Operators on Hilbert Spaces

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Abstract

The concepts of arbitrary vector spaces can be generalised to inner product spaces and complete inner product spaces, called **Hilbert spaces**. The inner product is a generalisation of the dot product in $\mathbb{R}^n$. The dot product and orthogonality are important in many applications. The theory of inner product spaces and Hilbert spaces is richer than that of general normed and Banach spaces.

We recall that a Banach space is a complete normed vector space. Hence, every Hilbert space is a Banach space but the converse is not generally true. A necessary and sufficient condition for a Banach space to be a Hilbert space is that the parallelogram equality, which will be discussed in a subsequent section, should hold.

The distinguishing features between Hilbert spaces and Banach spaces are

1. Representation of a Hilbert space as a direct sum of a closed subspace and its orthogonal complement.
2. Orthogonal sets and sequences and corresponding representations of the elements of the Hilbert space.
3. The Riesz representation of bounded linear functionals by inner products.
4. The Hilbert-adjoint operator $\mathbf{T}^*$ of the bounded linear operator $\mathbf{T}$.

Spectral theory is a very broad but important aspect of applied functional analysis which extends to the eigenvector and eigenvalue theory of a single square matrix. The name was introduced by David Hilbert in his original formulation of Hilbert space theory. It was later discovered that spectral theory could explain features of atomic spectral in quantum mechanics. In this essay, we will consider the spectrum of a Hermitian (or self-adjoint) operator.

Contents

1. Introduction

The equation $Lf = g$ occurs frequently in fields like mathematics, physics and engineering. Here g is a known function of space and time variables, L is a linear differential operator or mapping and f is a function which is unknown but required to satisfy some initial or boundary conditions. The major problem will be that of finding f. No matter the trick used, it is not certain to work for all equations. Moreover, it is unlikely to obtain a general solution in a form explicit enough to tell everything about the system. We will be faced with questions like, does every equation has a solution? If so, is it unique? In dealing with these questions, we might seek inspiration from linear algebra since $Lf = g$ resembles $Ax = b$, where A is a matrix and x, b are vectors. In finite dimensions, the solution exists and is unique if and only if the determinant of L is not equal to zero (det $L \neq 0$). However, the differences between finite and infinite dimensions are great. In infinite dimensions, determinants can only play a very limited role. Nineteenth-century analysts made great progress with these questions. In trying to present the result with simplicity and generality brought about the evolution of functional analysis.

In modern view, functional analysis is seen as the study of spaces of functions over the real or complex numbers. Hilbert spaces are the most useful spaces in practical applications of functional analysis. The notion of the Hilbert space was initiated by D. Hilbert, a German mathematician, in 1912. Completeness is an extremely important property of Hilbert spaces as we shall see. Complete spaces posses many useful properties that are absent in incomplete spaces. Hilbert spaces are useful in partial differential equations, quantum mechanics and signal processing.

Operators or mappings such as L in the equation above can be defined on Hilbert spaces. In particular, bounded linear operators can be defined on Hilbert spaces and this is what we seek to study in this essay. We shall acquaint ourselves with properties of normed and inner product spaces. We shall discuss mappings or operators from one space to another. We need at least two sets in order to define an operator. If the two sets are vector spaces, we can introduce the concept of a linear operator, if the sets are normed spaces, we can construct a theory of bounded linear operators on such spaces. Operators that map members of a specified space into the real or complex numbers are called functionals. We shall also discuss the Hilbert-adjoint operators as well as the self-adjoint, unitary and normal operators. Finally, we shall look at the spectral analysis of the Hermitian (self-adjoint) operators which is an important aspect of functional analysis.

2. Basic Concepts

2.1 Inner product spaces and Hilbert spaces

To make a definition that will be applicable to both real and complex vector spaces, we need to examine some properties of the complex plane $\mathbb{C}^n$. Recall that if $\lambda = a + bi$, where $a, b \in \mathbb{R}$, then the absolute value of λ is defined by $|\lambda| = \sqrt{a^2 + b^2}$ and the complex conjugate of λ is defined by $\overline{\lambda = a - bi}$. Also, $|\lambda|^2 = \lambda\overline{\lambda}$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we define the norm of z by $\|z\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. We want $\|z\|$ to be nonnegative hence we take the absolute values of its elements. Note that $\|z\|^2 = z_1\overline{z_1} + \cdots + z_n\overline{z_n}$. Also for any $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, the inner product of w with z is equal to the complex conjugate of the inner product of z with w.

Definition 2.1.1 *([Kre78], 3·1-1 (Inner product))*
An inner product on a vector space **X** *is a function that takes each ordered pair (x, y) of elements of* **X** *to number $\langle x, y\rangle$ of the elements of K (where K is the scalar field of* **X***). That is, with every pair of vector (x, y), there is associated a scalar $\langle x, y\rangle$ with the following properties:*

1. $\langle x, x\rangle \geq 0$ *for all* $x \in$ **X**,
2. $\langle x, x\rangle = 0$ *if and only if* $x = 0$,
3. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ *for all* $x, y, z \in$ **X**,
4. $\langle ay, z\rangle = a\langle y, z\rangle$ *for all* $a \in K$, *where K is the scalar field of* **X** *and* $y, z \in$ **X**,
5. $\langle y, z\rangle = \overline{\langle z, y\rangle}$ *for all* $y, z \in$ **X**.

An inner product on **X** defines a norm on **X** given by $\|x\| = \sqrt{\langle x, x\rangle} = \langle x, x\rangle^{\frac{1}{2}}$ and a metric on **X** given by $d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle}$.
An inner product space (or pre-Hilbert space) is a vector space with inner product $\langle x, y\rangle$ defined on it.

Definition 2.1.2 *([Kre78], 3·1-2 (Orthogonality))*
An element x of an inner product space **X** *is said to be orthogonal to an element $y \in$* **X** *if $\langle x, y\rangle = 0$. It is denoted by $x \perp y$ and we say that x and y are orthogonal. Similarly, for subsets* **A**, **B** $\subset$ **X**, *we write $x \perp$* **A** *if $x \perp a$ for all $a \in$* **A**, *and* **A** $\perp$ **B** *if $a \perp b$ for all $a \in$* **A** *and all $b \in$* **B**.

Definition 2.1.3 *(Hilbert space)*
A Hilbert space is a **complete** *inner product space.*

Example 2.1.4 *([Kre78], 3·1-3 (Examples of Hilbert spaces))*

1. *The Euclidean space $\mathbb{R}^n$ is a Hilbert space with inner product defined by $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$ where $x = (x_1, ..., x_n)$ and $y = (y_1, \cdots, y_n)$. We obtain that $\|x\| = \langle x, x\rangle^{\frac{1}{2}} = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$ and the metric defined on it is given by*

$$d(x, y) = \|x - y\| = \langle x - y, x - y\rangle^{\frac{1}{2}} = \left[(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2\right]^{\frac{1}{2}}.$$

2. *The unitary space $\mathbb{C}^n$: $\langle x, y\rangle = x_1\overline{y}_1 + \cdots + x_n\overline{y}_n$.*

3. *The Hilbert space $l^2 = \left\{\sum_{j=1}^{\infty} |x_j|^2 < \infty\right\}$, with inner product defined by $\langle x, y\rangle = \sum_{j=1}^{\infty} x_j\overline{y}_j$ and the norm $\|x\| = \langle x, x\rangle^{\frac{1}{2}} = (\sum_{j=1}^{\infty} |x_j|^2)^{\frac{1}{2}}$.*

2.2 Properties of inner product space

The norm on an inner product space **X** satisfies the following properties.

Theorem 2.2.1 *([Erd80], Lemma 1.4 (Parallelogram law))*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2). \tag{2.1}$$

Proof.
If $x, y \in \mathbf{X}$, then

$$\begin{aligned} \|x + y\|^2 + \|x - y\|^2 &= \langle x + y, x + y\rangle + \langle x - y, x - y\rangle \\ &= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|x\|^2 + \|y\|^2 - \langle x, y\rangle - \langle y, x\rangle \\ &= 2(\|x\|^2 + \|y\|^2). \end{aligned}$$

∎

Remark 2.2.2 *If a norm does not satisfy the parallelogram equality, then it cannot be obtained from an inner product space.*

Theorem 2.2.3 *([Erd80], Lemma 1.5 (Pythagorean theorem)) If x, y are orthogonal in an inner product space **X**, then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2. \tag{2.2}$$

Proof.
Suppose x, y are orthogonal vectors in **X**. Then

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y\rangle \\ &= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle \\ &= \|x\|^2 + \|y\|^2. \end{aligned} \tag{2.3}$$

∎

The Cauchy-Schwarz inequality gives one of the most important inequalities in Mathematics. Before we state and prove the theorem, let us examine how we can decompose two orthogonal vectors. If $x, y \in \mathbf{X}$, then we would like to write x as a scalar multiple of y plus a vector w orthogonal to y. This is called the orthogonal decomposition. Now let $a \in K$ (where K is the scalar field of $\mathbf{X}$). Then $x = ay + (x - ay)$. We choose a so that y is orthogonal to $(x - ay)$. That is, we want

$$0 = \langle x - ay, y\rangle = \langle x, y\rangle - a\|y\|^2 \text{ thus, } a = \frac{\langle x, y\rangle}{\|y\|^2}$$

(where $y \neq 0$). Hence we write

$$x = \frac{\langle x, y\rangle}{\|y\|^2} y + \left(x - \frac{\langle x, y\rangle}{\|y\|^2} y\right).$$

We will use this to prove the Cauchy-Schwarz inequality.

