Metrizability of Topological Spaces

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Abstract

It is well known that the distance function or metric defined on a metric space X induces a topology on that space X. The goal of this essay is to study well-known characterizations of the class of topologies that can be obtained from metrics in this way. We present some important criteria, which are necessary and sufficient, that topological spaces must possess in order to be metrizable.

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1. Introduction

In any metric space (X, d), one can define open sets as follows :

O is open in X if and only if for all $x \in O,$ there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset O.$ $\quad (*)$

If we put $\tau = \{O|\ O$ is open in X in the sense given by $(*)\}$, then τ is a topology on X called *metric topology* or the *topology induced by the metric* d, and (X, τ) is called *metric topological space*. This means that every metric space is a topological space. In comparaison with the obviousness of the last statement, in this essay, we study how is it possible to reverse that statement. This enables us to ask the question that given a topological space (X, τ), can we define a metric d on X such that the topology induced by d is τ ? if so, the space X is said to be *metrizable*.

The main purpose of this paper is to give the answer to that question, since unfortunately, not all topological spaces are metrizable.

The organisation of this paper is as follows. We begin with preliminaries in Chapter 2, in which we give an overview of some background on metric and topological spaces that is pertinent to our purposes, that is to say notions we use in the next chapter.

In Chapter 3, which is the main part of this essay, we present some main results on metrizability of topological spaces, particularly the results due to Urysohn [4], Nagata and Smirnov [14], R.H. Bing [4], H.A. Frink [14], and Alexandroff and Urysohn [14]. we conclude in chapter 4.

2. Metric and Topological Spaces

In this chapter, we only give a brief overview of some topological notions in metric and topological spaces which will be relevant to this essay. Thus, notions such as interior, derived set, boundary of the set,. . . , and complete and connected spaces for example will not be presented. The reader who is interested to these notions can find them in almost any book of general topology. (See for example [8], [6], [13] and [2]).

2.1 Metric Spaces

Definition 2.1.1. *Let X be any non-empty set. A function $d : X \times X \longrightarrow \mathbb{R}_+$ is said to be a metric or a distance on X if the following conditions are satisfied.*

(O_1) $d(x, y) \geq 0$ *and* $d(x, y) = 0 \Longleftrightarrow x = y$ *for all* $x, y \in X$ *(Positivity),*

(O_2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *(Symmetry),*

(O_3) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$ *(Triangle inequality).*

The pair (X, d) is called a *metric space* and the elements of X are called *points*.

Example 2.1.2.

- *Let* $X = \mathbb{R}^n$*. One can define on* X *the following distances: for any two points* $x = (x_1, x_2, ..., x_n)$ *and* $y = (y_1, y_2, ..., y_n)$ *of X;*

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|, \ d_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}, \text{ and } d_\infty(x, y) = \sup_{1 \leq i \leq n} |x_i - y_i|.$$

These three distances are equivalent, i.e. there are real constants k_1, k_2, k_3 *such that*

$$d_\infty(x, y) \leq k_1 d_2(x, y) \leq k_2 d_1(x, y) \leq k_3 d_\infty(x, y).$$

- *Let* $X = \mathbb{C}$ *the set of complex numbers. For* z_1, z_2 *in* $\mathbb{C}$*,*

$d(z_1, z_2) = |z_1 - z_2|$ *is a distance in* $\mathbb{C}$*.*

- *Let* X *be any arbitrary non-empty set and let us define for* $x, y \in X$

$$d(x, y) = \begin{cases} 1 & if \ x \neq y \\ 0 & if \ x = y \ . \end{cases}$$

One can see that d *is a metric in* X*, called discrete metric and* (X, d) *the discrete metric space.*

- *If (X, d) is any metric space and Y any subset of X, then (Y, d) can also be considered as a metric space using the same distance as X. Y is then called a subspace of the metric space (X, d).*

We define the distance from a point x of a metric space (X, d) to a subspace A of X by:

$$d(x, A) = \inf_{y \in A} d(x, y).$$

There are particularly interesting subsets in a metric space, namely the balls. One can define the open ball and the closed ball with center x and radius $r > 0$ respectively by:

$$B(x, r) = \{y \in X : d(x, y) < r\} \text{ and } \bar{B}(x, r) = \{y \in X : d(x, y) \leqslant r\}.$$

Proposition 2.1.3. *Given any ball $B(x, r)$ in a metric space and a point y in $B(x, r)$, there exists $\delta > 0$ such that $B(y, \delta) \subseteq B(x, r)$.*

Proof. Let $B(x, r)$ be a ball of a metric space X, $y \in B(x, r)$ and suppose $\delta = r - d(x, y) > 0$, then $B(y, \delta) \subseteq B(x, r)$. □

The ball $B(x, r)$ is called r-neighborhood of the point x in X.

Proposition 2.1.4. *Let (X_n, d_n), $n \in \mathbb{N}$, be a countable family of metric spaces. Then the product of X_n countably many times, $\prod_{n \in \mathbb{N}} X_n$, is a metric space.*

Proof. Let (X_n, d_n), $n \in \mathbb{N}$, be a countable family of metric spaces. We define a metric on the product $\prod_{n \in \mathbb{N}} X_n$ as follow: let x and y be any points of $\prod_{n \in \mathbb{N}} X_n$; let us denote by x_n and y_n the n^{th} coordinates of x and y respectively. Let us define

$$d'_n(x_n, y_n) = min\left(d_n(x_n, y_n), 1\right).$$

The metric d'_n thus defined on X_n is bounded by 1 and is equivalent to the metric d_n. The distance defined by

$$d(x, y) = \sum_{n \in \mathbb{N}} \frac{d'_n(x_n, y_n)}{2^n} \tag{2.1}$$

is a metric on $\prod_{n \in \mathbb{N}} X_n$. □

Remark 2.1.5. *The assumption that the metrics d'_n are bounded by 1 and the factors $\frac{1}{2^n}$ are needed only to guarantee that the series given by (2.1) is convergent. For a finite sequence X_1, X_2,...,X_n of metric spaces, one can define the distance between two points $x = (x_k)$ and $y = (y_k)$, $1 \le k \le n$, on the set $X = X_1 \times X_2 \times ... \times X_n$ by*

$$d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2) + ... + d_n(x_n, y_n).$$

Corollary 2.1.6. *The Hilbert cube $\prod_{n \in \mathbb{N}} [0, 1]$, which is the product of $[0, 1]$ countably many times, is a metric space.*

Open and Closed sets in metric spaces

Definition 2.1.7. *A subset U of a metric space X is open if and only if it is union of open balls, or U is open in X if given any $y \in U$, there exists $\epsilon > 0$ such that $B(y, \epsilon) \subset U$.*

Definition 2.1.8. *A subset F of a metric space X is closed if and only if its complement*[1] $X \setminus F$ *is open.*

2.2 Topological Spaces

Definition 2.2.1. *A topological space is a pair (X, τ) consisting of a non-empty set X and a family τ of subsets of X satisfying the following conditions,*

(T_1) *$\emptyset \in \tau$ and $X \in \tau$,*

(T_2) *The intersection of any two (and hence any finite number of) sets of τ is again in τ,*

(T_3) *The union of any family of sets of τ is again in τ.*

The family τ is called a *topology* for X, and its members are called open sets of X. Hence the statements "$U \in \tau$" and "U is open in τ" mean the same thing. Elements of X are called points.

Example 2.2.2.

- *Let X be any non-empty space and $\tau_1 = \{\emptyset, X\}$. Clearly, the axioms for a topology $(T_1), (T_2), (T_3)$ given above hold. $\tau_1 = \{\emptyset, X\}$ is a topology in X called the indiscrete topology.*

- *Let X be any non-empty set and let $\tau_2 = \mathcal{P}(E)$. Then τ_2 is a topology in X called the discrete topology.*

- *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, X\}$. It is easily verified that τ is a topology in X.*

- *The real line. Let $X = \mathbb{R}$. Define the family τ as follows:*

  $U \in \tau \Longleftrightarrow$ for any $x \in U, \exists \delta_x > 0$ such that $u \in U$ if $|x - u| < \delta_x$. τ is a topology called usual topology in $\mathbb{R}$.

Remark 2.2.3. *A subset U of a topological space X is a neighborhood of the point x if and only if there is an open set V such that $x \in V \subset U$. A neighborhood of a point need not to be an open set, but every open set is a neighborhood of each of its points. Thus we have the following characterization of open sets, like in the case of metric spaces.*

Proposition 2.2.4. *A subset U of a topological space X is open if and only if it is a neighborhood of all its points. That is for all $x \in U$, we can always find an open set O_x such that $x \in O_x \subset U$.*

[1] The complement of a subspace A of a metric space X is given by: $X \setminus A = \{y \in X : y \notin A\}$

The neighborhood system of a point is the family of all neighborhoods of that point.

Definition 2.2.5. *A subset F of a topological space (X, τ) is said to be closed if its complement $X \setminus F$ is open.*

Therefore, since closed sets are complement of open sets, then taking the complements of properties (T_1),(T_2) and (T_3) from Definition 2.2.1, we have the following topological space axioms of a family $\mathscr{F}$ of subsets of X called closed sets.

Proposition 2.2.6. *Let X be any set and suppose that $\mathscr{F}$ is a family of closed subsets of X. The $\mathscr{F}$ has the following properties,*

(F_1) $\emptyset \in \mathscr{F}$ *and* $X \in \mathscr{F}$,

(F_2) *the union of any two (and hence any finite number of) sets of $\mathscr{F}$ is again in $\mathscr{F}$,*

(F_3) *The intersection of any family of sets of $\mathscr{F}$ is again in $\mathscr{F}$.*

Remark 2.2.7. *From properties $(T_1), (T_2), (T_3)$ of open sets and $(F_1), (F_2), (F_3)$ of closed sets, we have the following result: in any topological space, the whole space X and the empty set $\emptyset$ are both open and closed.*

Definition 2.2.8. Subspace of a topological space and subspace topology.

Let (X, τ) be a topological space and $G \subset X$. Then the family $\mathscr{O}$ of all sets $G \bigcap U$ where U is open in X satisfies axioms for a topology $(T_1), (T_2), (T_3)$ of the definition 2.2.1 given above. Indeed,

(T_1) *Since X and $\emptyset$ are in τ, then $G \bigcap X = G$ and $G \bigcap \emptyset = \emptyset$ are open in G.*

(T_2) *Suppose U and V are open in G, then $U = G \bigcap U'$ and $V = G \bigcap V'$ where U' and V' are open subsets of X. We have*

$$U \bigcap V = \left(G \bigcap U'\right) \bigcap \left(G \bigcap V'\right) = G \bigcap \left(U' \bigcap V'\right).$$

But $U' \bigcap V'$ is an open subset of X; hence $U \bigcap V$ is an open subset of G.

(T_3) *Suppose $\{U_i\}$, $i \in I$, is a family of open subsets of G, then $U_i = G \bigcap U_i'$ where U_i' is an open subset of X for each $i \in I$, then*

$$\bigcup_{i \in I} U_i = \bigcup_{i \in I} \left(G \bigcap U_i'\right) = G \bigcap \left(\bigcup_{i \in I} U_i'\right).$$

Since $\bigcup_{i \in I} U_i'$ is the union of open sets, it is open, and therefore $\bigcup_{i \in I} U_i$ is open in G.

The family $\mathscr{O} = \{G \bigcap U : \ U$ is open in $X\}$ of open sets in G is a topology on G called *subspace topology* or *induced topology*. The set G with this topology is called a subspace of the topological space X.

Remark 2.2.9. *We can define by the same way a closed subset F of a subspace G of a topological space X as the intersection of G with a closed set F' in X, i.e.*

$$X \text{ is a topological space, } G \text{ a subspace of } X\text{. Then } F \subset G \text{ is closed in } G \iff F = G \bigcap F',$$

where F' is closed in X.

We shall define now some other families of sets in topological spaces.

Definition 2.2.10. Base of a topological space $(X, \mathscr{O})$.

A family $\mathscr{B} \subset \mathscr{O}$ is called a base of the topological space $(X, \mathscr{O})$ if every non-empty open subset of X can be writen as the union of the members of $\mathscr{B}$ i.e.

$\mathscr{B}$ base of $(X, \mathscr{O})$ if for every open set $U \subset X$, $U = \bigcup_{i \in I} O_i$, where $O_i \in \mathscr{O}$.

Definition 2.2.11. Subbase of a topological space $(X, \mathscr{O})$.

