The Prime Number Theorem

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June 7, 2007

Abstract

The number of primes less than or equal to a positive real x is asymptoticly equal to $x/\log x$ as x goes to infinity, the Prime Number Theorem conjectured in 18^{th} century by Gauss and Legendre, and proven in 1896 by de la Vallée Poussin and Hadamard. This work will present a modern proof of the Prime Number Theorem, which involves much use of complex analysis by the study of the Riemann zeta function denoted by $\zeta(s)$, with a complex variable s. We are going to see some properties of $\zeta(s)$ and derived from that the Prime Number Theorem. To proceed we will use an Analytic Theorem introduced by Newman that we are going to prove in this work.

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Introduction

Before you can count from 1,2 up to 9, they turn simple arithmetic into difficult mathematics. The mysterious primes whose sequences starts by 2,3,5,7,... and goes forever, has intrigued mathematicians for 25 centuries, from the age of Euclid and his proof of the existence of infinitely many primes. People though that they are just random points placed in the set of integers.

Because of the role that primes have played in everything from Pythagorean Triangle to public key cryptography, many mathematicians tried to generate a formula for the n^{th} prime, but no one succeeded. Fermat suggested a formula $2^{2^n}+1$ called n^{th} Fermat's Number, in 1650 he conjectured that they are all primes, unfortunately it was refuted when Euler showed in 1732 that the fith Fermat's Number is composite. By the end of the 18^{th} century many mathematician's tables of primes had been computed and studied by Gauss and Legendre. They have changed the question, instead of trying to find a formula for the n^{th} prime, they define the function $\pi(x)$ to be the number of primes not exceeding x, and they both conjectured the Prime Number Theorem or the P.N.T. In the middle of the 19^{th} century Chebyshev established the bounds $0.921\frac{x}{\log x} < \pi(x) < 1.106\frac{x}{\log x}$ for x sufficiently large.

The Prime Number Theorem was first proved by Jacques Hadamard and Charles-Jeans de la Vallée Poussin in 1896. Their proof was based on the fact that the Meromorphic function $\zeta(s)$, the Riemann Zeta function, does not have a zero in the domain $\{\Re(s) \geq 1\}$. Since then, many mathematicians have improved the proof of P.N.T, and at the same time tried to prove a new conjecture, the Riemann Hypothesis. The latter which is related to the PNT becomes a subject of works of many mathematicians such as Hardy who proved that $\zeta(s)$ has infinitely many zeros in the critical line $\{\Re(s) = \frac{1}{2}\}$, Landau, Littlewood, Wiener and many others. Later Selberg and Erdős gave a new version of the proof of the P.N.T in 1949, without any use of complex analysis but it was much less clear than the analytic one. A few years ago, Newman gave a short proof by using the Analytic Theorem that we are going to expose in this work.

1. Elementary Approach

In this chapter we are going to treat some theorems which can be proved without using complex analysis, and introduce some fundamental formulas on which we will focus till the end of this work.

Notation. In the set of positive integers those elements such as $2, \quad 3, \quad 5, \quad 7, \cdots$ which can not be divided by any positive integer except themself and 1 are called primes; generally we denote them by p, and p_n the n^{th} prime. For any positive real number, the number of prime not exceeding x is denoted by $\pi(x)$.

To make everything much easier we shall use the following notations:

For any function $f(n)$ defined on the set of integers, $\sum_n f(n), \quad \prod_n f(n)$ is respectively the sum and product over all positive integers and similarly $\sum_p f(p), \quad \prod_p f(p)$ is respectively the sum and product over the primes . The sum and product over an empty set will considered respectively as 0 and 1. The symbol $[x]$ is the integer part of x for x real ($[x] \leq x < [x]+1$).

1.1 Fundamental Theorems

In this section we are going to show some result which can help us to understand the distribution of the prime numbers in the range of positive integers.

1.1.1 Infinitely Many Primes

Theorem 1.1 (Euclid's Lemma). *There are infinitely many prime.*

Proof. For any positive integer n greater than 2, we have

$$\gcd(n, n+1) = 1$$

so the number $n(n+1)$ has at least 2 different primes factors. Similarly,

$$\gcd\{n(n+1), n(n+1)+1\} = 1.$$

Thus the product $n(n+1)(n(n+1)+1)$has at least 3 different primes divisor. Continuing this process we construct integers with arbitrarily many distinct prime factors. □

1.1.2 Bertrand's Postulate

Lemma 1.2. *For any real positive x*

$$\prod_{p \leq x} p \leq 4^{x-1}.$$

Proof. We can see that it suffices to deal with x an odd integer by the fact that if q is the greater odd integer less than x, then

$$\prod_{p\leq x} p = \prod_{p\leq q} p \leq 4^{q-1} \leq 4^{x-1}.$$

So we will prove by induction that the formula is truefor any positive integer.

First, $x = 2$ satisfy the inequality, since $2 \leq 4$. Now suppose that the formula is true for any positive integer less than $2n+1$, we have,

$$\prod_{p\leq 2n+1} p = \prod_{p\leq n+1} p \prod_{n+1<p\leq 2n+1} p$$

according to the hypothesis above,

$$\prod_{p\leq n+1} p \leq 4^n$$

while

$$\prod_{n+1<p\leq 2n+1} p \leq \binom{2n+1}{n} \leq 2^{2n}$$

since,

$$\begin{aligned} 2\binom{2n+1}{n} &= \binom{2n+1}{n} + \binom{2n+1}{n+1} \\ &\leq \sum_{k=0}^{2n+1} \binom{2n+1}{k} = 2^{2n+1} \end{aligned}$$

then we get

$$\prod_{p\leq 2n+1} p \leq 2^{2n}2^{2n} = 4^{2n}.$$

. □

Lemma 1.3 (Erdős)**.** *For $n \geq 3$ integer, any prime p, with $\frac{2}{3}n < p \leq n$, does not divide $\binom{2n}{n}$.*

Proof. First we have,

$$\binom{2n}{n} = \frac{(2n)!}{n!n!}.$$

Hence $3p > 2n$ implies that p and $2p$ are the only multiples of p which appear in the numerator; on the other hand $p \leq n$ means that p appear at least twice in the denominator. But if $n \geq 3$ then $p \geq 3$; so knowing that $\binom{2n}{n}$ is an integer, all p's appearing in the numerator will be cancelled by all the p's in the numerator. □

Theorem 1.4 (Bertrand's postulate)**.** *For any positive $n \geq 1$ there exists some primes p satisfying, $n < p \leq 2n$.*

Proof. The main point of this proof is to analyse the number $\binom{2n}{n}$ and to show that if all primes appearing in the decomposition of $\binom{2n}{n}$ are less or equal than n, then $\binom{2n}{n}$ will be "too small".

First, by the **Legendre theorem**[1], the number $\binom{2n}{n} = \frac{2n!}{n!n!}$ contains the prime factor p exactly

$$\sum_{k\geq 1}\left(\left[\frac{2n}{p^k}\right] - 2\left[\frac{n}{p^k}\right]\right)$$

times.

But we have from the definition of $[.]$ and for $k \geq 1$,

$$\left[\frac{2n}{p^k}\right] - 2\left[\frac{n}{p^k}\right] < \frac{2n}{p^k} - 2\left(\frac{n}{p^k} - 1\right) = 2$$

Then each summand of the sum below is at most 1, moreover if $p^k > 2n$ then the correspondant summand vanishes. Hence,

$$\sum_{k\geq 1}\left(\left[\frac{2n}{p^k}\right] - 2\left[\frac{n}{p^k}\right]\right) \leq \max\{r : p^r \leq 2n\}.$$

Now we can say that the largest power of p dividing $\binom{2n}{n}$ is less than $2n$. In particular if $p > \sqrt{2}n$ then p appear at most once.

Therefore, according to the Lemma 1.3 we have the relation, for $n \geq 3$,

$$\binom{2n}{n} \leq \prod_{p\leq\sqrt{2n}} 2n \prod_{\sqrt{2n}<p\leq\frac{2}{3}n} p \prod_{n<p\leq 2n} p.$$

While,

$$\frac{4^n}{2n} \leq \binom{2n}{n}$$

using the Newton binomial Formula, and noting that $\binom{2n}{n}$ is the biggest term in the sum $\sum_{k=0}^{2n}\binom{2n}{k} = 2^{2n}$.

Now suppose that for $n \geq 3$ there is no prime p where $n < p \leq 2n$. It implies that,

$$\prod_{n<p\leq 2n} p = 1.$$

From the formula described in the Lemma 1.2,

$$\prod_{\sqrt{2n}<p\leq\frac{2}{3}n} p \leq 4^{\frac{2}{3}n}.$$

[1] The number $n!$ contains the prime factor p exactly

$$\sum_{k\geq 1}\left[\frac{n}{p^k}\right]$$

times

There are at most $\sqrt{2n}$ prime not greater than $\sqrt{2n}$, so

$$\prod_{p\leq\sqrt{2n}} 2n \leq (2n)^{\sqrt{2n}}$$

therefore,

$$4^n \leq (2n)^{1+\sqrt{2n}} 4^{\frac{2}{3}n}$$

or

$$4^{\frac{n}{3}} \leq (2n)^{1+\sqrt{2n}}.$$

Taking the logarithm,

$$2n\frac{\log 2}{3} \leq (1+\sqrt{2n})\log(2n).$$

By studying the function $x^2\frac{\log 2}{3} - (1+x)\log(x^2)$, we get from computing the derivative that it is an increasing function for $x > 45$, and its value at the point 45 is positive ($\simeq 117.66$), then this function is positive for $x > 45$.

