Theoretical Description of Quasi-free A(a,a'b)B Reactions

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Abstract

The aim of this work is to present the theoretical formulation for nuclear reactions of general form A(a,a'b)B called quasi-free knockout reactions. The differential cross section formulation of the reaction is described following two approaches, the distorted wave impulse approximation (DWIA) and the plane wave impulse approximation (PWIA). Emphasis is made on several physical aspects and assumptions in the DWIA method and special attention is focused on the fact that the beam of incident particles is polarised.

Résumé

L'objet de ce travail est de pésenter la formulation théorique de réactions nucléaires de forme générale A(a,a'b)B appelées "knock-out" quasi-libres réactions. La formulation de la section efficace est réalisée suivant deux approches dans le cadre de l'approximation d'impulsion, soit par onde distordue, soit par onde plane. Divers aspects physiques ainsi que d'autres liens à l'approximation par onde distordue sont soulignés, tandis qu'une attention particulière est portée au cas d'un faisceau polarisé de particules incidentes.

Contents

1. Introduction

For many decades, the cluster structure of nuclei has been a rising research field in nuclear physics. Since a free alpha particle is a particularly stable configuration, it is tempting to predict the existence alpha-clusters in nuclei. One might ask whether these clusters are real entities or simply a way of carrying out calculations to describe observables appropriately for a many-body system [3].

To investigate alpha-clustering in nuclei experimentally, one needs to know whether four nucleons (two protons and two neutrons) in a nucleus can form an alpha-cluster. The most direct experimental method of studying ground-state alpha-clustering in nuclei is by means of a *knockout reaction* in which an incident particle knocks a bound alpha particle out. Quasi-free scattering, where a projectile interacts directly with a bound particle, is used to investigate cluster structure of the bound nucleon.

The theoretical framework for describing knockout reactions is the distorted wave impulse approximation (DWIA). The DWIA takes into account the effects of distortion of the incident and emitted wave functions by the residual nucleus [12]. The plane wave impulse approximation (PWIA) is a special case (all scattering potentials are zero) of the DWIA.

This project concerns the formulation for the differential cross section of the general knockout reaction $A(a, a'b)B$ using the DWIA formulation, where the nucleon or cluster b is ejected during inelastic scattering of the projectile a, hence resulting in three-body final states.

Chapter 2 describes the theoretical background of the formulation of the differential cross section. We will begin by using Fermi's Golden rule, which in turn will lead to the formulation of the energy density, the triple differential cross section, the transition amplitude, as well as the description of the optical potentials which distort the nucleon wave function.

Chapter 3 describes the plane wave impulse approximation (PWIA), and we will study the distorted wave impulse approximation (DWIA) in detail. Through examination of the DWIA approach, we are able to obtain the general formulation of the differential cross section, and the treatment of polarised incident particles will allow us to define the analysing power. The PWIA is a severe and simplistic approximation, and the only purpose of looking at such an unrealistic approximation is that in this way one gets a feeling for the important parts of the formulation. One gets a simple expression that consists of two separate parts, namely the two-body $a - b$ cross section and the wave function of the particle b bound in the target that gets knocked out. The DWIA contains these same basic components, but in a complicated way which is not as obvious.

Chapter 4 includes a summary of all approximations taken into account and summarises what we have achieved in the time available for this project. We then suggest possible avenues of future research.

2. Theoretical Background

Consider the general $(p, p\alpha)$ reaction as $A(a, a'b)B$ where A is the target nucleus, a the incoming proton, a' and b the outgoing particles and B the residual nucleus. The incident proton, a, knocks out a bound nucleon, b, from a specific orbital in the target nucleus, A, resulting in three particles in the final state, namely the recoil residual nucleus, B, and two outgoing particles, a' and b (Refer to 2.1). The prime serves to identify the particle a in the exit channel.

