ainsi que les hypothèses et restrictions usuelles; certaines de ces dernières se revèlent alors incertaines et souvent fausses ou non nécessaires.

Contents

Introduction

Stochastic portfolio theory is a mathematical methodology for constructing stock portfolios, analysing the behaviour of portfolios, and understanding the structure of equity markets. It distinguishes itself from other financial modelling approaches by the fact that it is descriptive rather than normative.

Normative approaches assume that certain norms or requirements are satisfied by the market we are modelling. The usual assumptions are non-arbitrage, market equilibrium and market completeness. On the other hand, a descriptive approach can model any market. In particular Stochastic Portfolio Theory is compatible with or without arbitrage, with equilibrium or desequilibrium, with market completeness or incompleteness.

Stochastic portfolio theory has both theoretical and practical applications: as a theoretical tool it can be used to construct examples of theoretical portfolios with specified characteristics, and to determine the distributional component of portfolio return. On a practical level, stochastic portfolio theory has been the basis for strategies used for over a decade by the institutional equity manager INTECH, where *Robert Fernholz* has served as chief investment officer.

This work starts by defining the basics concepts of financial modelling (stock, portfolio, market) within the framework of stochastic portfolio. The second chapter presents some characteristics peculiar to stochastic portfolio theory, relative return and market portfolio. These are then used to study the *long term* behaviour of portfolio and to define possible optimisation strategies. The last chapter presents some key concepts of the market and their use and implication.

1. Stocks and Portfolios

Throughout this work, all random processes are defined either on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ or when we consider a standard n-dimensional Brownian motion[1]

$$W_t = (W_t^1, W_t^2, .., W_t^n), \quad t \in [0, \infty),$$

on the induced filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t, t \in \mathbb{R}_+\}$ is the *natural filtration* generated by W and defined by

$$\mathcal{F}_t = \sigma(W(s); 0 \leq s < t).$$

Moreover, $\mathbb{F}$ will be the only filtration considered when we speak of adapted processes, martingales and other properties with respect to a filtration.

Remark. *The time, t, is always from the interval $[0, \infty)$ and not on an interval $[0, T]$.*

1.1 Stocks

Instead of the common *arithmetic model*, we shall use the *logarithmic model* for continuous time stock price processes because it outlines certain aspects of portfolio behaviour that remain obscure with the arithmetic representation[2].

Definition 1.1 (Stock price process). *Considering n a positive integer, a* stock price process *X subject to n sources of uncertainty is a random process that satisfies*

$$d \log X(t) \quad = \quad \gamma(t)dt + \sum_{\nu=1}^{n} \xi_\nu(t) dW_\nu(t), \quad t \in [0, \infty), \tag{1.1}$$

where $(W_1, W_2, \cdots, W_n)$ is an n-dimensional Brownian motion, γ and the $\xi_\nu, \nu = 1, 2, \cdots, n$ are measurable, adapted (i.e. they do not depend on future events), and satisfy

i. $\int_0^T \sum_{\nu=1}^{n} (\xi_\nu(t))^2 \, dt < \infty, \quad T \in [0, \infty)$, *a.s.*[3]*;*

ii. $\lim_{t \to \infty} t^{-1} \sum_{\nu=1}^{n} (\xi_\nu(t))^2 \log \log t = 0$*;*

iii. $\sum_{\nu=1}^{n} (\xi_\nu(t))^2 > 0, \quad t \in [0, \infty), a.s.$*;*

[1]See Appendix B.

[2]See Appendix A and 3.1.

[3]"*almost surely*" indicates that an event has probability one as measured by $\mathbb{P}$.

iv. $\int_0^T |\gamma(t)|dt < \infty$, $\forall T \in [0, \infty)$,*a.s.*

In the stochastic differential equation (1.1) $\gamma(t)dt$ is the *deterministic part* and the process γ is called the *growth rate* process of X while $\sum_{\nu=1}^{n} \xi_\nu(t)dW_\nu(t)$ is *the stochastic term* [2] and the ξ_ν are called the *volatility* processes of X. The process ξ_ν measures X sensitivity to the ν-th source of uncertainty, W_ν.

Assuming that $X_0 > 0$ is the initial value of the stock, integration of (1.1) gives

$$\log X(t) \quad = \quad \log X_0 + \int_0^t \gamma(s)ds + \int_0^t \sum_{\nu=1}^{n} \xi_\nu(s)dW_\nu(s), \quad t \in [0, \infty).$$

Hence

$$X(t) \quad = \quad X_0 \exp\left(\int_0^t \gamma(s)ds + \int_0^t \sum_{\nu=1}^{n} \xi_\nu(s)dW_\nu(s)\right), \quad t \in [0, \infty). \tag{1.2}$$

X is computable. Indeed condition $(iv.)$ (respectively $(i.)$) ensures that $\int_0^t \gamma(t)dt < \infty$ (respectively the variance of X_t, $\int_0^t \sum_{\nu=1}^{n} \xi_\nu(t)dW_\nu(t) < \infty$ i.e. $\|\xi\|$ is a.s. of bounded quadratic variation). Condition $(ii.)$ ensures that a stock volatility does not increase too quickly i.e. when studying X over time, the growth rate influence becomes insignificant with respect to the volatility impact. Condition $(iii.)$ ensures that the variance of $X(t)$ is non-degenerate i.e. $\exists i = 1, 2, \cdots, n$ such that $\xi_i > 0$, $\forall t$.

We immediately notice from equation (1.1) that $\log X(t)$ is a continuous semimartingale with bounded variation component

$$\gamma(t)dt, \quad t \in [0, \infty),$$

and local martingale component

$$\sum_{\nu=1}^{n} \xi_\nu(t)dW_\nu(t), \quad t \in [0, \infty).$$

Assumptions. *To simplify the presentation we make some standard assumptions [3]. The number of companies in the market is finite and fixed. Neither are there new companies founded nor do existing companies disappear (go bankrupt). Companies do not merge or break up. The total number of shares of each company is fixed. Trading is continuous in time as well is the number of stocks i.e. infinitesimally small fractions of stocks can be bought and sold. There are no trading cost or taxes; dividends are paid continuously rather than discretely. When not needed we assume that there are no dividends.*

To evaluate how well behaved the stock prices are, we study their variations through the induced covariance matrix. The covariance matrix is mainly used to measure the risk of achieving a particular goal inherent to the stocks as a whole.

Definition 1.2 (Covariance). *Let* $(X_i)_{1\leq i\leq n}$ *be a family of stock prices such that*

$$d\log X_i(t) = \gamma_i(t)dt + \sum_{\nu=1}^{n}\xi_{i\nu}(t)dW_\nu(t), \quad t\in[0,\infty), \tag{1.3}$$

i.e.

$$X_i(t) = X_0^i \exp\left(\int_0^t \gamma_i(s)ds + \int_0^t \sum_{\nu=1}^{n}\xi_{i\nu}(s)dW_{i\nu}(s)\right), \quad t\in[0,\infty), \tag{1.4}$$

where $X_0^i > 0$ *is the initial initial value of the* i*th stock.*

We set $\xi(t) = (\xi_{ij}(t))_{1\leq i,j\leq n}$ *and define the* covariance *process by*

$$\sigma(t) = \xi(t)\xi^T(t), \quad t\in[0,\infty). \tag{1.5}$$

For any $x\in\mathbb{R}^n$ and $t\in[0,\infty)$

$$x\sigma(t)x^T = x\xi(t)\xi^T(t)x^T = x\xi(t)\left(x\xi(t)\right)^T = \|x\xi(t)\|^2 \geq 0, \tag{1.6}$$

where $\|.\|$ is the *Euclidean norm* in $\mathbb{R}^n$.

Therefore $\sigma(t)$ is positive semidefinite $\forall t\in[0,\infty)$. The *covariance* of $X_i(t)$ and $X_j(t)$ is $\sigma_{ij}(t)$ while $\sigma_{ii}(t)$ is the *variance* of $X_i(t)$.

If $X(t)$ and $Y(t)$ are two real-valued continuous functions, then $\langle X,Y\rangle_t$ denotes their *cross-variation* function while $\langle X\rangle_t = \langle X,X\rangle_t$ is the *quadratic variation* function for $X(t)$.

The cross-variation differential of $\log X_i(t)$ and $\log X_j(t)$ is

$$\begin{aligned} d\langle\log X_i, \log X_j\rangle_t &\overset{(1.3)}{=} \langle\sum_{\nu=1}^{n}\xi_{i\nu}dW_\nu, \sum_{\iota=1}^{n}\xi_{j\iota}dW_\iota\rangle_t \\ &= \sum_{\nu,\iota=1}^{n}\xi_{i\nu}(t)\xi_{j\iota}(t)\delta_{\nu\iota}dt \\ &= \sum_{\nu=1}^{n}\xi_{i\nu}(t)\xi_{j\nu}(t)dt \\ &\overset{(1.5)}{=} \sigma_{ij}(t)dt, \end{aligned}$$

$t\in[0,\infty)$, a.s.

Finally,

$$d\langle\log X_i, \log X_j\rangle_t \overset{a.s.}{=} \sigma_{ij}(t)dt, \quad t\in[0,\infty), \quad \forall i,j = 1,2,\cdots,n. \tag{1.7}$$

1.2 Equity Market

Definition 1.3 (Market). *If we assume $\sigma(t)$ to be* invertible *i.e.* non-singular, *then the set of all stock prices $\mathcal{M}$ is the market:*

$$\mathcal{M} = \{X_1, X_2, \cdots, X_n\}.$$

The market $\mathcal{M}$ is said to be *non-degenerate* if there exists $\varepsilon > 0$ such that

$$x\sigma(t)x^T \geq \varepsilon\|x\|^2, \quad t \in [0, \infty), \quad \forall x \in \mathbb{R}^n, \quad a.s. \tag{1.8}$$

This means that the stock variance matrix is not degenerate i.e. a zero variance portfolio can not be built on the market $\mathcal{M}$.

The market $\mathcal{M}$ is said to be of *bounded variance* if there exists $M > 0$ such that

$$x\sigma(t)x^T \leq M\|x\|^2, \quad t \in [0, \infty), \quad \forall x \in \mathbb{R}^n, \quad a.s. \tag{1.9}$$

It means that it is not possible to build a portfolio with any given variance i.e. the set of attainable portfolio variance is bounded.

Remark. *Here because the number of stocks in our market is equal to the dimension of the Brownian motion, the market is said to be* complete. *However, in general we just need to have at least as many sources of uncertainty (dimension of Brownian motion) in the market as there are stocks. We have not included a riskless asset since we want to study the behaviour of stock portfolio only.*

Lemma 1.4. *For the market covariance process σ, $\sigma(t)$ is positive definite for all $t \in [0, \infty)$.*

Proof. From (1.6) $\forall t \in [0, \infty), \sigma(t)$ is positive semidefinite i.e. σ's eigenvalues are nonnegative. We assumed in Definition 1.3 that $\sigma(t)$ is invertible, then σ's eigenvalues are non zero; therefore, σ's eigenvalues are strictly positive and hence $\sigma(t)$ is positive definite for all $t \in [0, \infty)$, a.s. ∎

Example (A market: S&P 500). *The S&P 500 is a market made up of 500 stocks chosen for their market size and liquidity among other factors. It is designed to be a leading indicator of US corporations and is meant to reflect the risk/return characteristics of the large-cap[4] universe.*

1.3 Portfolio

Definition 1.5. *In the market $\mathcal{M}$, a* portfolio *is a measurable, adapted vector-valued process $\pi(t) = (\pi_1(t), \pi_2(t), \cdots, \pi_n(t))$ that is a.s. bounded with*

$$\sum_{i=1}^{n} \pi_i(t) \overset{a.s.}{=} 1.$$

[4]Market with capital of over \$ 10 billion.

The ith component, π_i, represents the proportion of the corresponding stocks in the portfolio. We allow short sales i.e. negative components in π. Note that π_i is not the number/fraction of the ith stock's share held in the portfolio but the proportion[5] of amount invested in the ith stock.

1.3.1 Trading Strategy

To build a portfolio agents in the market define a trading strategy

$$\psi(t) = \left(\psi_t^1, \psi_t^2, \cdots, \psi_t^n\right), \quad t \in [0,\infty),$$

where ψ_t^i is the number/fraction of the ith stock's share held in the portfolio at time t.

