Solving Singular Perturbation Problems Having Oscillatory Solutions via Adaptive Spline Methods

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Abstract

Two interesting classes of one-dimensional singular perturbation problems (SPPs) are those having solutions with layer and oscillatory behaviours, respectively. Such SPPs are well-known for the difficulties associated in solving them. Many researchers used a variety of numerical methods to solve the problems having solutions possessing boundary/interior layer(s) but relatively less success (if at all) is achieved in the research on the second class of problems when the solution is oscillatory. The aim here is to investigate whether the spline methods (which are very popular for engineering problems) can be used to solve these second class of SPPs. We use three splines, namely, cubic spline, B-spline and periodic cubic splines. Numerical examples are considered to demonstrate these methods whereas theoretical analysis of the numerical methods is currently under progress.

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1. Introduction

In the development of science and technology, many practical problems described by differential equations involving large or small parameters, such as mathematical boundary layer theory or approximation of solutions of such problems. It is natural to use asymptotic methods in the analysis of the small parameter problems which uses the boundary layer theory in various fields of applied mathematics.

In this chapter, firstly, we discuss a very important concept called "boundary layer". Secondly, we comment on asymptotic approaches to solve the underlying problem followed by a brief literature survey on the related work.

To begin with, consider a differential equation involving a small parameter ε and denote the associated problem by P_ε. Let its solution be denoted by y_ε. Setting $\varepsilon = 0$ in P_ε, we obtain the reduced problem P_0 whose solution is denoted by y_0. Then as $\varepsilon \to 0$, we check whether $y_\varepsilon \to y_0$? If yes then the problem is called a regular perturbation problem otherwise the problem is called a singular perturbation problem.

These problems arises in various fields of science and engineering, such as fluid mechanics, fluid dynamics, quantum mechanics, plasma-dynamics, chemical-reactor theory, reaction-diffusion processes and other domains of fluid motion. Some of such models are mentioned in [KP02b].

Let us now consider the singular perturbation problem

$$\left.\begin{array}{c} \varepsilon y^{''} + b(x)y = f(x), \\ y(0) = \alpha_0,\ y(1) = \alpha_1, \end{array}\right\} \tag{1.1}$$

where $b(x)$ and $f(x)$ are sufficiently smooth and $\alpha_0, \alpha_1 \in R$ and ε is a small parameter.

The solutions corresponding to this problem will either have layers (boundary and or interior) or they might be oscillatory depending upon the nature of $b(x)$. We consider $b(x) > 0$ in this essay in which case, then the solution is oscillatory. Below we describe some of these concepts with appropriate details.

1.1 The concept of boundary layer

A layer (boundary and interior) is a narrow region where the solution of singular perturbation problem changes rapidly. By definition, the thickness of a boundary layer must approach to 0 as $\varepsilon \longrightarrow 0$. The main concern with singular perturbation problems is that their solutions exhibit isolated (well-separated) narrow regions of rapid variation over a thick region (ones whose thickness does *not* vanish with ε).

There are two standard approximations that one makes in boundary-layer theory. In the outer region(away from a boundary layer) $y(x)$ is slowly varying, so it is valid to neglect any derivatives of $y(x)$ which are multiplied by ε. Inside a boundary layer the derivatives of $y(x)$ are large, but the boundary layer is so narrow that we may approximate the coefficient functions of the differential equations by constant. Thus, we can replace a single differential equation by a sequence of much simpler approximate equations in each of the several inner and outer regions. In every region the solution of the approximate equation will contain one or more unknown constants of integration, which can be determined from the boundary or initial conditions. Combining all these solutions suitably, one gets the solution of the original differential equation.

1.2 Dissipative vs dispersive phenomena

To construct an approximate solution to a singularly perturbed, we require one to match slowly varying outer solution to rapidly varying inner solutions. An outer solution remains smooth if we allow ε to approach $0+$. But in this limit an inner solution becomes discontinuous across the boundary layer because the thickness of the boundary layer tends to 0. We thus say that the solution suffers a local breakdown at the boundary layer as $\varepsilon \longrightarrow 0+$ (A local breakdown occurs where the approximate solution is exponentially increasing or decreasing). This kind of behavior is called $dissipative$ because the rapidly varying component of the solution decays exponentially away from the point of local breakdown. Some singular perturbation problems have solutions which exhibit a global breakdown. For example, the singular perturbation problem

$$\varepsilon y^{''} + y = 0, \qquad y(0) = 0, \ y(1) = 1, \tag{1.2}$$

has the exact solution

$$y(x) = \frac{\sin(x/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})}\,, \qquad \varepsilon \neq (n\pi)^{-2},$$

which becomes rapidly oscillatory (see the plots below in figures 1.1 and 1.2) for small ε and discontinuous where $\varepsilon \longrightarrow 0+$. The breakdown is global because it occurs throughout the interval $[0, 1]$. A global breakdown is associated with the rapidly oscillatory ($dispersive$) behavior. A dispersive solution is wavelike with very small and slowly changing wavelengths and slowly varying amplitudes as functions of x.

Dissipative and dispersive phenomena are both characterized by exponential behavior, where the exponent is real in the former case and imaginary in the latter case. Thus, for a singular perturbation problem that exhibits either or both type of behaviour, it is natural to seek an approximate solution of the form

$$y(x) \sim A(x)e^{S(x)/\delta}, \qquad \delta \longrightarrow 0+\,. \tag{1.3}$$

The phase $S(x)$ is assumed non-constant and slowly varying in a breakdown region. When $S(x)$ is real, there is a boundary layer of thickness δ, and when $S(x)$ is imaginary, there is a region of rapid oscillation characterized by waves having wavelength of order δ. When $S(x)$ is constant, the behavior of the solution $y(x)$ is expressed by the slowly varying amplitude function $A(x)$.

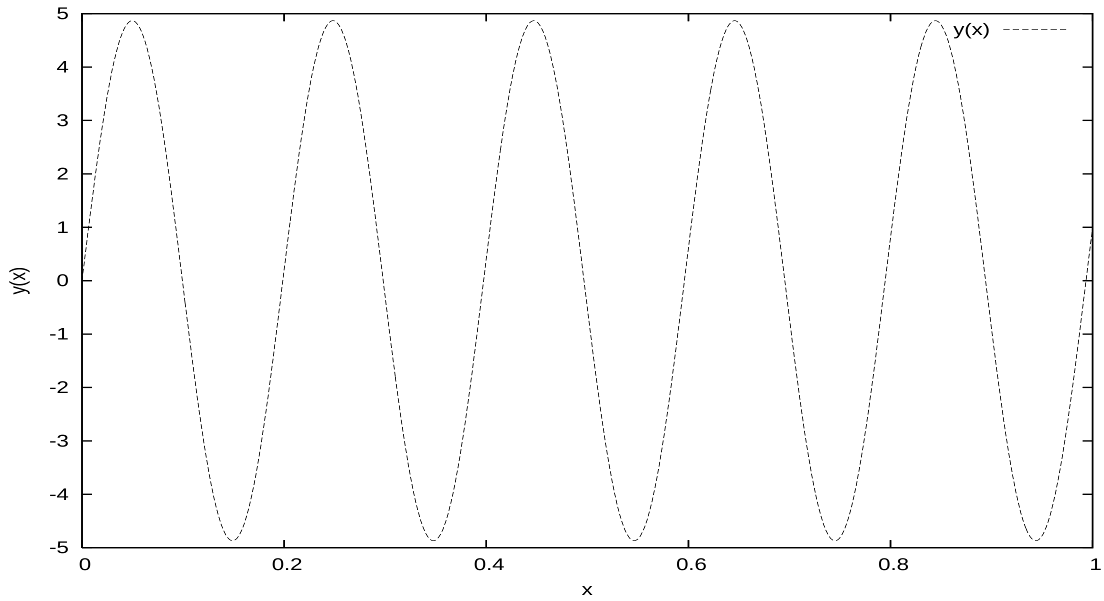

Figure 1.1: The exact solution for problem (1.2) with $\varepsilon = 0.001$

1.3 WKB approximation

The exponential approximation in equation (1.3) is conventionally known as a WKB approximation, named after Wentzel, Kramers, and Brillouin who popularized the theory [BO78]. WKB theory is a powerful tool for obtaining a global approximation to the solution of a linear singular perturbation problems. The exact solution of such problems may be some unknown function of overwhelming complexity. The WKB approximation consists of exponentials of elementary integrals of algebraic functions, and many special functions. WKB approximation is suitable for linear singular perturbation problems of any order, for boundary or initial value problems. The limitation of conventional WKB techniques is that they are only useful for linear equations[BO78].

In equation (1.3) we expand $A(x)$ and $S(x)$ as a series in powers of δ and combining these two series in a single exponential power series and obtain

$$y(x) \sim \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right], \text{ as } \delta \longrightarrow 0. \tag{1.4}$$

To explain the above WKB method, we consider the following example.