Theorem 2.2.4 *([Erd80], Theorem* 1.1 *(Cauchy-Schwarz inequality)) The Cauchy Schwarz inequality states that if* $x, y \in \mathbf{X}$*, then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|. \tag{2.4}$$

Proof.
Let $x, y \in \mathbf{X}$. If $y = 0$, then both sides of (2.4) will be equal to 0 and the inequality will hold. Suppose $y \neq 0$. Consider the orthogonal decomposition $x = \frac{\langle x, y\rangle}{\|y\|^2} y + w$, where w is orthogonal to y. By the Pythagorean theorem,

$$\|x\|^2 = \left\|\frac{\langle x, y\rangle}{\|y\|^2} y\right\|^2 + \|w\|^2 = \frac{|\langle x, y\rangle|^2}{\|y\|^2} + \|w\|^2 \geq \frac{|\langle x, y\rangle|^2}{\|y\|^2}. \tag{2.5}$$

Multiplying both sides of equation (2.5) by $\|y\|^2$ and then taking square roots yield the Cauchy-Schwarz inequality. ■

Remark 2.2.5 *Equation* (2.4) *is an equality if and only if* (2.5) *is an equality. This will happen if and only if* $w = 0$*. But* $w = 0$ *if and only if* x *is a multiple* of *y. Hence equation* (2.4) *is an equality if and only if one of* x *or* y *is a scalar multiple of the other.*

Theorem 2.2.6 *([Axl97],* 6.9 *(Triangular inequality)) The triangular inequality states that the length of any side of a triangle is less than the sum of the lengths of the other two sides. If* x,y $\in \mathbf{X}$*, then*

$$\|x + y\| \leq \|x\| + \|y\|. \tag{2.6}$$

This theorem can be used to show that the shortest path between two points is a straight line segment.

Proof.
Let $x, y \in \mathbf{X}$. Then

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \langle y,x\rangle \\ &= \langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \overline{\langle x,y\rangle} \\ &= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\langle x,y\rangle \\ &\leq \|x\|^2 + \|y\|^2 + 2|\langle x,y\rangle| \end{aligned} \tag{2.7}$$

and by Cauchy-Schwarz inequality,

$$\begin{aligned} &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

Therefore $$\|x+y\| \leq \|x\| + \|y\|.$$ ■

Theorem 2.2.7 *([Kre78], 3·2-2 Lemma (Continuity of inner product))*
In an inner product space, if $x_n \to x$ and $y_n \to y$, then

$$\langle x_n, y_n\rangle \to \langle x, y\rangle. \tag{2.8}$$

Proof.
The condition $\langle x_n, y_n\rangle \to \langle x, y\rangle$ is equivalent to $|\langle x_n, y_n\rangle - \langle x, y\rangle| \to 0$ as $n \to \infty$.

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x,y\rangle| &= |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x,y\rangle| \\ &\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \qquad \text{(By triangular inequality)} \\ &\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\| \qquad \text{(By Schwarz inequality)} \end{aligned}$$

Now (x_n) is convergent and hence bounded, so that $\|x_n\| \leq K$ for all $n \in \mathbb{N}$. Therefore

$$|\langle x_n, y_n\rangle - \langle x,y\rangle| \leq K\|y_n - y\| + \|x_n - x\|\|y\|.$$

Since $y_n - y \to 0$ and $x_n - x \to 0$, the last expression tends to zero as n tends to infinity. ■

2.3 Orthogonal complement and direct sum

In a metric space $\mathbf{X}$, the distance δ from an element $x \in \mathbf{X}$ to a nonempty subset $\mathbf{M} \subset \mathbf{H}$ is given by

$$\delta = \inf_{y \in \mathbf{M}} d(x,y). \tag{2.9}$$

In a normed space it becomes

$$\delta = \inf_{y \in \mathbf{M}} \|x - y\|. \tag{2.10}$$

Theorem 2.3.1 *([Kre78], [Red53], 3·3-1 Theorem (Minimising vector)) Let* $\mathbf{X}$ *be an inner product space and let* $\mathbf{M} \neq \emptyset$ *be a convex subset which is complete (in the metric induced by the inner product). Then for every given* $x \in \mathbf{X}$*, there exists a unique* $y \in \mathbf{M}$ *such that*

$$\delta = \inf_{\hat{y} \in \mathbf{M}} \|x - \hat{y}\| = \|x - y\|. \tag{2.11}$$

Proof.

Existence

We choose a sequence (y_n) in $\mathbf{M}$ such that $\delta_n \to \delta$, where $\delta_n = \|x - y_n\|$ (by definition of infimum). We will show that (y_n) is a Cauchy sequence and then we can make some deductions. Using the parallelogram law,

$$\begin{aligned} \|y_n - y_m\|^2 &= \|(y_n - x) - (y_m - x)\|^2 \\ &= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - \|(y_n - x) + (y_m - x)\|^2 \\ &= 2{\delta_n}^2 + 2{\delta_m}^2 - 2^2\|\frac{1}{2}(y_n + y_m) - x\|^2. \end{aligned}$$

But $\frac{1}{2}(y_n + y_m) \in \mathbf{M}$ so that $\|\frac{1}{2}(y_n + y_m) - x\|^2 \geq \delta^2$. As n and m tends to infinity, we have $\|y_n - y_m\| \geq 0$, which implies that (y_n) is a Cauchy sequence. Now since $\mathbf{M}$ is complete, the sequence (y_n) will converge to a limit say $y_0 \in \mathbf{M}$ so that $\|x - y_0\| \geq \delta$. Also

$$\begin{aligned} \|x - y_0\| &= \|x - y_n + y_n - y_0\| \leq \|x - y_n\| + \|y_n - y_0\| \\ &= \delta_n + \|y_n - y_0\|. \end{aligned}$$

As n tends to infinity, $\|y_n - y_0\|$ tends to zero and δ_n tends to δ. Hence $\|x - y_0\| \leq \delta$ and we conclude that $\|x - y_0\| = \delta$.

Uniqueness

We assume that $y_1 \in \mathbf{M}$ and $y_2 \in \mathbf{M}$ both satisfy $\|x - y_1\| = \delta$ and $\|x - y_2\| = \delta$. We show that $y_1 = y_2$. By the parallelogram equality,

$$\begin{aligned} \|y_1 - y_2\|^2 &= \|(y_1 - x) - (y_2 - x)\|^2 \\ &= 2\|y_1 - x\|^2 + 2\|y_2 - x\|^2 - \|(y_1 - x) + (y_2 - x)\|^2 \\ &= 2\delta^2 + 2\delta^2 - 2^2\|\frac{1}{2}(y_1 + y_2) - x\|^2 \end{aligned}$$

But $\frac{1}{2}(y_1 + y_2) \in \mathbf{M}$ so that $\|\frac{1}{2}(y_1 + y_2) - x\|^2 \geq \delta^2$, which implies that

$$2\delta^2 + 2\delta^2 - 2^2\|\frac{1}{2}(y_1 + y_2) - x\|^2 \leq 4\delta^2 - 4\delta^2.$$

Hence

$$\|y_1 - y_2\| \leq 0.$$

Clearly $\|y_1 - y_2\| \geq 0$, which means that $y_1 = y_2$. ■

Theorem 2.3.2 *([Kre78], 3·3-2 Lemma (Orthogonality)) In Theorem* 2.3.1*, let* **M** *be a complete subspace* **Y** *and* $x \in \mathbf{X}$ *fixed. Then* $z = x - y$ *is orthogonal to* **Y**.

Proof.
Suppose by contradiction that z is not orthogonal to **Y**. Then there would be a $y_1 \in \mathbf{Y}$ such that

$$\langle z, y_1 \rangle = \beta \neq 0. \tag{2.12}$$

$y_1 \neq 0$ since otherwise $\langle z, y_1 \rangle = 0$.
Now for any scalar α,

$$\begin{aligned} \|z - \alpha y_1\|^2 &= \langle z - \alpha y_1, z - \alpha y_1 \rangle \\ &= \langle z, z \rangle - \bar{\alpha}\langle z, y_1 \rangle - \alpha[\langle y_1, z \rangle - \bar{\alpha}\langle y_1, y_1 \rangle] \\ &= \langle z, z \rangle - \bar{\alpha}\beta - \alpha[\bar{\beta} - \bar{\alpha}\langle y_1, y_1 \rangle]. \end{aligned} \tag{2.13}$$

Let us choose $\alpha = \frac{\beta}{\langle y_1, y_1 \rangle}$, so that $\bar{\alpha} = \frac{\bar{\beta}}{\langle y_1, y_1 \rangle}$. Then the expression $\bar{\beta} - \bar{\alpha}\langle y_1, y_1 \rangle$ becomes zero. From Theorem 2.3.1, we have that $\|z\| = \|x - y\| = \delta$ for some $y \in \mathbf{Y}$, so that equation (2.13) becomes

$$\|z - \alpha y_1\|^2 = \|z\|^2 - \frac{|\beta|^2}{\langle y_1, y_1 \rangle} < \delta^2.$$

But this is impossible since $z - \alpha y_1 = x - y_2$, where $y_2 = y + \alpha y_1 \in \mathbf{Y}$ so that $\|z - \alpha y_1\| \geq \delta$, by definition of δ. Hence equation (2.12) cannot hold, and the lemma is proved. ∎

Definition 2.3.3 *([Erd80], Orthogonal complement)*
Given a subspace **M** *of* **H***, the orthogonal complement* $\mathbf{M}^\perp$ *is defined by*

$$\mathbf{M}^\perp = \{x \in \mathbf{X} : \langle x, m \rangle = 0 \text{ for all } m \in \mathbf{M}\}. \tag{2.14}$$

Lemma 2.3.4 *([Erd80], 1·9 Lemma) For subsets* **A** *and* **B** *of a Hilbert space* **H***, if* $\mathbf{B} \supset \mathbf{A}$*, then*

(i) $\mathbf{A} \subset \mathbf{A}^{\perp\perp}$,

(ii) $\mathbf{B}^\perp \subset \mathbf{A}^\perp$,

(iii) $\mathbf{A}^{\perp\perp\perp} = \mathbf{A}^\perp$.

Proof.
(i) Let $a \in \mathbf{A}$. Then $a \perp \mathbf{A}^\perp$ and by Definition 2.3.3 $a \in \mathbf{A}^{\perp\perp}$. Hence $\mathbf{A} \subset \mathbf{A}^{\perp\perp}$.