So n must be less than $\frac{45^2}{2} = 1012.5$. But the sequence of primes

$$2, \quad 3, \quad 5, \quad 7, \quad 13, \quad 23, \quad 43, \quad 83, \quad 163, \quad 317, \quad 631, \quad 1259$$

shows that it can not be true, because in this list each prime is less than the twice of the prime before it. □

1.2 Growth of $\pi(x)$

We have seen before, that $\pi(x) \to \infty$ as $x \to \infty$. But now we are going to see the asymptotic behaviour of $\pi(x)$.

1.2.1 Definition and Notation

We shall use the Landau signs O, o and $\sim$ for the "asymptotic quality" which mean:

- The functions $f(x) \sim g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$;
- $f(x) = O(g(x))$ if there exists a constant $C > 0$ such that for x sufficiently large $|f(x)| \leq C|g(x)|$;
- $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

Definition 1.5 (Chebyshev theta function)**.** For any real positive x we call $\vartheta(x)$ the function defined as

$$\vartheta(x) = \sum_{p\leq x} \log p.$$

Proposition 1.6. *For x sufficiently large, $\vartheta(x) \leq 2x\log 2$.*

Proof. Take the logarithm of the expression of the Lemma 1.2 □

Theorem 1.7. *The series $\sum_p \frac{1}{p}$ is divergent.*

Proof. We have, for $0 < x < 1$,

$$-\log(1-x) = x + \frac{x^2}{2} + x^2\varepsilon(x)$$

where $\varepsilon(x) \to 0$ when $x \to 0$. Then for any nonzero positive integer N,

$$\begin{aligned}\log\prod_{i=1}^{N}(1-p_i^{-1})^{-1} &= -\sum_{i=1}^{N}\log(1-p_i^{-1})\\ &= \sum_{i=1}^{N}\tfrac{1}{p_i} + \sum_{i=1}^{N}\tfrac{1}{p_i^2}\varepsilon_1(\tfrac{1}{p_i}).\end{aligned}$$

where $\varepsilon_1(x) \to \frac{1}{2}$ when $x \to 0$. But the product $\prod_{p_i}(1-p_i^{-1})^{-1}$ is divergent, since for $x > 1$

$$\begin{aligned}\prod_{p\leq x}(1-p^{-1})^{-1} &\geq \sum_{n=1}^{[x]}\frac{1}{n}\\ &\geq \sum_{n=1}^{[x]}\int_n^{n+1}\frac{1}{t}dt\\ &\geq \int_1^{[x]+1}\frac{1}{t}dt\\ &\geq \log x. \qquad (1)\end{aligned}$$

On the other hand, there exists a constant $C > 0$ such that

$$\sum_p \frac{1}{p^2}\varepsilon_1(\tfrac{1}{p}) \leq C\sum_{1\leq n}\frac{1}{n^2}$$

and the last series is convergent. Then it follows that $\sum_p \frac{1}{p}$ is divergent as a sum of two series divergent and convergent

$$\sum_p \tfrac{1}{p} = \sum_p \tfrac{1}{p^2}\varepsilon_1(\tfrac{1}{p}) - \sum_p \log(1-p^{-1}).$$

□

1.2.2 Chebyshev Theorems

Theorem 1.8 (Chebyshev). *As* $x \to \infty$, $\pi(x) = o(x)$

Proof. First we introduce the function $N_r(x,h)$, where r and h are positive integers and x real positive, the number of positive integers multiple of h less or equal than x, not divisible by any of the r first primes. Particularly, $N_0(x,h) = \left[\frac{x}{h}\right]$.

We have the induction relation

$$N_{r-1}(x,h) = N_r(x,h) + N_{r-1}(x, p_r h) \tag{1.1}$$

since,the multiple of h less or equal than x not divisible by any of the $r-1$ first primes may be divisible by p_r or not. If it divisible by p_r that forms all the elements counted in $N_{r-1}(x, p_r h)$; if not,it is counted in $N_r(x,h)$.

Then we have

$$N_r(x,h) = N_0(x,h) - \sum_{i \le r} N_0(x, p_i h) + \sum_{i<j\le r} N_0(x, p_i p_j h) + \cdots,$$

which we can prove easily by induction using Relation (1.1).

On the other hand, we have for $r \le x$

$$\pi(x) \le r + N_r(x,1)$$

because all primes between r and x are counted in $N_r(x,1)$.

Now, let us take $2 < \xi < x$, and choose r satisfying $p_r \le \xi < p_{r+1}$.

We have,

$$\begin{aligned}
\pi(x) &\le r + [x] - \sum_{i\le r}\left[\frac{x}{p_i}\right] + \sum_{i<j\le r}\left[\frac{x}{p_i p_j}\right] + \cdots \\
&\le r + 2^r + x\left(1 - \sum_{i \le r}\frac{1}{p_i} + \sum_{i<j\le r}\frac{1}{p_i p_j} + \cdots\right) \\
&\le r + 2^r + x\prod_{p\le\xi}(1 - \frac{1}{p}).
\end{aligned}$$

Since $|\alpha - [\alpha]| \le 1$, and there are less than 2^r terms from which we remove the square bracket. Moreover, $r \le 2^r \le 2^\xi$ then $r + 2^r \le 2^{\xi+1}$; and we can take ξ as a function of x, satisfying , $\xi \to \infty$ and $2^{\xi+1}/x \to 0$ as $x \to \infty$.

Remark 1.9. If we take $\xi(x) = c\log x$ with $0 < c < (\log x)^{-1}$, we will get the more precise result

$$\pi(x) = o\left(\frac{x}{\log\log x}\right)$$

using the relation (1), for $\prod_{p\le\xi}(1-\frac{1}{p})$.

$\square$

Theorem 1.10 (Chebyshev). *There exist two constants c_1 and c_2 such that*

$$c_1 \frac{x}{\log x} \leq \pi(x) \leq c_2 \frac{x}{\log x}.$$

Proof. We have

$$\binom{2n}{n} = \frac{2n!}{n!n!} = \frac{2n}{n} \frac{2n-1}{n-1} \cdots \frac{n+1}{1}$$

and every term of this product is greater than 2, so

$$2^n \leq \binom{2n}{n}.$$

We recall the result that we have seen in the proof of the Bertrand's postulate

$$\binom{2n}{n} \leq \prod_{p \leq \sqrt{2n}} 2n \prod_{\sqrt{2n} < p \leq 2n} p.$$

Therefore,

$$\begin{aligned} 2^n &\leq \prod_{p \leq \sqrt{2n}} 2n \prod_{\sqrt{2n} < p \leq 2n} p \\ &\leq (2n)^{\sqrt{2n}} \prod_{p \leq 2n} 2n. \end{aligned}$$

By taking the logarithm of this relation, we get

$$n \log 2 \leq \sqrt{2n} \log(2n) + \pi(2n) \log(2n)$$

or

$$\pi(2n) \log(2n) \geq n \log 2 - \sqrt{2n} \log(2n) = n \log 2 \left(1 - \frac{\sqrt{2} \log(2n)}{\sqrt{n} \log 2}\right)$$

When n goes to infinity, $\frac{\sqrt{2}\log(2n)}{\sqrt{n}\log 2}$ goes to zero, so there exists $C > 0$ such that for n sufficiently large $\pi(2n)\log(2n) \geq Cn$. If we take $2n \leq x < 2n+1$

$$\pi(x) \log(x) \geq \pi(2n) \log(2n) \geq Cn > C\left(\frac{x-1}{2}\right).$$

For all $x \geq 2$ the function $\pi(x)\log(x)$ is strictly greater than 1, and we can find $c_1 > 0$ such that for all $x \geq 2$, we have $\pi(x)\log(x) \geq c_1 x$. For the upper bound, we are going to assume a relation that we will prove later, which says that for any $\epsilon > 0$, and for x large enough

$$(1-\epsilon)\left(\frac{\pi(x)\log x}{x} - \frac{\log x}{x} O(x^{1-\epsilon})\right) \leq \frac{\vartheta(x)}{x}. \tag{1.2}$$

Using Proposition 1.6, the relation (1.2) becomes

$$(1-\epsilon)\left(\frac{\pi(x)\log x}{x} - \frac{\log x}{x}O(x^{1-\epsilon})\right) \leq 2\log 2.$$

For $\epsilon > 0$ we have $\lim_{x\to\infty} \dfrac{\log x}{x} O(x^{1-\epsilon}) = 0$. It implies that there exist two reals $M > 0$ and $B > 0$ such that for $x > B$ we have $\dfrac{\pi(x)\log x}{x} \leq M$

For $x \leq B$, we have $\dfrac{\pi(x)\log x}{x} \leq \log B$. Since for all $x \geq 2$, we have $\pi(x) \leq x$ then we can choose c_2 any positive real greater than $\max\{M, \log B\}$. □