Let $(X, \mathscr{O})$ be a topological space. A subcollection $\mathscr{S}$ of $\mathscr{O}$ is said to be a subbase for $\mathscr{O}$ if the set

$$\mathscr{B} = \{B \mid B \text{ is the intersection of finitely many members of } \mathscr{S}\}$$

is a base for $\mathscr{O}$.

Example 2.2.12. *In $\mathbb{R}$, the collection of all open intervals is a base for the usual topology. For a subbase, one can take the open intervals of the form $(-\infty, a)$ and (b, ∞), since any open interval is either one of these or else the intersection of the two of them, i.e. if $a < b$, $(a, b) = (-\infty, b) \bigcap (a, \infty)$; and any open set in $\mathbb{R}$ is a union of open intervals.*

Definition 2.2.13. *A collection of subsets $\mathscr{B}$ of a topological space X is said to be locally finite (respectively discrete) if for any $x \in X$, there is an open set U containing x which meets only finetely many sets (respectively at most one set) in $\mathscr{B}$ i.e.*

$$\left\{B \in \mathscr{B} \mid B \bigcap U \neq \emptyset\right\}$$

is a finite subcollection of $\mathscr{B}$ (respectively is either empty or contains exactly one subset from the collection).

A collection of subsets $\mathscr{B}$ is said to be σ-locally finite (respectively σ-discrete) if it is countable union of locally finite (respectively discrete) collection, i.e.

$$\mathscr{B} = \bigcup_{n=1}^{\infty} \mathscr{B}_n,$$

where each $\mathscr{B}_n$ is locally finite (respectively discrete).

Definition 2.2.14. Closure of a subset A of X

For any $A \in X$, let us consider the family $\mathscr{F}_A$ of all closed sets containing A. By axiom (F_1), $\mathscr{F}_A \neq \emptyset$, and from (F_3) it follows that the intersection $\bigcap \mathscr{F}_A$ is closed. It is the smallest closed set containing A called the closure of A and represented by $\bar{A}$.

Obviously, the subset F of X is closed if and only if $F = \bar{F}$.

The closure of a subset F satifies the following properties;

Property 2.2.15. *For any subsets A, B, C of a space X, we have;*

(P_1) $A \subset \bar{A}$.

(P_2) $A \subset B \Longrightarrow \bar{A} \subset \bar{B}$.

(P_3) $\bar{\bar{A}} = \bar{A}$, $\overline{A \bigcup B} = \bar{A} \bigcup \bar{B}$, $\overline{A \bigcap B} \subset \bar{A} \bigcap \bar{B}$.

Definition 2.2.16. Density

A subspace B of a topological space X is said to be dense in X if $\bar{B} = X$, in other words, every point of B is limit of a net[2] in X.

Example 2.2.17. *The set $\mathbb{Q}$ of rational numbers if dense in the set $\mathbb{R}$ of real numbers. Indeed, each real number is a limit of some sequence of rational numbers.*

2.3 Covering, Compact and Paracompact Spaces

Definition 2.3.1. Covering

A collection $\mathscr{C} = \{G_\alpha : \alpha \in I\}$ of subsets of a set X is said to be a cover of X if

$$\bigcup_{\alpha \in I} G_\alpha = X.$$

If $\mathscr{C}_1$ and $\mathscr{C}_2$ are two covers of X such that $\mathscr{C}_2 \subset \mathscr{C}_1$, then $\mathscr{C}_2$ is called a subcover of $\mathscr{C}_1$.

Example 2.3.2. *The collection $\mathscr{C}_1 = \{]-n, n[: n \in \mathbb{N}\}$ is a cover of the set $\mathbb{R}$ of real numbers and $\mathscr{C}_2 = \{]-2n, 2n[: n \in \mathbb{N}\}$ is a subcover of $\mathscr{C}_1$.*

- A cover $\mathscr{C}$ of (X, τ) is said to be τ-*open cover* of X if every member of $\mathscr{C}$ is a τ-open set,
- A cover $\mathscr{C}$ of X is said to be *finite* if $\mathscr{C}$ has only a finite number of members.

[2] A net in a topological space X is a function from some directed set D into X

Definition 2.3.3. Refinement.

Let $\mathscr{V}$ be a cover of X. The collection $\mathscr{U}$ is called a refinement of the cover $\mathscr{V}$, or $\mathscr{U}$ refines $\mathscr{V}$, and we write $\mathscr{U} < \mathscr{V}$, if and only if $\mathscr{U}$ is a cover of X and each member of $\mathscr{U}$ is a subset of a member of $\mathscr{V}$ i.e.

$$\mathscr{U} < \mathscr{V} \iff \text{for all } U \in \mathscr{U},\ U \subset V \text{ where } V \in \mathscr{V}.$$

Notice that any open subcover is a refinement, but a refinement is not necessary an open subcover. If $\mathscr{U}$ is a cover of X and $A \subset X$, the *star of A with respect to $\mathscr{U}$* is the set

$$St(A, \mathscr{U}) = \bigcup \left\{ U \in \mathscr{U} \mid A \bigcap U \neq \emptyset \right\}.$$

We say that $\mathscr{U}$ *star-refines* $\mathscr{V}$, or $\mathscr{U}$ is a star-refinement of $\mathscr{V}$, written $\mathscr{U}^* < \mathscr{V}$, if and only if for each $U \in \mathscr{U}$, there is some $V \in \mathscr{V}$ such that $St\,(U, \mathscr{U}) \subset V$. Let us recall that the star of a one-point set $\{x\}$ with respect to a cover $\mathscr{U}$ is called the star of the point x with respect to $\mathscr{U}$, and is denoted by $St(x, \mathscr{U})$.

Finally, $\mathscr{U}$ is a *barycentic-refinement of $\mathscr{U}$*, written $\mathscr{U} \Delta \mathscr{V}$, provided the sets $St(x, \mathscr{U})$, for each $x \in X$, refine $\mathscr{V}$. Clearly, every star-refinement is a barycentric refinement and every barycentric refinement is a refinement.

Definition 2.3.4. Compactness

A topological space (X, τ) is said to be compact if every τ-open cover of X has a finite subcover.

Example 2.3.5. *For all real numbers a and b such that $a < b$, the closed interval $[a, b]$ is compact, while the open interval $]a, b[$ is not (indeed, $\mathbb{R}$ itself is not compact).*

Definition 2.3.6. Paracompactness.

A topological space (X, τ) is said to be paracompact if X is T_2 and every open cover of X has a locally finite open refinement.

2.4 Continuity and Homeomorphism

Let $(X, \mathscr{O}_1)$ and $(Y, \mathscr{O}_2)$ be two topological spaces and f a map from $(X, \mathscr{O}_1)$ into $(Y, \mathscr{O}_2)$.

Definition 2.4.1. *The map $f{:}(X, \mathscr{O}_1) \longrightarrow (Y, \mathscr{O}_2)$ is continuous if the inverse image of any open subset of Y is open in X, i.e.*

$$f \text{ is continuous if } f^{-1}(U) \in \mathscr{O}_1 \text{ for any } U \in \mathscr{O}_2.$$

Since closed sets are complements of open sets, we have the following proposition.

Proposition 2.4.2. *The inverse image of a closed subset of Y through a continuous map is a closed set in X.*

Example 2.4.3.

* *If X is a discrete space, then any map from X into a topological space Y is continuous.*
* *Any map from a topological space into an indiscrete space is continuous.*
* *For any set A, $x \longmapsto d(x, A)$ is a continuous function.*

Definition 2.4.4. Homeomorphism

A map $f : X \longrightarrow Y$ is called a homeomorphism if the following conditions hold,

* *f is a continuous bijective map,*
* *the inverse map $f^{-1} : Y \longrightarrow X$ is also continuous.*

If there exists a homeomorphism from X into Y, we say that X and Y are *homeomorphic* or *topologically equivalent*. Hence, the statement '*X and Y are homeomorphic*' is an equivalence relation. In other words, the homeomorphisms preserve all the structures that topological spaces possess[3].

Proposition 2.4.5. *If f is a bijective map from $(X, \mathscr{O}_1)$ to $(Y, \mathscr{O}_2)$, then the following statements are equivalent;*

($\mathscr{S}_1$) *f is a homeomorphism,*

($\mathscr{S}_2$) *The subset F is open (respectively closed) in X if and only if $f(F)$ is open (respectively closed) in Y,*

($\mathscr{S}_3$) *The subset G is open (respectively closed) in Y if and only if $f^{-1}(B)$ is open (respectively closed) in X.*

Proof. The proposition is proved by using definitions 2.4.1 and 2.4.4, and the fact that if f is a homeomorphism, then its inverse f^{-1} is also a homeomorphism and we have

$$A = f^{-1}(f(A)) \text{ and } f(A) = \left(f^{-1}\right)^{-1}(A).$$

□

[3]Because of that property, we will later see that if a topological space X is homeomorphic to a metric space, then X is metrizable.

3. Metrizability of Topological Spaces

Any metric space (X, d) gives rise to a topological space (X, τ), where τ is defined to be the collection of all subsets which are open in the sense of Definition 2.1.7, Chapter 2. τ is then a topology on X called a *metric topology* induced by d. Suppose now instead of starting with a metric space, we are given a topological space (X, τ). We would like to know if there exits a metric d which can be defined on X such that the topology induced by d is τ? If so, the space X is said to be *metrizable*. Otherwise X is *nonmetrizable*.

The question now is; when is a topological space metrizable? A theorem which answers to that question is called a *Metrization Theorem*. One of the most important Metrization Theorems is the *Urysohn's Metrization Theorem* which gives the criteria which suffice for metrizability.

3.1 Urysohn's Metrization Theorem

Theorem 3.1.1. *[4]. Let X be a normal space which is second countable. Then X is metrizable.*

Let us first define the second axiom of countability.[1]

Definition 3.1.2. *A space X is said to be second countable if it has a countable base, i.e. there is a countable collection of open sets such that any open set can be expressed as a union of sets from this collection.*

Example 3.1.3. $\mathbb{R}^n$ *is second countable. We may take as a countable base, the collection of all balls of rational radius, centered at points having all coordinates rational.*

However not all metrizable spaces are second countable.

Example 3.1.4. *Let us consider an uncountable set with the discrete topology. This topology arises from a metric (e.g. take the distance between distinct points to be always 1). However any base for the discrete topology must contain all the singleton sets, and there are uncountably many of them.*

Proposition 3.1.5. *Any subspace of a second countable space is second countable.*

Proof. Let (X, τ) be a second countable space, $\mathscr{B} = \{B_i\}_{i \in \mathbb{Z}_+}$ a countable base for τ, and B a subspace of X. Then the family

$$\mathscr{B}_B = \left\{ B \bigcap O |\ O \in \mathscr{B} \right\}$$

is a countable base for the subspace topology on B. Then B is second countable. □

[1] A first-countable space is a topological space which has a countable base at each of its points.

The idea in the proof of Urysohn's Metrization Theorem is to construct a homeomorphism f between the given topological space X, which is normal and second countable, and a subspace of the Hilbert cube, which is metrizable (see Corolary 2.1.6, Chapter 2). Then $f(X)$ will be metrizable as a subspace of a metrizable space (see Example 2.1.2, Chapter 2) and so is X since homeomorphic to a metrizable space. Now constructing a map into a product is equivalent to constructing a map into each factor. Thus the following lemma plays a central role in the proof of Urysohn's Metrization Theorem:

3.1.1 Urysohn's Lemma.

Lemma 3.1.6. *Let A and B be disjoint closed subsets of a normal space X. Then we can find a continuous function*

$$f : X \longrightarrow [0,1]$$

such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Proof. Let X be a normal space, then given any closed subset F of X and an open set H which contains F, we can always find an open set G (See the proposition A.0.22 in the Appendix) such that

$$F \subset G \subset Cl(G) \subset H. \tag{3.1}$$

Let D denote the set of dyadic rational numbers, i.e. all numbers which can be written in the form

$$p = \frac{m}{2^n}, \text{ where } m \text{ and } n \text{ are positive integers such that } 0 \leq m \leq 2^n.$$

We shall associate to each such dyadic rational number p an open set $U_p \subseteq X$ such that:

(i**)** $A \subset U_p$,

(j**)** $B \bigcap U_p = \emptyset$, (3.2)

(k**)** if $p < q$, then $Cl(U_p) \subseteq U_q$.

We will construct the U_p's by induction by using the characterization (3.1) of normality.