Example 1.1. Consider

$$\left.\begin{aligned} \varepsilon y'' + b(x)y = 0, \quad b(x) \neq 0, \quad 0 < \varepsilon << 1, \\ y(0) = 1, \; y(1) = 1. \end{aligned}\right\} \tag{1.5}$$

Differentiating (1.4), we obtain

$$y' \sim \left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'\right)\exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n\right), \text{ as } \delta \longrightarrow 0, \tag{1.6}$$

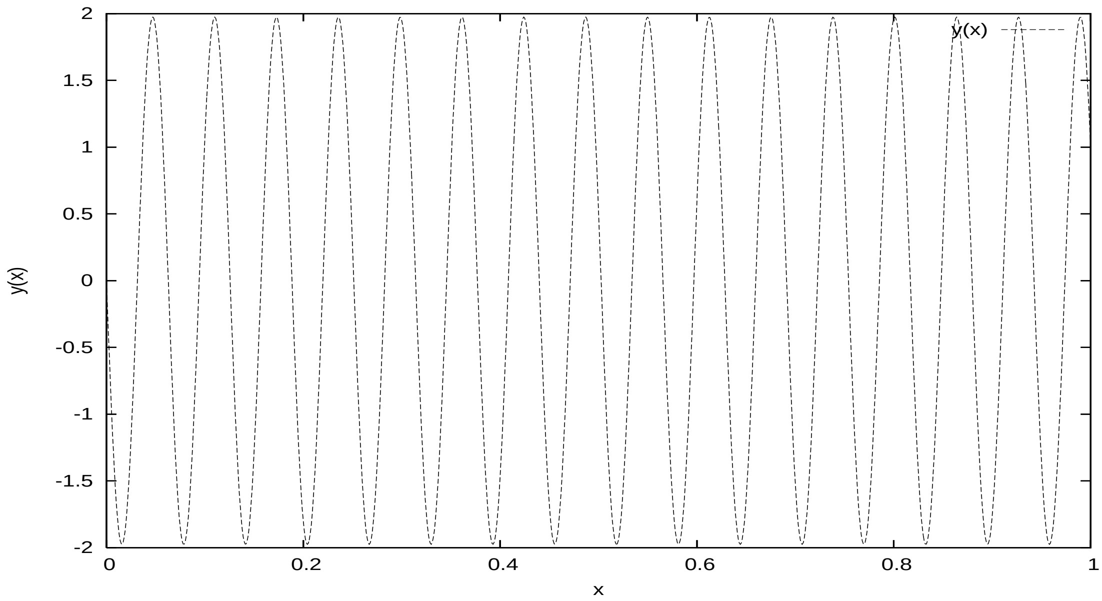

Figure 1.2: The exact solution for problem (1.2) with $\varepsilon = 0.0001$

and

$$y'' \sim \left[\frac{1}{\delta^2}\left(\sum_{n=0}^{\infty}\delta^n S_n'\right)^2 + \frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''\right]\exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n\right), \text{ as } \delta \longrightarrow 0. \tag{1.7}$$

By substitute (1.7) into (1.5) and simplifying, we get

$$\frac{\varepsilon}{\delta^2}S_0'^2 + \frac{2\varepsilon}{\delta}S_0'S_1' + \frac{\varepsilon}{\delta}S_0'' + \varepsilon S_1'^2 + \varepsilon S_1'' + \frac{\varepsilon}{\delta}S_2'' + ... = -b(x). \tag{1.8}$$

The largest term on the left side of (1.8) is $\varepsilon S_0'^2/\delta^2$. By dominant balance this term must have the same of magnitude as $b(x)$ on the write side. (With assumption that $b(x) \neq 0$.) Thus, δ is proportional to $\sqrt{\varepsilon}$ and for simplicity we choose $\delta = \sqrt{\varepsilon}$.

Setting $\delta = \sqrt{\varepsilon}$ in equation (1.8) and comparing powers of ε, we obtain a sequence of equations which determine $S_0, S_1, S_2, ...$

Then

$$S_0'^2 = -b(x), \tag{1.9}$$

$$2S_0'S_1' + S_0'' = 0, \tag{1.10}$$

$$2S_0'S_n' + S_{n-1}'' + \sum_{j=1}^{n-1} S_j'S_{n-j}' = 0, \quad n \geqslant 2. \tag{1.11}$$

The equation (1.9) for S_0 is called the *eikonal* equation whose solution is

$$S_0(x) = \pm i\int^x \sqrt{b(t)}dt. \tag{1.12}$$

The equation (1.10) for S_1 is called the $transport$ equation whose solution is

$$S_1(x) = -\frac{1}{4}\ln b(x). \tag{1.13}$$

Combining equations (1.12) and (1.13) gives a pair of approximate solutions to equation (1.5), one solution for each sign of S_0. As $\varepsilon \longrightarrow 0$, the two solutions are

$$y_1 \sim \exp\left[\frac{i}{\sqrt{\varepsilon}}\int_a^x \sqrt{b(t)}dt - \frac{1}{4}\ln(b(x))\right], \tag{1.14}$$

and

$$y_2 \sim \exp\left[-\frac{i}{\sqrt{\varepsilon}}\int_a^x \sqrt{b(t)}dt - \frac{1}{4}\ln(b(x))\right]. \tag{1.15}$$

The general solution is a linear combination of above two equations and is given by

$$\begin{aligned} y(x) \sim\ & c_1\,[b(x)]^{-1/4}\exp\left[\frac{i}{\sqrt{\varepsilon}}\int_a^x \sqrt{b(t)}dt\right] \\ &+ c_2\,[b(x)]^{-1/4}\exp\left[-\frac{i}{\sqrt{\varepsilon}}\int_a^x \sqrt{b(t)}dt\right], \quad \varepsilon \longrightarrow 0, \end{aligned} \tag{1.16}$$

where c_1 and c_2 are constants to be determined from boundary conditions and a is an arbitrary but fixed integration constant. This expression is the leading-order WKB approximation to the solution of (1.5) which differs from the exact solution by terms of order ε in regions where $b(x) \neq 0$.

Let us now consider the singular perturbation problem (1.5) with $b(x) = 1$. Even thought it is not possible to solve this problem using boundary-layer theory, the WKB approximation in (1.16), with $b(x) = 1$, gives

$$y^{WKB}(x) = c_1\exp(ix/\sqrt{\varepsilon}) + c_2\exp(-ix/\sqrt{\varepsilon}). \tag{1.17}$$

Imposing the boundary conditions, we have

$$y^{WKB}(x) = \left(\frac{1 - e^{(-i/\sqrt{\varepsilon})}}{e^{(i/\sqrt{\varepsilon})} - e^{(-i/\sqrt{\varepsilon})}}\right)e^{(ix/\sqrt{\varepsilon})} + \left(\frac{e^{(i/\sqrt{\varepsilon})} - 1}{e^{(i/\sqrt{\varepsilon})} - e^{(-i/\sqrt{\varepsilon})}}\right)e^{(-ix/\sqrt{\varepsilon})}, \tag{1.18}$$

which when simplified gives

$$y^{WKB}(x) = \cos(x/\sqrt{\varepsilon}) + \left(\frac{1 - \cos(1/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})}\right)\sin(x/\sqrt{\varepsilon}),$$

which is also the exact solution (fortunately in this case).

However, it should be noted that when the coefficient-functions are not constant, then the above approximation will be very tedious and the only way is to go for numerical methods which we will discuss in Chapter 2.

1.4 Literature Survey

The birth of singular perturbations occurred at the Third International Congress of Mathematicians in Heidelberg 1904. Since then, a lot of work has been done in this area. We briefly mention here few of the works. Other works can be found in the large survey articles by Kadalbajoo and Patidar [KP02a, KP03b]. Some of the notable books are Bender and Orszag [BO78], Chang and Howes [CH84], Eckhaus [Eck73], Kevorkian and Cole [KC81], Nayfeh [Nay81], O'Malley [O'M74], Roos et al. [RST96] and Van Dyke [Dyk64].

Sakai and Usmani [SU86] gave a new concept of B-spline bases for hyperbolic and trigonometric splines which are different from earlier known ones. It is proved that the hyperbolic and trigonometric B-splines are characterized by a convolution of some special exponential functions and a characteristic function on the interval [0,1]. On this basis one obtains properties of these hyperbolic and trigonometric B-splines similar to those of the polynomial ones.

Kadalbajoo and Reddy [KR86] considered linear SPPs in the form of two-point boundary value problems. The basic idea is to introduce a small deviating or retarded argument δ and to expand the second-order term in the BVP. This converts the problem into a first-order functional differential equation. This equation is then integrated by parts with the trapezoidal rule.