Theorem 1.11. *If one of the limits* $\lim_{x\to\infty} \frac{\pi(x)\log x}{x}$ *and* $\lim_{x\to\infty} \frac{\vartheta(x)}{x}$ *exists then they are both convergent and converge to the same limits.*

Proof. We have for x large enough,

$$\vartheta(x) \leq \pi(x)\log(x).$$

For $\epsilon > 0$,

$$\begin{aligned}
\vartheta(x) &\geq \sum_{x^{1-\epsilon}\leq p\leq x} \log p \\
&\geq \sum_{x^{1-\epsilon}\leq p\leq x} (1-\epsilon)\log x \\
&\geq (1-\epsilon)(\pi(x) - \pi(x^{1-\epsilon}))\log x.
\end{aligned}$$

Then we have the relation for any $\epsilon > 0$,

$$(1-\epsilon)\left(\frac{\pi(x)\log x}{x} - \frac{\log x}{x}O(x^{1-\epsilon})\right) \leq \frac{\vartheta(x)}{x} \leq \frac{\pi(x)\log x}{x}$$

This proves the relation (1.2), and the theorem follows by taking the limits $x \to \infty$ and $\epsilon \to 0$. □

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2. Proof of the Prime Number Theorem

2.1 Review of Complex Analysis

In this section, we are going to cite without giving any proof some important results that we will often use in the rest of the document. But for the basic concepts of Complex Analysis, we can find in any standard text such as [Riv07].

2.1.1 Some Useful Theorems

Theorem 2.1 (Cauchy's Formula). *If $f(z)$ is a holomorphic function in domain U and C a simple closed curve inside U, positively oriented, then for any z inside C*

$$f(z) = \frac{1}{2i\pi} \int_C \frac{f(\omega)}{\omega - z} d\omega.$$

Theorem 2.2 (Uniqueness of Analytic Function). *If two functions $f(z)$ and $g(z)$ are analytic in a domain U and coincide in a set which has an accumulation point in U, then they coincide in U.*

In this theorem the set which has an accumulation point can be replaced by an open set.

Theorem 2.3 (Cauchy Residues Formula). *Let $f(z)$ be meromorphic in a domain U with only poles $\{z_1, z_2, \ldots, z_n\}$ in U. Let C be a closed curve inside U that does not pass through any pole. Then*

$$\frac{1}{2i\pi} \int_C f(z)dz = \sum_{k=1}^{n} \operatorname{Ind}(C, z_k) \operatorname{Res}(f, z_k).$$

Where $\operatorname{Ind}(C, z_k)$ is the index of the point z_k in respect to the oriented closed curve C, and $\operatorname{Res}(f, z_k)$ is the residue of the meromorphic function $f(z)$ at the point z_k.

2.1.2 Analytic Theorem

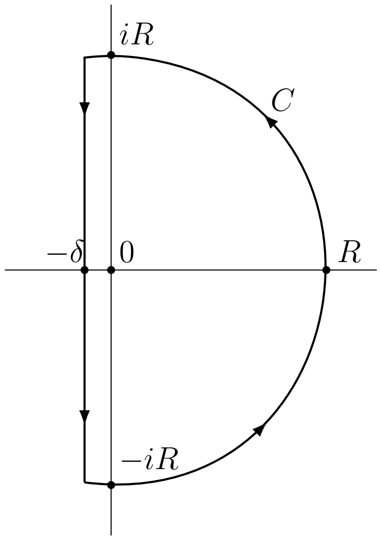
We will need the following. This is not a standard result, it is essentially due to Newman.

Theorem 2.4. *Let $f(t)$, for $t \geq 0$, be a bounded and locally integrable function and suppose that the function $g(z) = \int_0^\infty f(t)e^{-zt}dt$ extends holomorphically in the $\{\Re(z) \geq 0\}$. Then $\int_0^\infty f(t)dt$ exists (and equals $g(0)$).*

Proof. For $a > 0$, let $g_a(z) = \int_0^a f(t)e^{-zt}dt$. It is clear that for all $a > 0$, $g_a(z)$ is entire, what we need to show is that $\lim_{a\to\infty} g_a(0) = g(0)$.

For $R > 0$ their exist $\delta > 0$ (small and may depend on R), such that $g(z)$ is analytic in the region $\{z \in \mathbb{C} \big| |z| \leq R\} \cap \{-\delta \leq \Re(z)\}$.

So let us consider the path C, the boundary of the region $\{z \in \mathbb{C} \big| |z| \leq R\} \cap \{-\delta \leq \Re(z)\}$ oriented in the anti-clockwise direction, and $C_+ = C \cap \{\Re(z) > 0\}$, $C_- = C \cap \{\Re(z) < 0\}$.



Then by Cauchy formula, we have

$$g(0) - g_a(0) = \frac{1}{2i\pi} \int_C (g(z) - g_a(z)) \left(1 + \frac{z^2}{R^2}\right) e^{za} \frac{dz}{z}. \tag{2.1}$$

So

$$2\pi |g(0) - g_a(0)| \leq \left| \int_{C_+} (g(z) - g_a(z)) \left(1 + \frac{z^2}{R^2}\right) e^{za} \frac{dz}{z} \right| + \left| \int_{C_-} (g(z) - g_a(z)) \left(1 + \frac{z^2}{R^2}\right) e^{za} \frac{dz}{z} \right|.$$

But we know that

$$g(z) - g_a(z) = \int_a^\infty f(t) e^{-zt} dt$$

then,

$$|g(z) - g_a(z)| \leq M \int_a^\infty e^{-\Re(z)t} dt = M \frac{e^{-\Re(z)a}}{\Re(z)}$$

where

$$M = \max_{t \geq 0} |f(t)|$$

. On the other hand, for z in $\mathbb{C}$

$$\left| \frac{e^{za}}{z} \left(1 + \frac{z^2}{R^2}\right) \right| = \frac{e^{\Re(z)a}}{R} \left|1 + e^{2i \arg(z)}\right|.$$

Let denote $\theta = \arg(z)$, so we have

$$\begin{aligned} \left|1 + e^{2i\theta}\right| &= |1 + \cos(2\theta) + i\sin(2\theta)| \\ &= \sqrt{(1 + \cos(2\theta))^2 + \sin^2(2\theta)} \\ &= 2|\cos\theta| \\ &= 2\frac{|\Re(z)|}{R}. \end{aligned}$$

therefor, for $z \in C_+$

$$\left|(g(z) - g_a(z))\left(1 + \frac{z^2}{R^2}\right)\frac{e^{za}}{z}\right| \leq 2\frac{M}{R^2}$$

and it follows that the part of the integral on C_+ is bounded by $2M/R$.

In the domain $\{\Re(z) < 0\}$, we have

$$\left|\int_{C_-}(g(z) - g_a(z))\left(1 + \frac{z^2}{R^2}\right)e^{za}\frac{dz}{z}\right| \leq \left|\int_{C_-}g(z)\left(1 + \frac{z^2}{R^2}\right)e^{za}\frac{dz}{z}\right| + \left|\int_{C_-}g_a(z)\left(1 + \frac{z^2}{R^2}\right)e^{za}\frac{dz}{z}\right|.$$

Let us consider where $C'_- = \{z \in \mathbb{C}|\, |z| = R, \Re(z) < 0\}$, and knowing that $g_a(z)$ is entire then

$$\left|\int_{C_-}g_a(z)\left(1 + \frac{z^2}{R^2}\right)e^{za}\frac{dz}{z}\right| = \left|\int_{C'_-}g_a(z)\left(1 + \frac{z^2}{R^2}\right)e^{za}\frac{dz}{z}\right|$$

in addition we have the inequality

$$|g_a(z)| \leq M\int_{-\infty}^{a}|e^{-zt}|dt = M\frac{e^{-\Re(z)a}}{|\Re(z)|}.$$

So

$$\left|\int_{C_-}g_a(z)\left(1 + \frac{z^2}{R^2}\right)e^{za}\frac{dz}{z}\right| \leq \frac{2}{R}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}|g_a(e^{i\theta})\Re(e^{i\theta})|e^{\Re(e^{i\theta})a}d\theta \leq 2\pi\frac{M}{R}.$$

The function e^{az} converges uniformly to 0 in a compact set as $a \to \infty$.

It follows that

$$\lim_{a\to\infty}\left|\int_{C_-}g(z)\left(1 + \frac{z^2}{R^2}\right)e^{za}\frac{dz}{z}\right| = 0.$$

Therefore for any $R > 0$,

$$\lim_{a\to\infty}|g(0) - g_a(0)| \leq \frac{(\pi + 1)M}{\pi R}.$$

□

2.2 Riemann-Zeta Function

The Prime Number Theorem is related to the properties of the Riemann-Zeta function. So, first we are going to figure out some properties of this function, and apply them in the theory prime numbers.