(ii) Let $y \in \mathbf{B}^\perp$ and $a \in \mathbf{A}$. Then $\mathbf{A} \subset \mathbf{B}$ implies that $a \in \mathbf{B}$. Thus $y \perp a$. Hence $y \in \mathbf{A}^\perp$.

(iii) Applying (i) to $\mathbf{A}^\perp$ gives $\mathbf{A}^\perp \subset \mathbf{A}^{\perp\perp\perp}$. Also from (i), $\mathbf{A} \subset \mathbf{A}^{\perp\perp}$ and applying (ii) gives $\mathbf{A}^\perp \supset \mathbf{A}^{\perp\perp\perp}$. Hence $\mathbf{A}^{\perp\perp\perp} = \mathbf{A}^\perp$. ∎

Definition 2.3.5 *(([Kre78], 3·3-3 Definition (Direct Sum)) A vector space* $\mathbf{X}$ *is said to be the direct sum of two subspaces* $\mathbf{Y}$ *and* $\mathbf{Z}$ *of* $\mathbf{X}$ *if each* $x \in \mathbf{X}$ *has a unique representation*

$$x = y + z, \quad \text{with } y \ \in \mathbf{Y} \text{ and } z \in \mathbf{Z}. \tag{2.15}$$

We denote the direct sum of $\mathbf{Y}$ *and* $\mathbf{Z}$ *by*

$$\mathbf{X} = \mathbf{Y} \oplus \mathbf{Z}. \tag{2.16}$$

Our main interest is to represent a Hilbert space $\mathbf{H}$ as a direct sum of a closed subspace $\mathbf{Y}$ and its orthogonal complement $\mathbf{Y}^{\perp} = \{z \in \mathbf{H} \ |z \perp \mathbf{Y}\}$. This leads us to the next theorem which is sometimes called the projection theorem. Before we proceed, we will state and prove two useful lemmas.

Lemma 2.3.6 *If* $\mathbf{A}$ *is a subspace of a Hilbert space* $\mathbf{H}$*, then* $\mathbf{A}^{\perp}$ *is closed.*

Proof. Clearly, $\mathbf{A}^{\perp}$ is a vector subspace. To show that it is closed, let $t_n \in \mathbf{A}^{\perp}$ be a sequence converging to t. Then by the continuity of inner product, Theorem 2.2.7, for all $m \in \mathbf{A}$, $\langle t, m\rangle = lim_{n\to\infty}\langle t_n, m\rangle = 0$ so that $t \in \mathbf{A}^{\perp}$. ■

Lemma 2.3.7 *([Erd80], Corollary 1·8) If* $\mathbf{N}$ *is a closed subspace of a Hilbert space* $\mathbf{H}$*, then*

$$\mathbf{N}^{\perp} = \{0\} \Leftrightarrow \mathbf{N} = \mathbf{H}. \tag{2.17}$$

Proof.
Clearly if $\mathbf{N} = \mathbf{H}$ then $\mathbf{N}^{\perp} = \{0\}$. Now suppose that $\mathbf{N}^{\perp} = \{0\}$ and $\mathbf{N} \neq \mathbf{H}$. Take $h \notin \mathbf{N}$. Then there exists $n_0 \in \mathbf{N}$ such that $d(h, \mathbf{N}) = \|h - n_0\| \neq 0$. Moreover by Theorem 2.3.2, $0 \neq h - n_0 \perp \mathbf{N}$, so that $\mathbf{N}^{\perp} \neq \{0\}$ which contradicts our assumption that $\mathbf{N}^{\perp} = \{0\}$. Hence $\mathbf{N} = \mathbf{H}$. ■

Theorem 2.3.8 *([Erd80], Theorem 1·11) Let* $\mathbf{N}$ *be any closed subspace of a Hilbert space* $\mathbf{H}$*. Then*

$$\mathbf{H} = \mathbf{N} \oplus \mathbf{N}^{\perp}. \tag{2.18}$$

Proof.
If $\mathbf{N}$ is a closed subspace of $\mathbf{H}$, then by Theorem 2.3.8, $\mathbf{N} \oplus \mathbf{N}^{\perp}$ is also a closed subspace of $\mathbf{H}$. Also if $x \in (\mathbf{N} \oplus \mathbf{N}^{\perp})^{\perp}$ then $x \in \mathbf{N}^{\perp} \cap \mathbf{N}^{\perp\perp}$ so that $x = \{0\}$. Hence from Theorem 2.3.7, $\mathbf{N} \oplus \mathbf{N}^{\perp} = \mathbf{H}$. ■

Corollary 2.3.9 *A subspace* $\mathbf{A}$ *of a Hilbert space is closed if and only if* $\mathbf{A} = \mathbf{A}^{\perp\perp}$.

Proof.
Let $\mathbf{A}$ be a closed subspace of a Hilbert space $\mathbf{H}$. Then by Lemma $2.3.4$ (i), $\mathbf{A} \subset \mathbf{A}^{\perp\perp}$.
Let $x \in \mathbf{A}^{\perp\perp}$. Since $\mathbf{A}$ is a closed subspace of $\mathbf{H}$, then by Theorem $2.3.8$, $\mathbf{H} = \mathbf{A} \oplus \mathbf{A}^{\perp}$ and for some $y \in \mathbf{A}$ and $z \in \mathbf{A}^{\perp}$ we have that $x = y + z$. Now since $\mathbf{A} \subset \mathbf{A}^{\perp\perp}$, we have that $y \in \mathbf{A}^{\perp\perp}$ and $z = x - y \in \mathbf{A}^{\perp\perp}$. But $z \in \mathbf{A}^{\perp}$, so that $z = 0$. Hence $x = y \in \mathbf{A}$ and $\mathbf{A}^{\perp\perp} = \mathbf{A}$.

Conversely let $\mathbf{A} = \mathbf{A}^{\perp\perp}$. Since $\mathbf{A}^{\perp} \subset \mathbf{H}$, by Lemma $2.3.6$ we have that $(\mathbf{A}^{\perp})^{\perp} = \mathbf{A}$ is closed. ■

3. Operators

3.1 Functionals

Definition 3.1.1 *([Red53], 3·2 (Linear functional)) A linear functional on a Hilbert space* **H** *is a linear map from* **H** *to* $\mathbb{C}$*. That is*

$$\varphi : \mathbf{H} \to \mathbb{C}. \tag{3.1}$$

Definition 3.1.2 *([Red53], 5·2 (Bounded linear functional)) A linear functional* φ *is bounded, or continuous, if there exists a constant* K *such that*

$$|\varphi(x)| \leq K\|x\| \ \text{ for all } x \in \mathbf{H}. \tag{3.2}$$

.

The norm of a bounded linear functional is

$$\|\varphi\| = \sup_{\|x\|=1} |\varphi(x)|. \tag{3.3}$$

If $y \in \mathbf{H}$, then

$$\varphi_y(x) = \langle y, x\rangle \tag{3.4}$$

is a bounded linear functional on **H**, with $\|\varphi_y\| = \|y\|$.

Example 3.1.3 *The function* $\varphi : L^2(a,b) \to \mathbb{R}$ *defined by*

$$\varphi(u) = \int_a^b \varphi \ u(x)dx. \tag{3.5}$$

is a functional. Where $L^2(a,b) = \left\{u \text{ integrable} \mid \int_a^b |\varphi|^2 \leq +\infty\right\}$ *and* $\langle\varphi, u\rangle = \int_a^b \varphi \ u(x)dx$ *defines an inner product on* L^2*. Thus for every* $u \in L^2(a,b)$*,* $\|\varphi\|_{L^2} = \sqrt{\int_a^b \varphi^2 u^2(x)dx}$*. Clearly, for* $\varphi = 1$*,* φ *is a linear functional since* $\langle\varphi, \alpha u + \beta v\rangle = \alpha\langle\varphi, u\rangle + \beta\langle\varphi, v\rangle$*. Further more, applying the Cauchy-Schwarz inequality on* L^2 *we have*

$$|\langle\varphi, u\rangle| = \left|\int_a^b 1 \cdot u(x)dx\right| \leq \|1\|_{L^2}\|u\|_{L^2} = |b-a|\|u\|_{L^2}.$$

Hence φ *is bounded.*

Remark 3.1.4 *One of the fundamental facts about Hilbert spaces is that all bounded linear functionals are of the form given in equation* (3.4)*. This is the basis for the next theorem.*

3.2 Riesz Representation of functionals

Theorem 3.2.1 *([Red53], 5·4 Theorem (Riesz representation theorem)) Every bounded linear functional φ on a Hilbert space **H** can be represented in terms of the inner product*

$$\varphi(x) = \langle x, u \rangle, \tag{3.6}$$

where u depends on φ and is uniquely determined by φ and has norm $\|\varphi\| = \|u\|$.

We want to prove that
(a) φ has a representation $\varphi(x) = \langle x, u \rangle$,
(b) u is unique,
(c) $\|\varphi\| = \|u\|$ holds.

Proof.
If $\varphi = 0$ then (a), (b) and (c) hold, if we take $u = 0$. So suppose that $\varphi \neq 0$.

(a) Now $\mathbf{N}(\varphi)$ is a closed subspace of **H**. Furthermore since $\varphi \neq 0$, we have that $\mathbf{N}(\varphi) \neq \mathbf{H}$, which implies that $\mathbf{N}(\varphi)^{\perp} \neq 0$ by Theorem 2.3.7. Hence there must be at least one nonzero element, say u_0 in $\mathbf{N}(\varphi)^{\perp}$.