At time t, if $\pi(t)$ is the portfolio corresponding to this strategy and $Z_\pi(t)$ is the total amount invested, then the amount invested in the ith asset is given by either $\pi_i(t)Z_\pi(t)$ or $\psi_t^i X_i(t)$, hence

$$\pi_i(t)Z_\pi(t) = \psi_t^i X_i(t), \quad t \in [0,\infty). \tag{1.10}$$

Remark. *Note that*

$$Z_\pi(t) = \sum_{i=1}^{n} \psi_t^i X_i(t), \quad t \in [0,\infty).$$

Example (A portfolio). *Let us consider the set of portfolios*

$$\pi^\nu(t) = \left(\delta_{1\nu}, \delta_{2\nu}, \cdots, \delta_{n\nu}\right), \quad 1 \le \nu \le n,$$

for $t \in [0,\infty)$ and where δ_{ij} is the Kronecker delta *symbol.*

The portfolio $\pi^\nu(t)$ is the constant vector with all zero components except the ν-th component which is 1. This means that all the investment is made in a unique stock, the ν-th stock; the portfolio value fluctuations are similar to the changes in the ν-th stock price.

The trading strategy corresponding to $\pi^\nu(t)$ is defined by

$$\psi^\nu(t) = \left(\delta_{1\nu}, \delta_{2\nu}, \cdots, \delta_{n\nu}\right), \quad 1 \le \nu \le n,$$

for $t \in [0,\infty)$.

Example (A portfolio-Bis). *Let us consider the trading strategy*

$$\psi(t) = \left(1, 1, \cdots, 1\right), \quad t \in [0,\infty).$$

It is the portfolio made of one share of each of the stocks in the market. The corresponding portfolio is

$$\pi(t) = \left(\frac{X_1(t)}{\sum_{i=1}^{n} X_i(t)}, \frac{X_2(t)}{\sum_{i=1}^{n} X_i(t)}, \cdots, \frac{X_n(t)}{\sum_{i=1}^{n} X_i(t)} \right), \quad t \in [0,\infty).$$

[5] With respect to the total amount invested in the portfolio.

1.3.2 Portfolio Properties

For a portfolio defined at time t by $\pi(t)$ with corresponding trading strategy $\psi(t)$ and value $Z_\pi(t)$, the amount invested in the ith stock is $\pi_i(t)Z_\pi(t)$. A change of value dX_i in the ith stock price induces a change of

$$\psi_t^i dX_i(t) \overset{(1.10)}{=} \pi_i(t)Z_\pi(t)\frac{dX_i(t)}{X_i(t)}$$

in the portfolio value. Therefore, the instantaneous portfolio return can be written as

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^{n} \pi_i(t)\frac{dX_i(t)}{X_i(t)}, \qquad (1.11)$$

which is the weighted average of instantaneous component returns.

We are interested in the solutions to this equation which can be computed from the following proposition.

Proposition 1.6. *If we consider a portfolio π in $\mathcal{M}$, then the portfolio value Z_π satisfies*

$$d\log Z_\pi(t) = \gamma_\pi(t)dt + \sum_{i,j=1}^{n} \pi_i(t)\xi_{ij}(t)dW_j(t), \qquad (1.12)$$

where

$$\gamma_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right). \qquad (1.13)$$

Proof. Let us apply Itô formula[6] with $F(x) = \log x$ and $x = X(t)$

$$d\log(X(t)) = \frac{dX(t)}{X(t)} - \frac{1}{2X^2(t)}d\langle X\rangle_t.$$

Taking $X(t) = Z_\pi(t)$ we have

$$\begin{aligned} d\log(Z_\pi(t)) &= \frac{dZ_\pi(t)}{Z_\pi(t)} - \frac{1}{2Z_\pi^2(t)}d\langle Z_\pi\rangle_t \\ &= \frac{dZ_\pi(t)}{Z_\pi(t)} - \frac{1}{2}\langle \frac{dZ_\pi}{Z_\pi}\rangle_t \end{aligned}$$

i.e.

$$d\log(Z_\pi(t)) \overset{(1.11)}{=} \sum_{i=1}^{n} \pi_i(t)\frac{dX_i(t)}{X_i(t)} - \frac{1}{2}\langle \frac{dZ_\pi}{Z_\pi}\rangle_t. \qquad (1.14)$$

[6] See Appendix C.

We will now compute the terms on the right hand side of equation (1.14). Let us apply Itô formula to $X_i(t) = \exp(\log X_i(t))$

$$dX_i(t) = X_i(t) d\log X_i(t) + \frac{1}{2} X_i(t) d\langle \log X_i \rangle_t$$

However, from equation (1.3) and because $d\langle \log X_i \rangle_t \overset{(1.7)}{=} \sigma_{ii}(t)dt$

$$\frac{dX_i(t)}{X_i(t)} = \left(\gamma_i(t) + \frac{1}{2}\sigma_{ii}(t)\right) dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t) dW_\nu(t), \quad t \in [0, \infty).$$

Combining this with (1.11), we have

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^{n} \pi_i(t)\gamma_i(t)dt + \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t)dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)dW_\nu(t). \tag{1.15}$$

Therefore,

$$\begin{aligned}
\langle \frac{dZ_\pi}{Z_\pi} \rangle_t &\overset{(1.15)}{=} d\langle \sum_{i,\nu=1}^{n} \pi_i \xi_{i\nu} dW_\nu \rangle_t \\
&= d\langle \sum_{i,\nu=1}^{n} \pi_i \xi_{i\nu} dW_\nu, \sum_{j,\beta=1}^{n} \pi_j \xi_{j\beta} dW_\beta \rangle_t \\
&= \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t) \sum_{j,\beta=1}^{n} \pi_j(t)\xi_{j\beta}(t)\delta_{\nu\beta} dt \\
&= \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t) \sum_{j=1}^{n} \pi_j(t)\xi_{j\nu}(t) dt \\
&= \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t) \left(\sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t)\right) dt \\
&= \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t)dt.
\end{aligned}$$

With the notation

$$\gamma_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right),$$

substituting back in (1.14) we have

$$d\log Z_\pi(t) = \gamma_\pi(t)dt + \sum_{i,j=1}^{n} \pi_i(t)\xi_{ij}(t)dW_j(t).$$

■

The exponential form of (1.12) is

$$Z_\pi(t) \;=\; Z_\pi(0)\exp\left(\int_0^t \gamma_\pi(s)ds + \int_0^t \sum_{i,j=1}^n \pi_i(s)\xi_{ij}(s)dW_j(s)\right). \tag{1.16}$$

Since all γ_i, π_i and ξ_{ij} have good properties[7] Z_π is a semimartingale because of (1.12).

The covariance can also be defined with respect to a portfolio. In this case, it still measures the portfolio variation or risk involved when trying to achieve a goal. The covariance process is given by

$$\sigma_{\pi\pi}(t) \;=\; \int_0^t \langle \log Z_\pi\rangle_s ds \quad [3]. \tag{1.17}$$

However,

$$\begin{aligned} d\langle \log Z_\pi\rangle_t &\overset{(1.12)}{=} d\langle \sum_{i,j=1}^n \pi_i\xi_{ij}dW_j\rangle_t \\ &= \langle \frac{dZ_\pi}{Z_\pi}\rangle_t \\ &= \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t)dt. \end{aligned}$$

Therefore,

$$d\langle \log Z_\pi\rangle_t \;=\; \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t)dt. \tag{1.18}$$

Equation (1.17) and the integration of (1.18) give

$$\sigma_{\pi\pi}(t) \;=\; \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t).$$

Remark. *Note that if $\mathcal{M}$ is of bounded variance, then $\sigma_{\pi\pi}$ is a.s. bounded for any π.*

We can now rewrite (1.13) as

$$\gamma_\pi(t) \;=\; \sum_{i=1}^n \pi_i(t)\gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sigma_{\pi\pi}(t)\right),$$

where the term

$$\gamma_\pi^*(t) \;=\; \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sigma_{\pi\pi}(t)\right) \tag{1.19}$$

[7] See Definition 1.1.

is the *excess growth rate* process of π. Hence,

$$\gamma_\pi(t) \quad = \quad \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t), \quad t \in [0, \infty). \tag{1.20}$$

From (1.12) we obtain

$$d \log Z_\pi(t) \quad = \quad \sum_{i=1}^{n} \pi_i(t) d \log X_i(t) + \gamma_\pi^*(t) dt. \tag{1.21}$$

Let us examine this equation. The instantaneous logarithmic portfolio return is the weighted average of the component stocks *plus* the excess growth rate. The excess growth rate is half the difference between the average component variance and the portfolio variance. Therefore we can interpret γ_π^* as a measure of diversification efficiency since it reduces the volatility of Z_π compared to its components [3]. In classic portfolio theory it is well known that diversification can lower portfolio volatility. But here we see that *diversification also influences the portfolio growth rate*. As we shall see later[8], for a portfolio without short sales this excess growth rate is always non-negative and is strictly positive for such portfolios that do not concentrate their holdings in just one stock. This behaviour is not apparent if we are looking at the arithmetic return of the portfolio which is the weighted average of its stocks arithmetic return.

1.3.3 Dividend Rate

Companies' gains are either reinvested in the business or shared to its shareholders. This is usually done through dividend rates.

Definition 1.7. *A* dividend rate *process is a measurable, adapted process δ such that*

$$\int_0^t |\delta(s)| ds \quad < \quad \infty, \quad t \in [0, \infty), \quad a.s.$$

Let X be a stock with associated dividend rate process δ, the corresponding *total return* process $\widehat{X}$ is given by

$$\widehat{X}(t) \quad = \quad X(t) \exp\left(\int_0^t \delta(s) ds\right), \quad t \in [0, \infty). \tag{1.22}$$

It is the value of an investment in X with all dividends continuously reinvested. Notice that if $\delta = 0$, then $\widehat{X}(t) = X(t)$. Furthermore, equation (1.22) implies that $\widehat{X}(0) = X(0)$ and

$$d \log \widehat{X}(t) \quad = \quad d \log X(t) + \delta(t) dt, \quad t \in [0, \infty). \tag{1.23}$$

The *augmented growth rate* is defined by

$$\rho(t) \quad = \quad \gamma(t) + \delta(t), \quad t \in [0, \infty). \tag{1.24}$$

[8] Proposition 3.3

Notice that if the stock price X satisfies

$$d\log X(t) = \gamma(t)dt + \sum_{\nu=1}^{n} \xi_\nu(t)dW_\nu(t), \quad t \in [0,\infty), \tag{1.25}$$

then the total return process $\widehat{X}$ satisfies

$$d\log \widehat{X}(t) = \rho(t)dt + \sum_{\nu=1}^{n} \xi_\nu(t)dW_\nu(t), \quad t \in [0,\infty), \tag{1.26}$$

which is similar[9] to (1.25).

Assuming $\mathcal{M}$ is a market made of the stocks X_i, $1 \le i \le n$, with respective dividend rate $\delta_i(t)$, $1 \le i \le n$. From (1.21), for any portfolio π its investment $Z_\pi(t)$ satisfies

$$d\log Z_\pi(t) = \sum_{i=1}^{n} \pi_i(t)d\log X_i(t) + \gamma_\pi^*(t)dt.$$

Therefore from (1.23), if we reinvest all dividends of X_i in the market proportionally with respect to π_i across the entire portfolio, $d\log Z_\pi(t)$ increases of $\pi_i(t)\delta_i dt$. The *total investment* $\widehat{Z}_\pi(t)$ is defined by

$$\begin{aligned} d\log \widehat{Z}_\pi(t) &= \sum_{i=1}^{n} \pi_i(t)d\log X_i(t) + \gamma_\pi^*(t)dt + \sum_{i=1}^{n} \pi_i(t)\delta_i(t)dt \\ &= d\log Z_\pi(t) + \sum_{i=1}^{n} \pi_i(t)\delta_i(t)dt, \end{aligned}$$

for $t \in [0,\infty)$. Using the notation $\delta_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\delta_i(t)$, we have

$$\begin{aligned} & d\log \widehat{Z}_\pi(t) = d\log Z_\pi(t) + \delta_\pi(t)dt \\ \Leftrightarrow \quad & \log \widehat{Z}_\pi(t) = \log Z_\pi(t) + \int_0^t \delta_\pi(s)ds, \end{aligned}$$

for $t \in [0,\infty)$. Hence

$$\widehat{Z}_\pi(t) = Z_\pi(t)\exp\left(\int_0^t \delta_\pi(s)ds\right), \quad t \in [0,\infty).$$

The dividend rate process for the portfolio π is defined by

$$\delta_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\delta_i(t), \tag{1.27}$$

which is the weighted average of the stocks' dividend rate. Similarly to (1.24), we defined the *augmented growth rate* of the portfolio π by

$$\rho_\pi(t) = \gamma_\pi(t) + \delta_\pi(t), \quad t \in [0,\infty). \tag{1.28}$$

[9] Except that we have $\rho(t)$ instead of $\gamma(t)$ which is the growth rate.