We start the induction by taking $U_1 = X \setminus B$ and applying (3.1) to the inclusion $A \subseteq U_1$, we obtain an open set U_0 satisfying

$$A \subseteq U_0 \subseteq Cl(U_0) \subseteq U_1.$$

Similarly, using again (3.1) to the inclusions $A \subseteq U_0$ and $Cl(U_0) \subseteq U_1$, there exist open sets $U_{\frac{1}{2}}$ and $U_{\frac{3}{4}}$ satisfying

$$A \subseteq U_{\frac{1}{2}} \subseteq Cl(U_{\frac{1}{2}}) \subseteq U_0,$$

$$Cl(U_0) \subseteq U_{\frac{3}{4}} \subseteq Cl(U_{\frac{3}{4}}) \subseteq U_1.$$

We now have the following inclusions

$$A \subseteq U_{\frac{1}{2}} \subseteq Cl(U_{\frac{1}{2}}) \subseteq U_0 \subseteq Cl(U_0) \subseteq U_{\frac{3}{4}} \subseteq Cl(U_{\frac{3}{4}}) \subseteq U_1.$$

We will continue by finding the U_p's by induction on n, the exponent of 2 in $p = \frac{m}{2^n}$. Let us notice that we have already defined U_p for $n = 1$ and $n = 2$. Now assume we have defined U_p for n. We now define U_p for $n+1$ (and thus for $m = 1, 3, \ldots, 2^{n+1} - 1$). Note that the definition of U_p needs to be given only for odd n; because if n were even, the numerator and denominator of p could be divided by 2. Because the U_p's have already been constructed for $p = \frac{m}{2^n}$, m odd, and since

$$Cl(U_{\frac{m-1}{2^{n+1}}}) = Cl(U_{\frac{\frac{m-1}{2}}{2^n}}) \text{ , which has already been defined,}$$

we have

$$Cl(U_{\frac{m-1}{2^{n+1}}}) \subseteq U_{\frac{m+1}{2^{n+1}}} \text{ , because of the property } (k) \text{ of } (3.2).$$

Applying (3.1), we obtain an open set, which we let be $U_{\frac{m}{2^{n+1}}}$, such that

$$Cl(U_{\frac{m-1}{2^{n+1}}}) \subseteq U_{\frac{m}{2^{n+1}}} \subseteq Cl(U_{\frac{m}{2^{n+1}}}) \subseteq U_{\frac{m+1}{2^{n+1}}}.$$

We thus have an inductive definition of U_p for each p in $D \bigcap [0,1]$ as described. By construction, the collection of U_p have properties (i) through (k) given by (3.2).

We extend the definition to dyadic rationals outside the interval $[0,1]$ by defining

$$U_p = \emptyset \text{ if } p < 0 \qquad\qquad U_p = X \text{ if } p > 1,$$

and (3.2) obviously still holds.

We now define the function $f : X \longrightarrow [0,1]$ by $f(x) = \inf\{p \in D |\ x \in U_p\}$.

Note that since $A \subseteq U_p$ for all p, then if $a \in A$,

$$f(a) = \inf\{p \in D |\ p \geq 0\} = 0.$$

Similarly, since $U_p \subseteq U_1$ for all p and $U_p \bigcap B = \emptyset$ for all p, then if $b \in B$

$$f(b) = \inf\{p \in D |\ p > 1\} = 1.$$

All that remains to be proved is that f is continuous, and to do that, we will use the following lemma.

Lemma 3.1.7. *Let P be a subset of the real numbers and let S be a set. Suppose that for each $p \in P$, there is associated a subset $F_p \subseteq S$ and that*

$$\bigcup_{p \in P} F_p = S.$$

If we define a function $f : S \longrightarrow \mathbb{R}$ by

$$f(s) = \inf\{p \in P |\ s \in F_p\},$$

then for any real number c

$$f^{-1}((-\infty, c)) = \bigcup_{p < c} F_p.$$

If we assume in addition that

(a) P is dense in $\mathbb{R}$,

(b) $F_p \subset F_q$ whenever $p < q$,

then we also have $f^{-1}((-\infty, c]) = \bigcap_{p>c} F_p$.

Proof. Since for any set of real numbers T

$$\inf T < x \Longleftrightarrow \exists t \in T \text{ such that } t < x,$$

we have

$$\begin{aligned} f^{-1}((-\infty, c)) &= \{s | \inf \{p \in P | s \in F_p\} < c\} \\ &= \{s | \exists \text{p such that } s \in F_p \text{ and } p < c\} \\ &= \bigcup_{p<c} F_p. \end{aligned}$$

For the second part, we have:

$$\begin{aligned} f^{-1}((-\infty, c]) &= f^{-1}\left(\bigcap_{d>c}(-\infty, d)\right) \\ &= \bigcap_{d>c} f^{-1}\left((-\infty, d)\right) \\ &= \bigcap_{d>c}\left(\bigcup_{p<d} F_p\right). \end{aligned}$$

Thus it remains to show that

$$\bigcap_{d>c}\left(\bigcup_{p<d} F_p\right) = \bigcap_{q>c} F_q.$$

First of all given any $d > c$, by density of P, we can find a $q_0 \in P$ such that $c < q_0 < d$. Thus

$$\bigcap_{q>c} F_q \subseteq F_{q_0} \subset \bigcup_{p<d} F_p.$$

Since this is true for all d, it follows that

$$\bigcap_{q>c} F_q \subseteq \bigcap_{d>c}\left(\bigcup_{p<d} F_p\right).$$

On the other hand, given any $q \in P$ such that $q > c$, we have by (b):

$$\bigcap_{d>c}\left(\bigcup_{p<d} F_p\right) \subseteq \bigcup_{p<q} F_p \subseteq F_q.$$

The two inclusions thus establish

$$\bigcap_{d>c}\left(\bigcup_{p<d} F_p\right) = \bigcap_{q>c} F_q.$$

This conclude the proof of the lemma. □

Let us go back to the proof of Urysohn's Lemma. To check that $f : X \longrightarrow [0,1]$ is continuous we think of f as a function $f : X \longrightarrow \mathbb{R}$. Thus we have to check that the inverse images under f of the subbasic open sets $(-\infty, c)$ and (c, ∞) are open in X for all c. By the preceding lemma, we have

$$f^{-1}((-\infty, c)) = \bigcup_{p<c} U_p,$$

which is obviously open.

We also have

$$f^{-1}((c,\infty)) = f^{-1}(\mathbb{R} \setminus (-\infty, c]) = X \setminus f^{-1}((-\infty, c]).$$

Now by the preceding lemma

$$f^{-1}((-\infty, c]) = \bigcap_{p>c} U_p,$$

and it suffices to show that this is closed in X. This will follow if we can show that

$$\bigcap_{p>c} U_p = \bigcap_{p>c} Cl(U_p).$$

Clearly the left hand side is contained in the right hand side, i.e. $\bigcap_{p>c} U_p \subset \bigcap_{p>c} Cl(U_p)$. To show the reverse inclusion, let us choose for each $p > c$, another dyadic rational $\lambda(p)$ such that $c < \lambda(p) < p$. Then we have

$$Cl(U_{\lambda(p)}) \subseteq U_p,$$

from which follows

$$\bigcap_{p>c} Cl(U_p) \subseteq \bigcap_{p>c} Cl(U_{\lambda(p)}) \subseteq \bigcap_{p>c} U_p.$$

This conclude the proof of Urysohn's Lemma. □

Remark 3.1.8.

- *It is a common error to read Urysohn's Lemma as saying that*

$$A = f^{-1}(\{0\}) \qquad\qquad B = f^{-1}(\{1\}).$$

This however is wrong[2]*. All Urysohn's Lemma says is that*

$$A \subseteq f^{-1}(\{0\}) \qquad\qquad B \subseteq f^{-1}(\{1\}).$$

[2] In order to have equalities, the subsets A and B have to satisfy some additional conditions given in the next Lemma 3.1.9.

- *Conversely, if a T_1-space X has the property that for each pair A and B of disjoint closed subsets there exists such a function, then X is normal. Indeed the sets $U_0 = \left\{x \mid 0 \leq x < \frac{1}{2}\right\}$ and $V_0 = \left\{x \mid \frac{1}{2} < x \leq 1\right\}$ are two disjoint open subsets of $[0,1]$; therefore, since f is continuous, then the sets $U = f^{-1}(U_0)$ and $V = f^{-1}(V_0)$ are two disjoint open subsets of X such that $A \subset U$ and $B \subset V$.*

Let us give another lemma which will be useful in the proof of Urysohn's Metrization Theorem.

Lemma 3.1.9. *If S is a subset of a normal space X, then S is the preimage of a point under a continuous map $f : X \longrightarrow \mathbb{R}$ if and only if S is both closed and G_δ.*

Proof. First off all, let us suppose that X is normal, $f : X \longrightarrow \mathbb{R}$ is continuous and S a subset of X such that $S = f^{-1}(\{c\})$, $c \in \mathbb{R}$. Then S being the continuous preimage of the closed set $\{c\}$ is also closed. Furthermore, we have

$$\begin{aligned} S &= f^{-1}(\{c\}) \\ &= f^{-1}\left(\bigcap_{n=1}^{\infty}\left(c - \frac{1}{n}, c + \frac{1}{n}\right)\right) \\ &= \bigcap_{n=1}^{\infty} f^{-1}\left(\left(c - \frac{1}{n}, c + \frac{1}{n}\right)\right). \end{aligned}$$

i.e. S is G_δ. Conversely let us suppose that S is closed and G_δ. Then

$$S = \bigcap_{n=1}^{\infty} O_n,$$

for some open sets O_n. By Urysohn's Lemma, we can find continuous functions $g_n : X \longrightarrow [0,1]$, such that

$$S \subseteq g_n^{-1}(\{0\}) \quad \text{and} \quad X \setminus O_n \subseteq g_n^{-1}(\{1\}). \qquad \text{(see Remark 3.1.8)}.$$

Then we can define a continuous map into the Hilbert cube:

$$\begin{aligned} g : X &\longrightarrow \prod_{n=1}^{\infty} [0,1] \\ x &\longmapsto (g_1(x), g_2(x), g_3(x), \ldots). \end{aligned}$$

It is clear that the inverse image under g of $(0,0,0,\ldots)$ is S.

Now let us compose g with the continuous map

$$\tau : \prod_{n=1}^{\infty} [0,1] \longrightarrow \mathbb{R},$$

where τ denotes the distance from the point $(0,0,0,\ldots)$. Then

$$f = \tau \circ g$$

and clearly

$$\begin{aligned} f^{-1}(\{0\}) &= (\tau \circ g)^{-1}(\{0\}) \\ &= g^{-1}\left[\tau^{-1}(\{0\})\right] \\ &= g^{-1}(0,0,0,\ldots) \\ &= S. \end{aligned}$$

□

Remark 3.1.10. *In the proof of Urysohn's Metrization Theorem, we can choose $f : X \longrightarrow [0,1]$ rather than $\mathbb{R}$, with $f^{-1}(\{0\}) = S$.*

Remark 3.1.11. *The Urysohn's Metrization Theorem, which says that if the space X is T_4 and second countable then X is metrizable can now be stated in a stronger form and we have the following theorem.*

Theorem 3.1.12. *If the space X is T_3 and second countable, then X is metrizable.*

The connection between these two forms of Urysohn's Metrization Theorem is provided by the following theorem, due to Tychonoff.