Uzelac and Surla [US88] considered a collocation cubic spline difference schemes for a singularly perturbed two-point boundary value problem of second-order. This family of schemes includes the Ilin finite-difference scheme, which is uniformly convergent with first-order of accuracy.

Sakai and Usmani [SU89] applied simple exponential splines to solve SPPs They proved that the limiting case of their collocation method reduces to the collocation method with the usual quadratic spline. An estimate for the approximation error is given along with some numerical experiments.

Gartland [GJ88] considered a linear singularly perturbed two-point boundary value problem without turning points. Under minimal hypotheses on the coefficients in the differential equation, he gave an elegant proof that the El Mistikawy and Werle discretization on a regular mesh is second-order accurate, uniformly in the perturbation parameter. Also under slightly weaker assumptions for an equivalent PetrovGalerkin formulation, he proved the global uniform O(h) convergence.

Gartland [GJ89] established the strong uniform stability for a model singular perturbation problem. He used the finite-difference discretization of the continuous problem, and also some general stability results are verified for the schemes. He concluded that the finite-difference schemes (i) can be convergent without being close to the exact scheme, (ii) can be consistent without being stable, and vice versa, (iii) can be convergent without satisfying certain stability conditions. Further, stability of the finite-difference schemes can be achieved in ways other than using exponential fitting in regions away from layers of the solution, with the exception of the situations where the possible locations of all layer are not known a priori; in these cases the use of exponential fitting fully justified and desired.

Kadalbajoo and Patidar [KP01] described a numerical method based on cubic spline on nonuniform grid for a singularly perturbed two-point boundary value problems having turning point and boundary layers at one or both ends of the underlying interval. They have shown that the scheme derived is uniformly convergent with second-order of accuracy. They solved several numerical examples to demonstrate the applicability of their method.

Kadalbajoo and Patidar [KP02a] solved the singularly perturbed two-point boundary value problems by using spline in tension. They have shown that by making use of the continuity of the first derivatives of the spline function, the resulting spline difference scheme gives a tridiagonal system. They considered three types of problems. First they analyze the problems in which the second derivative term and the function term are present while the term containing the first derivative is absent. Secondly, they consider the problems having the second and first derivative terms but lacking the function term. The third case deals with the most general problem. They have shown that the schemes derived are second-order accurate. They solved six numerical examples to demonstrate the applicability of the method.

Kadalbajoo and Patidar [KP03a] used spline in compression for solving singular perturbation problems. These schemes are second-order accurate.

Recently, Lubuma and Patidar [LP07] constructed and analyzed non-standard finite difference methods for a class of singularly perturbed differential equation. They considered two types of problems, (i) those having solutions with layer behavior and (ii) those having solutions with oscillatory behavior. Since no fitted mesh method can be designed for the latter type of problems, they treated the problems using the non-standard finite difference methods (NSFDMs).

As far as the splines are concerned, some notable books are [ANW67, dB78, N̈89] and [Sch81].

The rest of the essay is organized as follows. In Chapter 2, we describe some spline methods to solve the problem under consideration. The results obtained with these spline methods are presented in Chapter 3. We discuss the pitfalls and any new plan in Chapter 4. The essay ends with a list of references.

2. Splines Methods

There are a number of methods available in the literature to solve SPPs like equation (1.1), but we are interested to use splines methods to solve these problems. In this chapter, we present these splines methods, first in general and next in context of our problem.

Consider the $n+1$ points

$$x_0,\ x_1, ...,\ x_n$$

in the segment $[a,b]$ which satisfy a grid

$$a = x_0 < x_1 < ... < x_n = b.$$

These points are called knots. The points x_0 and x_n are called end (boundary) knots. The grid above is called uniform if a distance between every two neighboring knots is the same [SP95].

A function $S(x)$ given on segment $[a,b]$ is called a spline of degree p if this function consists of piecewise polynomials on every segment

$$\triangle_j = [x_j, x_{j+1}],\ j = 0,\ 1, ...,\ n-1,$$

i.e., we can write $S(x)$ in the form

$$S(x) = S_j(x) = \sum_{k=0}^{p} a_k^{(j)} (x - x_j)^k, \quad j = 0,\ 1, ...,\ n-1, \tag{2.1}$$

which is $p-1$ times continuously differentiable on segment $[a,b]$. This means that $S(x) \in C^{p-1}[a,b]$. Here the index (j) of coefficient $a_k^{(j)}$ indicates for every partial segment $\triangle_j$ a system of numbers of the function $S(x)$ [SP95].

The above spline function is plugged into the differential equation under consideration in some specific ways which depends on the type of spline being used as is discussed below. (The systems of equations obtained via various splines methods are solved using Octave).

2.1 Cubic Spline

This is the most common spline interpolant. Given a function $y(x)$ defined on $[a,b]$ and a set of knots

$$a = x_0 < x_1 < ... < x_n = b,$$

a cubic spline interpolant, S, for $y(x)$ is a function that satisfies the following conditions [BF93]:

(a) S is a cubic polynomial denoted by S_j on the subinterval $[x_j, x_{j+1}]$ for $j = 0,\ 1, ...,\ n-1$,

(b) $S(x_j) = y(x_j)$ for $j = 0,\ 1, ...,\ n$,

(c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for $j = 0,\ 1, ...,\ n-2$,

(d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for $j = 0, 1, ..., n-2$,

(e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for $j = 0, 1, ..., n-2$,

(f) one of the following set of end (boundary) conditions is satisfied

- $S''(x_0) = S''(x_n) = 0$, (free or natural boundary),
- $S'(x_0) = y'(x_0)$ and $S'(x_n) = y'(x_n)$, (clamped boundary).

When the free boundary conditions are used, the spline is called a natural spline. On the other hand, when clamped boundary conditions are used, the spline is called a clamped cubic spline. In general clamped splines are more accurate approximations since they include more information about the function.

Why do we need the end conditions? To answer this, we note that in each interval we need to find four coefficients to specify the cubic polynomials, and we have n intervals. We therefore need to find a total of $4n$ unknowns. The conditions (b) give $n+1$ independent equations, and the conditions (c), (d) and (e) give $3 \times (n-1)$ independent equations. So we have $4n$ unknowns and $4n-2$ equations. There are two missing equations, and that is why the end (boundary) conditions (f) are required. The conditions (b) are called the interpolation conditions, and the conditions (c), (d) and (e) are called the continuity conditions.

Now we drive the equation for $S_j(x)$ on the interval $[x_j, x_{j+1}]$.

Define the numbers

$$z_j = S''(x_j).$$

These $z_{j'}s$ exist for $0 \leq j \leq n$ and satisfy

$$\lim_{x \to x_j^-} S''(x) = z_j = \lim_{x \to x_j^+} S''(x), \qquad (1 \leq j \leq n-1), \tag{2.2}$$

because S'' is continuous at each interior knots [KC91].

Since S_j is a cubic polynomial on $[x_j, x_{j+1}]$, S'' is a linear function satisfying

$$S''_j(x_j) = z_j$$

and

$$S''_j(x_{j+1}) = z_{j+1}$$

and is therefore given by the straight line between z_j and z_{j+1}

$$S''_j(x) = \frac{z_j}{h_j}(x_{j+1} - x) + \frac{z_{j+1}}{h_j}(x - x_j), \tag{2.3}$$

where $h_j = x_{j+1} - x_j$.

Integrating equation (2.3) twice, we obtain

$$S_j(x) = \frac{z_j}{6h_j}(x_{j+1} - x)^3 + \frac{z_{j+1}}{6h_j}(x - x_j)^3 + C(x - x_j) + D(x_{j+1} - x), \tag{2.4}$$

where C and D are the constants of integration.

Now the interpolation conditions

$$S_j(x_j) = y_j$$

and

$$S_j(x_{j+1}) = y_{j+1}$$

can be imposed on S_j to determine C and D, where we use the notation y_j to denote $y(x_j)$.

This gives

$$S_j(x) = \frac{z_j}{6h_j}(x_{j+1}-x)^3 + \frac{z_{j+1}}{6h_j}(x-x_j)^3 + \left(\frac{y_{j+1}}{h_j} - \frac{z_{j+1}h_j}{6}\right)(x-x_j) + \left(\frac{y_j}{h_j} - \frac{z_j h_j}{6}\right)(x-x_j). \tag{2.5}$$

To determine $z_1, z_2, ..., z_{n-1}$, we use the continuity conditions for S'. At the interior knots x_j, we have $S'_{j-1}(x_j) = S'_j(x_j)$. Substituting $x = x_j$ in equation (2.5), we get

$$S'_j(x_j) = -\frac{h_j}{3}z_j - \frac{h_j}{6}z_{j+1} - \frac{y_j}{h_j} + \frac{y_{j+1}}{h_j}. \tag{2.6}$$

Using equation (2.5) to obtain S_{j-1}, we have

$$S'_{j-1}(x_j) = \frac{h_{j-1}}{6}z_{j-1} + \frac{h_{j-1}}{3}z_j - \frac{y_{j-1}}{h_{j-1}} + \frac{y_j}{h_{j-1}}. \tag{2.7}$$

By virtue of equation (2.3) and equation (2.5) the functions S''_j and $S_j(x)$ are continuous on $[a,b]$.