2. Relative Return and Market Portfolio

Frequently in investment practice one measures stock performance and portfolio performance with respect to a given benchmark portfolio or index. A natural index is the *market portfolio* consisting of all the stocks in the market which we will define in section (2.2).

2.1 Relative Return

When building a portfolio, agents aim to outbeat the market: They try to build the portfolio with the greatest interest rate which is the "best" interest rate achievable. The more a portfolio's interest rate is close to the "best" interest rate, the "better" the portfolio is. However, generally we can't get the portfolio with the "best" interest rate. An alternative is to study a portfolio with respect to another portfolio through its relative return.

Definition 2.1. *For a stock $X_i, 1 \leq i \leq n$, and a benchmark portfolio η, the* relative return *process of X_i versus η is*

$$\log \frac{X_i(t)}{Z_\eta(t)}, \quad t \in [0, \infty). \tag{2.1}$$

Given that

$$\log \frac{X_i(t)}{Z_\eta(t)} = \log X_i(t) - \log Z_\eta(t),$$

the relative return process is a continuous semimartingale. Furthermore,

$$\begin{aligned} \langle \log \frac{X_i}{Z_\eta}, \log \frac{X_j}{Z_\eta} \rangle_t &= \langle \log X_i - \log Z_\eta, \log X_j - \log Z_\eta \rangle_t \\ &= \langle \log X_i, \log X_j \rangle_t - \langle \log X_i, \log Z_\eta \rangle_t - \langle \log Z_\eta, \log X_j \rangle_t + \langle \log Z_\eta \rangle_t, \end{aligned}$$

for $t \in [0, \infty)$. Let us define

$$\sigma_{i\eta}(t) = \sum_{j=1}^{n} \eta_j(t) \sigma_{ij}(t), \quad t \in [0, \infty), \tag{2.2}$$

for $i = 1, 2, \cdots, n$, and

$$\sigma_{\eta\eta}(t) = \eta(t) \sigma(t) \eta^T(t), \quad t \in [0, \infty). \tag{2.3}$$

We have

$$\begin{aligned} d\langle \log X_i, \log Z_\eta\rangle_t &= \langle \sum_{\nu=1}^n \xi_{i\nu} dW_\nu, \sum_{j,\varsigma=1}^n \eta_j \xi_{j\varsigma} dW_\varsigma\rangle_t \\ &= \sum_{\nu=1}^n \sum_{j,\varsigma=1}^n \eta_j(t)\xi_{i\nu}(t)\xi_{j\varsigma}(t)\delta_{\nu\varsigma} dt \\ &= \sum_{j=1}^n \eta_j(t) \sum_{\nu=1}^n \xi_{i\nu}(t)\xi_{j\nu}(t) dt \\ &= \sum_{j=1}^n \eta_j(t)\sigma_{ij}(t) dt \\ &= \sigma_{i\eta}(t) dt, \end{aligned}$$

for $t \in [0, \infty)$. We also have

$$\langle \log Z_\eta\rangle_t \overset{(1.18)}{=} \sigma_{\eta\eta}(t), \quad t \in [0, \infty).$$

Therefore,

$$\begin{aligned} d\langle \log \frac{X_i}{Z_\eta}, \log \frac{X_j}{Z_\eta}\rangle_t &= d\langle \log X_i, \log X_j\rangle_t - d\langle \log X_i, \log Z_\eta\rangle_t - d\langle \log Z_\eta, \log X_j\rangle_t + d\langle \log Z_\eta\rangle_t \\ &= [\sigma_{ij}(t) - \sigma_{i\eta}(t) - \sigma_{j\eta}(t) + \sigma_{\eta\eta}(t)]\, dt \end{aligned}$$

for $t \in [0, \infty)$. With the notation

$$\tau_{ij}^\eta(t) = \sigma_{ij}(t) - \sigma_{i\eta}(t) - \sigma_{j\eta}(t) + \sigma_{\eta\eta}(t), \quad t \in [0, \infty), \tag{2.4}$$

we get

$$d\langle \log \frac{X_i}{Z_\eta}, \log \frac{X_j}{Z_\eta}\rangle_t = \tau_{ij}^\eta(t) dt, \quad t \in [0, \infty), \tag{2.5}$$

and

$$d\langle \log \frac{X_i}{Z_\eta}\rangle_t = \tau_{ii}^\eta(t) dt, \quad t \in [0, \infty).$$

The *relative covariance* process τ^η is the matrix-valued process

$$\tau^\eta(t) = \left(\tau_{ij}^\eta(t)\right)_{1\le i,j\le n}, \quad t \in [0, \infty).$$

Let us point out some properties of the relative covariance process, starting by the next lemma.

Lemma 2.2. *For a benchmark portfolio η, $\tau^\eta(t)$ is positive semidefinite with rank $n-1$, for $t \in [0, \infty)$, a.s., and the null space of $\tau^\eta(t)$ is spanned by $\eta(t)$.*

Proof. Considering $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \backslash \{0\}$, from (2.4) we get

$$\begin{aligned}
x\tau^\eta(t)x^T &= \sum_{i,j=1}^n x_i x_j \tau^\eta(t)_{ij} \\
&= \sum_{i,j=1}^n x_i x_j \sigma_{ij}(t) - \sum_{i,j=1}^n x_i x_j \sigma_{i\eta}(t) - \sum_{i,j=1}^n x_i x_j \sigma_{j\eta}(t) + \sum_{i,j=1}^n x_i x_j \sigma_{\eta\eta}(t) \\
&\overset{(2.3)}{=} x\sigma(t)x^T - 2\sum_{i,j=1}^n x_i x_j \sigma_{i\eta}(t) + \eta(t)\sigma(t)\eta^T(t)\sum_{i,j=1}^n x_i x_j \\
&\overset{(2.2)}{=} x\sigma(t)x^T - 2\sum_{i,j=1}^n x_i x_j \sum_{k=1}^n \eta_k(t)\sigma_{ik}(t) + \eta(t)\sigma(t)\eta^T(t)\sum_{i,j=1}^n x_i x_j \\
&= x\sigma(t)x^T - 2\sum_{j=1}^n x_j \left(\sum_{i,k=1}^n x_i \eta_k(t)\sigma_{ik}(t)\right) + \eta(t)\sigma(t)\eta^T(t)\left(\sum_{i=1}^n x_i\right)^2 \\
&= x\sigma(t)x^T - 2x\sigma(t)\eta^T(t)\sum_{i=1}^n x_i + \eta(t)\sigma(t)\eta^T(t)\left(\sum_{i=1}^n x_i\right)^2 .
\end{aligned}$$

Recall that $\sigma(t)$ is positive definite (Lemma 1.4). If $\sum_{i=1}^n x_i = 0$ then $x\tau^n(t)x^T = x\sigma(t)x^T > 0$. If on the other hand $\sum_{i=1}^n x_i = a \neq 0$, we consider the vector $y = \frac{x}{a}$ that satisfies $\sum_{i=1}^n y_i = 1$ and observe that

$$\begin{aligned}
y\tau^\eta(t)y^T &= y\sigma(t)y^T - 2y\sigma(t)\eta^T(t) + \eta(t)\sigma(t)\eta^T(t) \\
&= y\sigma(t)y^T - \eta\sigma(t)y^T(t) - y\sigma(t)\eta^T(t) + \eta(t)\sigma(t)\eta^T(t), \text{ since } \sigma \text{ is symmetric} \\
&= (y - \eta(t))\,\sigma(t)\,(y - \eta(t))^T ,
\end{aligned}$$

meaning

$$y\tau^\eta(t)y^T = (y - \eta(t))\,\sigma(t)\,(y - \eta(t))^T . \tag{2.6}$$

On the other hand

$$y\tau^\eta(t)y^T = a^{-2}x\tau^\eta(t)x^T,$$

therefore $x\tau^\eta(t)x^T = 0$ if and only if $y = \eta(t)$, or equivalently $x = a\eta(t)$. ■

Definition 2.3 (Relative variance). *The relative variance process of π versus η is defined for $t \in [0, \infty)$ by*

$$\tau^\eta_{\pi\pi}(t) = \pi(t)\tau^\eta(t)\pi^T(t).$$

We have

$$\tau^{\eta}_{\pi\pi} = \tau^{\pi}_{\eta\eta}.$$

Proof. By definition $\sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} \eta_i = 1$ thus as in the previous proof, we have

$$\begin{aligned} \pi(t)\tau^{\eta}(t)\pi^T(t) &\overset{(2.6)}{=} (\pi(t) - \eta(t))\,\sigma(t)\,(\pi(t) - \eta(t))^T \\ &= (\eta(t) - \pi(t))\,\sigma(t)\,(\eta(t) - \pi(t))^T \\ &\overset{(2.6)}{=} \eta(t)\tau^{\pi}(t)\eta^T(t). \end{aligned}$$

■

Example (Index portfolio). *The* Dow Jones Industrial Average *(DJIA) is one of several stock market indices created by* Charles Dow *who was both* Wall Street Journal *editor and* Dow Jones & Company *co-founder. It was built to measure the performance of the industrial component of America's stock markets and is one of the oldest U.S. market index still in use. Nowadays the Dow Jones consists of* 30 *of the largest and most widely held public companies in the United States. A list of the portfolio's weight is published daily.*

2.2 Market Portfolio

We will now study one of the most used portfolio benchmark, the market portfolio.

Definition 2.4 (Market Portfolio). *Let us have* $\psi(t) = (1, 1, \cdots, 1)$ *for* $t \in [0, \infty)$ *as trading strategy. The corresponding portfolio* $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ *is given by*

$$\mu_i(t) = \frac{X_i(t)}{\sum_{j=1}^{n} X_j(t)}, \quad t \in [0, \infty),$$

for $i = 1, 2, \cdots, n$. *The portfolio* μ *is the* market portfolio *and the* μ_i, $i = 1, 2, \cdots, n$ *are called market weights.*

It is straightforward to show that $0 < \mu_i(t) < 1$ and that the market portfolio agrees with the definition of a portfolio as specified above[1]. Investing in the portfolio μ amounts to owning the entire market, in proportion of course to the initial investment. For this reason, we call μ the *market portfolio* [6]. Since μ is of such importance we write $\tau_{ij} = \tau^{\mu}_{ij}$ and $\tau_{\pi\pi} = \tau^{\mu}_{\pi\pi}$. Note that the weight processes are the relative returns of the X_i against μ. From now on, μ always represents the market portfolio and Z_μ its value process.

[1] Definition 1.5.

The differential of the respective returns' crossvariation is again $\tau_{ij}(t)$:

$$d\langle \log \mu_i, \log \mu_j \rangle_t \overset{(2.5)}{=} \tau_{ij}(t). \tag{2.7}$$

Applying Itô's formula to $\mu_i(t) = \exp(\log \mu_i(t))$ for $t \in [0, \infty)$ we get

$$\begin{aligned} d\mu_i(t) &= \mu_i(t) d \log \mu_i(t) + \frac{1}{2}\mu_i(t) d\langle \log \mu_i \rangle_t & (2.8) \\ &\overset{(2.7)}{=} \mu_i(t) d \log \mu_i(t) + \frac{1}{2}\mu_i(t)\tau_{ii}(t)dt. & (2.9) \end{aligned}$$

Hence

$$\begin{aligned} d\langle \mu_i, \mu_j \rangle_t &= d\langle \mu_i \log \mu_i, \mu_j \log \mu_j \rangle_t \\ &= \mu_i(t)\mu_j(t) d\langle \log \mu_i, \log \mu_j \rangle_t \\ &\overset{(2.7)}{=} \mu_i(t)\mu_j(t)\tau_{ij}(t)dt, \end{aligned}$$

for $t \in [0, \infty)$.

The crossvariation of two market weights is the product of these weights and their corresponding relative covariance component. This and equation (2.7) are unique to the market portfolio [3].