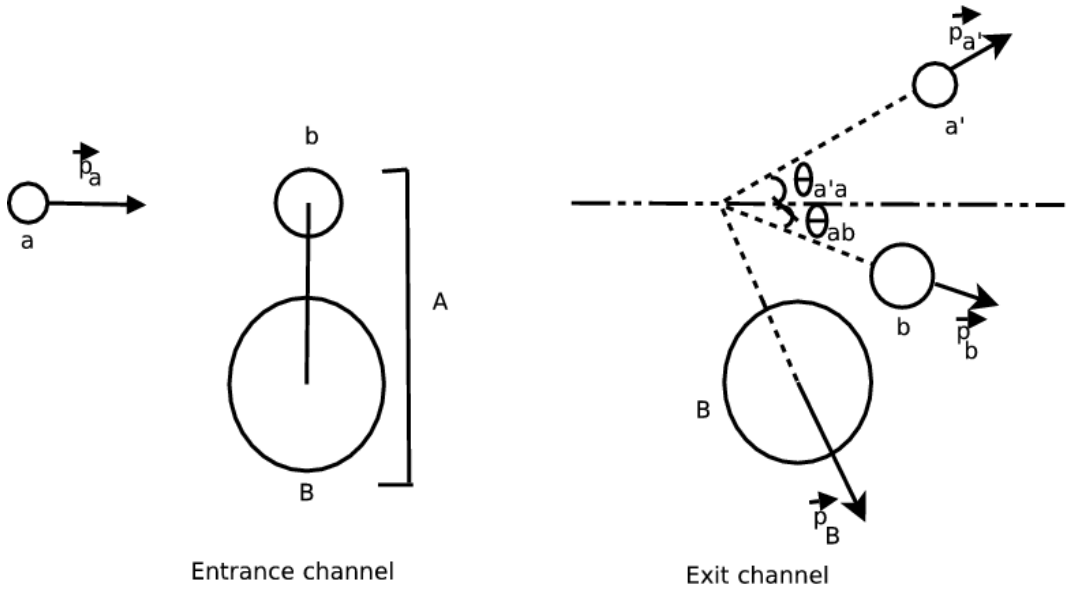

Figure 2.1: Schematic representation for the coplanar $\mathbf{A}(\mathbf{a}, \mathbf{a'b})\mathbf{B}$ reaction.

2.1 Cross Section

In nuclear and particle physics, the concept of a *cross section* is used to express the likelihood of interaction between particles.The cross section σ is defined by

$$\sigma = \frac{R}{I} \tag{2.1}$$

where I the number of incident particles per unit time per unit area and R the number of reactions per unit time per nucleus. It therefore characterise the probability that a particular nuclear reaction will take place, or the probability per target nucleus that the incident particle will be scattered into a given unit solid angle.

A differential cross section is defined as the probability to observe a scattered particle in a given quantum state per unit solid angle, such as within a given cone of observation. If the target is

irradiated by a flux of one particle per surface unit, the differential cross section is defined as

$$\frac{d\sigma}{d\Omega} = \frac{\text{Scattered flux/Unit of solid angle}}{\text{Incident flux/Unit of surface}}. \tag{2.2}$$

Fermi's Golden rule states that the differential cross section for this reaction is given by [10]

$$\sigma_{fi} = \frac{2\pi}{\hbar v}|T_{fi}|^2\omega_f \ , \tag{2.3}$$

where v is the relative velocity of a and A in the entrance channel, ω_f is the energy density of final states and T_{fi} is the transition amplitude.

2.1.1 Energy Density

The energy density is the number of quantum states available to the final system per unit interval of total energy. For particles in finite space, we enclose the system in a very large but finite cubic box and impose periodic boundary conditions on the wave functions at the walls of the box. Hence, if we set the length of an edge of the box to L, the momentum eigenvalues are given by

$$\vec{p} = \frac{2\pi\hbar}{L}(n_1, n_2, n_3), \quad n_1, n_2, n_3 = 0, \pm1, \pm2, ... \tag{2.4}$$

We introduce momentum space, whose rectangular coordinates are the components of momentum (p_x, p_y, p_z). Any momentum vector $\vec{p}$ is represented by a point in this space, and the momentum eigenvalues (2.4) form a cubic lattice of points with lattice spacing $2\pi\hbar/L$. It follows that the volume in momentum space per lattice point, i.e. per state, is $(2\pi\hbar/L)^3$, and the number of states in a volume $d^3\vec{p}$ of momentum space is

$$\rho(p)d^3\vec{p} = \left(\frac{L}{2\pi\hbar}\right)^3 d^3\vec{p}. \tag{2.5}$$

If the momentum interval $p \leqslant |\vec{p}| \leqslant p + dp$ corresponds to the energy range $(E, E + dE)$, then the density of states $\omega(E)$ is given by

$$\omega(E)dE = \frac{d^3\vec{p}}{(2\pi\hbar)^3} \ \text{ in units where } L = 1. \tag{2.6}$$

Let us assume that the detection system specifies the energy and direction of the particle a' and the direction of the particle b. Particle B is undetected. Since the momentum of the particle B is fixed by momentum conservation, we can write the energy density of final states in the laboratory system as

$$\omega_f = \frac{dn}{dE} = \frac{d^3\vec{p}_{a'}}{(2\pi\hbar)^3}\frac{d^3\vec{p}_b}{(2\pi\hbar)^3}\frac{1}{dE}, \tag{2.7}$$

where E is the total energy and p_i is the momentum of the two outgoing particles.