Now set $z = \varphi(x)u_0 - \varphi(u_0)x$, then apply φ to give $\varphi(z) = \varphi(x)\varphi(u_0) - \varphi(u_0)\varphi(x) = 0, \text{ and so } z \in \mathbf{N}(\varphi)$. Also, since $u_0 \in \mathbf{N}(\varphi)^{\perp}$ we have,

$$\begin{aligned} 0 &= \langle z, u_0 \rangle = \langle \varphi(x)u_0 - \varphi(u_0)x, u_0 \rangle \\ &= \varphi(x)\langle u_0, u_0 \rangle - \varphi(u_0)\langle x, u_0 \rangle \end{aligned}$$

where

$$\langle u_0, u_0 \rangle = \|u_0\|^2 \neq 0,$$

which implies that

$$\varphi(x) = \frac{\varphi(u_0)\langle x, u_0 \rangle}{\|u_0\|^2}.$$

If we set

$$u = \frac{\overline{\varphi(u_0)}u_0}{\|u_0\|^2},$$

then $\varphi(x) = \langle x, u \rangle$. Hence (a) is proved.

(b) To prove that u is unique, suppose that for all $x \in \mathbf{H}$ we have,

$$\varphi(x) = \langle x, u_1 \rangle = \langle x, u_2 \rangle.$$

Then

$$\langle x, u_1 - u_2 \rangle = 0 \text{ for all } x.$$

If we choose x such that $x = u_1 - u_2$, then

$$\langle x, u_1 - u_2 \rangle = \langle u_1 - u_2, u_1 - u_2 \rangle = \|u_1 - u_2\|^2 = 0.$$

Hence $u_1 - u_2 = 0$, which implies that $u_1 = u_2$. Hence u is unique.

(c) Let $\varphi \neq 0$. Then $u \neq 0$. From (a) with $x = u$,

$$\|u\|^2 = \langle u, u \rangle = \varphi(u) \leq \|\varphi\| \|u\|$$

which implies that

$$\|u\| \leq \|\varphi\|. \tag{3.7}$$

By Cauchy-Schwarz inequality and (a),

$$|\varphi(x)| = |\langle x, u \rangle| \leq \|x\| \|u\|$$

which implies that $\|\varphi\| = \sup\limits_{\|x\|=1} |\langle x, u \rangle| \leq \|u\|$.

Hence

$$\|\varphi\| \leq \|u\|. \tag{3.8}$$

Combining equations (3.7) and (3.8) gives

$$\|u\| = \|\varphi\|, \tag{3.9}$$

which yields (c). ■

Remark 3.2.2 *The Riesz theorem is named after Frigyes Riesz a Hungarian mathematician. This representation is important in the theory of operators on Hilbert spaces. In particular it is important in the representation of the Hilbert-adjoint operator of a bounded linear operator. In the mathematical treatment of quantum mechanics, the theorem can be seen as a justification for the popular bra-ket notation. When the theorem holds, every ket $|\psi\rangle$ has a corresponding bra $\langle\psi|$.*

3.3 Sesquilinear functionals and Riesz representation

Definition 3.3.1 *([Kre78], 3·8-3 Definition (Sesquilinear functional)) Let* **X** *and* **Y** *be vector spaces over the same field* **K** *($\mathbb{R}$ or $\mathbb{C}$). A sesquilinear form (or sesquilinear functional) is defined as a mapping*

$$h : \mathbf{X} \times \mathbf{Y} \to K \tag{3.10}$$

such that for all $x, x_1, x_2 \in \mathbf{X}$ and $y, y_1, y_2 \in \mathbf{Y}$ and all scalars α and β,

1. $h(x_1 + x_2, y) = h(x_1, y) + h(x_2, y)$,

2. $h(x, y_1 + y_2) = h(x, y_1) + h(x, y_2)$,
3. $h(\alpha x, y) = \alpha h(x, y)$,
4. $h(x, \beta y) = \bar{\beta} h(x, y)$.

Remark 3.3.2 *Notice that h is linear in the first argument and conjugate linear in the second one. If* **X** *and* **Y** *are real then (4) becomes $h(x, \beta y) = \beta h(x, y)$. Clearly, h is bilinear in both arguments.*

Definition 3.3.3 *([Swa97], Definition 8 (Bounded sesquilinear functional)) A sesquilinear functional h is bounded if there exists $K > 0$ such that*

$$|h(x, y)| \leq K\|x\|\|y\|, \quad \text{for every } x \in \mathbf{X}, y \in \mathbf{Y}. \tag{3.11}$$

The norm of of h is defined to be

$$\|h\| = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|h(x, y)|}{\|x\|\|y\|} = \sup_{\substack{\|x\|=1\\ \|y\|=1}} |h(x, y)|. \tag{3.12}$$

It follows that $|h(x, y)| \leq \|h\|\|x\|\|y\|$ for all $x \in \mathbf{X}$, $y \in \mathbf{Y}$. The inner product is sesquilinear and it is bounded.

Lemma 3.3.4 *([Kre78], 3·8-2 Lemma (Equality))*
If $\langle x_1, y\rangle = \langle x_2, y\rangle$ for all y in an inner product space **X***, then $x_1 = x_2$. In particular if $\langle x_1, w\rangle = 0$ for all $w \in \mathbf{X}$, then $x_1 = 0$.*

Proof.
For all y,

$$\langle x_1 - x_2, y\rangle = \langle x_1, y\rangle - \langle x_2, y\rangle = 0.$$

If we choose $y = x_1 - x_2$, then $\|x_1 - x_2\|^2 = 0$. Hence we have $x_1 - x_2 = 0$ implying that $x_1 = x_2$. In particular, if $\langle x_1, y\rangle = 0$ with $y = x_1$ we get $\|x_1\|^2 = 0$, giving $x_1 = 0$. ■

Theorem 3.3.5 *([Kre78], 3·8-4 Theorem (Riesz representation))*
Let $\mathbf{H}_1$, $\mathbf{H}_2$ be Hilbert spaces and let $h : \mathbf{H}_1 \times \mathbf{H}_2 \to \mathbf{K}$ be a bounded sesquilinear form. Then h has a representation

$$h(x, y) = \langle \mathbf{S}x, y\rangle \text{ for } x \in \mathbf{H}_1,\ y \in \mathbf{H}_2 \tag{3.13}$$

where $\mathbf{S} : \mathbf{H}_1 \to \mathbf{H}_2$ is a bounded linear operator and **S** *is uniquely determined and has norm $\|\mathbf{S}\| = \|h\|$.*

We will apply Riesz Theorem 3.2.1 in the proof of this theorem. We will show that
(a) h has a representation $h(x,y) = \langle \mathbf{S}x, y\rangle$,
(b) $\|h\| = \|\mathbf{S}\|$, and
(c) $\mathbf{S}$ is unique.

Proof.
(a) Let us take a fixed x and then we will show that $\phi : y \to \overline{h(x,y)}$ is a linear map. Suppose $\phi(y) = \overline{h(x,y)}$, then by Definition 3.3.1,

$$\begin{aligned}\phi(y_1 + y_2) &= \overline{h(x, y_1 + y_2)} \\ &= \overline{h(x, y_1)} + \overline{h(x, y_2)} \\ &= \phi(y_1) + \phi(y_2).\end{aligned}$$

Moreover,

$$\begin{aligned}\phi(\alpha y) &= \overline{h(x, \alpha y)} \\ &= \overline{\overline{\alpha} h(x, y)} \\ &= \alpha \overline{h(x, y)} \\ &= \alpha \phi(y).\end{aligned}$$

Hence $\overline{h(x,y)}$ is linear.
Applying Theorem 3.2.1 yields a representation $\overline{h(x,y)} = \langle y, z\rangle$. Hence

$$h(x,y) = \langle z, y\rangle. \tag{3.14}$$

Our $z \in \mathbf{H}_2$ is unique but depends on our fixed $x \in \mathbf{H}_1$. Equation (3.14) with variable x defines an operator $\mathbf{S} : \mathbf{H}_1 \to \mathbf{H}_2$ given by $z = \mathbf{S}x$. Hence equation (3.14) becomes $h(x,y) = \langle \mathbf{S}x, y\rangle$, giving (3.13).
Now let us show that $\mathbf{S}$ is linear. From equation (3.13), and the sesquilinearity we obtain,

$$\begin{aligned}\langle \mathbf{S}(\alpha x_1 + \beta x_2), y\rangle &= h(\alpha x_1 + \beta x_2, y) \\ &= \alpha h(x_1, y) + \beta h(x_2, y) \\ &= \alpha \langle \mathbf{S}x_1, y\rangle + \beta \langle \mathbf{S}x_2, y\rangle \\ &= \langle \alpha \mathbf{S}x_1 + \beta \mathbf{S}x_2, y\rangle\end{aligned}$$

for all y in $\mathbf{H}_2$, so that by Lemma 3.3.4,

$$\mathbf{S}(\alpha x_1 + \beta x_2) = \alpha \mathbf{S}x_1 + \beta \mathbf{S}x_2.$$

Hence $\mathbf{S}$ is linear.
(b) We show that $\mathbf{S}$ is bounded. Leaving the trivial case $\mathbf{S} = 0$, we have

$$\|h\| = \sup_{\substack{x\neq 0 \\ y \neq 0}} \frac{|\langle \mathbf{S}x, y\rangle|}{\|x\|\|y\|} \geq \sup_{\substack{x\neq 0 \\ \mathbf{S}x \neq 0}} \frac{|\langle \mathbf{S}x, \mathbf{S}x\rangle|}{\|x\|\|\mathbf{S}x\|} = \sup_{x\neq 0} \frac{\|\mathbf{S}x\|}{\|x\|} = \|\mathbf{S}\|. \tag{3.15}$$

Now by Cauchy-Schwarz inequality,

$$\sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|\langle \mathbf{S}x, y\rangle|}{\|x\|\|y\|} \leq \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{\|\mathbf{S}x\|\|y\|}{\|x\|\|y\|} = \|\mathbf{S}\|, \tag{3.16}$$

so that

$$\|h\| \leq \|\mathbf{S}\|. \tag{3.17}$$

Combining equations (3.15) and (3.17) gives $\|h\| = \|\mathbf{S}\|$.
(c) Let us assume that $\mathbf{T} : \mathbf{H}_1 \to \mathbf{H}_2$ is a linear operator such that for all $x \in \mathbf{H}_1$ and $y \in \mathbf{H}_2$, we have

$$h(x, y) = \langle \mathbf{S}x, y\rangle = \langle \mathbf{T}x, y\rangle.$$

Then, by Lemma $3.3.4$, $\mathbf{S}x = \mathbf{T}x$ for all $x \in \mathbf{H}_1$. Hence $\mathbf{S} = \mathbf{T}$ by definition. ■

Remark 3.3.6 *An important consequence of the Riesz representation theorem is the existence of the adjoint of a bounded operator on a Hilbert space.*

3.4 Hilbert-adjoint and self-adjoint operators

Definition 3.4.1 *([Kre78], 3·9-1 Definition (Hilbert-adjoint operator, $\mathbf{T}^*$))*
Given two Hilbert spaces, $\mathbf{H}_1$ and $\mathbf{H}_2$, let $\mathbf{T} : \mathbf{H}_1 \to \mathbf{H}_2$ be a bounded linear operator. Then the Hilbert-adjoint operator $\mathbf{T}^*$ of $\mathbf{T}$ is the operator $\mathbf{T}^* : \mathbf{H}_2 \to \mathbf{H}_1$ such that for all $x \in \mathbf{H}_1$ and $y \in \mathbf{H}_2$,

$$\langle \mathbf{T}x, y\rangle = \langle x, \mathbf{T}^*y\rangle. \tag{3.18}$$

Let us show that there exists such an operator $\mathbf{T}^*$.