Let us recall (1.21):

$$d \log Z_\pi(t) = \sum_{i=1}^{n} \pi_i(t) d \log X_i(t) + \gamma_\pi^*(t)dt \text{ a.s., for } \quad t \in [0, \infty),$$

thus

$$\begin{aligned} d \log Z_\pi(t) - d \log Z_\mu(t) &= \sum_{i=1}^{n} \pi_i(t) d \log X_i(t) - d \log Z_\mu(t) + \gamma_\pi^*(t)dt \\ &= \sum_{i=1}^{n} \pi_i(t) d \log X_i(t) - \sum_{i=1}^{n} \pi_i(t) d \log Z_\mu(t) + \gamma_\pi^*(t)dt \\ &= \sum_{i=1}^{n} \pi_i(t) d \log \frac{X_i(t)}{Z_\mu(t)} + \gamma_\pi^*(t)dt, \end{aligned}$$

i.e.

$$d \log \frac{Z_\pi(t)}{Z_\mu(t)} = \sum_{i=1}^{n} \pi_i(t) d \log \frac{X_i(t)}{Z_\mu(t)} + \gamma_\pi^*(t)dt, \tag{2.10}$$

a.s., for $\quad t \in [0, \infty)$.

Given that $\log \mu_i(t) = \log \frac{X_i(t)}{Z_\mu(t)}$, we can now describe the relative return process of a portfolio π against the market portfolio:

$$d \log \frac{Z_\pi(t)}{Z_\mu(t)} \overset{a.s.}{=} \sum_{i=1}^{n} \pi_i(t) d \log \mu_i(t) + \gamma_\pi^*(t)dt, \quad t \in [0, \infty). \tag{2.11}$$

We can represent the relative return of a portfolio versus the market portfolio in terms of the changes in the market weights.

3. Portfolio Behaviour and Optimisation

Markowitz, in his well known paper *"Portfolio Selection"* [8], stated that a portfolio was fully characterised by its return and variance. Let us see what happens if we subtract the growth rate from a portfolio value process:

$$L = \lim_{T\to\infty} \frac{1}{T}\left(\log Z_\pi(T) - \int_0^T \gamma_\pi(t)dt\right). \tag{3.1}$$

3.1 Portfolio Behaviour

Let us first state a lemma which will be useful afterwards.

Lemma 3.1. *Let M be a continuous local martingale such that*

$$\lim_{t\to\infty} t^{-2}\langle M\rangle_t \log\log t = 0, \quad a.s.$$

Then

$$\lim_{t\to\infty} t^{-1}M(t) = 0, \quad a.s. \quad [3]$$

The value of L is given by the following proposition.

Proposition 3.2. *For any portfolio π in $\mathcal{M}$,*

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\pi(T) - \int_0^T \gamma_\pi(t)dt\right) = 0, a.s.$$

Proof. From (1.16) we have

$$\log\frac{Z_\pi(t)}{Z_\pi(0)} = \int_0^t \gamma_\pi(s)ds + \int_0^t \sum_{i,j=1}^n \pi_i(s)\xi_{ij}(s)dW_j(s),$$

for $t \in [0,\infty)$.

The process $V(t)$ defined by

$$\begin{aligned} V(t) &= \log\frac{Z_\pi(t)}{Z_\pi(0)} - \int_0^t \gamma_\pi(s)ds \\ &= \int_0^t \sum_{i,j=1}^n \pi_i(s)\xi_{ij}(s)dW_j(s) \end{aligned}$$

is a continuous martingale since ξ_{ij}, are continuous martingales for $i, j = 1, 2, \cdots, n$.

We have

$$\begin{aligned} d\langle V\rangle_t &= \langle\sum_{i,j=1}^{n}\pi_i\xi_{ij}dW_j\rangle_t \\ &\stackrel{(1.12)}{=} \langle\log Z_\pi\rangle_t \\ &= \sigma_{\pi\pi}(t)dt. \end{aligned}$$

Therefore,

$$\langle V\rangle_t = \int_0^t \sigma_{\pi\pi}(s)ds, \quad t\in[0,\infty), \text{ a.s.}$$

Now let us recall condition (ii) of Definition (1.1):

$$\lim_{t\to\infty} t^{-1}\sum_{\nu=1}^{n}(\xi_{i\nu}(t))^2\log\log t = 0, \quad \forall 1\le i\le n.$$

Let us have

$$\begin{aligned} A(t) &= |\sigma_{\pi\pi}(t)| \\ &= \left|\sum_{i,j=1}^{n}\pi_i(t)\pi_j(t)\sum_{\nu=1}^{n}\xi_{i\nu}(t)\xi_{j\nu}(t)\right| \\ &\le \sum_{i,j=1}^{n}|\pi_i(t)|\cdot|\pi_j(t)|\sum_{\nu=1}^{n}|\xi_{i\nu}(t)|\cdot|\xi_{j\nu}(t)| \\ &\le \frac{1}{2}\sum_{i,j=1}^{n}|\pi_i(t)|\cdot|\pi_j(t)|\sum_{\nu=1}^{n}\left(\xi_{i\nu}^2(t)+\xi_{j\nu}^2(t)\right) \end{aligned}$$

because

$$\begin{aligned} &(|a|-|b|)^2\ge 0, \quad \forall a,b\in\mathbb{R} \\ \Leftrightarrow\quad &|a|\cdot|b|\le\frac{1}{2}(a^2+b^2), \quad \forall a,b\in\mathbb{R}. \end{aligned}$$

Combining the two last equations and because π is bounded we have

$$\lim_{t\to\infty} t^{-1}|\sigma_{\pi\pi}(t)|\log\log t = 0.$$

Hence

$$\lim_{t\to\infty} t^{-1}\sigma_{\pi\pi}(t)\log\log t = 0$$

and from Lemma 3.1 we have $L=0$. ■

In Proposition 3.2 the term in brackets is a continuous martingale with bounded quadratic variation and since $L=0$ we should consider growth rate rather than return rate for long-term investments. Hence the growth rate characterises the long term behaviour of a portfolio which is not the case for its return rate [4].

If we consider the portfolio consisting of a unique stock of price $X(t)$ and growth rate $\gamma(t)$, Proposition 3.2 gives

$$\lim_{T\to\infty}\frac{1}{T}\left(\log X(T)-\int_0^T\gamma(t)dt\right) \;=\; 0, \quad a.s. \tag{3.2}$$

3.1.1 More Portfolio Properties

Equation (1.19) tells us that the portfolio growth rate is the weighted average of stocks growth rates plus the excess growth rate, which is half the difference between the average component variance and the portfolio variance. However, it is obvious from (2.10) that γ_π^* does not depend on μ. Furthermore, straightforward calculations show that γ_i^* is *numeraire invariant* i.e. it is not function of the choice of the benchmark portfolio η. It can always be calculated as

$$\begin{aligned}
\gamma_\pi^*(t) &= \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\tau_{ii}^\eta(t)-\sum_{i,j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}^\eta(t)\right)\\
&= \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t)-2\sum_{i=1}^n \pi_i(t)\sigma_{i\eta}(t)+\sigma_{\eta\eta}\right.\\
&\quad -\sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t)+\sum_{i=1}^n \pi_i(t)\sigma_{i\eta}(t)\\
&\quad \left.+\sum_{j=1}^n \pi_j(t)\sigma_{j\eta}(t)-\sigma_{\eta\eta}\right)\\
&= \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t)-\sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right).
\end{aligned}$$

The first line is helpful for the calculation of the excess growth rate while from the last line we see that the excess growth rate is independent of η. Notice that

$$\begin{aligned}
\sum_{i,j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}^\eta(t) &= \pi(t)\tau^\eta(t)\pi^T(t)\\
&= (\pi(t)-\eta(t))\,\sigma(t)\,(\pi(t)-\eta(t))^T
\end{aligned}$$

because $\sum_{i=1}^n \pi_i(t) = 1$ (See proof of Lemma 2.2). Assuming π is the benchmark portfolio i.e $\eta=\pi$, we get

$$\sum_{i,j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}^\eta(t) \;=\; 0.$$

Therefore,

$$\gamma_\pi^*(t) = \frac{1}{2}\sum_{i=1}^{n}\pi_i(t)\tau_{ii}^{\pi}(t). \tag{3.3}$$

From Lemma 2.2, $\tau_{ii}^{\pi}(t)$ is non-negative and $\tau^{\pi}(t)$ is positive semi-definite thus the following proposition in true.

Proposition 3.3. *Let π be a portfolio with no short sale. Then*

$$\gamma_\pi^*(t) \geq 0, \quad t \in [0,\infty),\ a.s.$$

If $\exists i,j, i \neq j$ such that $\pi_i(t), \pi_j(t) > 0$, then

$$\gamma_i^*(t) > 0, \quad t \in [0,\infty),\ a.s.$$

3.2 Optimisation

The first and most commonly used approach to portfolio optimisation is due to Markowitz. Given an expected return rate, α_0, we aim to build the portfolio with return rate α_0 and minimum risk or variance.

3.2.1 "Markowitz" Approach

Assuming η is a portfolio benchmark, we optimise the portfolio π by minimising the variance of the portfolio with respect to η

$$\sum_{i,j=1}^{n}\pi_i(t)\pi_j(t)\tau_{ij}^{\eta},$$

under the constraints

$$\alpha_\pi = \sum_{i=1}^{n}\pi_i(t)\alpha_i(t) \geq \alpha_0,$$

and $\sum_{i=1}^{n}\pi_i(t) = 1$ with $\pi_i(t) \geq 0$, for $i = 1, 2, \cdots, n$. The function $\alpha_i(t)$ is the rate of return of the ith stock.

Thus we are looking for the portfolio with no short sales that has the minimum variance and a return rate of at least α_0.

As the growth rate is more appropriate in Stochastic Portfolio Theory, sometimes we fix a constraint on the growth rate rather than on the return rate.

3.2.2 Growth Rate Approach

Considering an expected rate of return γ_0, we optimise the portfolio π by minimising the variance of the portfolio relative to η

$$\sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\tau_{ij}^{\eta},$$

under the constraints

$$\gamma_\pi = \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\eta}(t) - \sum_{i=1}^{n} \pi_i(t)\pi_j(t)\tau_{ij}^{\eta}(t)\right) \geq \gamma_0$$

and $\sum_{i=1}^{n} \pi_i(t) = 1$ with $\pi_i(t) \geq 0$, for $i = 1, 2, \cdots, n$.

Similarly to the previous optimisation scheme, we are looking for the portfolio with no short sales that has the minimum variance and a growth rate of at least γ_0.

4. Market Behaviour and Diversity

In the previous chapters we studied the portfolio and related topics, now we will study the market and some related topics. We will first define the concept of *coherent* market and give some key lemmas. Later on we will introduce the concept of diversity and give some related properties.

Always keep in mind that $\mu(t) = (\mu_1(t), \mu_2(t), \cdots, \mu_n(t))$ is the market portfolio and $Z_\mu(t)$ is its value at time t.

4.1 Market Behaviour

Just as *arbitrage* is a key characteristic in standard financial modelling, market coherence is central in stochastic portfolio theory.

Definition 4.1 (Coherent Market). *The market $\mathcal{M}$ is* coherent *if*

$$\lim_{t\to\infty} \frac{\log \mu_i(t)}{t} = \lim_{t\to\infty} \frac{\log X_i(t) - \log Z_\mu(t)}{t} \tag{4.1}$$

$$\overset{a.s.}{=} 0 \tag{4.2}$$

for $i = 1, 2, \cdots, n$.

We have $\mu_i(t) < 1$ thus $\log \mu_i(t) < 0$ and (4.1) implies that none of the stocks declines too rapidly.

Proposition 4.2. *Let $\mathcal{M}$ be a market with stocks $X_1, X_2, \cdots, X_n$. Then the following statements are equivalent*

$i.$ *$\mathcal{M}$ is coherent;*

$ii.$ $\lim_{T\to\infty} \frac{1}{T} \int_0^T (\gamma_i(t) - \gamma_\mu(t))\, dt \overset{a.s.}{=} 0$, *for $i = 1, 2, \cdots, n$;*

$iii.$ $\lim_{T\to\infty} \frac{1}{T} \int_0^T (\gamma_i(t) - \gamma_j(t))\, dt \overset{a.s.}{=} 0$, *for $i, j = 1, 2, \cdots, n$.*

This means that the stocks and the market portfolio all have the same time average growth rate. However, it does not imply that the time average growth rates of the stocks exist.