Substituting $d^3\vec{p} = p^2 dp d\Omega$ yields

$$\omega_f = \frac{p^2 dp_b d\Omega_b}{(2\pi\hbar)^3} \frac{p_{a'}^2 dp_{a'} d\Omega_{a'}}{(2\pi\hbar)^3} \frac{1}{dE}, \tag{2.8}$$

where Ω_i is the solid-angle element within which scattered particle i is detected.
Using $E_{a'}^2 = p_{a'}^2 c^2 + m_{a'}^2 c^2 \Longrightarrow E_{a'} dE_{a'} = c^2 p_{a'} dp_{a'}$ in equation(2.8) yields

$$\omega_f = \frac{p_{a'} E_{a'} dE_{a'} d\Omega_{a'} p_b^2 dp_b d\Omega_b}{(2\pi\hbar)^6 c^2 dE} \ , \tag{2.9}$$

where c is the velocity of light and p_i and E_i are respectively the momentum and the energy for the particle i (i=a, a',b, B).
Since energy is conserved, the total energy E is given by

$$E = E_{a'} + E_b + E_B. \tag{2.10}$$

Using conservation of momentum,

$$\vec{p_a} = \vec{p_{a'}} + \vec{p_b} + \vec{p_B} \ , \tag{2.11}$$

So E can now be rewritten as

$$E = \sqrt{p_{a'}^2 c^2 + m_{a'}^2 c^4} + \sqrt{p_b^2 c^2 + m_b^2 c^4} + \sqrt{\left(\vec{p_a} - \vec{p_{a'}} - \vec{p_b}\right)^2 c^2 + m_B^2 c^4} \ , \tag{2.12}$$

where m_i is the mass of the particle i. Fixing $E_{a'}$ and taking the differential of E results in

$$dE = \frac{c^2 p_b dp_b}{E_b} + \frac{c^2 (p_b - p_a \cos\theta_{ab} + p_{a'} \cos\theta_{a'b}) dp_b}{E_B} \ , \tag{2.13}$$

where $\theta_{a'b}$ and θ_{ab} are respectively the angles between $\vec{p}_{a'}, \vec{p}_b$ and $\vec{p_a}, \vec{p}_b$ and

$$\left(\vec{p_a} - \vec{p_{a'}} - \vec{p_b}\right)^2 c^2 = p_a^2 c^2 + p_{a'}^2 c^2 + p_b^2 c^2 + 2 p_{a'} p_b \cos\theta_{a'b} - 2 p_a p_b \cos\theta_{ab}$$

$$-2 p_a p_{a'} \cos\theta_{a'a}. \tag{2.14}$$

Substituting equation(2.13) into equation(2.9), we find that

$$\omega_f = \frac{p_{a'} E_{a'} p_b^2 dp_b}{(2\pi\hbar)^6 c^2} \left(\frac{c^2 p_b dp_b}{E_b}\right)^{-1} \left[1 + \frac{E_b}{E_B}\left(1 - \frac{p_a}{p_b}\cos\theta_{ab} + \frac{p_{a'}}{p_b}\cos\theta_{a'b}\right)\right]^{-1}$$

$$\times d\Omega_{a'} d\Omega_b dE_{a'}. \tag{2.15}$$

The momentum and the total energy of the particle a are given by

$$\vec{p_a} = \gamma m_a \vec{v}_a \text{ and } \ E_a = \gamma m_a c^2 \ , \tag{2.16}$$

with $\gamma = 1/\sqrt{1 - v_a^2/c^2}$, we find then

$$\vec{v}_a = \frac{\vec{p}_a c^2}{E_a}. \tag{2.17}$$

This leads to the following expression for the energy density

$$\omega_f = \frac{v\hbar E_a E_{a'} E_b}{(\hbar c)^7 (2\pi)^5} \frac{p_{a'} p_b c}{p_a} \left[1 + \frac{E_b}{E_B}\left(1 - \frac{p_a}{p_b}\cos\theta_{ab} + \frac{p_{a'}}{p_b}\cos\theta_{a'b}\right)\right]^{-1} d\Omega_{a'} d\Omega_b dE_{a'}. \tag{2.18}$$

2.1.2 Triple Differential Cross Section

Equation(2.15) can be written now in the form of the *triple differential cross section*, defined by

$$\frac{d^3\sigma}{d\Omega_{a'}d\Omega_b dE_{a'}} \equiv \frac{E_a E_{a'} E_b}{(\hbar c)^7(2\pi)^5}\frac{p_{a'}p_b c}{p_a}\left[1+\frac{E_b}{E_B}\left(1-\frac{p_a}{p_b}\cos\theta_{ab}+\frac{p_{a'}}{p_b}\cos\theta_{a'b}\right)\right]^{-1}\times|T_{fi}|^2. \quad (2.19)$$