2.2.1 Notation and Definition

The Riemann-Zeta function is the function defined for any $s \in \{\Re(s) > 1\}$ as

$$\zeta(s) = \sum_{1\leq n} \frac{1}{n^s}.$$

We will now introduce another function that we will often use. The function defined on the same domain as the Riemann-Zeta function:

$$\Phi(s) = \sum_{p} \frac{\log(p)}{p^s}.$$

The power x^s, for x and s real or complex, is the same as $e^{s\log(x)}$ and $\log$ is the branch of natural logarithm that coincides with the usual $\log$ on $\mathbb{R}_+\backslash\{0\}$.

Remark 2.5. These functions are absolutely and locally uniformly convergent in the half-plane $\{\Re(s) > 1\}$. So they define a holomorphic functions in this domain. See [Riv07].

2.2.2 Properties of the Riemann Zeta Function and Φ

Proposition 2.6 (Euler Product)**.** *For any s in $\{\Re(s) > 1\}$,*

$$\zeta(s) = \prod_{p}(1 - p^{-s})^{-1}$$

Proof. We know that any positive integer n greater than 2 can be written in a unique way in the form $\prod_{i=1}^{k} p_i^{\alpha_k}$ with p_i prime and $p_i < p_j$ if $i < j$.

We have

$$(1 - p^{-s})^{-1} = 1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots$$

So

$$(1 - p_1^{-s})^{-1}(1 - p_2^{-s})^{-1} = \sum_{i,j\in\mathbb{N}} (p_1^{-i}p_2^{-j})^s.$$

Then for N positive integer, we have

$$\prod_{i=1}^{N}(1 - p_i^{-s})^{-1} = \sum_{n\in S(N)} n^{-s}$$

where $S(N) := \{\prod_{i=1}^{N} p_i^{\alpha_i} | \alpha_i \in \mathbb{N}\}$. The set $S(N)$ satisfies the relation for all N,

$$S(N) \subset S(N+1) \text{ and } \bigcup_{N\in\mathbb{N}} S(N) = \mathbb{N}.$$

Therefore,

$$\lim_{N\to\infty} \prod_{i=1}^{N}(1 - p_i^{-s})^{-1} = \sum_{n\in\mathbb{N}} n^{-s} = \zeta(s).$$

□

Remark 2.7. Relation (2.6) explains the link between primes and $\zeta(s)$, and we can conclude directly that $\zeta(s)$ does not have a zero in the region $\{\Re(s) > 1\}$ because the analytic function $\zeta(s)$ can be represented as an infinite product and every term is non-zero. See [Riv07].

Proposition 2.8. *The function $\zeta(s)$ can be extended meromorphically on the half plane $\{\Re(s) > 0\}$ with the only pole $s = 1$ which has a residue 1.*

Proof. The proposition is equivalent to saying that the function $\zeta(s) - \dfrac{1}{s-1}$ can be extended holomorphically to the half plane $\{\Re(s) > 0\}$.

First, we have

$$\begin{aligned}\zeta(s) - \frac{1}{s-1} &= \sum_{1\le n} \frac{1}{n^s} - \int_1^\infty \frac{1}{x^s} dx \\ &= \sum_{1\le n} \frac{1}{n^s} - \sum_{1\le n} \int_n^{n+1} \frac{1}{x^s} dx \\ &= \sum_{1\le n} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx \qquad (2.2)\end{aligned}$$

and

$$\begin{aligned}\left|\int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx\right| &= \left| s \int_n^{n+1} \int_n^x \frac{dt}{t^{s+1}} dx \right| \\ &\le \max_{n\le t\le n+1} \left|\frac{s}{t^{s+1}}\right| = \frac{|s|}{n^{\Re(s)+1}}.\end{aligned}$$

Then for any s with $\{\Re(s) > 0\}$ there exists $\delta > 0$, $\Re(s) > \delta > 0$, such that the series (2.2) is uniformly convergent in a neighbourhood of s included in the interior of the domain $\{\Re(s) > \delta\}$, and every summand is an analytic function. Then the sum (2.2) is an analytic function. See [Riv07]. □

Lemma 2.9. *For s, with $\Re(s) > 1$*

$$-\frac{\zeta'(s)}{\zeta(s)} = \Phi(s) + \sum_p \frac{\log(p)}{p^s(p^s - 1)}$$

Proof. We have from the remark 2.7 $\zeta(s)$ has no zeros in the half-plane so we can define the determination of logarithms and

$$\log \zeta(s) = -\sum_p \log\left(1 - p^{-s}\right)$$

Then

$$
\begin{aligned}
(-\log \zeta(s))' &= -\frac{\zeta'(s)}{\zeta(s)} \\
&= \sum_p \frac{(1-p^{-s})'}{1-p^{-s}} \\
&= \sum_p \frac{\log p}{p^s - 1}.
\end{aligned}
$$

The last series is locally uniformly convergent for s in $\{\Re(s) > 1\}$, and

$$
\begin{aligned}
\sum_p \frac{\log p}{p^s - 1} - \Phi(s) &= \sum_p \left(\frac{\log p}{p^s - 1} - \frac{\log p}{p^s} \right) \\
&= \sum_p \frac{\log p}{p^s (p^s - 1)}.
\end{aligned}
$$

□

Theorem 2.10. *The function* $\zeta(s) \neq 0$ *in* $\{\Re(s) \geq 1\}$ *and the function* $\Phi(s) - \dfrac{1}{s-1}$ *extends holomorphically to this domain.*

Proof. According to Lemma 2.9, we have the formula for $\Re(s) > 1$

$$
-\frac{\zeta'(s)}{\zeta(s)} = \Phi(s) + \sum_p \frac{\log(p)}{p^s(p^s - 1)}. \tag{2.3}
$$

The series on the right hand side is absolutely convergent for $s \in \{\Re(s) > \frac{1}{2}\}$; then the function Φ is meromorphic and has poles at $s = 1$ and at zeros of $\zeta(s)$.

Now suppose that there exists a zero $\sigma_1 = 1 + i\alpha$ of order r_1 of $\zeta(s)$.

Let $\sigma_2 = 1 + 2i\alpha$ be a zero of $\zeta(s)$ of order r_2 (r_2 can be a zero if σ_2 is not really a zero).

Noting that $\overline{\zeta(s)} = \zeta(\overline{s})$ for $\Re(s) > 1$,then $\overline{\sigma_1}$ and $\overline{\sigma_2}$ are also zeros of $\zeta(s)$ with order respectively r_1 and r_2.

If s_0 is a zero of order r_0 of $\zeta(s)$, then $\zeta(s)$ can be written as $\zeta(s) = (s - s_0)^{r_0} \zeta_1(s)$ with $\zeta_1(s_0) \neq 0$, and

$$
\frac{\zeta'(s)}{\zeta(s)} = \frac{r_0}{(s - s_0)} \zeta_1(s) + \frac{\zeta_1'(s)}{\zeta_1(s)}.
$$

In addition the function $(s-1)\zeta(s)$ can be extended holomorphically to the half-plane $\{\Re(s) > 0\}$. Let $f(s)$ be this extension; we can say that in a neighbourhood of 1, $f(s) \neq 0$. So for $s \neq 1$

$$
\frac{f'(s)}{f(s)} = \frac{1}{s-1} + \frac{\zeta'(s)}{\zeta(s)}.
$$

Then we have the following limits

$$\lim_{x\to 1^+}(x-1)\Phi(x) = 1 \tag{2.4}$$

$$\lim_{x\to 1^+}(x-1)\Phi(x\pm i\alpha) = -r_1 \tag{2.5}$$

$$\lim_{x\to 1^+}(x-1)\Phi(x\pm 2i\alpha) = -r_2 \tag{2.6}$$

On the other hand,

$$\begin{aligned}\sum_{k=-2}^{2}\binom{4}{2+k}\Phi(x+ik\alpha) &= \sum_{k=-2}^{2}\sum_{p}\binom{4}{2+k}\frac{\log p}{p^{x+ik\alpha}}\\ &= \sum_{p}\frac{\log p}{p^x}\sum_{k=-2}^{2}\binom{4}{2+k}\left(\frac{1}{p^{i\alpha}}\right)^k\\ &= \sum_{p}\frac{\log p}{p^x}\left(\frac{1}{p^{-i\alpha}}+\frac{1}{p^{i\alpha}}\right)^4\\ &= \sum_{p}16\frac{\log p}{p^x}\left[\cos(\alpha\log p)\right]^4\\ &\geq 0.\end{aligned}$$

Hence,

$$\lim_{x\to 1^+}(x-1)\sum_{k=-2}^{2}\binom{4}{2+k}\Phi(x+ik\alpha) = 6-8r_1-2r_2.$$

It follows that $6-8r_1-2r_2 \geq 0$, which means that r_1 must be 0.

The Limit 2.4 proves that the extended $\Phi(s)$ has a residue 1 at the pole $s=1$, and that completes the proof. □

2.2.3 Deducing the Prime Number Theorem

Lemma 2.11. *The functions*

$$f(t) = \vartheta(e^t)e^{-t}-1 \quad \textit{and} \quad g(z) = \frac{\Phi(z+1)}{z+1}-\frac{1}{z}$$

satisfy the hypothesis of the Theorem 2.4.