Proof. To prove the equivalence, we will prove the cyclic implication.
$i. \Rightarrow ii.)$ Let us recall Proposition 3.2:

$$\lim_{T\to\infty} \frac{1}{T} \left(\log Z_\mu(T) - \int_0^T \gamma_\mu(t) dt \right) \overset{a.s.}{=} 0 \tag{4.3}$$

and (3.2)

$$\lim_{T\to\infty}\frac{1}{T}\left(\log X_i(T)-\int_0^T\gamma_i(t)dt\right)\overset{a.s.}{=}0,\ \text{for } i=1,2,\cdots,n \tag{4.4}$$

By doing (4.4)-(4.3) + (4.1) we get $ii.$

$ii. \Rightarrow iii.)$ From $ii.$ we have for both $i,j=1,2,\cdots,n$

$$\lim_{T\to\infty}\int_0^T(\gamma_i(t)-\gamma_\mu(t))\,dt\overset{a.s.}{=}0,$$

and

$$\lim_{T\to\infty}\int_0^T(\gamma_j(t)-\gamma_\mu(t))\,dt\overset{a.s.}{=}0$$

the difference of the previous equations gives $iii.$

$iii. \Rightarrow i.)$ Let us remind (3.2) and Proposition 4.2.$iii.$

$$\lim_{T\to\infty}\frac{1}{T}\left(\log X_i(T)-\int_0^T\gamma_i(t)dt\right)\overset{a.s.}{=}0; \tag{4.5}$$

$$\lim_{T\to\infty}\int_0^T(\gamma_i(t)-\gamma_j(t))\,dt\overset{a.s.}{=}0, \tag{4.6}$$

for $i,j=1,2,\cdots,n.$

From now on, let us assume that j is fixed. The sum of (4.5) and (4.6) gives

$$\lim_{T\to\infty}\frac{1}{T}\left(\log X_i(T)-\int_0^T\gamma_j(t)dt\right)\overset{a.s.}{=}0, \tag{4.7}$$

for $i,j=1,2,\cdots,n.$ Therefore

$$\lim_{T\to\infty}\frac{1}{T}\left(\max_{1\le i\le n}(\log X_i(T))-\int_0^T\gamma_j(t)dt\right)\overset{a.s.}{=}0. \tag{4.8}$$

Since the function $\log$ is strictly increasing

$$\lim_{T\to\infty}\frac{1}{T}\left(\log\left(\max_{1\le i\le n}X_i(T)\right)-\int_0^T\gamma_j(t)dt\right)\overset{a.s.}{=}0.$$

On the other hand because X_i is nonnegative, we have a.s. for $t\in[0,\infty)$

$$X_j(t)\le\sum_{i=1}^n X_i(t)\le n\max_{1\le i\le n}X_i(t)$$

i.e.

$$\log X_j(t) \leq \log Z_\mu(t) \leq \log n + \log\left(\max_{1\leq i\leq n} X_i(t)\right). \tag{4.9}$$

Combining (4.7), (4.8) and (4.9), we have a.s.

$$\lim_{T\to\infty}\frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \gamma_j(t)dt\right) \leq \lim_{T\to\infty}\frac{1}{T}\log n + \lim_{T\to\infty}\frac{1}{T}\left(\log\left(\max_{1\leq i\leq n} X_i(T)\right) - \int_0^T \gamma_j(t)dt\right) = 0,$$

because n is a constant

$$\lim_{T\to\infty}\frac{1}{T}(\log n) \quad = \quad 0.$$

Hence

$$\lim_{T\to\infty}\frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \gamma_j(t)dt\right) \quad = \quad 0.$$

The last result and (4.7) gives

$$\lim_{T\to\infty}\frac{1}{T}(\log X_i(T) - \log Z_\mu(T)) \quad \stackrel{a.s.}{=} \quad 0.$$

Thus the market $\mathcal{M}$ is coherent. ■

The following lemmas will be useful to prove some later results.

Lemma 4.3. *Let π be a portfolio in a nondegenerate market. Then there exists an $\varepsilon > 0$ such that for $i = 1, 2, \cdots, n$*

$$\tau_{ii}^\pi(t) \quad \geq \quad \varepsilon\left(1 - \pi_{max}(t)\right)^2, \quad t \in [0,\infty), \qquad a.s.$$

with $\pi_{max}(t) = \underset{i}{max}\ \pi_i(t)$.

Proof. From (1.8) because the market is nondegenerate, there is a $\varepsilon > 0$ such that

$$x\sigma(t)x^T \quad \stackrel{a.s.}{\geq} \quad \varepsilon\|x\|^2, \quad t \in [0,\infty), \quad \forall x \in \mathbb{R}^n.$$

Especially for $x_\nu(t) = (\pi_1(t) - \delta_{1\nu}, \pi_2(t) - \delta_{2\nu}, \cdots, \pi_n(t) - \delta_{n\nu})$, with $\nu = 1, 2, \cdots, n$, we have

$$\begin{aligned}
x_\nu(t)\sigma(t)x_\nu^T(t) \quad &= \quad \sum_{i,j=1}^n \sigma_{ij}(t)(\pi_i(t) - \delta_{i\nu})(\pi_j(t) - \delta_{j\nu}) \\
&= \quad \sum_{i,j=1}^n \sigma_{ij}(t)\pi_i(t)\pi_j(t) - \sum_{i,j=1}^n \sigma_{ij}(t)\pi_i(t)\delta_{j\nu} - \sum_{i,j=1}^n \sigma_{ij}(t)\delta_{i\nu}\pi_j(t) + \sum_{i,j=1}^n \sigma_{ij}(t)\delta_{i\nu}\delta_{j\nu} \\
&= \quad \sigma_{\pi\pi}(t) - \sum_{i=1}^n \sigma_{i\nu}(t)\pi_i(t) - \sum_{j=1}^n \pi_j(t)\sigma_{\nu j}(t) + \sigma_{\nu\nu}(t) \\
&= \quad \sigma_{\pi\pi}(t) - 2\sigma_{\nu\pi}(t) + \sigma_{\nu\nu}(t), \quad \text{as } \sigma \text{ is symmetric } \sigma_{i\nu}(t) = \sigma_{\nu i}(t) \\
&\stackrel{(2.4)}{=} \quad \tau_{\nu\nu}^\pi(t) \\
&\stackrel{a.s.}{\geq} \quad \varepsilon\|x_\nu(t)\|^2, \quad t \in [0,\infty).
\end{aligned}$$

Thus

$$\tau^{\pi}_{\nu\nu}(t) \overset{a.s.}{\geq} \varepsilon \|x_\nu(t)\|^2, \quad t \in [0,\infty)$$

Now notice that by definition of $x_\nu(t)$ and $\|\cdot\|$ we have $\|x_\nu(t)\|^2 \geq (\pi_i(t) - \delta_{i\nu})^2$ with $i = 1, 2, \cdots, n$ and $t \in [0,\infty)$. For $i = \nu$ we have $\|x_\nu(t)\|^2 \geq (\pi_i(t) - 1)^2$ with $t \in [0,\infty)$. Combining this and the previous result then considering a maximisation with respect to ν leads to the lemma. ■

Combining this lemma with (3.3) and taking $\alpha = \frac{\varepsilon}{2}$, we have the following lemma.

Lemma 4.4. *Let π be a portfolio with nonnegative weights in a nondegenerate market. Then there exists an $\alpha > 0$ such that a.s. for $t \in [0,\infty)$,*

$$\gamma^*_\pi(t) \geq \alpha \left(1 - \pi_{max}(t)\right)^2.$$

This implies that $\gamma^*_\pi(t)$ is bounded away from zero if $\pi_{max}(t)$ is bounded away from 1. The next Lemma states the converse.

Lemma 4.5. *Let π be a portfolio in a market with bounded variance such that for $i = 1, 2, \cdots, n$, $0 \leq \pi_i(t) < 1$, for all $t \in [0,\infty)$, a.s. Then there exists a number $\varepsilon > 0$ such that*

$$\pi_{max}(t) \overset{a.s.}{\leq} 1 - \varepsilon\gamma^*_\pi(t), \quad t \in [0,\infty). \tag{4.10}$$

Proof. Now that the market has bounded variance, from (1.9) there exists $M > 0$ such that

$$x\sigma(t)x^T \leq M\|x\|^2, \quad t \in [0,\infty), \quad \forall x \in \mathbb{R}^n, \quad a.s. \tag{4.11}$$

Taking $x = x_\nu = (\delta_{1\nu}, \delta_{2\nu}, \cdots, \delta_{n\nu})$, for $\nu = 1, 2, \cdots, n$, (4.11) gives

$$\sigma_{\nu\nu}(t) \leq M, \quad t \in [0,\infty), \quad a.s. \tag{4.12}$$

Let us define for k such that $1 \leq k \leq n$ and $\pi_k(t) < 1$

$$\eta_i(t) = \begin{cases} \dfrac{\pi_i(t)}{1 - \pi_k(t)} & \text{if } i \neq k \\ 0 & \text{if } i = k, \end{cases} \tag{4.13}$$

for $t \in [0,\infty)$ and $i = 1, 2, \cdots, n$. By definition of a portfolio[1] and because the market is diverse, $\pi_k(t) < 1 \Rightarrow 0 < 1 - \pi_k(t)$ and $\pi_i(t) \geq 0$ for $i = 1, 2, \cdots, n$ thus $\eta_i(t) \geq 0$ for $t \in [0,\infty)$ and $i = 1, 2, \cdots, n$.

Moreover, given that $\sum_{i=1}^{n} \pi_i(t) = 1$ and $0 \leq \pi_i(t) < 1$ for $t \in [0,\infty)$ and $i = 1, 2, \cdots, n$,

$$\pi_i(t) + \pi_k(t) \leq 1 \Rightarrow \frac{\pi_i(t)}{1 - \pi_k(t)} = \eta_i(t) \leq 1.$$

[1] Definition 1.5.

We also have

$$\begin{aligned}\sum_{i=1}^{n}\eta_i(t) &= \sum_{\substack{i=1\\ i\neq k}}^{n}\eta_i(t)+\eta_k(t)\\ &= \frac{\sum_{\substack{i=1\\ i\neq k}}^{n}\pi_i(t)}{1-\pi_k(t)}+0\\ &= 1,\ \text{because}\ \sum_{i=1}^{n}\pi_i(t)=1.\end{aligned}$$

Hence $\eta(t)$ is a portfolio having nonnegative weights for $t \in [0,\infty)$ a.s. Therefore given that $\sigma(t)$ is positive definite for $t \in [0,\infty)$

$$\begin{aligned}\sum_{i=1}^{n}\eta_i(t)\sigma_{ii}(t)-\sigma_{\eta\eta}(t) &\leq \sum_{i=1}^{n}\eta_i(t)\sigma_{ii}(t), as \quad \sigma_{\eta\eta}(t)\geq 0\\ &\overset{(4.12)}{\leq} M, \text{since} \quad \sum_{i=1}^{n}\eta_i(t)=1.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n}\eta_i(t)\sigma_{ii}(t)-\sigma_{\eta\eta}(t) \quad \leq \quad M, \quad t\in[0,\infty) \tag{4.14}$$

Let us have

$$\begin{aligned}x_k(t) &= (\eta_1(t)-\delta_{1k},\eta_2(t)-\delta_{2k},\cdots,\eta_n(t)-\delta_{nk})\\ &= (\eta_1(t),\eta_2(t),\cdots,\eta_{k-1}(t),-1,\eta_{k+1}(t),\cdots,\eta_n(t)).\end{aligned}$$