3.1.2 Tychonoff's Theorem.

Theorem 3.1.13. *If the space X is T_3 and second countable, then X is T_4.*

Proof. Let X be a T_3-space which is second countable with a countable base $\mathscr{B} = \{B_i\}_{i\in\mathbb{Z}_+}$. Let A and B be two disjoint closed subsets of X. If $x \in A$, then $X \setminus B$ is an open set which contains x. Then, there is a neighborhood U of x such that $x \in U \subset X \setminus B$. Since X is T_3, then we can find a neighborhood V of x such that $Cl(V) \subset U$ (see Proposition A.0.17 in the Appendix). Finally, there is a member B_i of $\mathscr{B}$ containing x such that $B_i \subset V$. Now

$$Cl(B_i) \subset Cl(V) \subset U \subset X \setminus B. \quad \text{so } Cl(B_i) \bigcap B = \emptyset.$$

If we repeat this procedure for each $x \in A$, we will have a countable subcollection $\mathscr{C} = \{C_j\}_{j\in\mathbb{Z}_+}$ of $\mathscr{B}$ which covers A and which satisfies

$$Cl(C_j) \bigcap B = \emptyset \text{ for all } j.$$

Similarly, if $x \in B$, we can find a countable subcollection $\mathscr{D} = \{D_k\}_{k\in\mathbb{Z}_+}$ of $\mathscr{B}$ which covers B and which satisfies

$$Cl(D_k) \bigcap A = \emptyset \text{ for all } k.$$

It is clear that

$$A \subset \bigcup_{j\in\mathbb{Z}_+} C_j \qquad \text{and} \qquad B \subset \bigcup_{k\in\mathbb{Z}_+} D_k.$$

Let us notice that $\bigcup\limits_{j\in\mathbb{Z}_+} C_j$ and $\bigcup\limits_{k\in\mathbb{Z}_+} D_k$ can be considered as the neighborhoods of A and B respectively, but the problem is that their intersections may not be disjoint. For this reason, we need to construct new collections whose intersections will be disjoint. Thus, for each j and k, let us define

$$C'_j = C_j - \bigcup_{k=1}^{j} Cl(D_k) \qquad \text{and} \qquad D'_k = D_k - \bigcup_{j=1}^{k} Cl(C_j).$$

which are open since equal to an open set minus a closed set. Let us consider

$$W = \bigcup_{j\in\mathbb{Z}_+} C'_j \qquad \text{and} \qquad Z = \bigcup_{k\in\mathbb{Z}_+} D'_k.$$

Then W and Z are open sets since they are unions of open sets.

Let us suppose $a \in A$, then $a \in C_j$ for some j since $A \subset \bigcup\limits_{j\in\mathbb{Z}_+} C_j$, and

$$C'_j = C_j - \bigcup_{k=1}^{j} Cl(D_k).$$

But for all k, $Cl(D_k) \bigcap A = \emptyset$, this implies $a \notin Cl(D_k)$ for all k i.e. $a \notin \bigcup\limits_{k=1}^{j} Cl(D_k)$.

Finally, if $a \in A$, then $a \in C_j$ and $a \notin \bigcup\limits_{k=1}^{j} Cl(D_k)$ i.e. we have

$a \in C_j - \bigcup_{k=1}^{j} Cl(D_k) = C'_j \subset \bigcup\limits_{j\in\mathbb{Z}_+} C'_j = W$. Hence $A \subset W$.

Similarly, one can easily show that $B \subset Z$.

To conclude the proof, it remains to show that W and Z are disjoint.

Let us suppose that $x \in W \bigcap Z$, then $x \in C'_j \bigcap D'_k$ for some j and k. This implies that $x \in C'_j$ and $x \in D'_k$ for some j and k.

But $x \in C'_j \Longrightarrow x \in C_j$ and $x \in D'_k \Longrightarrow x \notin Cl(C_j) \supset C_j$. We have a contradiction.

Thus $W \bigcap Z = \emptyset$. Finally, W and Z are two disjoint open sets which contain A and B respectively. X is then T_4. □

We also have the criterion of metrizability of topological spaces with a weaker separation axiom together with some additional conditions.

Lemma 3.1.14. *A compact Hausdorff space is normal, i.e.*

if X is T_2 and compact, then X is T_4.

Proof. First of all, we show that

if X is T_2 and compact, then X is T_3.

Let us suppose that X is T_2 and compact, A a closed subset of X and $x \notin A$. X being T_2, then for each $a \in A$, there are disjoint open sets U_a and V_a containing x and a respectively. Then $\{V_a\}_{a\in A}$ is an open cover of A, which is also compact (being closed in X). Thus there is a finite subcover of $\{V_a\}_{a\in A}$ such that

$$A \subseteq \bigcup_{i=1}^{k} V_{a_i}.$$

Then $\bigcap_{i=1}^{k} U_{a_i}$ is an open set containing x which is disjoint from $\bigcup_{i=1}^{k} V_{a_i}$. Thus X is T_3.

We next prove that

if X is T_3 and compact, then X is T_4.

Let us suppose now X is T_3 and compact, A and B two disjoint closed subsets of X.Let $x \notin A$,then X being T_3, there are disjoint open sets U and V such that $x \in U$ and $A \subset V$. But for each $x \notin A$, the family $\{U_x\}_{x\in X\setminus A}$ such that $U_x \bigcap V = \emptyset$ is an open cover of B which is compact. Thus there is a finite subcover of $\{U_x\}_{x\in X\setminus A}$ such that

$$B \subseteq \bigcup_{k=1}^{p} U_{x_k}.$$

$\bigcup_{k=1}^{p} U_{x_k}$ and V are two open disjoint subsets of X containing B and A respectively.

Thus X is T_4 and this concludes the proof. □

Hence, by combining the above Lemma with the Theorem 3.1.1, we have the following result for the metrizability of Hausdorff spaces,

Theorem 3.1.15. *If X is a second countable compact Hausdorff space, then X is metrizable.*

Before giving the next result, lets us define another notion which is similar to compactness, but more closely related to second countability.

Definition 3.1.16. *A topological space X is said to be Lindelöf if every open cover of X has a countable subcover. A topological space X is said to be hereditarily[3] Lindelöf if every subspace of X is also Lindelöf.*

Then we have the following lemma.

Lemma 3.1.17. *A topological space X which is normal and hereditarily Lindelöf is perfectly normal i.e.*

If X is T_4 and hereditarily Lindelöf, then X is T_6.

[3] A property of a space is hereditary if each of its subspaces possesses this property.For example being second countable is a hereditary property.

Proof. Let A be a closed subset of a normal hereditarily Lindelöf space X. For each $x \in X \setminus A$, let us choose two open disjoint sets U_x and V_x containing x and A respectively. Then the family $\{V_x\}_{x \in X\setminus A}$ is an open cover of $X \setminus A$. But $X \setminus A$ is also Lindelöf , being a subpace of X which is hereditarily Lindelöf. Thus there is a countable subcover of $\{V_x\}_{x\in X\setminus A}$ such that

$$X \setminus A = \bigcup_{i=1}^{\infty} V_{x_i}.$$

Then

$$A = X \setminus \left(\bigcup_{i=1}^{\infty} V_{x_i} \right) = \bigcap_{i=1}^{\infty} U_{x_i},$$

Where the U_{x_i}'s are open. Thus A is G_δ i.e. X is T_6. □

From the above lemma, we notice that: We used the weaker T_3 separation axiom, instead of T_4, together with the hereditary Lindelöf condition to prove that all closed sets are G_δ. But let us recall that T_6 requires T_4 as part of the definition. However Tychonoff's Lemma, alluded to above actually shows that

If X is T_3 and Lindelöf, then X is T_4,

so that indeed T_4 can be replaced by T_3 in the above lemma.

Lemma 3.1.18. *Any second countable space X is hereditarily Lindelöf.*

Proof. Let X be a second countable space. Since any subspace of a second countable space is second countable, it suffices to show that X is Lindelöf.

Let $\{U_n\}_{n\in\mathbb{Z}_+}$ be a countable base for X and let $\{V_i\}_{i\in I}$ be any open cover of X. Let us next consider the subcollection $\left\{U_{n_j}\right\}_{j\in\mathbb{Z}_+}$ of the base consisting of those U_n's which are contained in some open set V_i, in the given open cover. Then for each such U_{n_j}, let us pick a particular $V_{i(n_j)}$ containing it (i.e. U_{n_j}). Then

$$X = \bigcup_{i\in I} V_i = \bigcup_{j=1}^{\infty} U_{n_j} \subseteq \bigcup_{j=1}^{\infty} V_{i(n_j)}.$$

It follows that $\left\{V_{i(n_j)}\right\}_{j\in\mathbb{Z}_+}$ is a countable subcover of X, then X is Lindelöf. □

Let us now prove the Urysohn Metrization Theorem :

Proof. Urysohn Metrization Theorem.

Let $\{U_n\}_{n\in\mathbb{Z}_+}$ be a countable base for the topology on X. Then by the above lemmas we have shown that

$$\begin{aligned} \text{If } X \text{ is } T_4 \text{ and second countable} &\Longrightarrow X \text{ is } T_4 \text{ and hereditarily Lindelöf} \\ &\Longrightarrow X \text{ is } T_6. \end{aligned}$$

Thus the closed sets $X \setminus U_n$ are G_δ and we can find continuous functions (see Lemma 3.1.9)

$$f_n : X \Longrightarrow [0,1]\,,$$

such that

$$f_n^{-1}(\{0\}) = X \setminus U_n.$$

Let us consider the function

$$\begin{aligned} f :& X \longrightarrow \prod_{n=1}^{\infty} [0,1] \\ & x \longmapsto (f_1(x), f_2(x), f_3(x), \ldots)\,. \end{aligned}$$

It is clear that f is continuous since each of its coordinate functions f_n is continuous.

f is bijective. Indeed, let $x, y \in X$ such that $x \neq y$. Then we can find a basic open set U_n such that $x \in U_n$ and $y \notin U_n$. Then $f_n(x) > 0$ while $f_n(y) = 0$. Thus $f(x) \neq f(y)$, since they differ in their n^{th} coordinates.

In order to show that the restriction of f

$$f : X \longrightarrow f(X)$$

is a homeomorphism, it suffices to show that for any open set $V \subset X$, the direct image $f(V)$ is open in $f(X)$. Since direct images take unions to unions, it suffices to prove this when V is one of the basic open sets U_m. Let us consider the projection

$$p_m : \prod_{n=1}^{\infty} [0,1] \longrightarrow [0,1]$$

into the m^{th} coordinate. Then we have

$$U_m = f_m^{-1}((0,1]) = (p_m \circ f)^{-1}((0,1]) = f^{-1}\left(p_m^{-1}((0,1])\right).$$

Applying f to this equation, we have

$$f(U_m) = f\left(f^{-1}\left(p_m^{-1}((0,1])\right)\right) = f(X) \bigcap p_m^{-1}((0,1])\,,$$

which is an open set in $f(X)$.

Thus X is homeomorphic to the subset $f(X)$ of the Hilbert cube $\prod\limits_{n=1}^{\infty} [0,1]$ which is metrizable. It follows that X is also metrizable and the topology on X arises from the metric $\bar{d}$ given by

$$\bar{d}(x,y) = d(f(x), f(y))\,, \text{ for x,y} \in X,$$

where d is the distance application on the Hilbert cube. This conclude the proof of Urysohn's Metrization Theorem. □

The Urysohn's Metrization Theorem gives only sufficient conditions for the metrizability of topological spaces, but does not give necessary conditions. The next metrization theorem, proved independently by Nagata and Smirnov in 1950, gives a complete answer to the metrization thoerem. But before giving the Nagata-Smirnov Metrization Theorem, let us first prove a theorem due to A.H. Stone, and after define the generalized Hilbert space, which will be used in the proof of Nagata-Smirnov Metrization Theorem.

Theorem 3.1.19. *[3].* **The Stone's Theorem**

Every open cover of a metrizable space has an open refinement which is both locally finite and σ-discrete i.e. every metrizable space is paracompact and has a σ-discrete refinement.

Proof. Let X be a metrizable space and $\{U_s\}_{s\in S}$ be an open cover of X. Let us take a metric d on X and a well-ordering relation $<$ on the set S. Let us define families $\mathcal{V}_i = \{V_{s,i}\}_{s\in S}$ of subsets of X by letting

$$V_{s,i} = \bigcup B(c, \frac{1}{2^i}),$$

where the union is taken over all points $c \in X$ satisfying the following conditions:

$*$ s is the smallest element of S such that $c \in U_s$, (1)

$*$ $c \notin V_{t,j}$ for $j < i$ and $t \in S$, (2)

$*$ $B(c, \frac{3}{2^i}) \subset U_s$. (3)

It follows from the definition that the sets $V_{s,i}$ are open, and (3) implies that $V_{s,i} \subset U_s$.

Let x be a point of X; let us take the smallest $s \in S$ such that $x \in U_s$ and a natural number i such tat $B(x, \frac{3}{2^i}) \subset U_s$. Clearly, we either have $x \in V_{t,j}$ for a $j < i$ and $t \in S$ or $x \in V_{s,i}$. Hence, the union $\mathcal{V} = \bigcup\limits_{i=1}^{\infty} \mathcal{V}_i$ is an open refinement of the cover $\{U_s\}_{s\in S}$.

We shall prove that for every i

$$\text{if } x_1 \in V_{s_1,i} \ , \ x_2 \in V_{s_2,i} \text{ and } s_1 \neq s_2, \text{ then } d(x_1, x_2) > \frac{1}{2^i} \qquad (4),$$

and this will show that the families $\mathcal{V}_i$ are discrete, because every $\frac{1}{2^i}$-ball meets at most one member of $\mathcal{V}_i$.