Using equations (2.6) and (2.7), we see that the continuity of S'_j at x_j yields

$$h_{j-1}z_{j-1} + 2(h_j + h_{j-1})z_j + h_j z_{j+1} = \frac{6}{h_j}(y_{j+1} - y_j) - \frac{6}{h_{j-1}}(y_j - y_{j-1}), \tag{2.8}$$

where $1 \le j \le n-1$. This gives us a system of $n-1$ linear equations for the $n+1$ unknowns $z_0, z_1, ..., z_n$. We can set $z_0 = 0$ and $z_n = 0$ which correspond to placing simple supports at the end [ANW67]. The resulting system can be solved to obtain $z_1, z_2, ..., z_{n-1}$. This is the natural cubic spline [KC91].

The linear system of equations (2.8) with $z_0 = 0$ and $z_n = 0$ is symmetric, tridiagonal, diagonally dominant, and can be written as

$$\begin{bmatrix} u_1 & h_1 & & & & \\ h_1 & u_2 & h_2 & & & \\ & h_2 & u_3 & h_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-3} & u_{n-2} & h_{n-2} \\ & & & & h_{n-2} & u_{n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-2} \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{bmatrix} \tag{2.9}$$

where

$h_j = x_{j+1} - x_j,$

$u_j = 2(h_j + h_{j-1})$,

$b_j = \frac{6}{h_j}(y_{j+1} - y_j),$

$v_j = b_j - b_{j-1}.$

We adopt the above general approach in the following manner for our problem.

Let the approximate solution of problem (1.1) is given in the form of the cubic spline $S(x)$, which is denoted by $S_j(x)$ on each subinterval $[x_j, x_{j+1}]$ for $j = 0,\ 1, ...,\ n-1$, gives the equation

$$\left.\begin{aligned} \varepsilon S''(x_j) + b(x_j)S(x_j) = f_j,\quad 0 \leqslant x_j \leqslant n \\ S(0) = \alpha_0,\ S(1) = \alpha_1. \end{aligned}\right\} \tag{2.10}$$

where $b_j = b(x_j)$, and $f_j = f(x_j)$. Then we have

$$z_j = S_j''(x_j) = \frac{1}{\varepsilon}\left[f_j - b_j S_j(x_j)\right] = \frac{1}{\varepsilon}\left[f_j - b_j y_j\right], \tag{2.11}$$

where $S \approx y$. Substituting z_j in equations (2.8), we obtain for $1 \leq j \leq n-1$:

$$\frac{1}{\varepsilon}h_{j-1}\left[f_{j-1} - b_{j-1}y_{j-1}\right] + \frac{2}{\varepsilon}(h_j + h_{j-1})\left[f_j - b_j y_j\right] + \frac{1}{\varepsilon}h_j\left[f_{j+1} - b_{j+1}y_{j+1}\right]$$
$$= \frac{6}{h_j}y_{j+1} - \frac{6}{h_j}y_j - \frac{6}{h_{j-1}}y_j + \frac{6}{h_{j-1}}y_{j-1}. \tag{2.12}$$

$$\Rightarrow \left[\frac{-b_{j-1}h_{j-1}}{\varepsilon} - \frac{6}{h_{j-1}}\right]y_{j-1} + \left[\frac{-2(h_j + h_{j-1})b_j}{\varepsilon} + \frac{6}{h_j} + \frac{6}{h_{j-1}}\right]y_j + \left[\frac{-b_{j+1}h_j}{\varepsilon} - \frac{6}{h_j}\right]y_{j+1}$$
$$= -\frac{h_{j-1}}{\varepsilon}f_{j-1} - \frac{-2(h_j + h_{j-1})}{\varepsilon}f_j - \frac{h_j}{\varepsilon}f_{j+1}.$$

Multiplying both sides by $-\varepsilon$, we obtain

$$\left[b_{j-1}h_{j-1} + \frac{6\varepsilon}{h_{j-1}}\right]y_{j-1} + \left[2(h_j + h_{j-1})b_j - \frac{6\varepsilon}{h_j} - \frac{6\varepsilon}{h_{j-1}}\right]y_j + \left[b_{j+1}h_j + \frac{6\varepsilon}{h_j}\right]y_{j+1}$$
$$= h_{j-1}f_{j-1} + 2(h_j + h_{j-1})f_j + h_j f_{j+1}. \tag{2.13}$$

When the mesh is uniform with spacing h, equation (2.13) becomes

$$\left[b_{j-1}h + \frac{6\varepsilon}{h}\right]y_{j-1} + \left[4hb_j - \frac{12\varepsilon}{h}\right]y_j + \left[b_{j+1}h + \frac{6\varepsilon}{h}\right]y_{j+1}$$
$$= hf_{j-1} + 4hf_j + hf_{j+1}, \tag{2.14}$$

$$\Rightarrow \gamma_j^- y_{j-1} + \gamma_j^c y_j + \gamma_j^+ y_{j+1} = F_j \tag{2.15}$$

where

$\gamma_j^- = b_{j-1}h + \frac{6\varepsilon}{h}$,

$\gamma_j^c = 4hb_j - \frac{12\varepsilon}{h}$,

$\gamma_j^+ = b_{j+1}h + \frac{6\varepsilon}{h}$,

$F_j = hf_{j-1} + 4hf_j + hf_{j+1}$.

Equation (2.15) gives a system of $n-1$ linear equations for the unknowns $y_1, y_2, ..., y_{n-1}$ with $y_0 = \alpha_0$ and $y_n = \alpha_1$:

$$\begin{bmatrix} \gamma_1^c & \gamma_1^+ & & & & \\ \gamma_2^- & \gamma_2^c & \gamma_2^+ & & & \\ & \gamma_3^- & \gamma_3^c & \gamma_3^+ & & \\ & & \ddots & \ddots & \ddots & \\ & & & \gamma_{n-2}^- & \gamma_{n-2}^c & \gamma_{n-2}^+ \\ & & & & \gamma_{n-1}^- & \gamma_{n-1}^c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} F_1 - \gamma_1^- y_0 \\ F_2 \\ F_3 \\ \vdots \\ F_{n-2} \\ F_{n-1} - \gamma_{n-1}^+ y_n \end{bmatrix} \tag{2.16}$$

The relevant results are presented in Chapter 3.

2.2 B-Spline

In general, given n distinct knots $x_1 < x_2 < ... < x_n$ in the open interval (a, b) and an integer $k \geq 1$, the space $S^k(x)$ is the space of functions of class C^{k-1} over $[a, b]$ which coincide with polynomials of degree at most k on each interval $[x_j, x_{j+1}]$, for $1 \leq j \leq n$, with $x_0 = a$ and $x_{n+1} = b$. The space $S^k(x)$ is called the space of splines of degree k [Sch02].
A function of the form

$$B_j^{(0)}(x) = \begin{cases} 1, & x \in [x_j, x_{j+1}] \\ 0, & x \notin [x_j, x_{j+1}] \end{cases}$$

is called a B-spline function of degree zero defined on segment $[a, b]$ and is depicted in Figure 2.1.

The B-spline function of degree $k \geqslant 1$ defined on segment $[a, b]$ is constructed by the recurrent relation

$$B_j^{(k)}(x) = \frac{x - x_j}{x_{j+1} - x_j} B_j^{(k-1)}(x) + \frac{x_{j+k+1} - x}{x_{j+k+1} - x_{j+1}} B_{j+1}^{(k-1)}(x). \tag{2.17}$$

We can give an explicit formula for B-splines of degree 1, i.e., $B_j^1(x)$ as

$$B_j^1(x) = \frac{x - x_j}{x_{j+1} - x_j} B_j^0(x) + \frac{x_{j+2} - x}{x_{j+2} - x_{j+1}} B_{j+1}^0(x).$$