Then

$$\begin{aligned}\|x\|^2 &= \sum_{\substack{i=1\\ i\neq k}}^{n}\eta_i^2(t)+1\\ &= \frac{\sum_{\substack{i=1\\ i\neq k}}^{n}\pi_i^2(t)}{(1-\pi_k)^2}+1\\ &\leq 2,\end{aligned}$$

for $k = 1, 2, \cdots, n$, because

$$\begin{aligned}(1-\pi_k)^2 &= \left(\sum_{\substack{i=1\\i\neq k}}^{n} \pi_i(t)\right)^2 \\ &= \sum_{\substack{i=1\\i\neq k}}^{n} \pi_i^2(t) + \sum_{\substack{i,j=1\\i,j\neq k}}^{n} \pi_i(t)\pi_j(t) \\ &\geq \sum_{\substack{i=1\\i\neq k}}^{n} \pi_i^2(t),\end{aligned}$$

for $k = 1, 2, \cdots, n$, since $\pi_i(t) \geq 0$ for $i = 1, 2, \cdots, n$. So

$$\|x\|^2 \leq 2. \tag{4.15}$$

On other hand, we have

$$\begin{aligned}x_k(t)\sigma(t)x_k^T(t) &= \sum_{i,j=1}^{n} \sigma_{ij}(t)(\eta_i(t)-\delta_{ik})(\eta_j(t)-\delta_{jk}) \\ &= \sum_{i,j=1}^{n} \sigma_{ij}(t)\eta_i(t)\eta_j(t) - \sum_{i,j=1}^{n} \sigma_{ij}(t)\eta_i(t)\delta_{jk} - \sum_{i,j=1}^{n} \sigma_{ij}(t)\delta_{ik}\eta_j(t) + \sum_{i,j=1}^{n} \sigma_{ij}(t)\delta_{ik}\delta_{jk} \\ &= \sigma_{\eta\eta}(t) - \sum_{i=1}^{n} \sigma_{ik}(t)\eta_i(t) - \sum_{j=1}^{n} \sigma_{kj}(t)\eta_j(t) + \sigma_{kk}(t) \\ &= \sigma_{\eta\eta}(t) - 2\sigma_{k\eta}(t) + \sigma_{kk}(t) \\ &\overset{(4.11)}{\leq} M\|x_k\|^2 \\ &\overset{(4.15)}{\leq} 2M.\end{aligned}$$

Thus,

$$\sigma_{\eta\eta}(t) - 2\sigma_{k\eta}(t) + \sigma_{kk}(t) \leq 2M, \quad k = 1, 2, \cdots, n. \tag{4.16}$$

Let us recall (1.19):

$$\gamma_\pi^*(t) = \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sigma_{\pi\pi}(t)\right).$$

Therefore,

$$
\begin{aligned}
2\gamma_\pi^*(t) &= \sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \sigma_{ij}(t)\pi_i(t)\pi_j(t) \\
&= \pi_k(t)\sigma_{kk}(t) + \sum_{\substack{i=1 \\ i\neq k}}^{n} \pi_i(t)\sigma_{ii}(t) - \pi_k^2(t)\sigma_{kk}(t) \\
&\quad - \sum_{\substack{i=1,\cdots,n \\ j=k}} \sigma_{ik}(t)\pi_i(t)\pi_k(t) - \sum_{\substack{j=1,\cdots,n \\ i=k}} \sigma_{kj}(t)\pi_k(t)\pi_j(t) - \sum_{\substack{i,j=1 \\ i,j\neq k}}^{n} \sigma_{ij}(t)\pi_i(t)\pi_j(t) \\
&\overset{(4.13)}{=} \pi_k(t)\sigma_{kk}(t) + (1-\pi_k(t))\sum_{i=1}^{n} \eta_i(t)\sigma_{ii}(t) - \pi_k^2(t)\sigma_{kk}(t) \\
&\quad -2\pi_k(t)(1-\pi_k(t))\sum_{i=1}^{n} \sigma_{ik}(t)\eta_i(t) - (1-\pi_k(t))^2 \sum_{i,j=1}^{n} \sigma_{ij}(t)\eta_i(t)\eta_j(t) \\
&= \pi_k(t)(1-\pi_k(t))\sigma_{kk}(t) + (1-\pi_k(t))\sum_{i=1}^{n} \eta_i(t)\sigma_{ii}(t) \\
&\quad -2\pi_k(t)(1-\pi_k(t))\sigma_{k\eta}(t) - (1-\pi_k(t))^2\sigma_{\eta\eta}(t) \\
&= \pi_k(t)(1-\pi_k(t))\sigma_{kk}(t) - 2\pi_k(t)(1-\pi_k(t))\sigma_{k\eta}(t) + \pi_k(t)(1-\pi_k(t))\sigma_{\eta\eta}(t) \\
&\quad +(1-\pi_k(t))\sum_{i=1}^{n} \eta_i(t)\sigma_{ii}(t) - (1-\pi_k(t))\sigma_{\eta\eta}(t) \\
&= \pi_k(t)(1-\pi_k(t))\left(\sigma_{kk}(t) - 2\sigma_{k\eta}(t) + \sigma_{\eta\eta}(t)\right) \\
&\quad +(1-\pi_k(t))\left(\sum_{i=1}^{n} \eta_i(t)\sigma_{ii}(t) - \sigma_{\eta\eta}(t)\right) \\
&\overset{(4.16)\&(4.14)}{\leq} \pi_k(t)(1-\pi_k(t))2M + (1-\pi_k(t))M \\
&\leq (1-\pi_k(t))3M, \text{ as } \pi_k(t) < 1.
\end{aligned}
$$

Taking $\varepsilon = \frac{2}{3M}$, we get (4.10). ∎

Furthermore, it can be shown that for any *constant-weighted* portfolio $\pi(t) = \pi(0)$ in $\mathcal{M}$,

$$\liminf_{T\to\infty} \frac{1}{T} \log \frac{Z_\pi(T)}{Z_\mu(T)} > 0, \quad \forall T \quad a.s. \tag{4.17}$$

Therefore, constantly weighted portfolios outperform the market portfolio with probability one.

Proof. of (4.17)
Let π be a constant-weighted portfolio with

$$\pi_i(t) = p_i \geq 0, \quad t \in [0,\infty),$$

for $i = 1, 2, \cdots, n$.

As $\mathcal{M}$ is nondegenerate, from Lemma 4.4 $\exists \varepsilon > 0$ such that

$$\gamma_\pi^*(t) \overset{a.s.}{\geq} \varepsilon\,(1-p)^2\,, \quad t \in [0,\infty),$$

with $p = \max_i p_i$. Thus

$$\frac{1}{T}\int_0^T \gamma_\pi^*(t)dt \overset{a.s.}{\geq} \varepsilon\,(1-p)^2\,, \quad T \in [0,\infty). \tag{4.18}$$

Let us recall (2.11):

$$d\log\frac{Z_\pi(t)}{Z_\mu(t)} \overset{a.s.}{=} \sum_{i=1}^n \pi_i(t) d\log\mu_i(t) + \gamma_\pi^*(t)dt, \quad t \in [0,\infty).$$

Therefore,

$$\begin{aligned}
&\int_0^T d\log\frac{Z_\pi(t)}{Z_\mu(t)} - \int_0^T \gamma_\pi^*(t)dt \overset{a.s.}{=} \int_0^T \sum_{i=1}^n p_i d\log\mu_i(t) \\
\Leftrightarrow\quad & \log\frac{Z_\pi(T)}{Z_\mu(T)} - \log\frac{Z_\pi(0)}{Z_\mu(0)} - \int_0^T \gamma_\pi^*(t)dt \overset{a.s.}{=} \sum_{i=1}^n p_i\left(\log\mu_i(T) - \log\mu_i(0)\right) \\
\Leftrightarrow\quad & \log\frac{Z_\pi(T)}{Z_\mu(T)} - \int_0^T \gamma_\pi^*(t)dt \overset{a.s.}{=} \sum_{i=1}^n p_i\log\mu_i(T) - \sum_{i=1}^n p_i\log\mu_i(0) + \log\frac{Z_\pi(0)}{Z_\mu(0)}.
\end{aligned}$$

Let us define the time constant

$$A = \log\frac{Z_\pi(0)}{Z_\mu(0)} - \sum_{i=1}^n p_i\log\mu_i(0).$$

We have

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}\left(\log\frac{Z_\pi(T)}{Z_\mu(T)} - \int_0^T \gamma_\pi^*(t)dt\right) &\overset{a.s.}{=} \lim_{T\to\infty}\frac{1}{T}\left(\sum_{i=1}^n p_i\log\mu_i(T)\right) + \lim_{T\to\infty}\frac{1}{T}A \\
&= \lim_{T\to\infty}\frac{1}{T}\left(\sum_{i=1}^n p_i\log\mu_i(T)\right), \text{ since } A \text{ is constant} \\
&= \sum_{i=1}^n p_i\left(\lim_{T\to\infty}\frac{1}{T}\log\mu_i(T)\right) \\
&= 0, \text{ because } \mathcal{M} \text{ is coherent.}
\end{aligned}$$

With (4.18), we deduce that

$$\lim_{T\to\infty}\frac{1}{T}\log\frac{Z_\pi(T)}{Z_\mu(T)} \overset{a.s.}{>} \varepsilon\,(1-p)^2 > 0.$$

■

4.2 Market Diversity

Roughly speaking, a market is *"diverse"* if it avoids concentration of all its capital into a few big companies. The *diversity* of a market measures how uniformly the capital is spread among the stocks. Since all major economically developed countries have some kind of antitrust legislation we assume that a mild form of diversity is always maintained.

Definition 4.6 (Diverse Market). *A market $\mathcal{M}$ is* diverse *if there exists a number $0 < \Delta < 1$ such that*

$$\mu_i(t) \leq 1 - \Delta, \quad t \in [0, \infty), a.s.$$

for $i = 1, 2, \cdots, n$.

Definition 4.7 (Weakly diverse Market). *A market is* weakly diverse *on $[0, T]$ if there exists a number $0 < \Delta < 1$ such that*

$$\frac{1}{T}\int_0^T \mu_i(t)dt \leq 1 - \Delta, \quad t \in [0, \infty), a.s.$$

for $i = 1, 2, \cdots, n$.

Thus a market is diverse if at no time there is a stock that accounts for more than a fraction[2] of the market portfolio. It is weakly diverse if this holds on average. Note that actual equity market are diverse [3].

From the previous definitions we have the next implication.

Proposition 4.8. *If the market $\mathcal{M}$ is nondegenerate and diverse, then there is a $\Delta > 0$ such that*

$$\gamma^*_\mu(t) \quad \geq \Delta, \quad t \in [0, \infty), a.s. \tag{4.19}$$

Moreover if (4.19) holds and $\mathcal{M}$ has bounded variance, then $\mathcal{M}$ is diverse.

Proof.
$\Rightarrow$)Supposing $\mathcal{M}$ is nondegenerate and diverse, by Definition 4.6 there exist $0 < \Delta < 1$ such that

$$\mu_i(t) \quad \leq 1 - \Delta, \quad t \in [0, \infty), a.s. \tag{4.20}$$

for $i = 1, 2, \cdots, n$. Let us take $\mu_{max}(t) = \max\limits_{1 \leq i \leq n} \mu_i(t)$, then

$$\mu_{max}(t) \leq 1 - \Delta, \quad t \in [0, \infty), a.s.$$

[2] Eventually large.

From Lemma 4.4 since $\mathcal{M}$ is nondegenerate, $\exists\varepsilon$ such that

$$\begin{aligned}\gamma_\mu^*(t) &\geq \varepsilon\left(1-\mu_{max}(t)\right)^2 \quad a.s.\\ &\overset{(4.20)}{\geq} \varepsilon\Delta^2\end{aligned}$$

for $t \in [0,\infty)$.
$\Leftarrow$) Let us now assume that $\mathcal{M}$ has bounded variance and $\exists\Delta > 0$ such that (4.19) is verified. Because $\mathcal{M}$ has bounded variance, Lemma 4.5 implies that there is $\varepsilon > 0$ such that

$$\begin{aligned}\pi_{max}(t) &\leq 1-\varepsilon\gamma_\pi^*(t) \quad a.s.\\ &\overset{(4.19)}{\leq} 1-\varepsilon\Delta,\end{aligned}$$

for $t \in [0,\infty)$, therefore $\mathcal{M}$ is diverse. ■

Combining this result and those of section 4.1, we see that in a diverse market all stocks do not have the same growth rate, otherwise we have a contradiction. Indeed from (1.13), we have

$$\gamma_\mu(t) = \sum_{i=1}^n \pi_i(t)\gamma_i(t) + \gamma_\mu^*(t), \quad t \in [0,\infty), a.s.$$

In particular, if the stocks have the same growth rate $\gamma(t)$, because $\sum_{i=1}^n \pi_i(t) = 1$, we have

$$\gamma_\mu(t) = \gamma(t) + \gamma_\mu^*(t), \quad t \in [0,\infty), a.s.$$

Since the market $\mathcal{M}$ is coherent, from Proposition 4.2-ii.

$$\begin{aligned}\lim_{T\to\infty}\int_0^T \left(\gamma(t)-\gamma_\mu(t)\right)dt &= \lim_{T\to\infty}\int_0^T \gamma_\mu^*(t)dt\\ &= 0 \quad a.s.\end{aligned}$$

This means that asymptotically, on average the excess growth rate is zero. This is in contradiction with the market diversity, especially the consequence stated in Proposition 4.8.