We define the kinematic factor F_K as

$$F_K = \frac{E_a E_{a'} E_b}{(\hbar c)^7(2\pi)^5}\frac{p_{a'}p_b c}{p_a}\left[1+\frac{E_b}{E_B}\left(1-\frac{p_a}{p_b}\cos\theta_{ab}+\frac{p_{a'}}{p_b}\cos\theta_{a'b}\right)\right]^{-1}. \quad (2.20)$$

2.2 Transition Amplitude

2.2.1 Antisymmetrisation of the State Vector

Consider one particle in the state $|\Phi\rangle$ and another in the state $|\Psi\rangle$. The state vector of the system is written as $|\Phi\Psi\rangle$. For identical particles, we cannot distinguish between $|\Phi\Psi\rangle$ and $|\Psi\Phi\rangle$. They are therefore two state vectors describing the same state, so they must be proportional.

$$|\Phi\Psi\rangle = P(\Phi,\Psi)|\Psi\Phi\rangle \text{ for some } P(\Phi,\Psi)\in\mathbb{C}. \quad (2.21)$$

Group theory considerations imply that, in three spatial dimensions, $P(\Phi,\Psi)$ does not depend on the states Ψ,Φ; it is just a constant P depending only on the type of particle. Therefore

$$|\Phi\Psi\rangle = P|\Psi\Phi\rangle = P(P|\Phi\Psi\rangle) = P^2|\Phi\Psi\rangle\ , \quad (2.22)$$

so $P=\pm1$. Particles which have $P=+1$ are called bosons, and particles which have $P=-1$ are called fermions. Therefore, bosonic statevectors must be symmetric under interchange of particles and fermionic statevectors must be antisymmetric under interchange of particles. So, we cannot use tensor products

$$|\Phi\Psi\rangle \neq |\Phi\rangle|\Psi\rangle. \quad (2.23)$$

In general, we must use symmetric tensor product state vectors

$$|\Phi\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\Phi\rangle|\Psi\rangle + |\Psi\rangle|\Phi\rangle\right) \text{ for Bosons} \quad (2.24)$$

and antisymmetric tensor product state vectors

$$|\Phi\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\Phi\rangle|\Psi\rangle - |\Psi\rangle|\Phi\rangle\right) \text{ for Fermions.} \quad (2.25)$$

Note that if $|\Phi\rangle$ and $|\Psi\rangle$ are the same, our equation (2.25) for the antisymmetric state gives the zero vector, which cannot be a state vector as it cannot be normalised. In other words, in a

fermionic state the particles cannot occupy the same state. This known as the Pauli exclusion principle.

Suppose now we have N particles in states Φ_1, Φ_2, ..., Φ_N.
If the particles are bosons, they occupy a totally symmetric state, which is symmetric under the exchange of any two particles' labels :

$$|\Phi_1\Phi_2...\Phi_N\rangle = \sqrt{\frac{\prod_j N_j!}{N!}} \sum_p |\Phi_{p(1)}\rangle|\Phi_{p(2)}\rangle...|\Phi_{p(N)}\rangle. \tag{2.26}$$

Here, the sum is taken over all possible permutations P acting on N elements. The square root on the right hand side is a normalising constant. The quantity N_j stands for the number of times each of the single-particle state appears in the N particle state.
In the same way, fermions occupy totally antisymmetric states :

$$|\Phi_1\Phi_2...\Phi_N\rangle = \frac{1}{\sqrt{N!}} \sum_p \text{Sgn(p)}|\Phi_{p(1)}\rangle|\Phi_{p(2)}\rangle...|\Phi_{p(N)}\rangle. \tag{2.27}$$

Here, Sgn(p) is the signature of each permutation (i.e. $+1$ if p is composed of an even number of transitions, and -1 if odd). Note that we have omitted the $\prod_j N_j$ term, because each single-particle state can appear only once in a fermionic state.