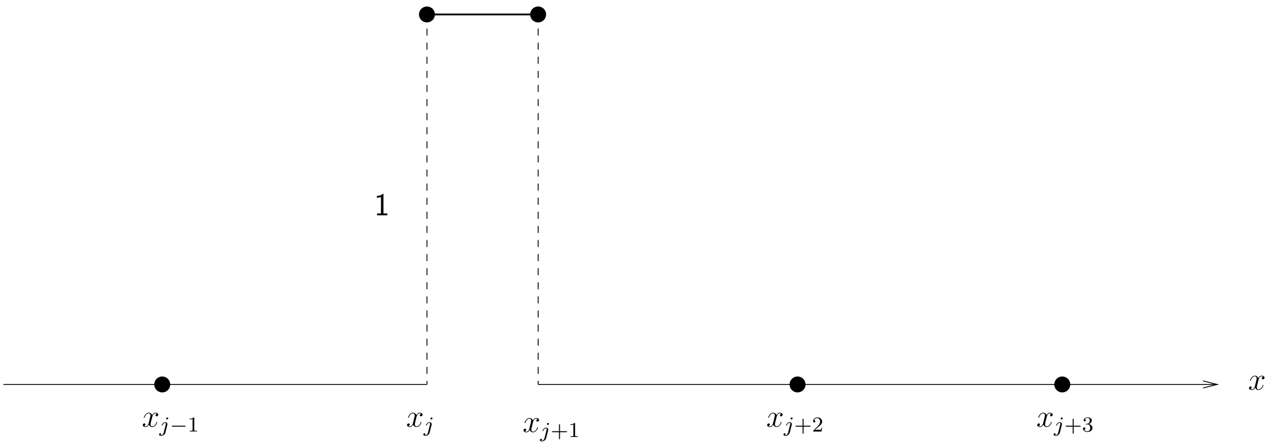

Figure 2.1: The B-spline B_j^0 of degree 0.

which gives

$$B_j^1(x) = \begin{cases} \frac{x_{j+2}-x}{x_{j+2}-x_{j+1}}, & \text{if } x \in [x_{j+1}, x_{j+2}] \\ \frac{x-x_j}{x_{j+1}-x_j}, & \text{if } x \in [x_j, x_{j+1}] \\ 0, & \text{otherwise.} \end{cases}$$

It is visualized in Figure 2.2.

By introducing some special linear functions

$$V_j^k(x) = \frac{x - x_j}{x_{j+k} - x_j},$$

we can write the recurrence relation in the following more elegant form

$$B_j^k = V_j^{k-1} + (1 - V_{j+1}^k) B_{j+1}^{k-1}.$$

Since B_j^0 is a piecewise polynomial of degree 0, and since V_j^k is linear, B_j^1 is a piecewise polynomial of degree $\leqslant 1$. The same reasoning shows that, in general, B_j^k will be a piecewise polynomial of degree $\leqslant k$.

We adopt the above general approach in the following manner for our problem. Let the approximate solution of problem (1.1) be given in the form of the B-spline. We subdivide the interval $[0, 1]$, and we choose piecewise uniform mesh points represented by

$$x_0,\ x_1,\ x_2, ...,\ x_n,$$

such that $x_0 = 0$ and $x_n = 1$ and h is the uniform spacing.

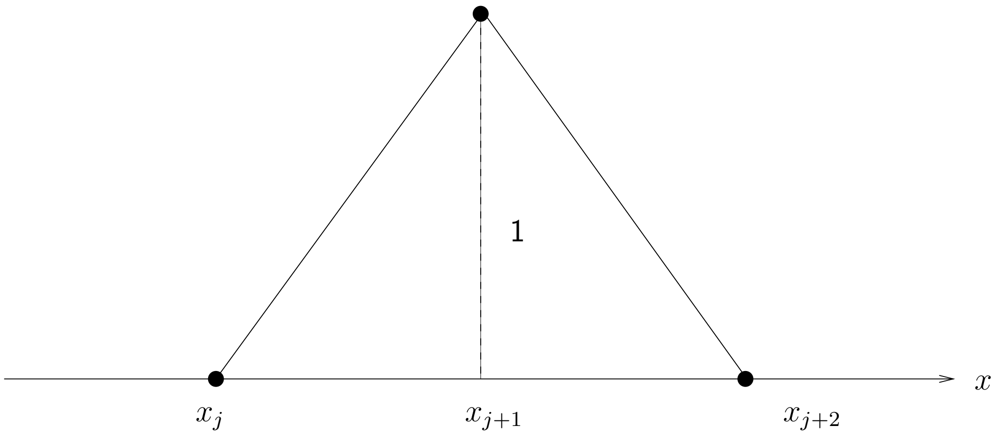

Figure 2.2: The B-spline B_j^1 of degree 1.

Now we define the B-spline for $i = 1, 2, ..., n$ as

$$B_i(x) = \begin{cases} \left(\frac{x-x_{i-2}}{h}\right)^3, & \text{if} \quad x \in [x_{i-2}, x_{i-1}], \\ 1 + 3\left(\frac{x-x_{i-1}}{h}\right) + 3\left(\frac{x-x_{i-1}}{h}\right)^2 - 3\left(\frac{x-x_{i-1}}{h}\right)^3, & \text{if} \quad x \in [x_{i-1}, x_i], \\ 1 + 3\left(\frac{x_{i+1}-x}{h}\right) + 3\left(\frac{x_{i+1}-x}{h}\right)^2 - 3\left(\frac{x_{i+1}-x}{h}\right)^3, & \text{if} \quad x \in [x_i, x_{i+1}], \\ \left(\frac{x_{i+2}-x}{h}\right)^3, & \text{if} \quad x \in [x_{i+1}, x_{i+2}], \\ 0, & \text{otherwise.} \end{cases} \tag{2.18}$$

We introduce four additional knots as $x_{-2} < x_{-1} < x_0$ and $x_{n+2} > x_{n+1} > x_n$.

From equation (2.18) we can notice that each of the functions $B_i(x)$ is twice continuously differentiable on the entire real line. Moreover

$$B_i(x_j) = \begin{cases} 4, & \text{if } i = j, \\ 1, & \text{if } i - j = \pm 1, \\ 0, & \text{if } i - j = \pm 2, \end{cases} \tag{2.19}$$

and that $B_i(x) = 0$ for $x \geqslant x_{i+2}$ and $x \leqslant x_{i-2}$.

Furthermore, we can prove that

$$B_i'(x_j) = \begin{cases} 0, & \text{if } i = j, \\ \pm\frac{3}{h}, & \text{if } i - j = \pm 1, \\ 0, & \text{if } i - j = \pm 2, \end{cases} \tag{2.20}$$

and

$$B_i''(x_j) = \begin{cases} \frac{-12}{h^2}, & \text{if } i = j, \\ \frac{6}{h^2}, & \text{if } i - j = \pm 1, \\ 0, & \text{if } i - j = \pm 2. \end{cases} \tag{2.21}$$

Now, let

$$\Omega = \{B_{-1},\, B_0,\, B_1, ...,\, B_{n+1}\}.$$

The functions in Ω are linearly independent on $[0, 1]$.

Define

$$S(x) = \sum_{i=-1}^{n+1} c_i B_i(x) \tag{2.22}$$

and assume that $S(x)$ satisfies the equation (1.1). Thus we have

$$\left.\begin{array}{ll} \varepsilon S''(x_j) + b(x_j)S(x_j) = f_j, & 0 \leqslant x_j \leqslant n \\ S(0) = \alpha_0,\; S(1) = \alpha_1. & \end{array}\right\} \tag{2.23}$$

Therefore, we obtain

$$\varepsilon \sum_{i=-1}^{n+1} c_i B_i''(x_j) + b_j \sum_{i=-1}^{n+1} c_i B_i(x_j) = f_j, \quad b_j = b(x_j)$$

Solving the above equation (note that the support of the function $B_i(x)$ is the segment $[x_{i-2}, x_{i+2}]$) we get

$$\begin{aligned} & c_{j-1}(\varepsilon B_{j-1}''(x_j) + b_{j-1}B_{j-1}(x_j)) + c_j(\varepsilon B_j''(x_j) + b_j B_j(x_j)) \\ & + c_{j+1}(\varepsilon B_{j+1}''(x_j) + b_{j+1}B_{j+1}(x_j)) = f_j, \quad \forall\ j = 0,\, 1, ...,\, n. \end{aligned} \tag{2.24}$$

Using equations (2.19) and (2.21) we get

$$(6\varepsilon + b_j h^2)c_{j-1} + (-12\varepsilon + 4b_j h^2)c_j + (6\varepsilon + b_j h^2)c_{j+1} = h^2 f_j,\ \forall j = 0,\, 1, ...,\, n. \tag{2.25}$$