Let us assume that $s_1 < s_2$. By definition of $V_{s_1,i}$ and $V_{s_2,i}$ there exist points c_1 and c_2 satisfying $(1)-(3)$ such that $x_k \in B(c_k, \frac{1}{2^i}) \subset V_{s_k,i}$ for $k = 1, 2$. From (3) it follows that $B(c_1, \frac{3}{2^i}) \subset U_{s_1}$ and from (1) we see that $c_2 \notin U_{s_1}$, so that $d(c_1, c_2) \geq \frac{3}{2^i}$. Hence,

$$d(c_1, c_2) \leq d(c_1, x_1) + d(x_1, x_2) + d(x_2, c_2) \ i.e.$$
$$d(x_1, x_2) \geq d(c_1, c_2) - d(c_1, x_1) - d(c_2, x_2) > \frac{1}{2^i},$$

which proves (4).

To conclude the proof of the theorem, it suffices to show that for every $t \in S$ and any pair k, j of natural numbers

$$\text{if } B(x, \frac{1}{2^k}) \subset V_{t,j} \text{ , then } B(x, \frac{1}{2^{i+k}}) \bigcap V_{s,i} = \emptyset \text{ for } i \geq j + k \text{ and } t \in S, \qquad (5)$$

because for every $x \in X$ there exist j, k and t such that $B(x, \frac{1}{2^k}) \subset V_{t,j}$ and thus the ball $B(x, \frac{1}{2^{j+k}})$ meets at most $j + k - 1$ members of $\mathcal{V}$ and this will show that the families $\mathcal{V}_i$ are locally finite.

It follows from (2) that the points c in the definition of $V_{s,i}$ do not belong to $V_{t,j}$ whenever $i \geq j + k$; since $B(x, \frac{1}{2^k}) \subset V_{t,j}$, we have $d(x, c) \geq \frac{1}{2^k}$ for any such c. The inequalities $j + k \geq k + 1$ and $i \geq k + 1$ imply that $B(x, \frac{1}{2^{j+k}}) \bigcap B(c, \frac{1}{2^i}) = \emptyset$, and this yields (5). □

Let us now define the generalized Hilbert space[4]

Definition 3.1.20. *Let τ be an infinite cardinal number. The generalized Hilbert space of weight τ, H^τ, is described as follows:*

Let A be an index set of cardinal τ. Then H^τ consists of all functions $x : A \longrightarrow \mathbb{R}$ such that

a) $x(a) = x_a \neq 0$ *for at most countably many* $a \in A$,

b) $\sum_{a \in A} x_a^2$ *converges.*

Note that the sum (b) makes sense, since it is really a countable sum. The distance function in H^τ is defined, just as it was in the Hilbert space H, by

$$d(x, y) = \sqrt{\sum (x_a - y_a)^2}. \text{ [14]}$$

3.2 Nagata-Smirnov Metrization Theorem.

Theorem 3.2.1. *[14] , [10]. Let X be a topological space. Then the following two conditions are equivalent:*

(a) *X is metrizable.*

(b) *X is T_3 and has a σ-locally finite base.*

Proof. Let X be a metrizable space and for each $n \in \mathbb{N}$, let $\mathscr{C}_n = \{B(x, \frac{1}{2^n}) | x \in X\}$ the open cover of X by $\frac{1}{2^n}$-balls about x. Then, according to the Stone's Theorem, X is paracompact. Let $\mathcal{V}_n$ be a locally finite open refinement of $\mathscr{C}_n$. One can easily verify that the family $\mathcal{D} = \bigcup_{k=1}^{\infty} \mathcal{V}_k$ is a σ-locally finite base for X. Since every metric space is T_3, then necessity is proved.

[4]The real Hilbert space H is the collection of all real sequences $x = (x_1, x_2, \ldots)$ such that $\sum x_k^2 < \infty$, with the metric $d(x, y) = \sqrt{\sum (x_k - y_k)^2}$.

Conversely, let us consider a T_3-space which has a σ-locally finite base $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$, where the $\mathcal{B}_n$'s are locally finite. It is apparent that X is paracompact, since every open cover of X has a σ-locally finite refinement consisting of basis elements, and hence X is normal. (see Theorem A.0.24 in the Appendix.)

Next we show that X is perfectly normal. Let G be open in X. By regularity, for each $x \in G$, there is a basis element B_x such that $Cl(B_x) \subset G$. Let

$$B_n = \bigcup \{Cl(B_x) |\ B_x \in \mathcal{B}_n\}.$$

Then B_n is the union of a locally finite collection of closed sets and hence is closed, and $G = \bigcup_{n=1}^{\infty} B_n$. Thus every open set in X is F_σ, so X is perfectly normal.

Now each basis element $B_{n\alpha}$ has the property that for some continuous function $f_{n\alpha} : X \longrightarrow [0,1]$,

$$B_{n\alpha} = \{x \in X |\ f_{n\alpha}(x) \neq 0\}, \text{by perfect normality.}$$

Let τ be the cardinal number of the base $\mathcal{B}$, and let H^τ be the generalized Hilbert space of weight τ; we can use the pairs n, α as the index set A in the definition of H^τ. Let us define $F : X \longrightarrow H^\tau$ by giving coordinate functions $F_{n\alpha}(x) = [F(x)]_{n\alpha}$ as follows:

$$F_{n\alpha}(x) = \frac{1}{(\sqrt{2})^n} \frac{f_{n\alpha}(x)}{\sqrt{1 + \sum_\beta f_{n\beta}^2(x)}}.$$

The denominator here makes sense because for any x in X, $x \in B_{n\alpha}$ for only finitely many $B_{n\alpha} \in \mathcal{B}_n$, so that $f_{n\alpha}(x) \neq 0$ for only finitely many α, if n is fixed. This also shows that $F_{n\alpha}(x) \neq 0$ for only countably many pairs n, α. Since

$$\sum_\alpha F_{n\alpha}^2(x) < \frac{1}{2^n},$$

we find that

$$\sum_{n,\alpha} F_{n\alpha}^2(x) < \sum_n \frac{1}{2^n} = 1,$$

so that $F(x)$ is indeed an element of H^τ. We claim F is a homeomorphism of X with a subset of H^τ.

First, if $x \neq y$ in X, then for some $B_{n\alpha} \in \mathcal{B}$, $x \in B_{n\alpha}$ and $y \notin B_{n\alpha}$. Then $f_{n\alpha}(x) \neq 0$ and $f_{n\alpha}(y) = 0$, from which it follows that $F_{n\alpha}(x) \neq F_{n\alpha}(y)$, and hence $F(x) \neq F(y)$.
Thus F is bijective.

Let us now prove that F is continuous. First, note that each $F_{n\alpha}$ is continuous as a map of X into $\mathbb{R}$. Now let $x_0 \in X$ and $\epsilon > 0$ be given. Choose N so large such that

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\epsilon^2}{4}.$$

Now let U be a neighborhood of x_0 meeting only finitely many $B_{n\alpha}$ for $n \leq N$; say, U meets $B_{n_1\alpha_1}, B_{n_2\alpha_2}, \ldots, B_{n_k\alpha_k}$. Let $V \subset U$ be a neighborhood of x_0 such that for $x \in V$,

$$\mid F_{n_i\alpha_i}(x) - F_{n_i\alpha_i}(x_0) \mid < \frac{\epsilon}{\sqrt{2k}},$$

for $i = 1, \ldots, k$. Now for $x \in V$ and any pair n, α other than n_i, α_i for $i = 1, \ldots, k$, we have $F_{n\alpha}(x) = F_{n\alpha}(x_0) = 0$. Hence, for $x \in V$,

$$\sum_{n \leq N} \sum_{\alpha} \mid F_{n\alpha}(x) - F_{n\alpha}(x_0) \mid^2 = \sum_{i=1}^{k} \mid F_{n_i\alpha_i}(x) - F_{n_i\alpha_i}(x_0) \mid^2 < \frac{\epsilon^2}{2}.$$

But we also have

$$\begin{aligned} \sum_{n > N} \sum_{\alpha} \mid F_{n\alpha}(x) - F_{n\alpha}(x_0) \mid^2 &\leq \sum_{n>N} \sum_{\alpha} \left(F_{n\alpha}^2(x) + F_{n\alpha}^2(x_0) \right) \\ &< \sum_{n>N} \left(\frac{1}{2^n} + \frac{1}{2^n} \right) = 2 \sum_{n>N} \frac{1}{2^n} < \frac{\epsilon^2}{2} \end{aligned}$$

by choice of N. It follows that, for $x \in V$,

$$\sum_{n,\alpha} \mid F_{n\alpha}(x) - F_{n\alpha}(x_0) \mid^2 < \epsilon^2.$$

Hence, for $x \in V$, $d(F(x), F(x_0)) = \sqrt{\sum_{n,\alpha} \mid F_{n\alpha}(x) - F_{n\alpha}(x_0) \mid^2} < \epsilon$, proving the continuity of F.

Finally, we show that F is closed, i.e. the direct image by F of a closed set A is closed. If A is closed in X, we assert $F(A) = Cl(F(A))$. Indeed, let us suppose $F(x) \notin F(A)$ i.e. $x \notin A$. Then for some $n\alpha, x \in B_{n\alpha}$ and $B_{n\alpha} \bigcap A = \emptyset$. Hence $f_{n\alpha}(x) \neq 0$ and $f_{n\alpha}(A) = 0$. It follows that $F_{n\alpha}(x) \neq 0$ and $F_{n\alpha}(A) = 0$ and then, obviously, $d(F(x), F(A)) > 0$ so that $F(x) \notin Cl(F(A))$. Thus $Cl(F(A)) \subset F(A)$. Furthermore, we always have $F(A) \subset Cl(F(A))$. So $F(A) = Cl(F(A))$ i.e. $F(A)$ is closed.

The homeomorphism F imbeds X into a subspace of the generalized Hilbert Space H^τ which is metrizable, then X is metrizable. □

Remark 3.2.2. *Any finite collection of sets is evidently locally finite. Thus any countable collection of sets is σ-locally finite. Therefore a second countable space has a σ-locally finite base, and thus the Urysohn Metrization Theorem is subsumed by the Nagata-Smirnov Metrization Theorem.*

There is a variant of the Nagata-Smirnov Metrization Theorem, proved independently by R.H. Bing in 1951:

3.3 Bing's Metrization Theorem

Theorem 3.3.1. *The following two conditions on a topological space are equivalent:*

(a) X *is metrizable.*

(b) X *is* T_3 *and has a* σ*-discrete base.*

Proof. Necessity follows from the Stone theorem. If $\mathscr{B}_i$ is an open σ-discrete refinement of the open cover of X consisting of all $\frac{1}{i}$-balls about $x \in X$, then $\mathscr{B} = \bigcup_{i=1}^{\infty} \mathscr{B}_i$ is a σ-discrete base for X. (Recall that every metric space is T_3.)

Sufficiency follows from the fact that a σ-discrete family is σ-locally finite and the Nagata-Smirnov Metrization Theorem implies that X is metrizable. □

We shall now introduce some notions related to the notion of a cover, which will be used to prove the next metrization theorems.

Definition 3.3.2. *A normal sequence in a space* X *is a sequence* $\mathscr{U}_1, \mathscr{U}_2, \ldots$ *of open covers of* X *such that* $\mathscr{U}_{n+1}$ *star-refines* $\mathscr{U}_n$*, for* $n = 1, 2, \ldots$*.*

It will be called compatible normal sequence in X *if and only if* $\{St\,(x, \mathscr{U})\,|n = 1, 2, \ldots\}$ *is a neighborhood base at* x*, for each* $x \in X$*.*

Any open cover of X *which is* $\mathscr{U}_1$ *in some normal sequence in* X *will be called a normal cover. (Thus, every cover in a normal sequence is a normal cover.)*

3.4 Pseudometrization Theorem.

All the previous metrization theorems are satisfied only for topological spaces which are at least regular. But what about T_0-spaces? The following theorem answers that question:

Theorem 3.4.1. *[14]. A topological space* X *is pseudometrizable*[5] *if and only if it has a compatible normal sequence. (Hence, a* T_0*-space is metrizable if and only if it has a compatible normal sequence).*

Proof. If X is pseudometrizable, its topology is generated by the pseudometric ρ. Let us define $\mathscr{U}_n = \left\{B_\rho(x, \frac{1}{3^n})|x \in X\right\}$. Then the sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$ is a compatible normal sequence in X. The sets $St(x, \mathscr{U}_n)$ form a neighborhood base at x, for each x in X. It is also clear that

$$St\left(B_\rho(x, \frac{1}{3^n}), \mathscr{U}_n\right) \subset B_\rho(x, \frac{1}{3^{n-1}}),$$

[5]A pseudometric is a real-valued function ρ on X satisfying $\rho(x,y) \geq 0$, $\rho(x,x) = 0$, $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for every $x, y, z \in X$. A pseudometric is not required to satisfy $\rho(x,y) > 0$ for $x \neq y$ like a metric.

so that $\cdots < \mathscr{U}_3^* < \mathscr{U}_2^* < \mathscr{U}_1$ i.e. $\mathscr{U}_{n+1}$ star-refines $\mathscr{U}_n$, for $n = 1, 2, \ldots$.