Theorem 3.4.2 *([Kre78], 3·9-2 Theorem (Existence)) The Hilbert-adjoint operator $\mathbf{T}^*$ of $\mathbf{T}$ exist, is unique and is a bounded linear operator with norm*

$$\|\mathbf{T}^*\| = \|\mathbf{T}\|. \tag{3.19}$$

Proof.
Consider

$$h(y, x) = \langle y, \mathbf{T}x\rangle. \tag{3.20}$$

We will show that h is sesquilinear. Now h is linear in the first argument and conjugate linear in the second argument since,

$$\begin{aligned} h(y, \alpha x_1 + \beta x_2) &= \langle y, \mathbf{T}(\alpha x_1 + \beta x_2)\rangle \\ &= \langle y, \alpha \mathbf{T} x_1 + \beta \mathbf{T} x_2\rangle \\ &= \bar{\alpha}\langle y, \mathbf{T} x_1\rangle + \bar{\beta}\langle y, \mathbf{T} x_2\rangle \\ &= \bar{\alpha} h(y, x_1) + \bar{\beta} h(y, x_2). \end{aligned}$$

Hence since the inner product is sesquilinear, we conclude that h is sesquilinear. By the Cauchy-Schwarz inequality,

$$|h(y,x)| = |\langle y, \mathbf{T}x\rangle| \leq \|y\|\|\mathbf{T}x\| \leq \|\mathbf{T}\|\|x\|\|y\|$$

which implies that $\frac{|h(y,x)|}{\|x\|\|y\|} \leq \|\mathbf{T}\|$ and by equation (3.12) we have that

$$\|h\| \leq \|\mathbf{T}\|. \tag{3.21}$$

Also,

$$\|h\| = \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|\langle y, \mathbf{T}x\rangle|}{\|y\|\|x\|} \geq \sup_{\substack{x \neq 0 \\ \mathbf{T}x \neq 0}} \frac{|\langle \mathbf{T}x, \mathbf{T}x\rangle|}{\|\mathbf{T}x\|\|x\|} = \|\mathbf{T}\|. \tag{3.22}$$

Combining equations (3.21) and (3.22) gives $\|h\| = \|\mathbf{T}\|$. From Theorem $3.3.5$, substituting $\mathbf{T}^*$ for $\mathbf{S}$, we have

$$h(y,x) = \langle \mathbf{T}^* y, x\rangle, \tag{3.23}$$

where $\mathbf{T}^* : \mathbf{H}_2 \to \mathbf{H}_1$ is a uniquely determined, bounded linear operator with norm

$$\|\mathbf{T}^*\| = \|h\| = \|\mathbf{T}\|. \tag{3.24}$$

Combining (3.20) and (3.23), we get

$$\langle y, \mathbf{T}x\rangle = \langle \mathbf{T}^* y, x\rangle.$$

Taking the conjugate gives equation (3.18). ■

Lemma 3.4.3 *([Kre78], 3·9-3 Lemma (Zero operator)) Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be inner product spaces and* $\mathbf{T} : \mathbf{X} \to \mathbf{Y}$ *a bounded linear operator. Then:*

1. $\mathbf{T} = 0$ *if and only if* $\langle \mathbf{T}x, y\rangle = 0$ *for all* $x \in \mathbf{X}$ *and* $y \in \mathbf{Y}$.

2. *For* $\mathbf{X}$ *a complex vector space, if* $\mathbf{T} : \mathbf{X} \to \mathbf{X}$ *and* $\langle \mathbf{T}x, x\rangle = 0$ *for all* $x \in \mathbf{X}$, *then* $\mathbf{T} = 0$.

Proof.
1. If $\mathbf{T} = 0$, then for all $x \in \mathbf{X}$, $\mathbf{T}x = 0$, and for any $u \in \mathbf{X}$ we have,

$$\langle \mathbf{T}x, y\rangle = \langle 0, y\rangle = 0\langle u, y\rangle = 0.$$

Now let

$$\langle \mathbf{T}x, y\rangle = 0 \text{ for all } x \in \mathbf{X},\ y \in \mathbf{Y}.$$

Then by Lemma 3.3.4, $\mathbf{T}x = 0$ for all $x \in \mathbf{X}$ and $\mathbf{T} = 0$.
2. If $\langle \mathbf{T}x, x\rangle = 0$ for all $x \in \mathbf{X}$, then for $w = \alpha x + y \in \mathbf{X}$ we have

$$\begin{aligned} \langle \mathbf{T}w, w\rangle &= \langle \mathbf{T}(\alpha x + y), \alpha x + y\rangle \\ &= |\alpha|^2\langle \mathbf{T}x, x\rangle + \langle \mathbf{T}y, y\rangle + \alpha\langle \mathbf{T}x, y\rangle + \bar{\alpha}\langle \mathbf{T}y, x\rangle. \end{aligned} \tag{3.25}$$

Now if we choose $\alpha = 1$, then (3.25) becomes

$$\langle \mathbf{T}w, w\rangle = \langle \mathbf{T}x, x\rangle + \langle \mathbf{T}y, y\rangle + \langle \mathbf{T}x, y\rangle + \langle \mathbf{T}y, x\rangle. \tag{3.26}$$

Now $\langle \mathbf{T}x, x\rangle$ and $\langle \mathbf{T}y, y\rangle$ are equal to zero by our assumption. Hence equation (3.26) becomes

$$\langle \mathbf{T}x, y\rangle + \langle \mathbf{T}y, x\rangle = 0. \tag{3.27}$$

If we choose $\alpha = i$, then $\bar{\alpha} = -i$ and equation (3.25) becomes

$$\langle \mathbf{T}x, y\rangle - \langle \mathbf{T}y, x\rangle = 0. \tag{3.28}$$

Adding equations (3.27) and (3.28) gives $\langle \mathbf{T}x, y\rangle = 0$ and $\mathbf{T} = 0$ follows from 1. ■

Theorem 3.4.4 *([Kre78], 3·9-4 Theorem (Properties of the Hilbert adjoint-operator)) Let* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *be Hilbert spaces,* $\mathbf{S} : \mathbf{H}_1 \to \mathbf{H}_2$ *and* $\mathbf{T} : \mathbf{H}_1 \to \mathbf{H}_2$ *be bounded linear operators and* β *any scalar. Then*

1. $\langle \mathbf{T}^* y, x\rangle = \langle y, \mathbf{T}x\rangle$ *for any* $x \in \mathbf{H}_1, y \in \mathbf{H}_2$,
2. $(\mathbf{S} + \mathbf{T})^* = \mathbf{S}^* + \mathbf{T}^*$,
3. $(\beta\mathbf{T})^* = \bar{\beta}\mathbf{T}^*$,
4. $(\mathbf{T}^*)^* = \mathbf{T}$,
5. $\|\mathbf{T}^*\mathbf{T}\| = \|\mathbf{T}\mathbf{T}^*\| = \|\mathbf{T}\|^2$,
6. $\mathbf{T}^*\mathbf{T} = 0$ *if and only if* $\mathbf{T} = 0$,
7. $(\mathbf{S}\mathbf{T})^* = \mathbf{T}^*\mathbf{S}^*$ *(if* $\mathbf{H}_2 = \mathbf{H}_1$*).*

Proof.
1. By Definition 3.4.1, we have

$$\langle \mathbf{T}^* y, x\rangle = \overline{\langle x, \mathbf{T}^* y\rangle} = \overline{\langle \mathbf{T}x, y\rangle} = \langle y, \mathbf{T}x\rangle. \tag{3.29}$$

2. By the definition of the Hilbert-adjoint operator in Definition (3.4.1), for all x and y,

$$\begin{aligned}\langle x, (\mathbf{S}+\mathbf{T})^* y\rangle &= \langle (\mathbf{S}+\mathbf{T})x, y\rangle \\ &= \langle \mathbf{S}x, y\rangle + \langle \mathbf{T}x, y\rangle \\ &= \langle x, \mathbf{S}^* y\rangle + \langle x, \mathbf{T}^* y\rangle \\ &= \langle x, (\mathbf{S}^* + \mathbf{T}^*)y\rangle.\end{aligned}$$

Thus it follows from Lemma 3.3.4 that, $(\mathbf{S}+\mathbf{T})^* y = (\mathbf{S}^*+\mathbf{T}^*)y$ for all $y \in \mathbf{H}_2$, so that $(\mathbf{S}+\mathbf{T})^* = \mathbf{S}^* + \mathbf{T}^*$.