The boundary conditions become

$$c_{-1} + 4c_0 + c_1 = \alpha_0, \tag{2.26}$$

and

$$c_{n-1} + 4c_n + c_{n+1} = \alpha_1. \tag{2.27}$$

The equations (2.25), (2.26) and (2.27) lead to a $(n+3) \times (n+3)$ tridiagonal system with $(n+3)$ unknowns $c_{-1},\, c_0,\, ...,\, c_{n+1}$. By eliminating c_{-1} from the first equation of (2.25) and (2.26), we get

$$-36\varepsilon\ c_0 = f_0 h^2 - \alpha_0(6\varepsilon + b_0 h^2). \tag{2.28}$$

Similarly, eliminating c_{n+1} from the last equation of (2.25) and (2.27), we get

$$-36\varepsilon\ c_n = f_n h^2 - \alpha_1(6\varepsilon + b_n h^2). \tag{2.29}$$

By putting the equations (2.28) and (2.29) with the $(n-1)$ remaining equations of (2.25), we get a system of $(n+1)$ linear equations in the unknowns $c_0,\ c_1, , ...,\ c_n$ of the form

$$\begin{bmatrix} -36\varepsilon & & & & & \\ \gamma_1 & \gamma_1^c & \gamma_1 & & & \\ & \gamma_2 & \gamma_2^c & \gamma_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \gamma_{n-1} & \gamma_{n-1}^c & \gamma_{n-1} \\ & & & & & -36\varepsilon \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} f_0h^2 - \alpha_0\gamma_0 \\ f_1h^2 \\ f_2h^2 \\ \vdots \\ f_{n-1}h^2 \\ f_nh^2 - \alpha_0\gamma_n \end{bmatrix} \tag{2.30}$$

where

$$\gamma_j = 6\varepsilon + b_j h^2,$$
$$\gamma_j^c = -12\varepsilon + 4b_j h^2.$$

Since $b(x) > 0$, we can see that the matrix associated with the system is strictly diagonally dominant and hence nonsingular. So we can solve the system for $c_0,\ c_1, ...,\ c_n$. Then we use the boundary equations (2.26) and (2.27) to obtain c_{-1} and c_{n+1}. Finally, solving equation (2.30), we obtains B-spline solution via equation (2.22) through equation (2.19).

The results corresponding to this spline are presented along with those obtained via cubic spline in Chapter 3.

2.3 Periodic Splines

In this case, we seek the spline $S(x)$ which is continuous together with its first and second derivatives on $[a, b]$, coincides with a cubic in each subinterval $x_{j-1} \leq x \leq x_j$ for $j = 1,\ 2, ...,\ n$, and satisfies $S_j(x) = y_j$, for $j = 0,\ 1, ...,\ n$.

The spline $S(x)$ is said to be periodic of period $(b-a)$ if the condition

$$S^{(p)}(a^+) = S^{(p)}(b^-), \qquad (p = 0, 1, 2) \tag{2.31}$$

is satisfied [ANW67].

If a periodic function which is at least twise continuously differentiable is to be represented on its interval of periodicty by an interpolating periodic spline function, the natural boundary conditions must be replaced by conditions that take into account the periodicity. The support points are arranged in increasing order $x_0 < x_1 < ... < x_n$. Equation (2.8) then gives $n-1$ linear equations in the quantities $z_1, ...,\ z_n$. In the case of periodic cupic spline, we require that the equation (2.8) be satisfied for $j = n$ as well. Here $y_n = y_0$, $z_n = z_0$, and $h_n = h_0$ and we prescribe $y_{n+1} = y_1$ and $z_{n+1} = z_1$ [ANW67].

The defining equations in this case are

$$\begin{bmatrix} u_1 & h_1 & & & & h_0 \\ h_1 & u_2 & h_2 & & & \\ & h_2 & u_3 & h_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-2} & u_{n-1} & h_{n-1} \\ h_0 & & & & h_{n-1} & u_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} \tag{2.32}$$

where

$$h_j = x_{j+1} - x_j,$$

$$u_j = 2(h_j + h_{j-1}),$$

$$b_j = \tfrac{6}{h_j}(y_{j+1} - y_j),$$

$$v_j = b_j - b_{j-1}.$$

To obtain the approximate solution for problem (1.1) using the periodic cubic spline, we require that the equation (2.15) should be satisfied for $j = n$ as well. When $f = 0$, we have $b_n = b_0$ and $b_{n+1} = b_1$, if not, we assume f as a periodic function, and hence we set $f_n = f_0$, $f_{n+1} = f_1$, $y_n = y_0$ and $y_{n+1} = y_1$. Thus we have a system of n linear equations for the unknowns $y_1, y_2, \ldots, y_n$:

$$\begin{bmatrix} (\gamma_1^c + \gamma_n^+) & \gamma_1^+ & & & & \gamma_n^- \\ \gamma_2^- & \gamma_2^c & \gamma_2^+ & & & \\ & \gamma_3^- & \gamma_3^c & \gamma_3^+ & & \\ & & \ddots & \ddots & \ddots & \\ & & & \gamma_{n-2}^- & \gamma_{n-2}^c & \gamma_{n-2}^+ \\ & & & & \gamma_{n-1}^- & \gamma_{n-1}^c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} F_1 + F_n - (\gamma_1^c + \gamma_n^+)y_0 \\ F_2 \\ F_3 \\ \vdots \\ F_{n-2} \\ F_{n-1} - \gamma_{n-1}^+ y_n \end{bmatrix} \tag{2.33}$$

where

$$\gamma_j^- = b_{j-1}h + \tfrac{6\varepsilon}{h},$$

$$\gamma_j^c = 4hb_j - \tfrac{12\varepsilon}{h},$$

$$\gamma_j^+ = b_{j+1}h + \tfrac{6\varepsilon}{h},$$

$$F_j = hf_{j-1} + 4hf_j + hf_{j+1}.$$

We are currently busy computing the results using this spline method.

3. Numerical Results

We solve the following examples using cubic spline and B-spline. The results corresponding to periodic cubic splines are still to be obtained and therefore not being included in this essay.

Example 3.1. Consider the boundary value problem

$$\left.\begin{aligned} \varepsilon y'' + y = 0 \\ y(0) = 1,\ y(1) = 1, \end{aligned}\right\} \tag{3.1}$$

in the interval $[0,1]$. The exact solution is given by

$$y(x) = \cos(x/\sqrt{\varepsilon}) + \left(\frac{1-\cos(1/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})}\right)\sin(x/\sqrt{\varepsilon}).$$

Example 3.2. Consider the boundary value problem

$$\left.\begin{aligned} \varepsilon y'' + y = 2 \\ y(0) = 1,\ y(1) = 1, \end{aligned}\right\} \tag{3.2}$$

the exact solution is given by

$$y(x) = 2 - \cos(x/\sqrt{\varepsilon}) + \left(\frac{\cos(1/\sqrt{\varepsilon})-1}{\sin(1/\sqrt{\varepsilon})}\right)\sin(x/\sqrt{\varepsilon})$$

We tabulate the maximum errors (maximum of the difference between the exact and the numerical solutions) for various values of $n(=1/h)$ and ε.

Table 3.1: Maximum error for Example 3.1 via cubic spline

ε	$n = 20$	$n = 40$	$n = 80$	$n = 160$	$n = 320$
1	3.24e-05	8.10e-06	2.03e-06	5.07e-07	1.27e-07
2^{-1}	1.65e-04	4.14e-05	1.03e-05	2.59e-06	6.47e-07
2^{-2}	1.20e-03	3.00e-04	7.51e-05	1.88e-05	4.69e-06
2^{-3}	4.74e-02	1.19e-02	2.99e-03	7.48e-04	1.87e-04
2^{-4}	1.76e-02	4.38e-03	1.09e-03	2.73e-04	6.84e-05
2^{-5}	5.46e-03	1.37e-03	3.44e-04	8.61e-05	2.15e-05
2^{-6}	5.13e-02	1.34e-02	3.39e-03	8.50e-04	2.13e-04
2^{-7}	9.68e-02	2.36e-02	5.88e-03	1.47e-03	3.67e-04
2^{-8}	2.58e+01	3.84e+00	6.87e-01	1.60e-01	3.94e-02
2^{-9}	8.01e+00	2.67e+00	4.25e-01	9.76e-02	2.39e-02
2^{-10}	1.82e+00	3.67e-01	1.01e-01	2.61e-02	6.62e-03
2^{-11}	2.64e+00	1.01e+00	3.54e-01	1.04e-01	2.74e-02
2^{-12}	7.44e+01	3.33e+00	8.36e-01	2.57e-01	7.10e-02
2^{-13}	3.40e+00	3.58e+01	4.02e+00	2.38e+00	1.15e+00
2^{-14}	2.55e+00	3.55e+00	4.34e+00	2.76e+00	1.41e+00
2^{-15}	1.21e+00	1.39e+00	2.20e+00	1.09e+01	6.49e+00