Conversely, Suppose we have a compatible normal sequence $(\mathscr{U}_n)$ for X. Define t on $X \times X$ as follows:

$$\begin{aligned} t(x,y) &= 0 && \text{if } y \in St(x, \mathscr{U}_n) \text{ for all n,} \\ t(x,y) &= 1 && \text{if } y \notin St(x, \mathscr{U}_1), \\ t(x,y) &= \frac{1}{2} && \text{if } y \in St(x, \mathscr{U}_1)\ ,\ y \notin St(x, \mathscr{U}_2), \\ t(x,y) &= \frac{1}{2^n} && \text{if } y \in St(x, \mathscr{U}_n)\ ,\ y \notin St(x, \mathscr{U}_{n+1}). \end{aligned}$$

Now for $x, y \in X$, let $\mathscr{S}(x,y)$ be all finite sequences $s = (x_1, \ldots, x_n)$ of points of X such that $x_1 = x, x_n = y$ or $x_1 = y, x_n = x$. Define

$$\rho(x,y) = \inf \left\{ \sum_{i=2}^{n} t(x_{i-1}, x_i) |\, \{x_1, \ldots, x_n\} \in \mathscr{S}(x,y) \right\}.$$

Then ρ is a pseudometric on X. Indeed, for all $x, y \in X$, $\rho(x,y) \geq 0$ and $\rho(x,y) = 0$ if $y \in St(x, \mathscr{U}_n)$ for all n, and in particular when $x = y$, we have $\rho(x,x) = 0$. Furthermore, for all $x, y \in X$, $\rho(x,y) = \rho(y,x)$ by the definition of $\mathscr{S}(x,y)$ and t. Moreover, ρ satisfies the triangle inequality, because for $x, y \in X$, then given any $\epsilon > 0$, there exist $x_1 = x$ and $x_n = y$ such that

$$\sum_{i=2}^{n} t(x_{i-1}, x_i) \leq \frac{\epsilon}{2} + \rho(x,y), \text{ this is by the definition of the infinimum.}$$

Using again the same argument for $y, z \in X$, there exist $x_1 = y$ and $x_n = z$ such that $\sum\limits_{i=2}^{n} t(x_{i-1}, x_i) \leq \frac{\epsilon}{2} + \rho(y,z)$. Finally,

$$\begin{aligned} \rho(x,z) &= \inf \left\{ \sum_{i=2}^{n} t(x_{i-1}, x_i) |\, \{x_1, \ldots, x_n\} \in \mathscr{S}(x,y) \right\} \\ &\leq \sum_{i=2}^{n} t(x_{i-1}, x_i) + \sum_{i=2}^{n} t(x_{i-1}, x_i) \\ &\leq \frac{\epsilon}{2} + \rho(x,y) + \frac{\epsilon}{2} + \rho(y,z). \end{aligned}$$

Then, by taking the limit when ϵ goes to zero, we have $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$.

It remains to show that ρ is compatible with the topology on X.

Let $\mathscr{V}_n$ be the cover of X by balls $B_\rho(x, \frac{1}{2^n})$. It will suffice to show that, for any n,

a) $\mathscr{U}_n < \mathscr{V}_{n-1}$,

b) $\mathscr{V}_n < \mathscr{U}_{n-1}$,

since it will then be clear that the topologies generated by the two sequences are the same.

a) Suppose $U \in \mathscr{U}_n$. Pick $x \in U$. If $y \in U$, then $y \in St(x, \mathscr{U}_n)$ so $t(x, y) \leq \frac{1}{2^n}$ and hence $\rho(x, y) \leq \frac{1}{2^n} < \frac{1}{2^{n-1}}$. Thus $y \in B_\rho(x, \frac{1}{2^{n-1}})$, so $U \subset B_\rho(x, \frac{1}{2^{n-1}})$ and thus $\mathscr{U}_n < \mathscr{V}_{n-1}$.

b) To show that $\mathscr{V}_n < \mathscr{U}_{n-1}$, it is enough to prove that whenever $\rho(x, y) < \frac{1}{2^n}$, then x and y lie together in some element of $\mathscr{U}_n$, since then

$$B_\rho(x, \frac{1}{2^n}) \subset St(x, \mathscr{U}_n) \subset U,$$

for some $U \in \mathscr{U}_{n-1}$.

Hence, suppose $\rho(x, y) < \frac{1}{2^n}$. Then

$$\inf_{s \in \mathscr{S}(x,y)} \sum_{i=2}^{k} t(x_{i-1}, x_i) < \frac{1}{2^n},$$

and consequently, for some sequence $\{x_1, \ldots, x_n\}$ from $\mathscr{S}(x, y)$,

$$\sum_{i=2}^{k} t(x_{i-1}, x_i) < \frac{1}{2^n}.$$

We proceed now by induction on the length k of this sequence. If $k = 2$, then $t(x, y) < \frac{1}{2^n}$ so that $y \in St(x, \mathscr{U}_m)$, $y \notin St(x, \mathscr{U}_{m+1})$ for some $m > n$. Hence, in particular, $y \in St(x, \mathscr{U}_{n+1})$, from which it follows that $x, y \in U$ for some $U \in \mathscr{U}_{n+1}$ in fact, so that certainly x, y lie together in some $U' \in \mathscr{U}_n$. (Recall, then, that if $t(x, y) < \frac{1}{2^n}$, we have x and y together in some element of $\mathscr{U}_{n+1}$; we will use this again.)

Suppose the result is true for sequences of length $< k$, and suppose $\sum_{i=2}^{k} t(x_{i-1}, x_i) < \frac{1}{2^n}$. Let j be the last number, $2 \leq j \leq k$, such that

$$\sum_{i=2}^{j} t(x_{i-1}, x_i) < \frac{1}{2^{n+1}}.$$

Then

$$\sum_{i=2}^{j+1} t(x_{i-1}, x_i) \geq \frac{1}{2^{n+1}}$$

so that

$$\sum_{i=j+2}^{k} t(x_{i-1}, x_i) < \frac{1}{2^{n+1}}.$$

Now by the inductive hypothesis x_1, x_j lie in some $U_1 \in \mathscr{U}_{n+1}$ while the argument above shows, since $t(x_j, x_{j+1}) < \frac{1}{2^n}$, that x_j, x_{j+1} lie in some $U_2 \in \mathscr{U}_{n+1}$, and finally, using the inductive hypothesis again, x_{j+1}, x_k lie in some $U_3 \in \mathscr{U}_{n+1}$. Then x_1 and x_k lie in $St(U_2, \mathscr{U}_{n+1}) \subset U$ for some $U \in \mathscr{U}_n$. This establishes our claim, by induction. X is then pseudometrizable. ☐

We use the above theorem to prove the following metrization theorem:

3.5 Neighborhood Metrization Theorem

Others criteria for a T_0-space to be metrizable are given by the following theorem:

Theorem 3.5.1. *[14]. A T_0-space X is metrizable if and only if each $x \in X$ possesses a countable neighborhood base $\{U_{xn} | n \in \mathbb{N}\}$ with the following properties:*

a) $y \in U_{xn} \Longrightarrow U_{yn} \subset U_{xn-1}$.

b) $y \notin U_{xn-1} \Longrightarrow U_{yn} \bigcap U_{xn} = \emptyset$.

Proof. Let X be a T_0 metrizable space, and let d be the distance defined on X.

Necessity is verified, since the properties (a) and (b) are obviously satisfied if we consider U_{xn} as the disk of radius $\frac{1}{2^n}$ about x. Indeed

a) If $y \in U_{xn} \Longrightarrow d(x, y) < \frac{1}{2^n} < \frac{1}{2^{n-1}}$. This means $y \in U_{xn-1}$ i.e. $U_{xn} \subset U_{xn-1}$.

b) If $y \notin U_{xn-1} \Longrightarrow d(x, y) \geq \frac{1}{2^{n-1}} > \frac{1}{2^n}$ i.e. $x \notin U_{yn}$ and $y \notin U_{xn}$ i.e. $U_{yn} \bigcap U_{xn} = \emptyset$.

Conversely, let $\mathscr{U}_n = \{U_{xn} | x \in X\}$ be a neighborhood base for each $x \in X$ satisfying properties (a) and (b). We claim that $St(U_{xn}, \mathscr{U}_n) \subset U_{xn-2}$, for any $n > 2$.

Indeed, suppose $U_{zn} \bigcap U_{xn} \neq 0$. Then by property (b), $z \in U_{xn-1}$. Hence, by property (a), $U_{zn} \subset U_{xn-2}$, and thus $St(U_{xn}, \mathscr{U}_n) \subset U_{xn-2}$ as asserted. It now follows that $\mathscr{U}_n$ star-refines $\mathscr{U}_{n-2}$ for any $n > 2$, so that $\mathscr{U}_1, \mathscr{U}_3, \ldots$ is a normal sequence. It also follows that $St(x, \mathscr{U}_n) \subset U_{xn-2}$ for any $n > 2$, so that $\mathscr{U}_1, \mathscr{U}_3, \ldots$ is compatible with X. Thus, by the above theorem (Theorem 3.4.1), X is metrizable. □

The next metrization theorem due to A.H. Frink is a direct consequence of the previuos Neighborhood Metrization Theorem.

3.6 Frink's Metrization Theorem

Theorem 3.6.1. *A T_1-space X is metrizable if and only if there is a neighborhood base $\{U_{xn} | n \in \mathbb{N}\}$ at each $x \in X$ such that:*

i) $\cdots \subset U_{x2} \subset U_{x1}$.

ii) for each $n \in \mathbb{N}$, there is some $m > n$ such that $U_{xm} \bigcap U_{ym} \neq \emptyset \Longrightarrow U_{xm} \subset U_{yn}$.

Proof. Necessity. Let X a metrizable space. Then one can take $U_{xn} = \left\{B(x, \frac{1}{2^n}) | x \in X\right\}$ and U_{xn} is a neighborhood base at $x \in X$ satisfying properties (i) and (ii) of the theorem.

Sufficiency. The property (ii) follows directly from properties (b) and (a) of the Neighborhood Metrization Theorem which give the metrizability of X. Indeed if $U_{xm} \bigcap U_{ym} \neq \emptyset$, then by (b)

we have $x \in U_{ym-1}$ and by (a) $U_{xm} \subset U_{ym-2} \subset U_{yn}$ for some n such that $m > n$ (because of the property (i)). □

We introduce now an idea which is obviously related to the notion of a compatible normal sequence which will be used in the next metrization theorem.

Definition 3.6.2. *A development for a space X is a sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$ of open covers of X such that $\mathscr{U}_n$ refines $\mathscr{U}_{n-1}$, and, at each $x \in X$, $\{St(x, \mathscr{U}_n)|n = 1, 2, \ldots\}$ is a neighborhood base.*

A space having a development is called *developable*. For example, a regular space having a development is called *Moore space*.

The following theorem on metrizability of developable spaces is due to *Alexandroff* and *Urysohn* in 1923. (See [14].)

3.7 Alexandroff-Urysohn's Metrization Theorem

Theorem 3.7.1. *A T_0-space X is metrizable if and only if it has a development $\mathscr{U}_1, \mathscr{U}_2, \ldots$ with the additional property that whenever $U, V \in \mathscr{U}_n$ and $U \bigcap V \neq \emptyset$, then $U \bigcup V \subset W$ for some $W \in \mathscr{U}_{n-1}$.*

Proof. Necessity. If X is a metrizable space, then one can take $\mathscr{U}_n$ as the collection of $\frac{1}{4^n}$-balls i.e. $\mathscr{U}_n = \left\{B(x, \frac{1}{4^n})|x \in X\right\}$.