Proof. The function f is continuous and bounded by $2\log 2$.

$$\begin{aligned}\int_0^\infty f(t)e^{-zt}dt &= \int_0^\infty \vartheta(e^t)e^{-(z+1)t}dt - \int_0^\infty e^{-zt}dt \\ &= \int_0^\infty \sum_{p\le e^t}\log(p)e^{-(z+1)t}dt - \frac{1}{z} \\ &= \sum_p \log p\int_{\log p}^\infty e^{-(z+1)t}dt - \frac{1}{z} \\ &= \sum_p \frac{\log p}{p^{z+1}(z+1)} - \frac{1}{z}.\end{aligned}$$

The Theorem 2.10 assures us that the function $\Phi(z+1)-\frac{1}{z}$ is holomorphic in $\{\Re(z)\ge 0\}$. The function $g(z)$ is holomorphic since,

$$g(z) = \frac{\Phi(z+1)-\frac{1}{z}-1}{z+1}.$$

□

Theorem 2.12. *As x goes to infinity, we have $\vartheta(x)\sim x$.*

Proof. First from Lemma 2.11 and the Analytic Theorem 2.4 we have

$$\int_0^\infty (\vartheta(e^t)e^{-t}-1)dt = \int_1^\infty \frac{\vartheta(x)-x}{x^2}dx.$$

by a change of the variable $x=e^t$, and the two integrals are both convergent. Now suppose that $\lim_{x\to\infty}\frac{\vartheta(x)}{x}\ne 1$, then there exists an $\epsilon>0$ such that the set $\{x>0\big||\frac{\vartheta(x)}{x}-1|>\epsilon\}$ is not bounded.

However, if $\frac{\vartheta(x)}{x}>1+\epsilon$, then

$$\int_x^{(1+\epsilon)x}\frac{\vartheta(t)-t}{t^2}dt \ge \int_x^{(1+\epsilon)x}\frac{(1+\epsilon)x-t}{t^2}dt = \int_1^{1+\epsilon}\frac{1+\epsilon-u}{u^2}du > 0$$

by a change of variable $u=\frac{t}{x}$.

Similarly, if $\frac{\vartheta(x)}{x}<1-\epsilon$ then

$$\int_{(1-\epsilon)x}^{x}\frac{\vartheta(t)-t}{t^2}dt \le \int_{(1-\epsilon)x}^{x}\frac{(1-\epsilon)x-t}{t^2}dt = \int_{1-\epsilon}^{1}\frac{1-\epsilon-u}{u^2}du < 0.$$

The last equation contradicts with the fact that $\int_1^\infty \frac{\vartheta(x)-x}{x^2}dx$ converges, when $x\to\infty$. □

The Prime Number Theorem follows by using one of the results that we have proved in the Theorem 1.11 at the end of the first chapter.

Corollary 2.13. *The function n^{th}-prime p_n is equivalent to $n\log n$ when n goes to infinity.*

Proof. By the definition of p_n,

$$\pi(p_n) = n.$$

Then, from Euclid's Lemma (Lemma 1.1)

$$\lim_{n\to\infty} \frac{\pi(p_n)}{\frac{p_n}{\log p_n}} = \lim_{n\to\infty} \frac{n\log p_n}{p_n} = 1.$$

and,

$$\lim_{n\to\infty} \frac{n\log(\frac{p_n}{n\log n} n\log n)}{\frac{p_n}{n\log n} n\log n} = \lim_{n\to\infty} \frac{\log(\frac{p_n}{n\log n})}{\frac{p_n}{n\log n}\log n} + \frac{\log(n\log n)}{\frac{p_n}{n\log n}\log n} = \lim_{n\to\infty} \frac{\log(\frac{p_n}{n\log n})}{\frac{p_n}{n\log n}\log n} + \frac{n\log n}{p_n} = 1$$

According to Chebyshev's theorem,

$$\pi(p_n) \leq c_2 \frac{p_n}{\log p_n}$$

where c_2 is a positive real number. Therefore

$$c_2 p_n \geq n\log p_n \geq n\log n \qquad (n < p_n)$$

or

$$\frac{p_n}{n\log n} \geq \frac{1}{c_2}.$$

But we know that the function $\frac{\log x}{x}$ for $x > 0$ is continuous and goes to zero when x goes to infinity. Thus there exists a constant $M > 0$ such that if $x \geq \dfrac{1}{c_2}$, then $\left|\dfrac{\log x}{x}\right| \leq M$. So we have

$$\lim_{n\to\infty} \left| \frac{\log(\frac{p_n}{n\log n})}{\frac{p_n}{n\log n}\log n} \right| \leq \lim_{n\to\infty} \frac{M}{\log n} = 0$$

. □

In the first chapter we have proved Bertrand's Postulate, which says that for any positive integer n, we can find some prime between n and $2n$. But knowing the Prime number Theorem we can get a more general result.

Corollary 2.14. *For any positive real c greater than 1, there is at most a finite number of intervals $(n, cn]$ not containing primes (n is a positive integer).*

Proof. First we have, $\pi(cn) \sim c\pi(n)$, since

$$\log(cn) = \log c + \log n$$

then

$$\frac{cn}{\log(cn)} \sim c\frac{n}{\log n}.$$

So,

$$\pi(cn) - \pi(n) \sim (c-1)\frac{n}{\log n}.$$

Therefore, for $c > 1$ fixed there exists $N > 0$ such that the number $\pi(cn) - \pi(n)$ is greater than 2 for $n > N$. While $\pi(cn) - \pi(n)$ is the number of primes in the interval $(n, cn]$. $\square$

It is always interesting to know more about how primes are distributed in the set of positive integers, and how dense they are with respect to some well known infinite subset of $\mathbb{N}$ where all the elements are given by a formula or by a recursive relation. Among them is the set of squares, we know that $\sum_n \frac{1}{n^2}$ is convergent and $\sum_p \frac{1}{p}$ is divergent. Then there are "more" primes than squares, nevertheless the question "is there always some primes between n^2 and $(n+1)^2$?" is still unsolved.

3. Further Theory of Zeta

The Riemann-Zeta function is one of the most beautiful discoveries in mathematics of the last two centuries. It is a sort of bridge between discrete and continuous mathematics. But there are still many interesting properties of $\zeta(s)$ which need to be understood, the most important of them is the Riemann Hypothesis, which has not yet been proven, which affects the zeros of $\zeta(s)$. But in this chapter, we are going to find a way to extend $\zeta(s)$ to a set as big as we can make it in the s-plane, and build a formula called the Functional Equation which can help us to understand the Riemann Zeta Function.

3.1 Analytic Continuation and Functional Equation

3.1.1 The Gamma Function

Definition 3.1. The gamma function, known as the factorial function, is defined by

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt \text{ for } \Re(s) > 0.$$

Remark 3.2. The gamma function is the extention of the factorial function defined for non zero positive integers as $1! = 1$ and $n! = n(n-1)!$.

Lemma 3.3. *For $s \in \mathbb{C}$ and $\Re(s) > 0$, we have $\Gamma(s+1) = s\Gamma(s)$.*

Proof. By definition,

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^s dt.$$

Doing an integration by parts we have for any $T > 0$

$$\int_0^T e^{-t} t^s dt = -T^s e^{-T} + s \int_0^T e^{-t} t^{s-1} dt.$$

The result follows by performing the limit $T \to \infty$. □

Proposition 3.4. *The function $\Gamma(s)$ has a meromorphic extension to $\mathbb{C}$ with the simple poles $s = 0, -1, -2, \ldots$, and its residues at $s = -k$ are given by $\frac{(-1)^{-k}}{k!}$, and $\Gamma(s) \neq 0$ for all $s \in \mathbb{C}$.*

Proof. Using the relation in the preceding lemma, we can define the value of $\Gamma(s)$ in the whole domain $\mathbb{C}$. We can see that,

$$\lim_{s \to 0} s\Gamma(s) = \Gamma(1) \neq 0.$$

For $s = -k$,

$$(s+k)\Gamma(s) = \frac{(s+k)\Gamma(s+k)}{s(s+1)\cdots(s+k-1)}$$

then,

$$\lim_{s \to -k} (s+k)\Gamma(s) = \frac{(-1)^k}{k!}.$$

□

Knowing that $\Gamma(s)$ is now defined on all of $\mathbb{C}$, we are going to assume some properties of that function, which we will use later.

Proposition 3.5. *For all $s \in \mathbb{C}$ as long as all terms are defined, we have the relations*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{3.1}$$

$$\Gamma(s)\Gamma(s+\frac{1}{2}) = 2^{1-2s}\pi^{\frac{1}{2}}\Gamma(2s). \tag{3.2}$$

For the proof see [Ahl53].

3.1.2 Extension of $\zeta(s)$

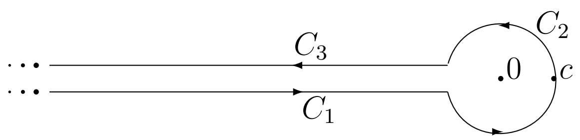
Proposition 3.6. *The function defined as*

$$I(s) := \int_C \frac{z^{s-1}}{e^{-z}-1} dz$$

is an entire function, where C is defined as the union of the paths C_1, C_2, C_3, where C_1 is the real negative line oriented in the positive direction from $-\infty$ to $-c$, C_2 the circle of centre 0 and radius c oriented in the anti-clockwise direction, and C_3 the real negative line oriented in the negative direction from $-c$ to $-\infty$, where c is a positive real number ($c < 2\pi$). The path C is given in the figure below.