Table 3.2: Maximum error for Example 3.1 via B-spline

ε	$n = 20$	$n = 40$	$n = 80$	$n = 160$	$n = 320$
1	3.24e-05	8.10e-06	2.03e-06	5.07e-07	1.27e-07
2^{-1}	1.65e-04	4.14e-05	1.03e-05	2.59e-06	6.47e-07
2^{-2}	1.20e-03	3.00e-04	7.51e-05	1.88e-05	4.69e-06
2^{-3}	4.74e-02	1.19e-02	2.99e-03	7.48e-04	1.87e-04
2^{-4}	1.76e-02	4.38e-03	1.09e-03	2.73e-04	6.84e-05
2^{-5}	5.46e-03	1.37e-03	3.44e-04	8.61e-05	2.15e-05
2^{-6}	5.13e-02	1.34e-02	3.39e-03	8.50e-04	2.13e-04
2^{-7}	9.68e-02	2.36e-02	5.88e-03	1.47e-03	3.67e-04
2^{-8}	2.58e+01	3.84e+00	6.87e-01	1.60e-01	3.94e-02
2^{-9}	8.01e+00	2.67e+00	4.25e-01	9.76e-02	2.39e-02
2^{-10}	1.82e+00	3.67e-01	1.01e-01	2.61e-02	6.62e-03
2^{-11}	2.64e+00	1.01e+00	3.54e-01	1.04e-01	2.74e-02
2^{-12}	7.44e+01	3.33e+00	8.36e-01	2.57e-01	7.10e-02
2^{-13}	3.40e+00	3.58e+01	4.02e+00	2.38e+00	1.15e+00
2^{-14}	2.55e+00	3.55e+00	4.34e+00	2.76e+00	1.41e+00
2^{-15}	1.21e+00	1.39e+00	2.20e+00	1.09e+01	6.49e+00

Table 3.3: Maximum error for Example 3.2 via Cubic spline

ε	$n=20$	$n=40$	$n=80$	$n=160$	$n=320$
1	3.24e-05	8.10e-06	2.03e-06	5.07e-07	1.27e-07
2^{-1}	1.65e-04	4.14e-05	1.03e-05	2.59e-06	6.47e-07
2^{-2}	1.20e-03	3.00e-04	7.51e-05	1.88e-05	4.69e-06
2^{-3}	4.74e-02	1.19e-02	2.99e-03	7.48e-04	1.87e-04
2^{-4}	1.76e-02	4.38e-03	1.09e-03	2.73e-04	6.84e-05
2^{-5}	5.46e-03	1.37e-03	3.44e-04	8.61e-05	2.15e-05
2^{-6}	5.13e-02	1.34e-02	3.39e-03	8.50e-04	2.13e-04
2^{-7}	9.68e-02	2.36e-02	5.88e-03	1.47e-03	3.67e-04
2^{-8}	2.58e+01	3.84e+00	6.87e-01	1.60e-01	3.94e-02
2^{-9}	8.01e+00	2.67e+00	4.25e-01	9.76e-02	2.39e-02
2^{-10}	1.82e+00	3.67e-01	1.01e-01	2.61e-02	6.62e-03
2^{-11}	2.64e+00	1.01e+00	3.54e-01	1.04e-01	2.74e-02
2^{-12}	7.44e+01	3.33e+00	8.36e-01	2.57e-01	7.10e-02
2^{-13}	3.40e+00	3.58e+01	4.02e+00	2.38e+00	1.15e+00
2^{-14}	2.55e+00	3.55e+00	4.34e+00	2.76e+00	1.41e+00
2^{-15}	1.21e+00	1.39e+00	2.20e+00	1.09e+01	6.49e+00

Table 3.4: Maximum error for Example 3.2 via B-spline

ε	$n=20$	$n=40$	$n=80$	$n=160$	$n=320$
1	3.24e-05	8.10e-06	2.03e-06	5.07e-07	1.27e-07
2^{-1}	1.65e-04	4.14e-05	1.03e-05	2.59e-06	6.47e-07
2^{-2}	1.20e-03	3.00e-04	7.51e-05	1.88e-05	4.69e-06
2^{-3}	4.74e-02	1.19e-02	2.99e-03	7.48e-04	1.87e-04
2^{-4}	1.76e-02	4.38e-03	1.09e-03	2.73e-04	6.84e-05
2^{-5}	5.46e-03	1.37e-03	3.44e-04	8.61e-05	2.15e-05
2^{-6}	5.13e-02	1.34e-02	3.39e-03	8.50e-04	2.13e-04
2^{-7}	9.68e-02	2.36e-02	5.88e-03	1.47e-03	3.67e-04
2^{-8}	2.58e+01	3.84e+00	6.87e-01	1.60e-01	3.94e-02
2^{-9}	8.01e+00	2.67e+00	4.25e-01	9.76e-02	2.39e-02
2^{-10}	1.82e+00	3.67e-01	1.01e-01	2.61e-02	6.62e-03
2^{-11}	2.64e+00	1.01e+00	3.54e-01	1.04e-01	2.74e-02
2^{-12}	7.44e+01	3.33e+00	8.36e-01	2.57e-01	7.10e-02
2^{-13}	3.40e+00	3.58e+01	4.02e+00	2.38e+00	1.15e+00
2^{-14}	2.55e+00	3.55e+00	4.34e+00	2.76e+00	1.41e+00
2^{-15}	1.21e+00	1.39e+00	2.20e+00	1.09e+01	6.49e+00

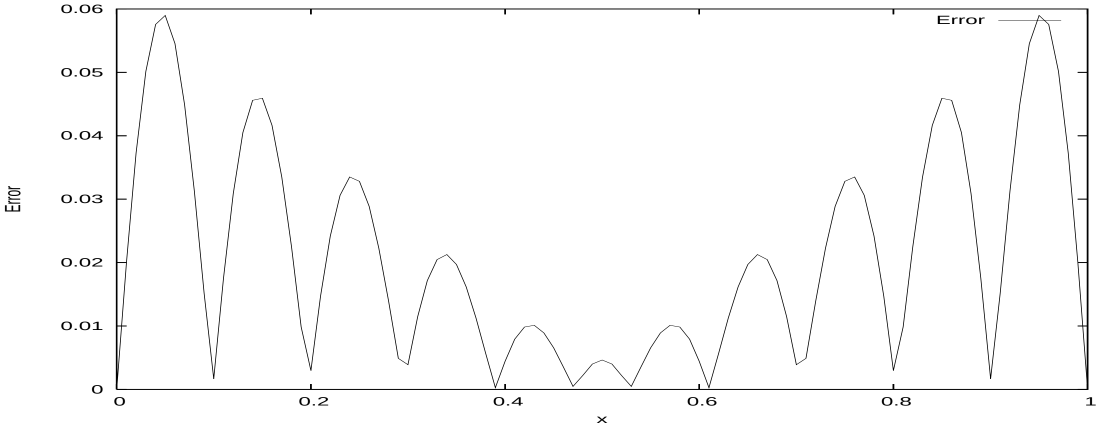

Figure 3.1: Difference between the exact and numerical solution obtained via cubic spline for Example 3.1: for n = 100, and $\varepsilon = 0.001$

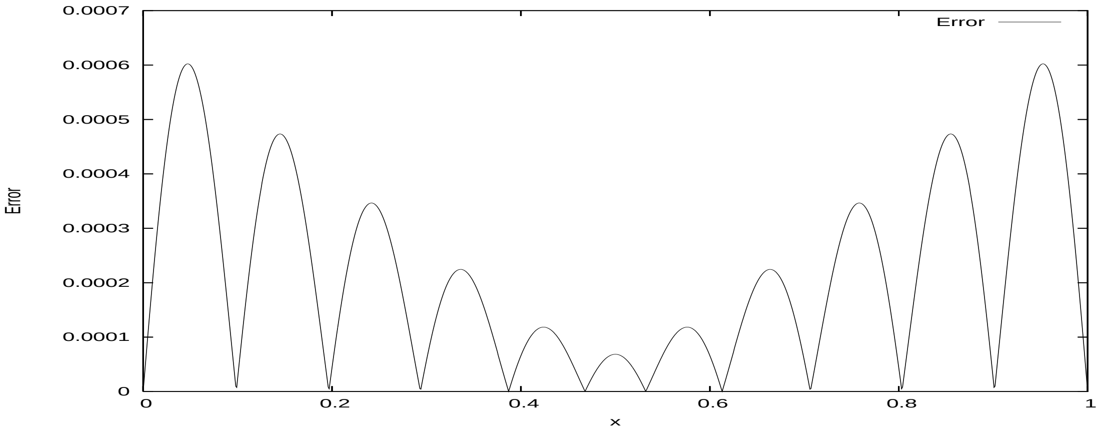

Figure 3.2: Difference between the exact and numerical solution obtained via cubic spline for Example 3.1: for n = 1000, and $\varepsilon = 0.001$