To prove sufficiency, we use the Neighborhood Metrization Theorem above. Let $\mathscr{U}_1, \mathscr{U}_2, \ldots$ be a development of X with the required property. Then, for each $n > 1$, we find that if $U \in \mathscr{U}_n$ and $x \in U$, then $St(U, \mathscr{U}_n) \subset St(x, \mathscr{U}_{n-1})$. Now for $n = 1, 2, \ldots$ and $x \in X$, define $U_{xn} = St(x, \mathscr{U}_n)$. Then we need only to verify properties $a)$ and $b)$ of the Neighborhood Metrization Theorem.

a) If $y \in U_{xn}$, then for some $V \in \mathscr{U}_n$, $x \in V$ and $y \in V$. But then

$$U_{yn} = St(y, \mathscr{U}_n) \subset St(V, \mathscr{U}_n) \subset St(x, \mathscr{U}_{n-1}) = U_{xn-1} \text{ i.e. } U_{yn} \subset U_{xn-1}.$$

b) If $U_{yn} \bigcap U_{xn} \neq \emptyset$, then for some $U, V \in \mathscr{U}_n$, $U \bigcap V \neq \emptyset$. But then $U \bigcup V \subset W$ for some $W \in \mathscr{U}_{n-1}$, and hence $y \in St(x, \mathscr{U}_{n-1}) = U_{xn-1}$. Thus if $y \notin U_{xn-1}$, then $U_{yn} \bigcap U_{xn} = \emptyset$. Thus, X is metrizable. □

Still in order to extend the class of metrizable spaces, we need another new terminology: the collectionwise normality, which together with the notion of development, will be used to establish further topological characterizations of the class of metrizable spaces. Collectionwise normality is another extension of normality, weaker than paracompactness.

Definition 3.7.2. *A topological space X is said to be collectionwise normal if X is a T_1-space and for every discrete family $\{F_s\}_{s\in S}$ of closed subsets of X, there exists a discrete family $\{V_s\}_{s\in S}$ of open subsets of X such that $F_s \subset V_s$ for every $s \in S$.*

It is clear that every collectionwise normal space is normal.

Theorem 3.7.3. *A T_1-space X is collectionwise normal if and only if for every discrete family $\{F_s\}_{s\in S}$ of closed subsets of X, there exists a discrete family $\{U_s\}_{s\in S}$ of open subsets of X such that $F_s \subset U_s$ for every $s \in S$ and $U_s \bigcap U_{s'} = \emptyset$ whenever $s \neq s'$.*

Proof. Necessity follows directly from the definition of collectionwise normality. To prove sufficiency, it suffices to show that any T_1-space X satisfying the given condition in the theorem is collectionwise normal. Let us suppose that these conditions are satisfied, then it is clear that X is normal, so that for a discrete family $\{F_s\}_{s\in S}$ of closed subsets of X, there is a family of pairwise disjoint open sets $\{U_s\}_{s\in S}$. Let us put $A = \bigcup\limits_{s\in S} F_s$ and $B = X \setminus \bigcup\limits_{s\in S} U_s$. Then the closed sets A and B are disjoint. Indeed, since we have $F_s \subset U_s$, then $\bigcup\limits_{s\in S} F_s \subset \bigcup\limits_{s\in S} U_s$ which means that $(X \setminus \bigcup\limits_{s\in S} U_s) \subset (X \setminus \bigcup\limits_{s\in S} F_s)$. This last inclusion implies $(X \setminus \bigcup\limits_{s\in S} U_s) \bigcap (\bigcup\limits_{s\in S} F_s) = \emptyset$. By the normality of X, we can find disjoint open sets U and V such that $A = \bigcup\limits_{s\in S} F_s \subset U$ and $B = (X \setminus \bigcup\limits_{s\in S} U_s) \subset V$. One can check that the family $\{V_s\}_{s\in S}$ where $V_s = U_s \bigcap U$ is discrete and contains F_s for every $s \in S$. ☐

Remark 3.7.4. *A collectionwise normal space can be defined as a space such that any discrete family of closed sets can be separated by a discrete family of open sets.*

We can now prove the metrisation theorem for Moore spaces due to Bring in 1951:

3.8 Bing's Metrization Theorem for Moore Spaces

Theorem 3.8.1. *[3]. A topological space is metrizable if and only if it is collectionwise normal and has a development; or in other words, a topological space is metrizable if and only if it is a collectionwise normal Moore space.*

Proof. Necessity. Let X be a metrizable space. Then, according to the Stone theorem, X is paracompact. The collectionwise normality of X is given by the following theorem:

Theorem 3.8.2. *Every paracompact space is collectionwise normal.*

Proof. Let $\{F_s\}_{s\in S}$ be a discrete family of closed subsets of a paracompact space X. For every $x \in X$, let us choose a neighborhood U_x of the point x whose closure meets at most one set F_s. Then X being paracompact, the open cover $\mathscr{U} = \{U_x\}_{x\in X}$ of X has a locally finite open refinement $\mathscr{W}$. For every $s \in S$, let $V_s = X \setminus \bigcup\{\overline{W} : W \in \mathscr{W}\ and\ \overline{W} \bigcap F_s = \emptyset\}$. Clearly, we have $F_s \subset V_s$. Indeed, suppose $y \notin V_s$ then $y \in \bigcup \{\overline{W} : W \in \mathscr{W}\ and\ \overline{W} \bigcap F_s = \emptyset\}$ which means that there is some $W \in \mathscr{W}$ suth that $y \in \overline{W}$ and $\overline{W} \bigcap F_s = \emptyset$ i.e. $y \notin F_s$. To conclude the proof, it suffices to show that every $W \in \mathscr{W}$ meets at most one element of the family $\{V_s\}_{s \in S}$. This, however, follows from the fact that $\overline{W}$ meets at most one set F_s. This establishes the collectionwise normality of X. □

Furthermore, the family $\mathscr{B}_n = \{B(x, \frac{1}{2^n}) | x \in X\}$ is a development of X.

Sufficiency. Let us consider a collectionwise normal space X having a development $\mathscr{B}_1, \mathscr{B}_2, \ldots$ We first prove that that X is paracompact.

Let us consider an open cover $\{U_s\}_{s \in S}$ of X and take a well-ordering relation $<$ on the set S. Let $F_{s,i} = X \setminus \left[St(X \setminus U_s, \mathscr{B}_i) \bigcup (\bigcup\limits_{s' < s} U_{s'}) \right]$. For all $s \in S$ and $i = 1, 2, \ldots$, $F_{s,i}$ are closed because complement of open sets. Furthermore these closed sets $F_{s,i}$ form a cover of the space X. Indeed, taking for any $x \in X$ the smallest element $s_x \in S$ such that $x \in U_{s_x}$, and a natural number i_x such that[6] $St(x, \mathscr{B}_{i_x}) \subset U_{s_x}$ we have $x \in F_{s_x, i_x}$. Moreover, the family $\mathscr{F}_i = \{F_{s,i}\}_{s \in S}$ is discrete, since for a fixed i, the neighborhood $U_{s_x} \bigcap St(x, \mathscr{B}_i)$ of the point x meets only one member of $\mathscr{F}_i$, namely the set $F_{s_x, i}$. By the collectionwise normality of X, there exists a discrete family $\{U_{s,i}\}_{s \in S}$ of open sets such that $F_{s,i} \subset U_{s,i}$. Since $F_{s,i} \subset U_s$, we can choose $U_{s,i}$ such that $U_{s,i} \subset U_s$ for $s \in S$ and $i = 1, 2, \ldots$ and we will have $F_{s,i} \subset U_{s,i} \subset U_s$. Hence $\{U_{s,i}\}_{s \in S}$, $i = 1, 2, \ldots$, is a σ-discrete and obviously σ-locally finite open refinement of $\{U_s\}_{s \in S}$. Then X is paracompact.

Now, for $i = 1, 2, \ldots$, let $\mathscr{D}_i$ be a σ-locally finite open refinement of the cover $\mathscr{B}_i$ (which is the element of the development of X), then the family $\mathscr{D} = \bigcup\limits_{i=1}^{\infty} \mathscr{D}_i$ is a base for the space X and the metrizability of X follows from the Nagata-Smirnov Metrization Theorem. □

[6]Since $\{St(x, \mathscr{B}_i) | i = 1, 2, \ldots\}$ is a neighborhood base at each $x \in X$.

4. Conclusion

The theory of metrizability of topological spaces is distinguished by its long history and the diversity of methods that have been used in its study. In this work, we have studied some important criteria for metrizability of topological spaces. The first metrization theorem was given in 1923 by Alexandroff and Urysohn (Theorem 3.7.1). Two years later, Urysohn gave a proof of metrizability of second countable normal space (Theorem 3.1.1). Unfortunately, Urysohn Metrization Theorem does not give a total answer because it only gives sufficient condition for metrizability. Topologists wanted to get general criteria which are both necessary and sufficient to establish the metrizability of topological spaces, like in Alexandroff-Urysohn Metrization Theorem case. The search of such conditions was long and was not satisfactorily concluded until the early 1950's when Nagata (in 1950) and Smirnov (in 1951) independently provided such conditions (Theorem 3.2.1). R.H. Bing provided the similar result in 1951 independently of works of Nagata and Smirnov (Theorem 3.3.1). Theorem 3.8.1 of metrizability of Moore spaces is also due to Bing. We owe the Uniform Metrization Theorem 3.4.1 to Alexandroff and Urysohn, and the Neighborhood Metrization Theorem 3.5.1 is a slight modification of results of Nagata; while Theorem 3.6.1, which is a consequence of Theorem 3.5.1, is due to A.H. Frink.

We have studied only a few well-known results on metrizability of topological spaces and the theory of metrizability continues to grow up and to be an active area of research. The normal Moore space conjecture for example which states that every normal Moore space is metrizable depends on the choice of set theory. For instance, F.B. Jones showed, assuming the continuum hypothesis, that every separable normal Moore is metrizable. Furthermore, Health and Silver (See [14], P. 309) states that it is consistent with the axioms of set theory through the axiom of choice to assume the existence of a non-metrizable separable normal Moore space.

Appendix A. The Separation Axioms

The definition of a topological space is very general; not many interesting theorems can be proved about all topological spaces. In this part, we study a certain classes of topological spaces in which we discuss axioms of separation which concern the ways of separating points and/or closed sets in topological spaces (see [3] and [5]). There are various degrees of separation and the first separation axiom is the following.

T_0-spaces

Definition A.0.3. *A topological space (X, τ) is said to be T_0 if given any two distinct points x and y of X, there exists a neighborhood of at least one point which does not contain the other point. T_0-spaces are also called Kolmogorov spaces.*

Example A.0.4. *Let (X, τ) be a topological space, where $X = \{0, 1\}$.*

- *If $\tau = \{\emptyset, X\}$ then X is not T_0, since there is no way of separating 0 from 1 or 1 from 0 i.e. there is no neighborhood of 0 which does not contain 1 and vice versa.*

- *But if $\tau = \{\emptyset, \{1\}, X\}$, then X becomes a T_0-space, since there is a neighborhood $\{1\}$ of 1 which does not contain 0.*

The following proposition gives another characterization of a T_0-space.

Proposition A.0.5. *A space (X, τ) is T_0 if given any two distinct points x and y of X, either $x \notin Cl\{y\}$ or $y \notin Cl\{y\}$, where $Cl\{x\} = \overline{\{x\}}$ is the closure of $\{x\}$.*

Proof. Suppose X is a T_0-space, x and y any two distinct points of X, then there is an open set U which contains x, but not y. U being open in X, $X \setminus U$ is a closed subset of X which contains y, but not x. Since $\{y\} \subset X \setminus U$ and $X \setminus U$ is closed, then $Cl\{y\} \subset X \setminus U$[1]; hence $x \notin Cl\{y\}$ (because $X \setminus U$ does not contain x). Interchanging the role of x and y and using the same argument, we show that $y \notin Cl\{x\}$. ☐

Corollary A.0.6. *In a T_0-space (X, τ), every neighborhood of x or y is a neighborhood of both x and y if $Cl\{x\} = Cl\{y\}$.*

Proof. If x and y are distinct points of a T_0-space (X, τ) such that $Cl\{x\} = Cl\{y\}$ then, since we always have $x \in Cl\{x\}$, $y \in Cl\{y\}$ and considering the fact that $Cl\{x\} = Cl\{y\}$, we have $x \in Cl\{y\}$ and $y \in Cl\{x\}$. Suppose U is a neighborhood of x which does not contain y, then $X \setminus U$ is a closed set which contains y, but not x and $Cl\{y\} \subset X \setminus U$; hence $x \notin Cl\{y\}$, a contradiction. This means that our assumption was wrong; there is no neighborhood of x which does not contain y. Similarly, there is no neighborhood of y which does not contain x. ☐

[1] because $Cl\{y\}$ is the smallest closed set containing $\{y\}$

The following definition gives a slightly stronger separation property than the one given by T_0-spaces.