For $s \in \mathbb{C}$ and $\Re(s) > 1$

$$I(s) = \frac{\sin(s\pi)}{\pi}\Gamma(s)\zeta(s).$$



Proof. Let us set $\Re(s) = \sigma$ and $\Im(s) = t$, then we have $s = \sigma + it$.

First we have to show that $I(s)$ is locally uniformly convergent.

Let δ be positive real, for $|s| < \delta$. The integral over the circle does not depend on the radius $c < 2\pi$ by Cauchy's formula. For the infinite path, let $z = re^{\pm i\pi}$, then

$$\left|\frac{z^{s-1}}{e^{-z}-1}\right| = \frac{r^{\sigma-1}e^{\pm t\pi}}{e^r - 1} < \frac{r^{\delta-1}e^{\delta\pi}}{e^r - 1} < e^{-\frac{r}{2}},$$

for r greater than some constant depending only on δ. Then the integral in the path C_1 and C_3 becomes a Riemann integral. Since $dz = e^{\pm i\pi}dr$ and by using the mean inequality, $I(s)$ converges uniformly in the disc $|s| < \delta$. It follows that $I(s)$ is an entire function.

Now to compute the value of $I(s)$, let us consider the function

$$g(z) := \frac{1}{e^{-z}-1}.$$

We parametrise z in C_1 by $z = re^{-i\pi}$, in C_3 by $z = re^{i\pi}$, and in C_2 by $z = ce^{-i\theta}$ where $r > c$ and $\theta \in [-\pi, \pi]$.

Then,

$$\begin{aligned}
2i\pi I(s) &= \int_{C_1} z^{s-1}g(z)dz + \int_{C_2} z^{s-1}g(z)dz + \int_{C_3} z^{s-1}g(z)dz \\
&= -\int_c^\infty r^{s-1}e^{-s\pi i}g(-r)dr + \int_{-\pi}^{\pi} c^s e^{s\theta i}g(ce^{\theta i})id\theta + \int_c^\infty r^{s-1}e^{s\pi i}g(-r)dr \\
&= (e^{s\pi i} - e^{-s\pi i})\int_c^\infty r^{s-1}g(-r)dr + \int_{-\pi}^{\pi} c^s e^{s\theta i}g(ce^{\theta i})id\theta \\
&= 2i\sin(s\pi)\int_c^\infty r^{s-1}g(-r)dr + ic^s\int_{-\pi}^{\pi} e^{s\theta i}g(ce^{\theta i})d\theta.
\end{aligned}$$

We can see that the function $zg(z)$ is continuous in the compact set $|z| \le \pi$. Then there exists $A_1 > 0$ such that for $|z| \le \pi$

$$|zg(z)| \le A_1.$$

We know that the value of $I(s)$ does not depend on $c < 2\pi$ so for $c \le \pi$,

$$\begin{aligned}
\left|ic^s\int_{-\pi}^{\pi} e^{s\theta i}g(ce^{\theta i})d\theta\right| &\le c^\sigma \int_{-\pi}^{\pi}\left|e^{s\theta i}g(ce^{\theta i})\right|d\theta \\
&\le A_1 c^{\sigma-1}\int_{-\pi}^{\pi} e^{-t\theta}d\theta.
\end{aligned}$$

We will see later on that,

$$\int_0^\infty r^{s-1}g(-r)dr = \Gamma(s)\zeta(s) \tag{3.3}$$

for $s \in \mathbb{C}$ with $\Re(s) > 1$. It follows that

$$\lim_{c\to 0} 2i\pi I(s) = 2i\sin(s\pi)\Gamma(s)\zeta(s)$$

since $I(s)$ does not depend on c. Therefore

$$I(s) = \frac{\sin(s\pi)}{\pi}\Gamma(s)\zeta(s)$$

for $\Re(s) > 1$. □

Theorem 3.7. *The function $\zeta(s)$ defined by $\sum_{1\leq n} n^{-s}$ for $\Re(s) > 1$ can be extended meromorphically to the whole domain $\mathbb{C}$.*

Proof. First we have the relation,

$$\int_0^\infty e^{-nx}x^{s-1}dx = \frac{\Gamma(s)}{n^s}.$$

For $\Re(s) > 1$ and n positive integer greater than 1, by a simple change of variable $x = nt$ on the definition of $\Gamma(s)$, we obtain

$$\begin{aligned}\Gamma(s)\sum_{1\leq n}\frac{1}{n^s} &= \sum_{1\leq n}\int_0^\infty e^{-nx}x^{s-1}dx \\ &= \int_0^\infty \frac{x^{s-1}}{e^x - 1}dx.\end{aligned}$$

We can interchange sum and integration because any integration and sum appearing in the equations are absolutely convergent for any s in $\mathbb{C}$ (this proves the relation 3.3). Therefore we get

$$\zeta(s) = \frac{\pi I(s)}{\sin(s\pi)\Gamma(s)} = \Gamma(1-s)I(s), \tag{3.4}$$

by using the equation (3.1).The function $I(s)$ is entire, and $\Gamma(s)$ a meromorphic function, then $\zeta(s)$ defined in the relation (3.4) is a meromorphic function in $\mathbb{C}$. □

Remark 3.8. We have seen that the function $\Gamma(s)$ has poles at $s = 0, -1, -2, \ldots$, then $\Gamma(1-s)$ has a pole at $s = 1,\ \ 2,\ \ 3, \cdots$. We know that $\zeta(s)$ has a pole at $s = 1$ and is well-defined at $s = 2,\ \ 3, \cdots$, thus $I(s)$ must be zero at those points. Hence the only pole of $\zeta(s)$ is $s = 1$ and it has the residue 1, this result is called the analytic continuation of $\zeta(s)$.

3.1.3 Functional Equation

Now we have defined the function $\zeta(s)$ for all $s \in \mathbb{C}$, but we still have no idea about the function $\zeta(s)$ for $\Re(s) < 1$. So we give in this section, the relation between $\zeta(1-s)$ and $\zeta(s)$, a relation that we call the *Functional Equation*.

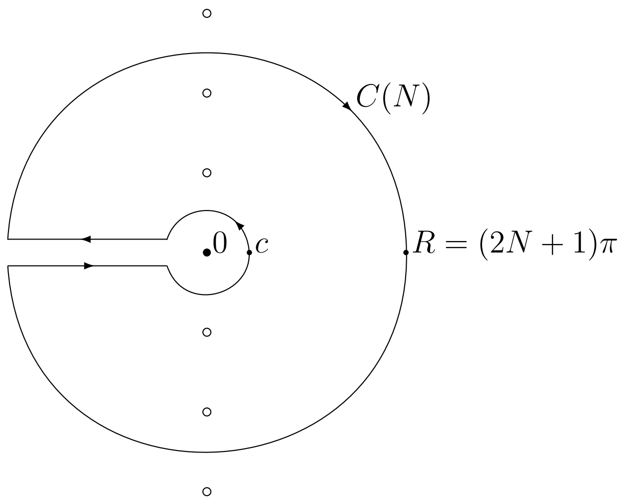
Theorem 3.9. *The function $\zeta(s)$ satisfies for all s the functional equation*

$$\zeta(1-s) = 2(2\pi)^{-s}\cos(\tfrac{1}{2}\pi s)\Gamma(s)\zeta(s). \tag{3.5}$$

Proof. Since $\{\Re(s) < 0\}$ is an open set in $\mathbb{C}$ and all functions in this equation are meromorphics, it suffices to prove Equation (3.5) for s with $\Re(s) < 0$ and use the Uniqueness of analytic function in Theorem 2.2. We are going to approximate the function $I(s)$ by the sequence of functions,

$$I_N(s) := \int_{C(N)} \frac{z^{s-1}}{e^{-z}-1}dz, \quad \text{for} N \in \mathbb{N},$$

for $s \in \mathbb{C}$ with $\Re(s) = \sigma < 0$ and where $C(N)$ is the path in the picture below.



Let us consider $S(\delta)$ the region which remains when we remove from $\mathbb{C}$ the set $\cup_{n\in\mathbb{Z}} D(2i\pi n, \delta)$, where $D(s_0, \delta)$ is the disc of centre s_0 and radius δ, and we should take $\delta < \pi$. Then, in $S(\delta)$ the function $\dfrac{1}{e^{-z}-1}$ is bounded, since it goes to zero if $|z| \to \infty$ and the pole of this function is $2in\pi$ with $n \in \mathbb{Z}$. Therefore we have for $\Re(s) < 0$, in the outer part of the path C, which is included in $S(\delta)$, and if $z = Re^{i\theta}$,

$$\left| \frac{z^{s-1}}{e^{-z}-1} \right| < R^{\sigma-1} e^{|t|\pi} A_2,$$

where A_2 is a real positive number, such that $\left| \dfrac{1}{e^{-z}-1} \right| < A_2$ for z in $S(\delta)$.