3. By Definition (3.4.1),

$$\begin{aligned}\langle (\beta\mathbf{T})^* y, x\rangle &= \langle y, (\beta\mathbf{T})x\rangle \\ &= \langle y, \beta(\mathbf{T}x)\rangle \\ &= \bar{\beta}\langle y, \mathbf{T}x\rangle \\ &= \bar{\beta}\langle \mathbf{T}^* y, x\rangle \\ &= \langle \bar{\beta}\mathbf{T}^* y, x\rangle.\end{aligned}$$

Hence by Lemma 3.4.3, $(\beta\mathbf{T})^* y = \bar{\beta}\mathbf{T}^* y$ for all $y \in \mathbf{H}_2$, which implies that $(\beta\mathbf{T})^* = \bar{\beta}\mathbf{T}^*$.

4. By Definition 3.4.1 and from 1 we have, $\langle (\mathbf{T}^*)^* x, y\rangle = \langle x, \mathbf{T}^* y\rangle = \langle \mathbf{T}x, y\rangle$ so that $\langle ((\mathbf{T}^*)^* - \mathbf{T})x, y\rangle$ and by Lemma 3.4.3, we have $(\mathbf{T}^*)^* = \mathbf{T}$.

5. We know that $\mathbf{T}^*\mathbf{T} : \mathbf{H}_1 \to \mathbf{H}_1$ and $\mathbf{T}\mathbf{T}^* : \mathbf{H}_2 \to \mathbf{H}_2$. By the Cauchy-Schwarz inequality and by the definition of the Hilbert-adjoint operator in Definition (3.4.1) we have

$$\|\mathbf{T}x\|^2 = \langle \mathbf{T}x, \mathbf{T}x\rangle = \langle \mathbf{T}^*\mathbf{T}x, x\rangle \le \|\mathbf{T}^*\mathbf{T}x\|\|x\| \le \|\mathbf{T}^*\mathbf{T}\|\|x\|^2.$$

Taking the supremum over all x of norm 1 we obtain $\|\mathbf{T}\|^2 \le \|\mathbf{T}^*\mathbf{T}\|$. Now by Theorem 3.4.2, we have $\|\mathbf{T}^*\mathbf{T}\| \le \|\mathbf{T}^*\|\|\mathbf{T}\| = \|\mathbf{T}\|^2$. Hence $\|\mathbf{T}^*\mathbf{T}\| = \|\mathbf{T}\|^2$. We substitute $\mathbf{T}^*$ for $\mathbf{T}$ to get $\|\mathbf{T}^{**}\mathbf{T}^*\| = \|\mathbf{T}^*\|^2 = \|\mathbf{T}\|^2$, But by 4 we have $(\mathbf{T}^*)^* = \mathbf{T}$, so that $\|\mathbf{T}\mathbf{T}^*\| = \|\mathbf{T}\|^2$.

6. From 5, if $\mathbf{T}^*\mathbf{T} = 0$, then $\mathbf{T} = 0$ and conversely if $\mathbf{T} = 0$, then $\mathbf{T}^*\mathbf{T} = 0$.

7. By Definition 3.4.1, $\langle x, (\mathbf{S}\mathbf{T})^* y\rangle = \langle (\mathbf{S}\mathbf{T})x, y\rangle = \langle \mathbf{T}x, \mathbf{S}^* y\rangle = \langle x, \mathbf{T}^*\mathbf{S}^* y\rangle$. Hence by Lemma 3.3.4 we obtain $(\mathbf{S}\mathbf{T})^* y = \mathbf{T}^*\mathbf{S}^* y$ for all $y \in \mathbf{H}_2$, so that $(\mathbf{S}\mathbf{T})^* = \mathbf{T}^*\mathbf{S}^*$. ■

Definition 3.4.5 *([Kre78], 3·10-1 (Self-adjoint operator)) A bounded linear operator* $\mathbf{T} : \mathbf{H} \to \mathbf{H}$ *on a Hilbert space* $\mathbf{H}$ *is said to be self-adjoint or Hermitian if*

$$\mathbf{T}^* = \mathbf{T}. \tag{3.30}$$

Equivalently, a bounded linear operator $\mathbf{T}$ is said to be self-adjoint if

$$\langle x, \mathbf{T}y\rangle = \langle \mathbf{T}x, y\rangle \quad \text{for all } x, y \in \mathbf{H}. \tag{3.31}$$

Example 3.4.6 *A linear map on $\mathbb{R}^n$ with matrix* $\mathbf{A}$ *is a self-adjoint if and only if* $\mathbf{A}$ *is symmetric* ($\mathbf{A} = \mathbf{A}^{\mathbf{T}}$)*. A linear map on $\mathbb{C}^n$ with matrix* $\mathbf{A}$ *is self-adjoint if and only if* $\mathbf{A}$ *is Hermitian* ($\mathbf{A} = \mathbf{A}^*$)*.*

Remark 3.4.7 *Self-adjoint operators on Hilbert spaces are used in quantum mechanics to represent physical observables such as position, momentum, angular momentum and spin. An important example is the Hamiltonian operator given by*

$$\mathbf{H}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi.$$

Definition 3.4.8 *([Kre78], 3·10-1 (Unitary operator)) A bounded linear operator* $\mathbf{T} : \mathbf{H} \to \mathbf{H}$ *on a Hilbert space* $\mathbf{H}$ *is said to be unitary if* $\mathbf{T}$ *is bijective and*

$$\mathbf{T}\mathbf{T}^* = \mathbf{T}^*\mathbf{T}. \tag{3.32}$$

Hence

$$\mathbf{T}^* = \mathbf{T}^{-1}. \tag{3.33}$$

Definition 3.4.9 *([Kre78], 3·10-1 (Normal operators)) A bounded linear operator* $\mathbf{T} : \mathbf{H} \to \mathbf{H}$ *on a Hilbert space* $\mathbf{H}$ *is said to be normal if*

$$\mathbf{T}\mathbf{T}^* = \mathbf{T}^*\mathbf{T}. \tag{3.34}$$

Remark 3.4.10 *If* $\mathbf{T}$ *is self-adjoint or unitary, then* $\mathbf{T}$ *is normal, but the converse is not generally true. For example if* $\mathbf{I} : \mathbf{H} \to \mathbf{H}$ *is the identity operator, then* $\mathbf{T} = 2i\mathbf{I}$ *is normal since* $\mathbf{T}^* = -2i\mathbf{I}$*, so that* $\mathbf{T}\mathbf{T}^* = \mathbf{T}^*\mathbf{T} = 4\mathbf{I}$ *but* $\mathbf{T}^* \neq \mathbf{T}$ *and* $\mathbf{T}^* \neq \mathbf{T}^{-1} = -\frac{1}{2}i\mathbf{I}$*.*

Theorem 3.4.11 *([Kre78], 3·10-3 Theorem (Self-Adjointness)) Let* $\mathbf{T} : \mathbf{H} \to \mathbf{H}$ *be a bounded linear operator on a Hilbert space* $\mathbf{H}$*. Then*

1. *If* $\mathbf{T}$ *is self-adjoint, then* $\langle \mathbf{T}x, x\rangle \in \mathbb{R}$ *for all* $x \in \mathbf{H}$*.*
2. *If* $\mathbf{H}$ *is complex and* $\langle \mathbf{T}x, x\rangle \in \mathbb{R}$ *for all* $x \in \mathbf{H}$*, then the operator* $\mathbf{T}$ *is self-adjoint.*

Proof.

1. If $\mathbf{T}$ is self-adjoint, then for all x,

$$\overline{\langle \mathbf{T}x, x\rangle} = \langle x, \mathbf{T}x\rangle. \tag{3.35}$$

By definition, $\langle \mathbf{T}x, y\rangle = \langle x, \mathbf{T}^*y\rangle$ and since $\mathbf{T}$ is self-adjoint, we have

$$\langle \mathbf{T}x, x\rangle = \langle x, \mathbf{T}x\rangle. \tag{3.36}$$

Combining equations (3.35) and (3.36) gives

$$\overline{\langle \mathbf{T}x, x\rangle} = \langle \mathbf{T}x, x\rangle.$$

Hence $\langle \mathbf{T}x, x\rangle$ is equal to its complex conjugate which implies that it is real.

2. If $\langle \mathbf{T}x, x\rangle \in \mathbb{R}$ for all $x \in \mathbf{H}$, then

$$\langle \mathbf{T}x, x\rangle = \overline{\langle \mathbf{T}x, x\rangle} = \overline{\langle x, \mathbf{T}^*x\rangle} = \langle \mathbf{T}^*x, x\rangle.$$

Hence

$$0 = \langle \mathbf{T}x, x\rangle - \langle \mathbf{T}^*x, x\rangle = \langle (\mathbf{T} - \mathbf{T}^*)x, x\rangle$$

and by Lemma $3.4.3$, $\mathbf{T} - \mathbf{T}^* = 0$. Therefore $\mathbf{T} = \mathbf{T}^*$.