2.2.2 Scattering of Clusters of Identical Fermions

In typical atomic and nuclear collisions, the projectile and/or the target are composite particles. Usually, there are clusters of fermions. In the case of nuclei, there are two different fermions, protons and neutrons. However, with the isotopic spin (isospin) formalism, they are treated as different states (isospin up for protons and isospin down for neutrons) of the same particle, the nucleon. In this way, the Pauli principle requires that the wave function be antisymmetric with respect to the exchange of any two identical fermions, either in the same cluster or in different ones. If the projectile contains N_p fermions with coordinates $r_1, r_2, ..., r_{N_p}$ and the target contains N_t particles with position vectors $r_{N_p+1}, r_{N_p+2}, ..., r_{N_p+N_t}$, the system's wave function can be written

$$\Psi(r_1, ..., r_{N_p}; r_{N_p+1}, ..., r_{N_p+N_t}) = \mathscr{A}\left\{\Phi_p(r_1, ..., r_{N_p})\Phi_t(r_{N_p+1}, ..., r_{N_p+N_t})\right\}. \tag{2.28}$$

Here, Φ_p and Φ_t are, respectively, the antisymmetrised intrinsic wave functions of the projectile and the target, and $N = N_p + N_t$. $\mathscr{A}$ is the many-body antisymmetrisation operator, which accounts the $N_p!N_t!/N!$ different ways of permuting fermions between the two clusters. It can be written

$$\mathscr{A} = \sum (-1)^{p_\alpha} p_\alpha \ , \tag{2.29}$$

where the operator p_α exchanges fermion coordinates until the permutation α is reached. The sign $(-1)^{p_\alpha}$ is positive for permutations obtained through an even number of fermions pair exchange and negative otherwise.[1]

2.2.3 Formulation

The transition reduced amplitude T_{fi} is given by

$$T_{fi} = \langle \Phi^{(-)}(\widetilde{B, a', b}) | V_{a'b} | \Psi^{(+)}(\widetilde{A, a}) \rangle, \tag{2.30}$$

where $\Psi^{(+)}(\widetilde{A, a})$ is the exact wave function for the initial system, $\Phi^{(-)}(\widetilde{B, a', b})$ represents the wave function of the exit channel in the absence of any interaction between outgoing particles a' and b and $V_{a'b}$ is the difference between the exact Hamiltonian and the Hamiltonian used to generate $\Phi^{(-)}(\widetilde{B, a', b})$.
Introducing the wave function $\Phi(\widetilde{A, a})$ for the initial state in the absence of V_{ab}, the interaction between the projectile a and the cluster b, and defining the transition operator t_{fi} as

$$V_{a'b} | \Psi^{(+)}(\widetilde{A, a}) \rangle = t_{fi} | \Phi^{(+)}(\widetilde{A, a}) \rangle. \tag{2.31}$$

So the reduced transition amplitude can be written as

$$T_{fi} = \langle \Phi^{(-)}(\widetilde{B, a', b}) | t_{fi} | \Phi^{(+)}(\widetilde{A, a}) \rangle. \tag{2.32}$$

To proceed, we next consider the problem of constructing antisymmetrised wave functions.
We can write

$$\Phi^{(+)}(\widetilde{A, a}) = \sqrt{\frac{A!a!}{(A+a)!}} \sum_{Q^r_{a,A}} (-1)^r Q^r_{a,A} \Phi(\widetilde{a}) \Phi(\widetilde{A}) \ , \tag{2.33}$$

where $Q^r_{a,A}$ is an operator which interchanges r particles between a and A. We see clearly that for each value of r, there are $\binom{a}{r}\binom{A}{r}$ distinct interchanges possible and hence $\binom{a}{r}\binom{A}{r}$ different operators $Q^r_{a,A}$. The normalisation constant follows from the fact that there are $\binom{A+a}{A}$ terms in all (including the identity $r = 0$).
In the similar way, the final state wave function is written as

$$\Phi^{(-)}(\widetilde{B, a', b}) = \sqrt{\frac{B!(a+b)!}{(A+a)!}} \sum_{Q^s_{a'+b,B}} (-1)^s Q^s_{a'+b,B} \Phi(\widetilde{a', b}) \Phi(\widetilde{B}) \ , \tag{2.34}$$

which may be substituted into equation (2.32) with the result

$$T_{fi} = \sqrt{\frac{B!(a+b)!}{(A+a)!}} \sum_{Q^s_{a'+b,B}} (-1)^s \langle Q^s_{a'+b,B} \Phi(\widetilde{a', b}) \Phi(\widetilde{B}) | t_{fi} | \Phi^{(+)}(\widetilde{A, a}) \rangle. \tag{2.35}$$