Furthermore, in a nondegenerate diverse market stocks' growth rate can not be constant.

Proposition 4.9. *Suppose that the market $\mathcal{M}$ is nondegenerate. If all the stock in $\mathcal{M}$ have constant growth rates, then $\mathcal{M}$ is not diverse.*

Proof. Let us first show that if we assume this situation possible, then the stocks with the highest growth rate will account for almost all market capital in finite time. From (3.2) for $i = 1, 2, \cdots, n$

$$\lim_{T\to\infty}\frac{1}{T}\left(\log X_i(T) - \int_0^T \gamma_i(t)dt\right) = 0, \quad a.s. \tag{4.21}$$

Since γ_i is constant

$$\lim_{T\to\infty} \frac{1}{T} (\log X_i(T) - T\gamma_i) \;=\; 0, \quad a.s. \tag{4.22}$$

Thus for T large enough, we have

$$\begin{aligned} &\log X_i(T) \approx T\gamma_i, \quad a.s. \\ \Leftrightarrow \quad & X_i(T) \approx e^{T\gamma_i}, \quad a.s. \end{aligned}$$

Let $\mathcal{N}$ be the set of stocks with the greatest growth rate γ_{max} in $\mathcal{M}$. If the ith stock is not in $\mathcal{N}$, then for T large enough

$$\begin{aligned} \mu_i(T) \;&\approx\; \frac{e^{T\gamma_i}}{\sum_{k=1}^{n} e^{T\gamma_k}} \\ &=\; \frac{e^{T(\gamma_i-\gamma_{max})}}{\sum_{\substack{k=1\\ k\in\mathcal{M}\setminus\mathcal{N}}}^{n} e^{T(\gamma_k-\gamma_{max})} + |\mathcal{N}|}, \end{aligned}$$

where $|\cdot|$ is the cardinal function. Therefore $\mu_i(T) \to 0$ as $T \to \infty$; hence the stocks without maximum growth rate are negligeable at long term i.e at long term $\mathcal{M}$ is equivalent to $\mathcal{N}$.

If $|\mathcal{N}| = 1$, then by definition $\mathcal{N}$ is not diverse. On the other hand, if $|\mathcal{N}| \neq 1$ then the submarket $\mathcal{N}$ is not diverse because all its stocks have the same growth rate. Hence $\mathcal{M}$ which is now equivalent to $\mathcal{N}$ is not diverse.

■

Considering this last result, to maintain market diversity companies have to redistribute capital in some manner. Usually they do it through dividend payments and other redistribution schemes can be assimilated to dividend payments that is why we will only consider this distribution form.

Proposition 4.10. *If all the stocks in the market $\mathcal{M}$ have non negative dividend rates and the same augmented rate. Then*

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma_\mu^*(t) - \delta_\mu(t)\big)\, dt \;\leq\; 0 \quad a.s. \tag{4.23}$$

This means that to maintain diversity the average dividend rate must be at least as high as the average excess growth rate of the market. From Proposition 4.8 it arises that the average market dividend should be at least Δ for the market to stay diverse.

Proof. Let ρ be the shared augmented growth rate. Recall that the total return process of the market portfolio is given by

$$\widehat{Z}_\mu(t) \;=\; \sum_{i=1}^{n} \widehat{X}_i(t), \quad t \in [0, \infty),$$

and

$$\widehat{Z}_\mu(0) = Z_\mu(0). \tag{4.24}$$

Because the growth rate of $\widehat{Z}_\mu(t)$ is $\rho(t)$ and $\rho(t)$ is to $\widehat{Z}_\mu(t)$ what $\gamma_\mu(t)$ is to $Z_\mu(t)$ (from (1.25) and (1.26)), a reasoning similar to the one in (3.1) gives

$$\lim_{T\to\infty} \frac{1}{T}\left(\log \widehat{Z}_\mu(T) - \int_0^T \rho(t)dt\right) \quad = \quad 0 \quad a.s. \tag{4.25}$$

Since we only have nonnegative dividend rates for $t \in [0,\infty)$

$$\begin{aligned}
& X_i(t) \le \widehat{X}_i(t), \quad \forall i, \quad a.s. \\
\Rightarrow\quad & Z_\mu(t) \le \widehat{Z}_\mu(t), \quad a.s. \\
\Leftrightarrow\quad & Z_\mu(t) - \rho(t) \le \widehat{Z}_\mu(t) - \rho(t), \quad a.s. \\
\Rightarrow\quad & Z_\mu(T) - Z_\mu(0) - \int_0^T \rho(t)dt \le \widehat{Z}_\mu(T) - \widehat{Z}_\mu(0) - \int_0^T \rho(t)dt, \quad a.s. \\
\overset{(4.24)}{\Leftrightarrow}\quad & Z_\mu(T) - \int_0^T \rho(t)dt \le \widehat{Z}_\mu(T) - \int_0^T \rho(t)dt, \quad a.s. \\
\Rightarrow\quad & \lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \rho(t)dt\right) \le \lim_{T\to\infty} \frac{1}{T}\left(\log \widehat{Z}_\mu(T) - \int_0^T \rho(t)dt\right) \overset{(4.25)}{=} 0, \quad a.s.
\end{aligned}$$

So

$$\limsup_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \rho(t)dt\right) \quad \le \quad 0, \quad a.s. \tag{4.26}$$

Let us now recall Proposition 3.2 with $\pi = \mu$:

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \gamma_\mu(t)dt\right) \quad = \quad 0, \quad a.s. \tag{4.27}$$

Equations (4.26) and (4.27) imply that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \left(\gamma_\mu(t) - \rho(t)\right) dt \quad \le \quad 0, \quad a.s. \tag{4.28}$$

From (1.28) the augmented growth rate of the market portfolio is

$$\rho_\mu(t) \quad \overset{a.s.}{=} \quad \gamma_\mu(t) + \delta_\mu(t), \quad t \in [0,\infty). \tag{4.29}$$

However,

$$\begin{aligned}
\rho_\mu(t) &\overset{a.s.}{=} \gamma_\mu(t) + \delta_\mu(t) \\
&\overset{(1.27)}{=} \gamma_\mu(t) + \sum_{i=1}^{n} \mu_i(t)\delta_i(t) \\
&\overset{(1.20)}{=} \sum_{i=1}^{n} \mu_i(t)\gamma_i(t) + \gamma_\mu^*(t) + \sum_{i=1}^{n} \mu_i(t)\delta_i(t) \\
&= \sum_{i=1}^{n} \mu_i(t)\left(\gamma_i(t) + \delta_i(t)\right) + \gamma_\mu^*(t) \\
&= \sum_{i=1}^{n} \mu_i(t)\rho(t) + \gamma_\mu^*(t), \text{ common augmented growth rate} \\
&= \rho(t) + \gamma_\mu^*(t) \\
&\overset{(4.29)}{=} \gamma_\mu(t) + \delta_\mu(t)
\end{aligned}$$

for $t \in [0, \infty)$. Hence,

$$\rho_\mu(t) \overset{a.s.}{=} \rho(t) + \gamma_\mu^*(t), \quad t \in [0, \infty),$$

so (4.23) results from (4.28). ■

4.3 Diversity Measurement

We will now investigate means by which the market diversity can be quantified. It is required from a *measure of diversity* to be a function of market weights that is positive, symmetric and concave [5]. The common diversity measure function introduced by Shannon [9] is the *entropy* function

$$S(x) = -\sum_{i=1}^{n} x_i(t) \log x_i(t), \quad t \in [0, \infty),$$

for all $x \in \triangle^n = \left\{ x \in (0,1)^n; \sum_{i=1}^{n} x_i(t) = 1 \right\}$. Another measure function is

$$D_p(\mu(t)) = \left(\sum_{i=1}^{n} \mu_i^p(t) \right)^{\frac{1}{p}}, \quad t \in [0, \infty),$$

where $0 < p < 1$. The following proposition is a direct consequence of the entropy function definition.

Definition 4.11 (Diverse market). *The market $\mathcal{M}$ is diverse if and only if there is an $\varepsilon > 0$ such that*

$$S(\mu(t)) \geq \varepsilon, \quad t \in [0, T] \quad a.s. \quad [3]$$

Let us now define a portfolio associated with the market entropy, the *entropy weighted portfolio.*

Definition 4.12 (Entropy weighted portfolio). *Let μ be the market portfolio. The entropy weighted portfolio is the portfolio π with weights*

$$\pi_i(t) \quad = \quad -\frac{\mu_i(t)\log\mu_i(t)}{S(\mu(t))}, \quad t \in [0,T], \quad i = 1,2,\cdots,n.$$

If $Z_\mu(t)$ and $Z_\pi(t)$ respectively denotes the market portfolio and the entropy weighted portfolio at time t then we have

$$d\log S(\mu(t)) \quad = \quad d\log\frac{Z_\pi(t)}{Z_\mu(t)} - \frac{\gamma_\mu^*(t)}{S(\mu(t))}dt \quad [3]. \tag{4.30}$$

If we assume market diversify on long term, then we need that

$$\lim_{T\to\infty}\frac{1}{T}\log S(\mu(t)) \quad \overset{a.s.}{=} \quad 0.$$

Let us expand (4.30) using (1.12)

$$d\log S(\mu(t)) \quad = \quad \left(\gamma_\pi(t) - \gamma_\mu(t) - \frac{\gamma_\mu^*(t)}{S(\mu(t))}\right)dt + \sum_{i,j=1}^{n}(\pi_i(t) - \mu_i(t))\,\xi_{ij}(t)dW_j(t).$$

From Proposition (4.2) it follows that

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\log S(\mu(t)) \quad &= \quad \lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\gamma_\pi(t) - \gamma_\mu(t) - \frac{\gamma_\mu^*(t)}{S(\mu(t))}\right)dt \\ &\overset{a.s.}{=} \quad 0.\end{aligned}$$

Therefore, in order to have stability over time the entropy weighted portfolio outperforms the market portfolio with probability one. If the market is diverse, then

$$\frac{Z_\pi(T)}{Z_\pi(0)} \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)} \quad a.s.$$

Conclusion

Some basics ideas of stochastic portfolio theory have been presented in this work, namely portfolios, markets, market diversity among others. Stochastic portfolio theory is an innovative approach to financial modelling. It is unique because it studies short and long time behaviour of markets without standard assumptions and restrictions like non-arbitrage and market equilibrium, while general financial doctrines build short time models and require non-arbitrage and market equilibrium. Therefore, stochastic portfolio theory widens financial modelling's scope of application and achieve a more accurate mathematical model of our market. We just started the presentation of the methodology here, there is still a lot to learn about in order to understand how to build our portfolios in this new doctrine. Among other concepts, we still have to learn about *portfolio generating functions* and *stable distributions* for equity markets.

Appendix A. Returns and Variance

The return is the ratio of our gains on our initial capital. It is used to *"measure"* the success of a trading strategy or portfolio in the market. While the return *"quantifies"* the performance of an investment scheme the variance *"quantifies"* the corresponding risk of not achieving the expected return. The variance is the standard deviation of the return sometimes refered to as *volatility*.

The two standard returns are the *arithmetic return* and the *geometric return*. Considering X_t a stock price and the time interval $[t_1, t_2]$, $t_2 > t_1$, its arithmetic return is

$$\frac{X_{t_2} - X_{t_1}}{X_{t_1}}$$

and its geometric return is

$$\log\left(\frac{X_{t_2}}{X_{t_1}}\right) = \log X_{t_2} - \log X_{t_1}$$

Considering $t_1 < t_2 < t_3$ we have

$$\log\left(\frac{X_{t_2}}{X_{t_1}}\right) + \log\left(\frac{X_{t_3}}{X_{t_2}}\right) = \log\left(\frac{X_{t_3}}{X_{t_1}}\right)$$

The sum of the geometric returns over two *"consecutive"* intervals is the geometric return over the union of the intervals. This is not the case for the arithmetic return. Therefore the geometric return is more appropriate for the study of the long term behaviour of an investment.

Let us now take $|t_1 - t_2| \to 0$. To represent this infinitesimal time interval we will use the expression dt. We end up with the instantaneous arithmetic return at time t, $\frac{dX(t)}{X(t)}$ and the instantaneous geometric return at time t, $d\log X(t)$, also known as the *log-return* or *continuous return*.