Then the value of the integral over the outer part is less than $R^{\sigma} e^{|t|\pi} A_2$. Since $\sigma < 0$,

$$\lim_{N\to\infty} I_N(s) = I(s). \tag{3.6}$$

Now by the Cauchy Residue Theorem,

$$\begin{aligned}
I_N(s) &= \sum_{n=1}^{N}\{(2in\pi)^{s-1} + (-2in\pi)^{s-1}\} \\
&= \sum_{n=1}^{N}(2n\pi)^{s-1}(i^{s-1} + (-i)^{s-1}) \\
&= \sum_{n=1}^{N}(2n\pi)^{s-1}(e^{i\frac{\pi}{2}(s-1)} + e^{-i\frac{\pi}{2}(s-1)}) \\
&= \sum_{n=1}^{N} 2(2n\pi)^{s-1}\cos(\tfrac{\pi}{2}(s-1)) \\
&= 2(2\pi)^{s-1}\sin(\tfrac{\pi}{2}s)\sum_{n=1}^{N} n^{s-1},
\end{aligned}$$

since the pole $2ni\pi$ where $0 < |n| \leq N$ is inside the region C_N and the function $\dfrac{z^{s-1}}{e^{-z}-1}$ has the residue $-(2ni\pi)^{s-1}$ at the pole $2ni\pi$.

In addition $\Re(1-s) = 1-\sigma > 1$, then using the limit (3.6)

$$I(s) = 2(2\pi)^{s-1}\sin(\tfrac{\pi}{2}s)\sum_{n=1}^{\infty} n^{s-1} = 2(2\pi)^{s-1}\sin(\tfrac{\pi}{2}s)\zeta(1-s).$$

By using the formula (3.4) we get

$$2(2\pi)^{s-1}\sin(\tfrac{\pi}{2}s)\zeta(1-s) = \frac{\sin(s\pi)}{\pi}\zeta(s)\Gamma(s).$$

Then using the trigonometric relation $\sin(s\pi) = 2\sin(\frac{\pi}{2}s)\cos(\frac{\pi}{2}s)$ we have for $\Re(s) < 0$ the functional equation

$$\zeta(1-s) = 2(2\pi)^{-s}\cos(\tfrac{1}{2}\pi s)\Gamma(s)\zeta(s).$$

□

The function $\zeta(1-s)$ is holomorphic in the set $\{\Re(s) < 0\}$, but we have seen from Proposition 3.4 that $\Gamma(s)$ has poles at $s = -1, \ -2, \ -3, \cdots$, and it follows that $\zeta(s)$ is necessarily zero at the points $s = -2, \ -4, \ -6, \ \cdots$.

Theorem 3.10. *The function*

$$\xi(s) := \tfrac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s)$$

is entire and satisfies, for all s, the equation

$$\xi(1-s) = \xi(s). \tag{3.7}$$

Proof. The function $\xi(s)$ is meromorphic, as a product of meromorphic functions is also meromorphic. We have seen that the function $\zeta(s)$ has a simple pole $s = 1$ but will be cancelled by the factor $(1 - s)$. The function $\Gamma(\frac{1}{2}s)$ has simple poles at $s = 0, \ -2, \ -4, \cdots$, but they will disappear since these points are zeros of $s\zeta(s)$. So the function ξ is entire.

For Equation (3.7), we have by definition

$$\begin{aligned}\xi(1-s) &= \tfrac{1}{2}s(s-1)\pi^{-\frac{1}{2}+\frac{1}{2}s}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\zeta(1-s)\\ &= \tfrac{1}{2}s(1-s)\pi^{-\frac{1}{2}s}\zeta(s)\pi^{-\frac{1}{2}}2^{1-s}\cos(\tfrac{1}{2}\pi s)\Gamma(s)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s),\end{aligned}$$

that comes from the functional equation (3.5)

$$\zeta(1-s) = 2(2\pi)^{-s}\cos(\tfrac{1}{2}\pi s)\Gamma(s)\zeta(s).$$

Then what we need to show is that

$$\Gamma(\tfrac{1}{2}s) = 2^{1-s}\pi^{\frac{-1}{2}} cos(\tfrac{1}{2}\pi s)\Gamma(s)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s). \tag{3.8}$$

But we know from the properties of the Gamma function in Proposition 3.5 by applying the equation (3.2) (after substituting s for $\frac{1}{2}s$) that

$$\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}) = 2^{1-s}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s) \tag{3.9}$$

and we can derive from Equation (3.1) the relation

$$\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s) = \frac{\pi}{cos(\frac{1}{2}\pi s)}. \tag{3.10}$$

Then Equation (3.8) follows from Equation (3.9) and Equation (3.10). □

3.2 Applications

We shall use the relations that we have seen up till now to find some results and the generalistions of some formulas that we knew before.

3.2.1 Bernoulli Numbers B_n

Definition 3.11. The Bernoulli Numbers are the sequence of numbers B_n, n positive integer, such that the Taylor expansion of the analytic function $z(e^z - 1)^{-1}$, near the point $z = 0$ is

$$\frac{z}{e^z-1} = \sum_{0\leq n}\frac{B_n z^n}{n!}. \tag{3.11}$$

We have a few values of B_n, for n even $B_0 = 1$; $B_2 = \frac{1}{6}$; $B_4 = \frac{-1}{30}$; $B_6 = \frac{1}{42}$ and $B_8 = \frac{-1}{30}$. For n odd $B_1 = \frac{-1}{2}$ and $B_3 = B_5 = \cdots = 0$. This last assertion is due to the fact that the function $h(z) = z(e^z - 1)^{-1} + \frac{1}{2}z$ is even, since

$$\begin{aligned} h(-z) - h(z) &= -z(e^{-z} - 1)^{-1} - \frac{1}{2}z - z(e^z - 1)^{-1} - \frac{1}{2}z \\ &= -z\left(\frac{e^z + e^{-z} - 2}{(e^z - 1)(e^{-z} - 1)} + 1\right) \\ &= 0. \end{aligned}$$

Proposition 3.12 (Formula for computing B_n)**.** *We have for m a positive integer*

$$B_0 = 1, \quad \sum_{j=0}^{m} \binom{m+1}{j} Bj = 0.$$

Proof. First, by multiplying Relation (3.11) by $e^z - 1$, we have

$$\begin{aligned} z &= \sum_{0 \le n} \frac{B_n}{n!} z^n (e^z - 1) \\ &= \sum_{0 \le n} \frac{B_n}{n!} z^n \sum_{1 \le m} \frac{z^m}{m!} \\ &= \sum_{0 \le m} \left[\sum_{j=0}^{m} \binom{m+1}{j} B_j\right] \frac{z^{m+1}}{(m+1)!} \end{aligned}$$

by the formula of **multiplication of power series.**[1]. Then the proof completes by the uniqueness of the Taylor expansion, which applies to the identity function z. □

3.2.2 Values of $\zeta(s)$ for s Integer

We recall that for $s \in \mathbb{C}$,

$$\zeta(s) = \frac{\pi I(s)}{\sin(s\pi)\Gamma(s)} = \Gamma(1-s)I(s) \tag{3.12}$$

and the functional equation

$$\zeta(1-s) = 2(2\pi)^{-s} \cos(\tfrac{1}{2}\pi s)\Gamma(s)\zeta(s). \tag{3.13}$$

Then we have the following theorems.

[1]The product of a power two series $\sum_{0 \le n} a_n z^n$ and $\sum_{0 \le n} b_n z^n$ is a power series $\sum_{0 \le n} c_n z^n$ where, for all n,

$$c_n = \sum_{j=0}^{n} b_j a_{n-j} = \sum_{j=0}^{n} a_j b_{n-j}.$$

Theorem 3.13 (values of $\zeta(s)$ in the set of negative integer). *For* $n = 0,\ \ 1,\ \ 2,\ \ 3, \cdots,$

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}.$$

Proof. From Equation (3.12) we have,

$$\begin{aligned}\zeta(-n) &= \Gamma(1+n)I(-n)\\ &= n!I(-n).\end{aligned}$$

To compute $I(-n)$ we shall consider the same path C in Proposition 3.6. Doing exactly the same computation for $I(-n)$ as we did for $I(s)$, we have, on the path C_1 and C_3 , $z = re^{\pm i\pi}$ and for c small enough

$$\begin{aligned}2\pi i I(-n) &= \int_{C_1} \frac{z^{-n-1}}{e^{-z}-1}dz + \int_{C_2} \frac{z^{-n-1}}{e^{-z}-1}dz + \int_{C_3} \frac{z^{-n-1}}{e^{-z}-1}dz\\ &= -(-1)^n \int_c^\infty \frac{r^{-n-1}}{e^r-1}dr + \int_{|z|=c} \frac{z^{-n-1}}{e^{-z}-1}dz + (-1)^n \int_c^\infty \frac{r^{-n-1}}{e^r-1}dr\\ &= \int_{|z|=c} \left(\sum_m \frac{B_m(-z)^m}{m!}\right) z^{-n-1}\frac{dz}{-z}\\ &= -\sum_m \frac{(-1)^m B_m}{m!} \int_{-\pi}^{\pi} iz^{m-n-1}d\theta\\ &= -2i\pi(-1)^{n+1}\frac{B_{n+1}}{(n+1)!}.\end{aligned}$$

Then it follows that

$$\zeta(-n) = -n!(-1)^{n+1}\frac{B_{n+1}}{(n+1)!} = (-1)^n \frac{B_{n+1}}{n+1}.$$

□

We have as a corollary a result that has already been seen.