Assuming that $Q^s_{a'+b,B}$ is Hermitian and must commute with $V_{a'b}$ which cannot involve specific nucleon labels, we can then rewrite equation (2.35) as

$$T_{fi} = \sqrt{\frac{(A+a)!}{B!(a+b)!}} \langle \Phi(\widetilde{a', b}) \Phi(\widetilde{B}) | t_{fi} | \Phi^{(+)}(\widetilde{A, a}) \rangle \ , \tag{2.36}$$

where we have made use of the identity

$$t_{fi}|\Phi(\widetilde{A,a})\rangle = V_{a'b}|\Psi^{(+)}(\widetilde{A,a})\rangle,$$

$$Q^s_{a'+b,B}\Psi^{(+)}(\widetilde{A,a}) = (-1)^s\Psi^{(+)}(\widetilde{A,a}), \tag{2.37}$$

and the fact that there are $\binom{A+a}{B}$ terms in the summation in equation (2.35).
If the particles in B are included among the A particles, so the operator $Q^r_{a,A}$ can be written as a product $Q^s_{a,B}Q^t_{a,b}$ of operators exchanging s particles between a and the core B and t particles between a and b. We thus have

$$\Phi^{(+)}(\widetilde{A,a}) = \sqrt{\frac{A!a!}{(A+a)!}}\sum_{Q^s_{a,B}}\sum_{Q^t_{a,b}}(-1)^{s+t}Q^s_{a,B}Q^t_{a,b}\Phi(\widetilde{a})\Phi(\widetilde{A}). \tag{2.38}$$

If we neglect exchange effects between the projectile a and the core B, the transition amplitude is modified as

$$T_{fi} = \sqrt{\frac{A!a!}{B!(a+b)!}}\sum_{Q^t_{a,b}}(-1)^t\langle\Phi(\widetilde{a',b})\Phi(\widetilde{B})|t_{fi}|Q^t_{a,b}\Phi(\widetilde{a})\Phi(\widetilde{A})\rangle. \tag{2.39}$$

$\Phi(\widetilde{a',b})$ can be written now as a product of antisymmetrised functions $\Phi(\widetilde{a'})\Phi(\widetilde{b})$ such that

$$\Phi(\widetilde{a',b}) = \sqrt{\frac{a!b!}{(a+b)!}}\sum_{Q^t_{a,b}}(-1)^tQ^t_{a,b}\Phi(\widetilde{a'})\Phi(\widetilde{b})\ , \tag{2.40}$$

and, defining

$$\mathscr{A}_{ab} = \sqrt{\frac{a!b!}{(a+b)!}}\sum_{Q^t_{a,b}}(-1)^tQ^t_{a,b}\ , \tag{2.41}$$

which is the two-body antisymmetrisation operator that accounts the $a!b!/(a+b)!$ different ways of permuting t particles between the two particles a and b, T_{fi} can finally be written as

$$T_{fi} = (A!/B!b!)^{1/2}\,\langle\Psi(\widetilde{B})\mathscr{A}_{ab}\eta^{(-)}_{Bab}\Psi(\widetilde{a})\Psi(\widetilde{b})|t_{fi}|\mathscr{A}_{ab}\eta^{(+)}_{Aa}\Psi(\widetilde{A})\Psi(\widetilde{a})\rangle, \tag{2.42}$$

where Ψ are internal wave functions for the various particles and $\eta^{(+)}_{Aa}$ and $\eta^{(-)}_{Bab}$ describe the relative motion of the centres of mass the particles in the entrance and exit channels, respectively.

2.3 Hamiltonian of the Reaction for a Channel

Consider the system $a+A$ in which a and A are the nuclei in reaction.
The time-independent Schrödinger equation of the entire system can be written as,

$$(H-E)\Psi = 0, \tag{2.43}$$

where E the total energy, Ψ the wave function of the system and H the Hamiltonian of the system. The Hamiltonian of the system can be written as,

$$H = H_\alpha + T_\alpha + V_\alpha, \tag{2.44}$$

where $H_\alpha = H_a + H_A$ refers only to the intrinsic structure of the nuclei and

$$V_\alpha = \sum_{i\in a, j\in A} V_{ij} \tag{2.45}$$

is the potential energy of the two nuclei which represents a sum of two-body interactions between nucleons of nuclei. T_α is the relative kinetic energy operator.

Let us call Ψ_α the initial state of the system, we have then

$$(E - H_\alpha - T_\alpha)\Psi_\alpha = V_\alpha \Psi_\alpha, \tag{2.46}$$

where E, H_i and T_α are the total energy of the system, the individual Hamiltonian for the particle i and the kinetic energy operator, respectively.
If there is no interaction V_α between the nuclei, the Schrödinger equation would be

$$(E - H_\alpha - T_\alpha)\Phi_\alpha = 0, \tag{2.47}$$

where Φ_α is the intrinsic state of two nuclei.