■

Theorem 3.4.12 *([Kre78], 3·10-5 Theorem (Sequence of self-adjoint operators)) Let $\mathbf{T}_n$ be a sequence of bounded self-adjoint linear operators $\mathbf{T}_n : \mathbf{H} \to \mathbf{H}$ on a Hilbert space $\mathbf{H}$. If $\mathbf{T}_n$ converges to $\mathbf{T}$, then $\mathbf{T}$ is a bounded self-adjoint linear operator.*

Proof.
If $\mathbf{T}_n \to \mathbf{T}$, then $\|\mathbf{T}_n - \mathbf{T}\| \to 0$. Now by Theorem $3.4.4$ and Theorem $3.4.2$ we have that $\|\mathbf{T}_n{}^* - \mathbf{T}^*\| = \|(\mathbf{T}_n - \mathbf{T})^*\| = \|\mathbf{T}_n - \mathbf{T}\|$, so that

$$\begin{aligned} \|\mathbf{T} - \mathbf{T}^*\| &\leq \|\mathbf{T} - \mathbf{T}_n\| + \|\mathbf{T}_n - \mathbf{T}_n^*\| + \|\mathbf{T}_n^* - \mathbf{T}^*\| \\ &= \|\mathbf{T} - \mathbf{T}_n\| + \|\mathbf{T}_n - \mathbf{T}\| = 2\|\mathbf{T}_n - \mathbf{T}\|. \end{aligned}$$

As n tends to infinity, $\|\mathbf{T}_n - \mathbf{T}\|$ tends to zero. Hence $\|\mathbf{T} - \mathbf{T}^*\| = 0$ which implies that $\mathbf{T}^* = \mathbf{T}$. Hence $\mathbf{T}$ is self-adjoint. ■

4. Spectral Analysis of a Self-Adjoint Operator

Definition 4.0.13 *([Aup91], 2·2·4 Definition (Spectrum of* $\mathbf{T}$*)) Let* $L(\mathbf{H})$ *denote the set of all bounded linear operators on* $\mathbf{H}$ *and let* $\mathbf{T} \in L(\mathbf{H})$*. We define the spectrum of* $\mathbf{T}$ *as the set of* $\lambda \in \mathbb{C}$ *such that* $\mathbf{T} - \lambda\mathbf{I}$ *is not invertible in* $L(\mathbf{H})$*. It is denoted by Sp* $\mathbf{T}$*. So* $\lambda \in Sp\mathbf{T}$ *if and only if at least one of the following conditions hold.*

1. *the range of* $\mathbf{T} - \lambda\mathbf{I}$ *is not all of* $\mathbf{H}$*, that is* $\mathbf{T} - \lambda\mathbf{I}$ *is not surjective.*
2. $\mathbf{T} - \lambda\mathbf{I}$ *is not injective.*

Hence

$$Sp\mathbf{T} = \{\lambda \in \mathbb{C} : \mathbf{T} - \lambda\mathbf{I} \text{ is not invertible in } L(\mathbf{H})\}.$$

Remark 4.0.14 *The kernel and range of* $\mathbf{T}$ *are denoted by* $N(\mathbf{T})$ *and* $R(\mathbf{T})$ *respectively. If* 2 *holds, then* λ *is called the eigenvalue of* $\mathbf{T}$ *and* $N(\mathbf{T} - \lambda\mathbf{I})$ *is the corresponding eigenspace which is the set of all* $x \in \mathbf{X}$ *such that* $\mathbf{T}x = \lambda x$*, where* $x \neq 0$ *is called an eigenvector corresponding to the the eigenvalue* λ*.*

Definition 4.0.15 *([Aup91]) The spectral radius of* $\mathbf{T} \in L(\mathbf{H})$ *is given by*

$$\rho(\mathbf{T}) = \max\{|\lambda| : \lambda \in Sp\mathbf{T}\}. \tag{4.1}$$

Theorem 4.0.16 *([Aup91], Theorem 2·3·1) Let* $\mathbf{H}$ *be a Hilbert space and let* $\mathbf{T} \in L(\mathbf{H})$*. Then*

1. $N(\mathbf{T}) = R(\mathbf{T}^*)^\perp$ *and* $N(\mathbf{T}^*) = R(\mathbf{T})^\perp$*,*
2. $\mathbf{I} + \mathbf{T}^*\mathbf{T}$ *is invertible in* $L(\mathbf{H})$*.*

Proof.
1. If we choose $x \in N(\mathbf{T})$, then for all $y \in \mathbf{H}$,

$$\langle 0, y\rangle = \langle \mathbf{T}x, y\rangle = \langle x, \mathbf{T}^*y\rangle = 0,$$

so that $x \in R(\mathbf{T}^*)^\perp$. Now applying the adjoint $*$ gives

$$N(\mathbf{T}^*) = R(\mathbf{T}^{**})^\perp,$$

but by Theorem $3.4.4$ (4), $\mathbf{T}^{**} = \mathbf{T}$ which implies that $N(\mathbf{T}^*) = R(\mathbf{T})^\perp$.

2.To show that $\mathbf{I} + \mathbf{T}^*\mathbf{T}$ is invertible in $\mathbf{L}(\mathbf{H})$, we show that $\mathbf{I} + \mathbf{T}^*\mathbf{T}$ is
(i) injective and
(ii) show that it is surjective by showing that the range of $\mathbf{I} + \mathbf{T}^*\mathbf{T}$ is the whole of $\mathbf{H}$.

(i) Consider $\mathbf{T}^*\mathbf{T}$ and $\mathbf{I} + \mathbf{T}^*\mathbf{T}$ which are self-adjoint.

$$\begin{aligned} \|(\mathbf{I} + \mathbf{T}^*\mathbf{T})x\|^2 &= \langle(\mathbf{I} + \mathbf{T}^*\mathbf{T})x, (\mathbf{I} + \mathbf{T}^*\mathbf{T})x\rangle \\ &= \langle(\mathbf{I} + \mathbf{T}^*\mathbf{T})(\mathbf{I} + \mathbf{T}^*\mathbf{T})x, x\rangle \qquad (4.2) \\ &= \langle(\mathbf{I} + \mathbf{T}^*\mathbf{T})^2x, x\rangle \\ &= \langle\mathbf{I} + 2\mathbf{T}^*\mathbf{T} + (\mathbf{T}^*\mathbf{T})^2x, x\rangle \\ &= \langle(x + (2\mathbf{T}^*\mathbf{T})x + (\mathbf{T}^*\mathbf{T})^2)x, x\rangle \\ &= \langle x, x\rangle + 2\langle\mathbf{T}^*(\mathbf{T}x), x\rangle + \langle(\mathbf{T}^*\mathbf{T})^2x, x\rangle \\ &= \|x\|^2 + 2\langle\mathbf{T}x, \mathbf{T}x\rangle + \langle\mathbf{T}^*\mathbf{T}x, \mathbf{T}^*\mathbf{T}x\rangle \\ &= \|x\|^2 + 2\|\mathbf{T}x\|^2 + \|\mathbf{T}^*\mathbf{T}x\|^2 \geq \|x\|^2. \qquad (4.3) \end{aligned}$$

It follows that if $(\mathbf{I} + \mathbf{T}^*\mathbf{T})x = 0$, then $x = 0$. Hence we conclude that $N(\mathbf{I} + \mathbf{T}^*\mathbf{T}) = \{0\}$, so that $\mathbf{I} + \mathbf{T}^*\mathbf{T}$ is injective.

(ii) We will first show that $R(\mathbf{I} + \mathbf{T}^*\mathbf{T})$ is closed. Let $((\mathbf{I} + \mathbf{T}^*\mathbf{T})x_n)$ be a Cauchy sequence in $R(\mathbf{I}+\mathbf{T}^*\mathbf{T})$. Then given $\epsilon \geq 0$, there exist N such that for $n, m \geq N$ we have $\|(\mathbf{I}+\mathbf{T}^*\mathbf{T})(x_n) - (\mathbf{I} + \mathbf{T}^*\mathbf{T})(x_m)\| \leq \epsilon$. Now from inequality (4.3),

$$\epsilon \geq \|(\mathbf{I} + \mathbf{T}^*\mathbf{T})(x_n) - (\mathbf{I} + \mathbf{T}^*\mathbf{T})(x_m)\| = \|(\mathbf{I} + \mathbf{T}^*\mathbf{T})(x_n - x_m)\| \geq \|x_n - x_m\|,$$

which implies that $\|x_n - x_m\| \leq \epsilon$. Hence x_n is also a Cauchy sequence. Now since $\mathbf{H}$ is complete, $x_n \to x \in \mathbf{H}$. Since $\mathbf{I} + \mathbf{T}^*\mathbf{T}$ is bounded, then it is continuous. Hence $\lim_{n\to\infty}(\mathbf{I} + \mathbf{T}^*\mathbf{T})(x_n) = (\mathbf{I} + \mathbf{T}^*\mathbf{T})(\lim_{n\to\infty}(x_n)) = (\mathbf{I} + \mathbf{T}^*\mathbf{T})x$. Hence $(\mathbf{I} + \mathbf{T}^*\mathbf{T})x \in R(\mathbf{I} + \mathbf{T}^*\mathbf{T})$ and we conclude that $R(\mathbf{I} + \mathbf{T}^*\mathbf{T})$ is closed.

Now by direct sum, Theorem $2.3.8$, $\mathbf{H} = R(\mathbf{I} + \mathbf{T}^*\mathbf{T}) \oplus R(\mathbf{I} + \mathbf{T}^*\mathbf{T})^\perp$. But by 1 we have that $R(\mathbf{I} + \mathbf{T}^*\mathbf{T})^\perp = N(\mathbf{I} + \mathbf{T}^*\mathbf{T}) = \{0\}$. Therefore $\mathbf{H} = R(\mathbf{I} + \mathbf{T}^*\mathbf{T})$. Hence $\mathbf{I} + \mathbf{T}^*\mathbf{T}$ is surjective.
We conclude from (i) and (ii) that $\mathbf{I} + \mathbf{T}^*\mathbf{T}$ is bijective and hence invertible. ■

Lemma 4.0.17 *If $\mathbf{S} \in L(\mathbf{H})$ where $\mathbf{H}$ a Hilbert space, $\mathbf{S} = \mathbf{S}^*$ and λ is an eigenvalue of $\mathbf{S}$, then $\lambda \in \mathbb{R}$.*

Proof.
If λ is an eigenvalue of $\mathbf{S}$, then by Remark $4.0.14$, $\mathbf{S} - \lambda\mathbf{I}$ is not injective. Hence $N(\mathbf{S} - \lambda\mathbf{I}) \neq \{0\}$. Hence for $x \in N(\mathbf{S} - \lambda\mathbf{I})$, $\mathbf{S}x = \lambda x$. Now since $\mathbf{S}$ is self-adjoint, then by Theorem $3.4.11$ (1), $\langle\mathbf{S}x, x\rangle \in \mathbb{R}$ for all $x \in \mathbf{H}$ and $\langle x, \mathbf{S}x\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\|x\|^2 \in \mathbb{R}$. Hence $\overline{\lambda} = \lambda$ and we conclude that $\lambda \in \mathbb{R}$. ■

Lemma 4.0.18 *Let* $\mathbf{S} \in L(\mathbf{H})$ *where* $\mathbf{H}$ *is a Hilbert space. Suppose that* $\mathbf{S}_\lambda = \lambda\mathbf{I} - \mathbf{S}$ *with* $\lambda = \alpha + i\beta$ *and* $\mathbf{S} = \mathbf{S}^*$. *If* $\|\mathbf{S}_\lambda x\| \geq |\beta|\|x\|$, *with* $\beta \neq 0$, *then* $\mathbf{S}_\lambda$ *is invertible.*

Proof.
(i) To show that it is injective, let $\|\mathbf{S}_\lambda x\| = 0$, then $|\beta|\|x\| \leq 0$ which implies that $x = 0$ for $\beta \neq 0$. Hence $N(\mathbf{S}_\lambda) = \{0\}$ and we conclude that $\mathbf{S}_\lambda$ is injective.