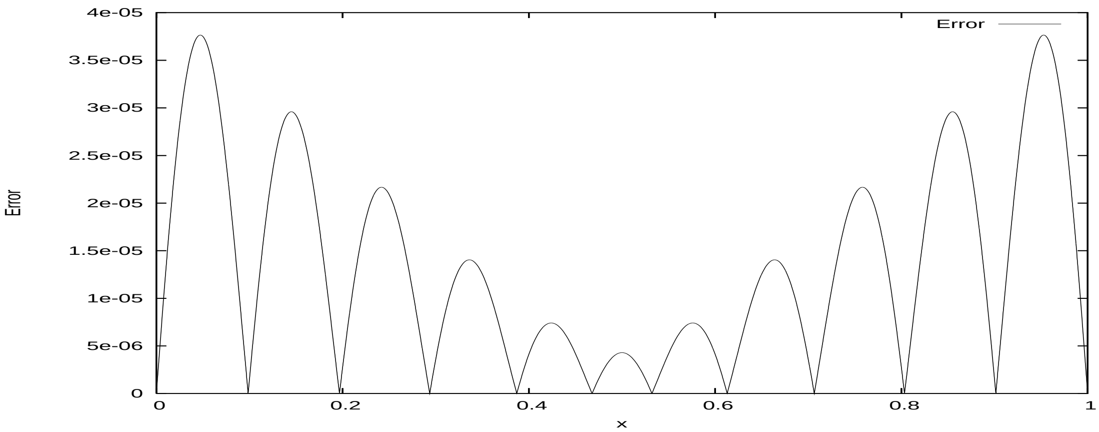

Figure 3.3: Difference between the exact and numerical solution obtained via cubic spline for Example 3.1: for n = 4000, and $\varepsilon = 0.001$

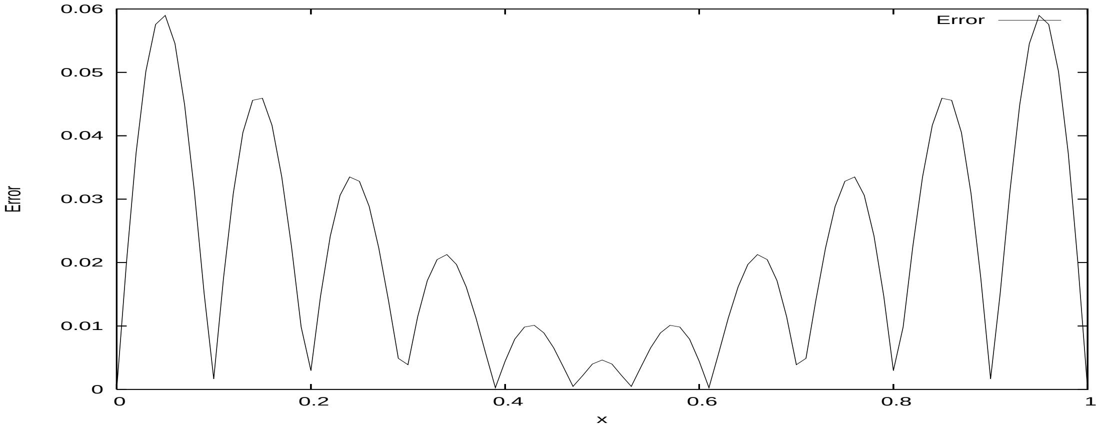

Figure 3.4: Difference between the exact and numerical solution obtained via cubic spline for Example 3.2: for n = 100, and $\varepsilon = 0.001$

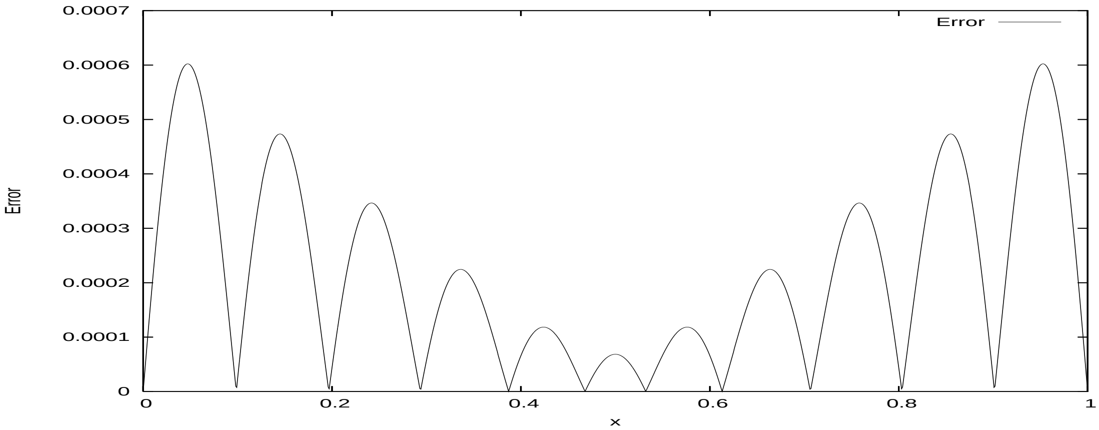

Figure 3.5: Difference between the exact and numerical solution obtained via cubic spline for Example 3.2: for n = 1000, and $\varepsilon = 0.001$

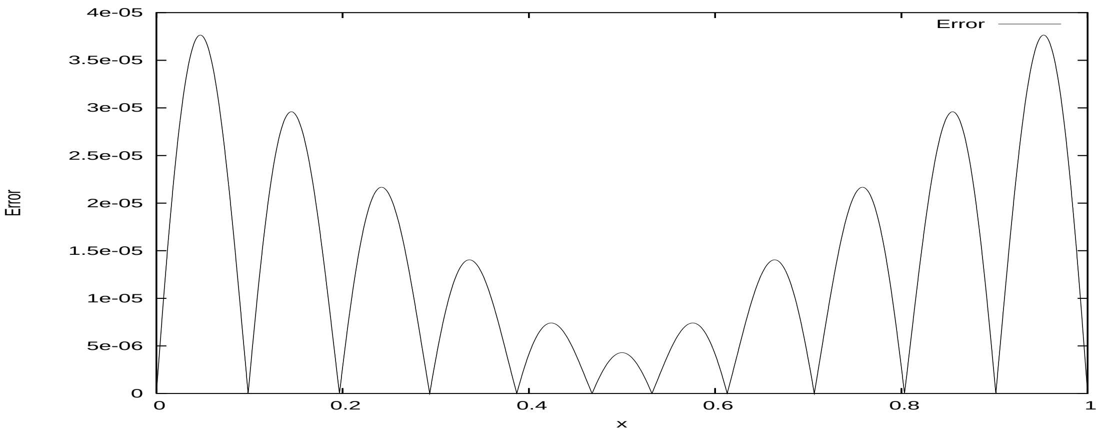

Figure 3.6: Difference between the exact and numerical solution obtained via cubic spline for Example 3.2: for n = 4000, and $\varepsilon = 0.001$

4. Conclusions and Future Plans

We have described some spline methods for solving singular perturbation problems whose solutions are oscillatory. We used Cubic spline, B-spline and periodic cubic splines.

The cubic spline and B-spline are tested for two examples (one is homogeneous and the other is a non-homogeneous boundary value problem) and results are presented in Chapter 3. Due to lack of time we could not finish computing the results for periodic cubic spline.

In Tables 3.1-3.4, we have tabulated the maximum errors for various values of h and ε. As can be seen form the tabular results, the cubic spline (as is expected) does provide good results when $h \leq \varepsilon$ but does not provide reliable results when the singular perturbation parameter ε becomes smaller than the step-size h. It also has the drawback that some of the numerical solutions are spurious.

The results obtained by B-spline are exactly the same for the two test problems. The mere reason for using B-spline was to find out whether this collocation approach performs better (if at all). By the way, in a collocation approach, one takes a set of basis functions and tests whether their linear combination satisfies the differential equation at some specific grid points. These grid points should be chosen carefully. Unfortunately, due to the oscillatory nature of the solution, locating such grid points is not obvious like the problems whose solutions have layer(s). Hence, we ended up getting the same results with the B-spline (on a uniform mesh) as those obtained with cubic spline.

We plot the graphs of the absolute values of the difference between the exact and numerical solution obtained via cubic spline. (Corresponding figures for B-spline solutions are not being included due to the similarity). One will notice that when h is larger than ε then the error is large (see, Figures 3.1 and 3.4); when h is of the order of ε then the error is reasonably small (see, Figures 3.2 and 3.5. However, when h is smaller than ε then the error decreases significantly (see, Figures 3.3 and 3.6). (Please note that the graphs look similar but the scale on the vertical axis is different). This is not surprising. This is how we expect the cubic spline to behave.

To check whether this dependence of h and ε can be circumvent, we decided to use the periodic cubic spline with the hope that it should give us better (if not the best) results due to the periodic nature of the theoretical solution. Both theoretical analysis and the numerical simulation to confirm this is currently under progress.

Our final goal through various approaches is to design a parameter uniform numerical method, i.e., a method which provides the numerical solution with the same error for any value of ε irrespective of the size of h.

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