2.4 Optical Model

We understand intuitively that two nuclei, as they approach each other, are deflected by the field generated by the mutual interaction of all nucleons in one nucleus with those of the other, both before and after any specific interaction exchanges energy or nucleons between them. We introduce thus a potential $U(r_\alpha)$ that depends only on the relative coordinate between two nuclei in the partition α[1]. This potential is therefore intended to describe the relative motion in this channel. We shall see below that we need to allow $U(r_\alpha)$ to be complex to accurately describe nuclear reactions.
The time-independent Schrödinger equation is rewritten as [5]

$$(H_\alpha + T_\alpha + U_\alpha - E)\Psi = 0. \tag{2.48}$$

The solution can be written as a product of the nuclear wave function $\Phi_\alpha = \Phi_a \Phi_A$ and a relative wave function $\eta_\alpha(r_\alpha)$ which satisfies the Schrödinger equation

$$(T_\alpha + U_\alpha - E_\alpha)\eta_\alpha(r_\alpha) = 0, \tag{2.49}$$

where $\Psi = \Phi_\alpha \eta_\alpha$ and $E = E_\alpha + \epsilon_\alpha$ and

$$(H_\alpha - \epsilon_\alpha)\Phi_\alpha = 0 \tag{2.50}$$

[1] α denotes either a typical channel or the ground state channel

and

$$(H_a - \epsilon_a)\Phi_a = 0, \tag{2.51}$$

where $\epsilon_\alpha = \epsilon_a + \epsilon_A$, ϵ_a and E_α are the eigenvalues of H_α and the kinetic energy of relative motion between the original nuclei $a + A$.
Regarding the equation(2.49), it obvious that

$$(T_\alpha + U_\alpha^* - E_\alpha)\eta_\alpha^*(r_\alpha) = 0. \tag{2.52}$$

Multiplying(2.49) by η_α^* and (2.52) by η_α and subtracting leads to

$$-\frac{\hbar^2}{2m_\alpha}(\eta_\alpha^* \bigtriangledown^2 \eta_\alpha - \eta_\alpha \bigtriangledown^2 \eta_\alpha^*) = (U_\alpha^* - U_\alpha)\eta_\alpha^*\eta_\alpha. \tag{2.53}$$

Recalling the definition of the quantum-mechanical current, this equation reads

$$-i\hbar\vec{\bigtriangledown}.\vec{j} = (U_\alpha^* - U_\alpha)\eta_\alpha^*\eta_\alpha \tag{2.54}$$

or

$$\hbar\vec{\bigtriangledown}.\vec{j} = 2\rho\mathrm{ImU}_\alpha, \tag{2.55}$$

where

$$\vec{j} = \frac{\hbar}{2m_\alpha i}(\eta_\alpha^*\vec{\bigtriangledown}\eta_\alpha - \eta_\alpha\vec{\bigtriangledown}\eta_\alpha^*) \tag{2.56}$$

is the quantum-mechanical current associated with a distorted wave function η_α and ρ the squared modulus of η_α.
Integrating this over a volume containing the scattering centre, we find that if U_α is real there is no net change of flux, whereas if ImU_α is negative there is a loss. Thus, for U to describe accurately the elastic cross section when nuclear reactions can also occur, it must be complex with a negative imaginary part.

2.5 Optical Potential

When applying the Optical Model to a reaction, one needs to start by establishing an appropriate potential shape. Since the nucleon-nucleon interaction between the projectile and the target decreases exponentially at large distances, the optical potential should exhibit the same behaviour. In 1954, the Woods-Saxon (WS) form was announced to be the most appropriate shape for the optical potential, because it decreases exponentially with increasing radius.
The general profile of the Woods-Saxon form factor used in the optical potential formalism can be written as

$$f(r, r_i, a_i) = \frac{1}{1 + \exp\left(\dfrac{r - r_i A^{1/3}}{a_i}\right)}, \tag{2.57}$$

where r is the distance between the centres of the projectile and the target, $R_i = r_i A^{1/3}$ is the radius where the nucleus potential dropped to the half of the central value and a_i is known as the diffuseness parameter, i.e. the width of the region where the function f is significantly different from 0 or 1.
In scattering problems, according to the optical model, the form of the interaction (or effective) potential is usually given by

$$U(r) = U_{coul}(r) + U_{CR}(r) + U_{CI}(r) + U_S(r). \tag{2.58}$$

The potential $U(r)$ includes a complex part that takes into account the absorption effects, i.e. inelastic scattering. The nuclear scattering is treated in a similar way as the scattering of light by a translucent glass sphere and the name of the model derives from this analogy. In the case of light, the absorption is included by using a complex refraction index.