(ii) Let us first show that $R(\mathbf{S}_\lambda)$ is closed. If $(\mathbf{S}_\lambda(x_n))$ is a Cauchy sequence so is (x_n). Now since $\mathbf{H}$ is complete, $x_n \to x \in \mathbf{H}$. Since $\mathbf{S}_\lambda$ is bounded, then it is continuous. Hence $\lim\limits_{n\to\infty}(\mathbf{S}_\lambda(x_n)) = (\mathbf{S}_\lambda)\lim\limits_{n\to\infty}(x_n) = (\mathbf{S}_\lambda)x$. Hence $(\mathbf{S}_\lambda)x \in R(\mathbf{S}_\lambda)$ and we conclude that $R(\mathbf{S}_\lambda)$ is closed.

Now by direct sum, Theorem 2.3.8, $\mathbf{H} = R(\mathbf{S}_\lambda) \oplus R(\mathbf{S}_\lambda)^\perp$. By Theorem 4.0.16 (1), we have that $R(\mathbf{S}_\lambda)^\perp = N(\mathbf{S}_\lambda{}^*)$. We will show that $\mathbf{S}_\lambda{}^*$ is injective and deduce that $R(\mathbf{S}_\lambda)^\perp = N(\mathbf{S}_\lambda{}^*) = \{0\}$.

Now by Theorem 4.0.16 (1), $R(\mathbf{S}_\lambda)^\perp = N(\mathbf{S}_\lambda{}^*)$. For $\beta \neq 0$, $N((\alpha\mathbf{I} + i\beta\mathbf{I} - \mathbf{S})^*) = N(\alpha\mathbf{I} - i\beta\mathbf{I} - \mathbf{S}) = \{0\}$ otherwise by Lemma 4.0.17, $\alpha - i\beta$ would be an eigenvalue. But $\alpha - i\beta \notin \mathbb{R}$, so that $\alpha - i\beta$ cannot be an eigenvalue. This implies that $N(\mathbf{S}_\lambda{}^*) = R(\mathbf{S}_\lambda)^\perp = \{0\}$. Hence $H = R(\mathbf{S}_\lambda)$, showing that $\mathbf{S}_\lambda$ is surjective. We conclude that $\mathbf{S}_\lambda$ is bijective hence invertible. ■

Theorem 4.0.19 *([Kre78], 7·5-5 (Theorem)) If* $\mathbf{T}$ *is a bounded linear operator on a complex Banach space, then*

$$\rho(\mathbf{T}) = \lim_{n\to\infty} \|\mathbf{T}^n\|^{\frac{1}{n}}. \tag{4.4}$$

Remark 4.0.20 *The proof of this theorem is not trivial and falls outside the scope of this essay.*

Corollary 4.0.21 *([Aup91], Corollary 2·3·2) Let* $\mathbf{H}$ *be a Hilbert space and let* $\mathbf{S}$ *be a self-adjoint operator on* $\mathbf{H}$. *Then*

1. $\|\mathbf{S}^2\| = \|\mathbf{S}\|^2$ *and consequently* $\|\mathbf{S}\| = \rho(\mathbf{S})$.
2. $Sp\ \mathbf{S} \subset \mathbb{R}$.

Proof.

1. From Theorem 3.4.4 (5), $\|\mathbf{S}^*\mathbf{S}\| = \|\mathbf{S}\|^2$. But $\mathbf{S}$ is self-adjoint. Hence $\|\mathbf{S}^2\| = \|\mathbf{S}\|^2$ and by induction we have $\|\mathbf{S}^{2^n}\| = \|\mathbf{S}\|^{2^n}$. Now by Theorem 4.0.19, we have that $\rho(\mathbf{S}) = \lim\limits_{n\to\infty}\|\mathbf{S}^n\|^{\frac{1}{n}} = \|\mathbf{S}\|$.
2. Let $\lambda = a + i\beta$ with $\alpha, \beta \in \mathbb{R}$ and let $\mathbf{S}_\lambda = \lambda\mathbf{I} - \mathbf{S}$ for all $\lambda \in \mathbb{C}$. Then

$$\begin{aligned} \mathbf{S}_\alpha + i\beta\mathbf{I} &= \alpha\mathbf{I} - \mathbf{S} + i\beta\mathbf{I} \\ &= (\alpha + i\beta)\mathbf{I} - \mathbf{S} \\ &= \lambda\mathbf{I} - \mathbf{S}. \end{aligned}$$

Therefore $\mathbf{S}_\alpha + i\beta\mathbf{I} = \mathbf{S}_\lambda$. Thus for all $x \in \mathbf{H}$ and $\mathbf{S}$ self-adjoint we have,

$$\begin{aligned}
\|\mathbf{S}_\lambda x\|^2 &= \langle \mathbf{S}_\lambda x, \mathbf{S}_\lambda x\rangle \\
&= \langle (\mathbf{S}_\alpha + i\beta\mathbf{I})x, (\mathbf{S}_\alpha + i\beta\mathbf{I})x\rangle \\
&= \langle \mathbf{S}_\alpha x, \mathbf{S}_\alpha x\rangle + \langle i\beta x, i\beta x\rangle + \langle \mathbf{S}_\alpha x, i\beta x\rangle + \langle i\beta x, \mathbf{S}_\alpha x\rangle \\
&= \langle \mathbf{S}_\alpha x, \mathbf{S}_\alpha x\rangle + \langle i\beta x, i\beta x\rangle - i\beta\langle \mathbf{S}_\alpha x, x\rangle + i\beta\langle \mathbf{S}_\alpha x, x\rangle \\
&= \langle \mathbf{S}_\alpha x, \mathbf{S}_\alpha x\rangle + |\beta|^2\|x\|^2 \\
&= \|\mathbf{S}_\alpha x\|^2 + |\beta|^2\|x\|^2 \geq |\beta|^2\|x\|^2.
\end{aligned}$$

Hence $\|\mathbf{S}_\lambda x\| \geq |\beta|\|x\|$, and by Lemma $4.0.18$, we conclude that for $\beta \neq 0$, $\mathbf{S}_\lambda = \lambda\mathbf{I} - \mathbf{S}$ is invertible and hence $Sp\ \mathbf{S} \subset \mathbb{R}$. ■

5. Conclusion and further work

The parallelogram equality gives Hilbert spaces an edge over Banach spaces. Moreover orthogonality, continuity of the inner product and completeness are also important features of Hilbert spaces.

Operators on Hilbert spaces is a basis for a comprehensive study of the spectral theory. In particular, the Hermitian or self-adjoint operators are very important in many applications. In this essay, we have seen that the minimising vector theorem, Riesz representation of functionals and direct sum play important roles in this study. Given an inner product space **X** and **M** is a nonempty complete convex subspace of **X** , the Minimising Vector Theorem (Theorem $2.3.1$) enables us to find a unique point $y \in$ **M** that is closer to $x \in$ **X** than any other point in **M**. The Riesz theorem, Theorem $3.2.1$ is important in the representation of Hilbert-adjoint operators and also in quantum physics. The direct sum theorem, Theorem $2.3.8$, makes it possible to represent a Hilbert space **H** as the sum of any closed subspace and its orthogonal complement (Lemma $2.3.7$).

The algebra of all $n \times n$ complex matrices is a special case of an algebra of bounded linear operators on a Hilbert space and so "Hermitian matrices" correspond to "self-adjoint operators". Moreover the spectrum of an $n \times n$ matrix (considered as an operator) consists of exactly all the eigenvalues of the matrix whereas the spectrum Sp**T** of the self-adjoint operator **T** contains other elements which are not eigenvalues of **T**. An element, $\lambda \in Sp$**T** of the spectrum of the self-adjoint operator is an eigenvalue of **T** if and only if **T** $- \lambda I$ is not injective. It is also well known that the eigenvalues of a Hermitian matrix are real numbers. In this essay, we have generalised this idea by showing that the spectrum of a self-adjoint operator on a Hilbert space also consists entirely of real values (Theorem $4.0.21$).

So far, we have studied a small aspect of the operators on Hilbert spaces. However, there are other interesting facts about the self-adjoint operators like the bounded resolution of the identity. Moreover, other aspects of the spectral theory includes

1. Integral equations, Fredholm theory, compact operators,
2. Sturm-Liouville theory, hydrogen atom,
3. Isospectral theory, Lax pairs,
4. Atiyah-Singer index theorem.

These are interesting facts about the spectral theory that can be studied in future.

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Bibliography

[Aup91] Bernard Aupetit, *A primer on spectral theory*, Springer, New York, 1991.

[Axl97] Sheldon Axler, *Linear algebra done right*, Undergraduate text in Mathematics, Springer, San Francisco State University, 1997.

[Erd80] John Erdos, *Operators on hilbert spaces*, King's College London, 1980.

[Kre78] Erwin Kreyszig, *Introductory functional analysis with applications*, John Wiley and sons, New York, 1978.

[Red53] B. Daya Reddy, *Introductory functional analysis with applications to boundary value problems and finite elements*, Text in Applied Mathematics, no. 27, Springer, University of Cape Town, 1953.

[Swa97] Charles Swartz, *An introduction to functional analysis*, Monographs and Textbooks, no. 157, Marcel Dekker, Inc. New York, 1997.