Corollary 3.14. *The function* $\zeta(s) = 0$ *for* $s = -2,\ \ -4,\ \ -6, \cdots,$ *and* $\zeta(0) = -1/2.$

Proof. Since B_n is zero for an odd value of n, except for $n = 1$, $B_1 = -1/2$. □

Theorem 3.15. *For* $n = 1,\ \ 2,\ \ 3, \cdots$ *we have*

$$\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2 \cdot (2n)!}. \tag{3.14}$$

Proof. By using the functional equation (3.5) for $s = 2n$ we have,

$$\zeta(1-2n) = 2(2\pi)^{-2n}\cos(\tfrac{1}{2}\pi 2n)\Gamma(2n)\zeta(2n).$$

But by Theorem 3.13

$$\zeta(1-2n) = \zeta(-(2n-1)) = (-1)^{2n-1}\frac{B_{2n}}{2n} = -\frac{B_{2n}}{2n}.$$

Then

$$2(-1)^n(2\pi)^{-2n}\zeta(2n)(2n-1)! = -\frac{B_{2n}}{2n}$$

which is equivalent to Equation (3.14). □

3.2.3 Riemann Hypothesis and the Prime Number Theorem

The Riemann hypothesis is the conjecture set by Riemann about the zeros of the meromorphic zeta function. We have seen in Corollary 3.14 that the points $-2, \quad -4, \quad -6, \cdots$ are zeros of $\zeta(s)$. But the ones that we are interested in are the those in $\{0 < \Re(s) < 1\}$. So let us call them the non-trivial zeros of zeta. The existence of those non-trivial zeros has already been proven, there are infinitely many of them, but we are not going to give the proof of this. We can find a proof in [Ing95].

Conjecture 3.16 (The Riemann Hypothesis)**.** *All non-trivial zeros of* $\zeta(s)$ *lie on the line* $\{\Re(s) = \frac{1}{2}\}$.

One of the key facts in the proof of the Prime Number Theorem was the fact that $\zeta(s)$ has no zero on the line $\{\Re(s) = 1\}$. But if the Riemann hypothesis is true then we should have the following result.

Theorem 3.17. *The Riemann Hypothesis is equivalent to*

$$\pi(x) = \mathrm{li}(x) + O(\sqrt{x}\log x),$$

where the $\mathrm{li}(x)$ *is the logarithmic integral function, defined for* $x > 1$ *as,*

$$\mathrm{li}(x) := \lim_{\epsilon\to 0^+}\left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^{x}\right)\frac{1}{\log t}dt.$$

We are not going to prove this statement, we can obtain a proof in [Ing95]. But at least we can prove that $\mathrm{li}(x)$ is well defined and

$$\pi(x) \sim \mathrm{li}(x).$$

For the convergence of $\mathrm{li}(x)$ it is sufficient to prove that $\mathrm{li}(2)$ exists, because if $\mathrm{li}(2)$ exists then for all $x > 1$ $\mathrm{li}(x) = \mathrm{li}(2) + \int_2^x \frac{1}{\log t}dt$. So we have for a small $\epsilon > 0$

$$\begin{aligned}\int_0^{1-\epsilon}\frac{1}{\log t}dt + \int_{1+\epsilon}^{2}\frac{1}{\log t}dt &= \int_{-1}^{-\epsilon}\frac{1}{\log(1+x)}dx + \int_{\epsilon}^{1}\frac{1}{\log(1+x)}dx\\ &= \int_{\epsilon}^{1}\frac{1}{\log(1-x)}dx + \int_{\epsilon}^{1}\frac{1}{\log(1+x)}dx\\ &= \int_{\epsilon}^{1}\left(\frac{1}{\log(1+x)} + \frac{1}{\log(1-x)}\right)dx\end{aligned}$$

There is no problem with the limit 1, because

$$\lim_{x \to 1^-} \left(\frac{1}{\log(1+x)} + \frac{1}{\log(1-x)} \right) = \frac{1}{\log 2}.$$

For the lower limit, we have for a small x,

$$\log(1+x) = x - \frac{x^2}{2} + x^2 o(x) \qquad \log(1-x) = -x - \frac{x^2}{2} + x^2 o(x).$$

Then

$$\begin{aligned} \frac{1}{\log(1+x)} + \frac{1}{\log(1-x)} &= \frac{\log(1+x) + \log(1-x)}{\log(1+x)\log(1-x)} \\ &= \frac{-x^2 + x^2 o(x)}{-x^2 + x^2 o(x)} \\ &= \frac{-1 + o(x)}{-1 + o(x)} \end{aligned}$$

which is an integrable function in a neighbourhood of 0 whatever the functions $o(x)$ are. Then the integral $\int_0^1 \left(\frac{1}{\log(1+x)} + \frac{1}{\log(1-x)} \right) dx$ is convergent.

In addition for any $1 > \epsilon > 0$ we have the relation, for x sufficiently large,

$$\frac{x-2}{\log x} \leq \int_2^x \frac{1}{\log t} dt \leq \int_2^{x^\epsilon} \frac{1}{\log t} dt + \frac{x - x^\epsilon}{\epsilon \log x} \leq \frac{x^\epsilon}{\log 2} + \frac{x - x^\epsilon}{\epsilon \log x}.$$

Therefore, for any ϵ, $1 > \epsilon > 0$

$$1 \leq \lim_{x \to \infty} \frac{\int_2^x \frac{1}{\log t} dt}{\frac{x}{\log x}} \leq \frac{1}{\epsilon}.$$

Then it follows that $\mathrm{li}(x) \sim \pi(x)$ when x goes to infinity, since $\mathrm{li}(x)$ differs from $\int_2^x \frac{1}{\log t} dt$ by $\mathrm{li}(2) \simeq 1.04$.

Actually, we can also approximate $\pi(x)$ just as well with $\int_2^x \frac{dt}{\log t}$.

Conclusion

We have proved the Prime Number Theorem by a deeper study of the Riemann Zeta Function $\zeta(s)$. We have seen that the Prime Number Theorem was derived from the fact that the meromorphic function $\zeta(s)$ has no zero in the line $\{\Re(s) = 1\}$, by the use of the Analytic Theorem. In the last Chapter, we introduce the link between the Riemann Hypothesis and the error of $\pi(x)$ in the logarithmic integral $\mathrm{li}(x)$.

We have below some values of the functions $\pi(x)$, $\mathrm{li}(x)$, and the ratio $\pi(x)/\,\mathrm{li}(x)$.

x	$\pi(x)$	$\mathrm{li}(x)$	$\pi(x)/\mathrm{li}(x)$
1000	168	178	0.94...
10,000	1,229	1,246	0.98...
50,000	5,133	5,167	0.993...
100,000	9,592	9,630	0.996...
500,000	41,538	41,606	0.9983...
1,000,000	78,498	78,628	0.9983...
2,000,000	148,933	149,055	0.9991...
5,000,000	348,513	348,638	0.9996...
10,000,000	664,579	664,918	0.9994...
20,000,000	1,270,607	1,270,905	0.9997...
90,000,000	5,216,954	5,217,810	0.99983...
100,000,000	5,761,455	5,762,209	0.99986...
1,000,000,000	50,847,534	50,849,235	0.99996...
10,000,000,000	455,052,512	455,055,614	0.999993...

Acknowledgements

I am specially grateful to my supervisor Dr Florian Breuer for providing me all the materials necessary for accomplishing this essay.

I would like also to thank everyone who helped me, specially the A.I.M.S staffs , all the lecturers, the tutors, and all the students.

Special acknowledgement is due to my family, friends, and specially my dear beloved Sylvia for their morale supports, and encouragements.

Bibliography

[Ahl53] Ahlfors L.V., *Complex Analysis*, Mc Graw Hill, 1953.

[AZ99] Aigner M. and Ziegler G.M., *Proofs from THE BOOK*, Springer, 1999.

[BD96] Bateman P.T. and Diamond H.G., *A Hundred Years of Prime Numbers*, The American Mathematical Monthly **103** (1996), no. 9, 729–741.

[Bre06] Breuer F., *Cours note from "topics in number theory". Honours Course*, Stellenbosch University, 2006.

[Ing95] Ingham A.E., *The Distribution of Prime Numbers*, Cambridge University Press, 1995.

[Jam03] Jameson G.J.O, *The Prime Number Theorem*, Cambridge University Press, 2003.

[New80] Newman D.J., *Simple Analytic Proof of the Prime Number Theorem*, The American Mathematical Monthly **87** (1980), no. 9, 693–696.

[Riv07] Vincent Rivasseau, *Advanced real and complex analysis. A course for the African Institute for Mathematical Sciences.*, Paris VI university at Orsay, 27 February 2007.

[Zag97] Zagier D., *Newman's Short Proof of the Prime Number Theorem*, The American Mathematical Monthly **104** (1997), no. 8, 705–708.