T_1-spaces

Definition A.0.7. *A topological space (X, τ) is said to be T_1 if given any two distinct points x and y of X, there exists a neighborhood of each of them which does not contain the other point. T_1-spaces are also called Fréchet spaces.*

Proposition A.0.8. *A topological space (X, τ) is T_1-space if and only if each one-point subset of X is closed.*

Proof. Let (X, τ) be a topological space and $x \in X$.Then the statement $\{x\}$ is closed in X is equivalent to the following statements:

* its complement $X \setminus \{x\}$ is open in X,
* each point $y \neq x$ has a neighborhood $(X \setminus \{x\})$ which does not contain x,
* X is a T_1-space. □

As consequence of the above proposition, we have the following result.

Corollary A.0.9. *If (X, τ) is T_1-space and $x \in X$, the $\{x\} = Cl\,\{x\}$.*

Example A.0.10. *Every metric space is T_1, because every one-point subset of a metric space is closed.*

Indeed, let (X, d) be any metric space and $x \in X$. Suppose $y \in X \setminus \{x\}$, then there is a ϵ- neighborhood of y which is disjoint from $\{x\}$ (otherwise y would be in $\{x\}$). Thus that ϵ-neighborhood of y lies completely in $X \setminus \{x\}$[2]. The point y being arbitrary chosen in $X \setminus \{x\}$, we say that $X \setminus \{x\}$ is a neighborhood of each of its points and then is open. Hence $\{x\}$ is closed.

Remark A.0.11. *Every T_1-space is a T_0-space, but not the inverse.*

There is a stronger separation property than the ones given by either T_0 or T_1-spaces due to Hausdorff.

T_2-spaces

Definition A.0.12. *A topological space (X, τ) is said to be T_2 if given any two distinct points x and y of X, there exist two open sets U and V such that $x \in U$, $x \in V$ and $U \bigcap V = \emptyset$. T_2-spaces are also called Hausdorff spaces.*

[2]since there $\epsilon > 0$ such that $B(y, \epsilon) \bigcap \{x\} = \emptyset$ and $B(y, \epsilon) \subset X \setminus \{x\}$, Cf Definition 2.1.7

Example A.0.13.

- *The set $\mathbb{R}$ of real numbers with the usual topology is a T_2-space.*

- *Suppose that the space (X, τ) is homeomorphic to the space (Y, τ') and that X is T_2, then Y also is T_2.*

Remark A.0.14. *Every T_2-space is a T_1-space.*

We now give some separation axioms which separate either a point from a closed subset or two closed subsets of a topological space.

T_3-spaces

Definition A.0.15. *A topological space (X, τ) is said to be T_3 if it is T_1 and given any closed subset A of X and any point x of X which is not in A, there exist two disjoint open sets U and V such that $x \in U$ and $A \subset V$. T_3-spaces are sometimes called Regular spaces.*

Remark A.0.16. *Every T_3-space is a T_2-space.*

Proposition A.0.17. *A topological space (X, τ) is T_3 if and only if given any $x \in X$ and any neighborhood U of x, there exists a neighborhood V of x such that $Cl(V) \subset U$, where $Cl(V)$ denotes the closure of V.*

Proof. Let us suppose that (X, τ) is T_3 and U an open neighborhood of x in X. Then the complement $X \setminus U$ of X is a closed subset of X which does not contain x. Then according to the definition of T_3-space, there exist disjoint open sets W and V such that $X \setminus U \subset W$ and $x \in V$. Since $X \setminus U \subset W$, we have $X \setminus W \subset X \setminus (X \setminus U) = U$. Furtheremore, since $W \bigcap V = \emptyset$, we have $V \subset X \setminus W \subset U$. So $X \setminus W$ is a closed set of X which contains V. $Cl(V)$ being the smallest closed set containing V, we finaly have

$$V \subset Cl(V) \subset X \setminus W \subset U.$$

Suppose now given any neighborhood U of $x \in X$, there exists a neighborhood V of x such that $Cl(V) \subset U$. Let $x \in X$ and let F be any closed subset of X which does not contain x. Then $X \setminus F$ is an open subset of X containing x i.e. a neighborhood of x; hence there exists a neighborhood V of x such that $Cl(V) \subset X \setminus F$. This implies $F \subset X \setminus Cl(V)$, i.e. $X \setminus Cl(V)$ is an open set which contains F, and V is an open set which contains x. Since we always have $V \subset Cl(V)$, then

$$(X \setminus Cl(V)) \bigcap V = \emptyset.$$

We finally have: $F \subset X \setminus Cl(V)$, $x \in V$ and $(X \setminus Cl(V)) \bigcap V = \emptyset$ which is similar to say that X is T_3. □

Example A.0.18.

- $\mathbb{R}^n$ *is a regular space.*

- *The circle* $\mathscr{C}$ *is a regular space (as subspace of* $\mathbb{R}^n$*).*

- *The torus* $\mathscr{T}$ *is regular, since* $\mathscr{T} = \mathscr{C} \times \mathscr{C} \subset \mathbb{R}^3$.

One of the most important of the separation axioms is given by the following definition.

T_4-spaces

Definition A.0.19. *A topological space* (X, τ) *is said to be* T_4 *if it is* T_1 *and given any two disjoint closed subsets* A *and* B *of* X*, there exist two disjoint open sets* U *and* V *such that* $A \subset U$ *and* $B \subset V$*.* T_4*-spaces are sometimes called Normal spaces.*

Remark A.0.20. *Every* T_4*-space is a* T_3*-space.*

Example A.0.21.

- *Each topological space carrying the discrete topology is* T_4*, but every topological space with the trivial topology is not normal, since no one-point subset of* X *is closed.*

- *Every metric space is normal.*

We now give another characterization of T_4-space which is very important.

Proposition A.0.22. *A topological space* (X, τ) *is normal if and only if given any closed subset* F *of* X *and any open set* U *of* X *such that* $F \subset U$*, there exists an open set* V *such that*

$$F \subset V \subset Cl(V) \subset U.$$

Proof. Suppose (X, τ) is normal, F and U two subsets of X such that $F \subset U$, F closed and U open. Then $X \setminus U$ is a closed set of X and $(X \setminus U) \bigcap F = \emptyset$. Hence there are two disjoint open sets W and V such that $X \setminus U \subset W$ and $F \subset V$. This implies $F \subset V \subset Cl(V) \subset U$.

Suppose now given any closed set F and any open set U which contains F, there exists an open set V such that $F \subset V \subset Cl(V) \subset U$. Let A and B be any two disjoint closed subsets of X. Then $X \setminus A$ is an open set which contains B. By hypothesis, then there is an open set V such that $B \subset V \subset Cl(V) \subset X \setminus A$. Hence, $A \subset X \setminus Cl(V)$ i.e. $X \setminus Cl(V)$ is an open set which contains A and is disjoint from V. We finally have $B \subset V$, V open, $A \subset X \setminus Cl(V)$ and $(X \setminus Cl(V)) \bigcap V = \emptyset$. Therefore X is normal. □

The following statement gives us the condition for a subset of a normal space to be normal.[3]

[3]for $k = 0, 1, 2, 3$, every subspace of a T_k-space is T_k. But for T_4-spaces, it is not true. However, each closed subset of a T_4-space is T_4.

Proposition A.0.23. *If Y is a closed subset of a normal space X, then Y is normal.*

Proof. Y is T_1 because subspace of a normal space X which is T_1. Since Y is closed, then if F_1 and F_2 are disjoint closed subsets of Y, they are also disjoint closed subsets of X since $Y \subset X$. There exist then two disjoint open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

Then $F_1 \subset U \bigcap Y$, $F_2 \subset V \bigcap Y$ where $U \bigcap Y$ and $V \bigcap Y$ are two disjoint open subsets of Y. Thus Y is normal. ☐

Theorem A.0.24. *Every paracompact space is nornal.*

Proof. We first establish regularity. Let us suppose A is closed set in a paracompact space X and $x \notin A$. For each $y \in A$. We find an open set V_y containing y such that $x \notin Cl\,(V_y)$. Then the sets V_y, $y \in A$, together with the set $X \setminus A$, form an open cover of X. Let $\mathscr{W}$ be an open locally finite refinement and let $V = \bigcup \{W \in \mathscr{W} |\ W \bigcap A \neq \emptyset\}$. Then V is open, contains A, and $Cl(V) = \{Cl(W)|\ W \bigcap A \neq \emptyset\}$. But each such set W is contained in some V_y, and hence $Cl(W)$ is contained in $Cl(V_y)$ and thus does not contain x. Hence $x \notin Cl(V)$. Thus x and A are separated by open sets in X.

Now let us suppose A and B are disjoint closed sets in X. For each $y \in A$, by regularity, we find open set V_y such that $y \in V_y$ and $Cl\,(V_y) \bigcap B = \emptyset$. Then proceeding exactly as above, we can produce an open set V such that $A \subset V$ and $Cl\,(V) \bigcap B = \emptyset$. Thus X is normal. ☐

T_5-spaces

Definition A.0.25. *A topological space (X, τ) is said to be T_5 if all subspaces of X are normal.*

T_5-spaces are sometimes called Completely normal spaces.

Remark A.0.26. *Every T_5-space is a T_4-space.*

T_6-spaces

Definition A.0.27. *A topological space (X, τ) is said to be T_6 if it is normal and all closed subsets of X are G_δ. T_6-spaces are also called Perfectly normal spaces.*

Definition A.0.28. *A subset B of a topological space is said to be G_δ if B is the intersection of countably many open sets i.e.*

$$B \text{ is } G_\delta \Longleftrightarrow B = \bigcap_{j=1}^{\infty} O_j \text{ where all the } O_j \text{ are open.}$$

Remark A.0.29. *Every T_6-space is a T_5-space.*

Proposition A.0.30. *A normal space X is perfectly normal if and only if every open subset A of X is F_σ i.e. A is the union of countably many closed sets. In other words,*

$$A \text{ is } F_\sigma \Longleftrightarrow A = \bigcup_{i=1}^{\infty} C_i \text{ where all the } C_i \text{ are closed.}$$

Remark A.0.31. *It is clear that F_σ sets are complement of G_δ sets. Then by taking complement from Definition A.0.28, we prove Proposition A.0.30.*

Finally, the separation axioms form the following hierarchy of conditions :

$$T_6 \Longrightarrow T_5 \Longrightarrow T_4 \Longrightarrow T_3 \Longrightarrow T_2 \Longrightarrow T_1 \Longrightarrow T_0.$$

Let us notice that any metric space satisfies all these conditions, from T_0 to T_6-spaces.

Indeed, let (X, d) be any metric space and A any closed subset of X. Let G_n denotes the family of open sets defined by:

$$G_n = \left\{ x : \ d(x, A) < \frac{1}{n} \right\} = \bigcup_{x \in A} B\left(x, \frac{1}{n}\right).$$

It is clear that

$$\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \bigcup_{x \in A} B\left(x, \frac{1}{n}\right) = A.$$

To prove that, let $a \in \bigcap_{n=1}^{\infty} \bigcup_{x \in A} B\left(x, \frac{1}{n}\right) \Longrightarrow \exists x_n \in A$ such that $a \in B(x_n, \frac{1}{n})$ for all n,

$$\Longrightarrow x_n \longrightarrow a \text{ when } n \longrightarrow \infty,$$
$$\Longrightarrow a \in \bar{A} = A\ ,$$

and this means that $\bigcap_{n=1}^{\infty} \bigcup_{x \in A} B\left(x, \frac{1}{n}\right) \subset A$. The inverse inclusion is obvious.

We conclude that $A = \bigcap_{n=1}^{\infty} \left(\bigcup_{x \in A} B\left(x, \frac{1}{n}\right) \right)$ i.e. A is G_δ.

Furthermore, for any disjoint closed subsets A and B of X , $f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}$ is a continuous real valued function on X such that $0 \leq f(x) \leq 1$,and $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$ i.e. $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$. We have $A \subset f^{-1}\left([0, \frac{1}{2})\right) = U$ and $B \subset f^{-1}\left((\frac{1}{2}, 1]\right) = V$, where U and V are two disjoint open subsets of X. This means that the metric space X is T_6 and G_δ, i.e. X is perfectly normal.

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