Remark. *(C.2) shows how to deduce $d\log X(t)$ from $\frac{dX(t)}{X(t)}$ and vice versa when X is a stochastic process.*

Appendix B. Brownian Motion

The concept of *Brownian motion* or *Wiener process* is the corner stone of stochastic modelling used for functions of *random* variations. Natural phenomenons are usually of unbounded local variations therefore they can not be simulated with standard modelling techniques. First a 19^{th} century physicist Brown noticed that dust particles floating on the surface of water had a random path. From above dust particles are seen to move *randomly* about in a manner similar to sawtooth pattern except that the motion can be in any 360° direction [1]. The phenomenon is due to random collisions of dust particles and is referred to as Brownian motion.

B.1 Intuitive Approach

From the observations a Brownian motion is characterised by three properties

a. The path of a Brownian motion has no jumps; it is a continuous process.

b. But, a sample path of a Brownian motion is not at all *smooth*; it is built from constant *zigzag* meaning that it is nowhere differentiable: contrary to a continuous and differentiable function no matter how small the time interval considered the corresponding sample path is "zigzagged".

c. Finally a Brownian motion is of *unbounded variation* which is not as obvious as the previous properties from observation.

To represent this sample path let us take a time interval $[0, T]$. The corresponding sample path is obtained by connecting the successive positions of the process. We then get a sketch like the one in Figure B.1

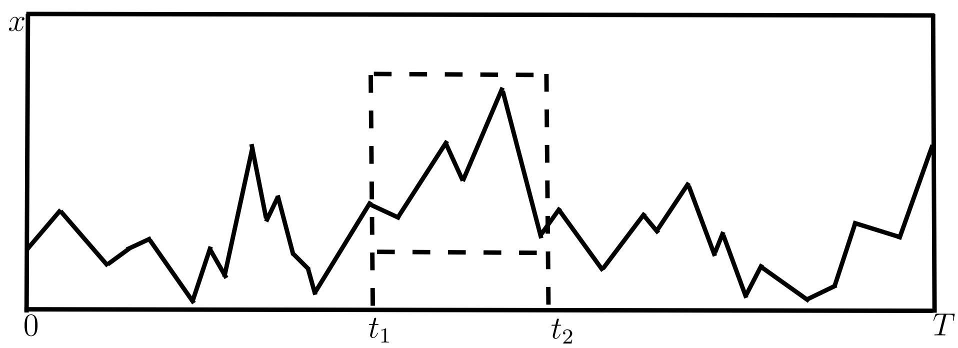

Figure B.1: Brownian motion sample path.

Now suppose that we amplify the path on the interval $[t_1, t_2]$. We will find something of the order of Figure B.2

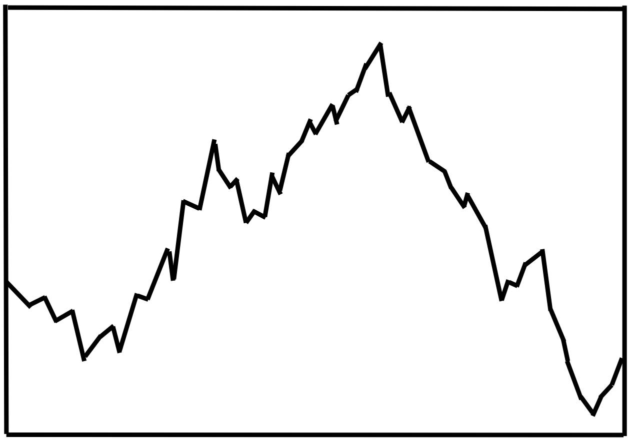

Figure B.2: Amplified Brownian motion sample path.

If we keep taking a segment and enlarge it, take another segment, enlarge it,$\cdots$ In the case of a differentiable function of bounded variation, after a certain number of iterations we will end up with a straight line as sample path. However, with a Brownian motion this will never happen; we will always get a jagged path.

B.2 Definition

Definition B.1 (Brownian motion [10]). *A real-valued stochastic process* $W(t), 0 \leq t \leq \infty$*, on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a Brownian motion, if*

1. $W(0) = 0$
2. $t \longmapsto W(t, \omega)$ *is a continuous function* $\mathbb{P}$*-a.s.*
3. *the increments* $W(t) - W(s)$ *are independent and have normal distribution* $N(0, t-s)$*, for any* $0 \leq s < t$

When we simultaneously have many sources of randomness, we use the multidimensional Brownian motion.

Definition B.2 (Multidimensional Brownian motion [7]). *Considering a positive integer* n*, the* n*-dimensional Brownian motion* W *on the filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *is the set of random vectors given by*

$$\{(W_1(t), W_2(t), .., W_n(t)), t \in [0, \infty)\},$$

where $W_1(t), W_2(t), .., W_n(t)$ *are independent Brownian motions on* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $\mathbb{F} = \{\mathcal{F}_t, t \in \mathbb{R}_+\}$ *is the filtration generated by* W*.*

Appendix C. Itô Process and Itô Formula

C.1 Definition

Definition C.1 (Itô process). *An* Itô process *or* stochastic integral *is a stochastic process on* $(\Omega, \mathcal{F}, \mathbb{P})$ *which can be represented in the form*

$$X(t) = X(0) + \int_0^t a(X(s), s)ds + \int_0^t b(X(s), s)dW(s), \qquad (C.1)$$

where a *and* b *are some processes in* L^2 *while* $W(t)$ *is a Brownian motion. As a shorthand* notation, *(C.1) can be written as*

$$dX(t) = a(X(t), t)dt + b(X(t), t)dW(t).$$

Theorem C.2 (Itô formula). *Let* $X(t)$ *be an Itô process of the form*

$$dX(t) = a(X(t), t)dt + b(X(t), t)dW(t), \qquad (C.2)$$

where $dW(t)$ *is a Brownian motion. Considering also a process* $y(t) = F(X(t), t)$*, then* $y(t)$ *is also an Itô process and satisfies the stochastic differential equation (SDE)*

$$dy(t) = \frac{\partial F}{\partial X}dX(t) + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX(t))^2. \qquad (C.3)$$

Notice that compared to standard differential equation, the stochastic differential equation has an additional term $\frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX(t))^2$ which is unique to SDE.

Combining (C.2) and (C.3)

$$\begin{aligned} dy(t) &= \frac{\partial F}{\partial x}(adt + bdW(t)) + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(adt + bdW(t))^2 \\ &= a\frac{\partial F}{\partial x}dt + b\frac{\partial F}{\partial x}dW(t) + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\left(a^2(dt)^2 + 2ab \cdot dt \cdot dW(t) + b^2(dW(t))^2\right) \end{aligned}$$

because $(dt)^2 = 0$, $dt \cdot dW(t) = 0$ and $(dW(t))^2 = dt$ [10]

$$dy(t) = \left(a\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 F}{\partial x^2}\right)dt + b\frac{\partial F}{\partial x}dW(t). \qquad (C.4)$$

Definition C.3 (Itô formula (Bis)). *Let* $X : [0, \infty) \to \mathbb{R}$ *be a continuous function with continuous quadratic variation* $\langle X\rangle_t$*, and* $F \in C^2(\mathbb{R})$ *a twice continuously differential real function. Then for any* $t \geq 0$

$$F(X(t)) = F(X(0)) + \int_0^t F'(X(s))dX(s) + \frac{1}{2}\int_0^t F''(X(s))d\langle X\rangle_s$$

where

$$\int_0^t F'(X_s)ds = \lim_{n\to\infty} \sum_{\substack{t_i\in\tau_n \\ t_i\le t}} F'(X_{t_i}) \left(X_{t_{i+1}-X_{t_i}}\right)$$

is Itô integral of $F'(X_t)$ *with respect to* X_t*. The fixed sequence* $(\tau_n), n = 1, 2, \cdots$ *is a sequence of finite partitions*

$$\tau_n = \{0 = t_0 < t_1 < \cdots < t_{i_n} < \infty\}$$

of $[0,\infty)$ *with* $t_{i_n} \underset{n}{\longrightarrow} \infty$ *and* $|\tau_n| = \sup_{t_i\in\tau_n} |t_{i+1} - t_i| \underset{n}{\longrightarrow} 0$*. [10]*

There is a very useful analogue of Itô formula in higher dimension.

Theorem C.4 (Multidimensional Itô formula). *Suppose* $dX(t) = a(X(t), t)dt + b(X(t), t)dW(t)$*, with vector* $a = (a_1, a_2, \cdots, a_n)$ *and matrix* $b = (b_{11}, b_{22}, \cdots, b_{nn})$ *have* H^2 *components and* W *is an* n*-dimensional Brownian motion. Let* $F(X(t), t) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ *be a twice continuously differentiable function. Then* $Y(t) = F(X(t), t)$ *is also an Itô process and*

$$dY(t) = \sum_{i=1}^{n} \frac{\partial F}{\partial X_i} dX_i(t) + \frac{\partial F}{\partial t} dt + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial X_i \partial X_j} dX_i(t)\cdot dX_j(t).$$

where $dX_i(t)\cdot dX_j(t)$ *is computed using the rules* $dt\cdot dt = dt\cdot dW_i = dW_i\cdot dt = 0, dW_i\cdot dWj = \delta_{ij}dt$ *for all* $i, j = 1, 2, \cdots$ *and* $H^2 = \{f \text{ such that } f, f', \text{ and } f'' \text{ are in } L^1\}$*. [2]*

Theorem C.5 (d-dimensional Itô formula). *For* $\mathbb{F} \in C^2(\mathbb{R}^d)$ *and* $X = (X^1, X^2, \cdots, X^d) \in \mathbb{R}^d$ *one has*

$$F(X(t)) = F(X(0)) + \int_0^t \nabla F(X(s))dX(s) + \frac{1}{2}\sum_{k,l=1}^{d} \int_0^t \frac{\partial^2 F}{\partial x_k \partial x_l} d\langle X^k, X^l\rangle_s$$

where

$$\nabla F(x) = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \cdots, \frac{\partial F}{\partial x_d}\right)(x)$$

is the gradient *of* F*. [10]*

C.2 Application

The arithmetic return of a stock price $X(t)$ is usually given by a stochastic differential equation (SDE) of the form

$$dX(t) = X(t)\left[\alpha dt + \sigma dW(t)\right]$$

where α is the rate of return of X, σ its variance and $W(t)$ is a Brownian motion.

We have $a(X(t),t) \equiv \alpha X(t)$, $b(X(t),t) \equiv \sigma X(t)$. To deduce the SDE giving the log return we consider $F(X(t),t) = \log X(t)$. We then have

$$\frac{\partial F}{\partial X(t)} = \frac{1}{X(t)}, \frac{\partial^2 F}{\partial X(t)^2} = -\frac{1}{X(t)^2}, \text{ and } \frac{\partial F}{\partial t} = 0.$$

From (C.4) we get

$$\begin{aligned} d\log X(t) &= \left[\alpha \frac{1}{X(t)} X(t) + 0 - \frac{1}{2}\frac{1}{(X(t))^2}(\sigma X(t))^2\right] dt + \frac{1}{X(t)}\sigma X(t) dW(t) \\ &= \left(\alpha - \frac{1}{2}\sigma^2\right) dt + \sigma dW(t). \end{aligned}$$

Bibliography

[1] Jean-Pierre Danthine. An intuitive overview of continuous time finance. *Lecture Notes*, 0:1–23, 2005.

[2] Jitka Dupacova, Jan Hurt, and Josef Stepan. *Stochastic Modeling in Economics and Finance*. Number 75 in Contemporary Mathematics. Kluver Academic Publishers, 2002.

[3] Robert Fernholz. *Stochastic Portfolio Theory*. Number 48 in Applications of Mathematics. Springer, 2002.

[4] Robert Fernholz. The application of stochastic portfolio theory to equity management. *Journal of Finance*, pages 1–12, 2003.

[5] Robert Fernholz. Stock market diversity. pages 1–11, 2005.

[6] Robert Fernholz and Ioannis Karatzas. Stochastic portfolio theory: An overview. *Handbook of Numerical Analysis*, 27:1–10, 2006.

[7] Genevieve Gauthier. Methodes stochastiques dans les sciences de la gestion. Available from `http://neumann.hec.ca/ p240/`, 2003.

[8] Harry Markowitz. Portfolio selection. *Journal of Finance*, 7:77–91, 1952.

[9] Claude Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27:379–423, 1948.

[10] Dieter Sondermann, editor. *Introduction to Stochastic Calculus for Finance*. Number 133 in Contemporary Mathematics. Springer, 1992.