A term corresponding to the Coulomb potential is added whenever the scattering involves charged particles. It has the form

$$U_{coul} = \begin{cases} \frac{Z_p Z_t e^2}{2R_c}(3 - \frac{r^2}{R_c^2}), & r < r_c A^{1/3} \\ \frac{Z_p Z_t e^2}{r}, & r \geq r_c A^{1/3} \end{cases}, \tag{2.59}$$

where it is assumed that the nucleus is a homogeneously-charged sphere of radius equal to $R_c = r_c A^{1/3}$. $Z_p e$ and $Z_t e$ represent the charge of the projectile and the charge of the target.
In general, the real part (which represents a nuclear well with U_o) of the *central potential*[2], U_{CR}, has a Woods-Saxon shape which can be represented as

$$U_{CR}(r) = -\frac{U_0}{1 + \exp\left(\frac{r - r_0 A^{1/3}}{a_0}\right)}, \tag{2.60}$$

where $R = r_0 A^{1/3}$ is the radius of the nucleus and a_0 measures the diffuseness of the potential. U_0, R and a_0 are treated as adjustable parameters.

Since the imaginary part of the central potential is related to the inelastic scattering process, U_{CI} is more responsive to the details of the interaction than the real potential U_{CR}. Besides the real part, a central potential has to have an imaginary part even for the elastic scattering case. This imaginary part is composed of two components: one is related to the absorption in the nucleus volume and the other component is associated with the absorption in the nucleus surface $U_{CI} = i(U'_{CI} + U''_{CI})$.
The volume component of the imaginary part could have WS form

$$U'_{CI} = -\frac{W_V}{1 + \exp\left(\frac{r - r_V A^{1/3}}{a_V}\right)}. \tag{2.61}$$

[2]Corresponding potential when the force acting between two interacting particles depends only on the magnitude of the distance between them.

The absorption in the nucleus surface for the imaginary part can be represented by a Woods-Saxon differential type surface potential

$$U''_{CI} = -4a_s W_s \frac{d}{dr} f(r, r_s, a_s). \tag{2.62}$$

A spin-orbit term is added to the optical potential. This term is usually written in the form

$$U_S(r) = \left(\frac{\hbar}{2m_\pi c^2}\right) V_S \frac{1}{r} \frac{d}{dr} f(r, r_{SO}, a_{SO}) \vec{S}.\vec{L} \ , \tag{2.63}$$

incorporating a normalisation factor that contains the mass of the pion m_π. $\vec{S}$ is the spin operator and $\vec{L}$ the angular orbital momentum operator. The spin-orbit interaction leads to asymmetric scattering due to the different signs of product $\vec{S}.\vec{L}$ as the projectile passes by one or the other side of the nucleus.
The values of V_S, r_{SO} and a_{SO} must be adjusted by experiment. The presence of the spin-orbit term is necessary to describe the effects of *polarisation* (non-random orientation of nuclear spin). Thus, the optical potential equation is given by [1]

$$U(r) = U_{coul} - U_0 f(r, r_0, a_0) - iW_V f(r, r_V, a_V) - iW_s 4a_s \frac{d}{dr} f(r, r_s, a_s)$$

$$+ \left(\frac{\hbar}{2m_\pi c^2}\right) V_S \frac{1}{r} \frac{d}{dr} f(r, r_{SO}, a_{SO}) \vec{S}.\vec{L}. \tag{2.64}$$

3. The Knockout Theory

3.1 Overview

The generalised nuclear reaction $A(a, a'b)B$ can be approached with two different approximations. The first is to assume that the incident particle does not interact with the target nucleus until it scatters off the bound nucleon, and neither of the scattered particles interact with the core. The second is to assume that interaction does occur between the target nucleus and the incoming particle and outgoing particles.
These two cases necessitate slightly different treatments, and give rise to the Plane Wave Impulse Approximation (PWIA) and the Distorted Wave Impulse Approximation (DWIA).

3.2 Plane Wave Impulse Approximation

The simplest theory for knockout reactions is the plane wave impulse approximation (PWIA), or spectator model. There is no interaction between the incoming particle and the target nucleus, and between the outgoing particles and the residual nucleus, hence the name spectator model.
We neglect the spins of particles for simplicity and we assume that the wave function for the target nucleus can be written as

$$\Psi_A = \Psi_b \Psi_B \Psi(\vec{r}_b - \vec{r}_B) \ ,$$

where Ψ_b and Ψ_B are the antisymmetrised internal wave functions for the cluster b and the residual nucleus B, and $\Psi(\vec{r}_b - \vec{r}_B)$ is the relative motion wave function for the two clusters.
The coordinate $\vec{r}_A = (m_b\vec{r}_b + m_B\vec{r}_B)/(m_b + m_B)$ is the centre of mass coordinate of the target nucleus which has momenta $\vec{k}_A$, and $\vec{s} = \vec{r}_b - \vec{r}_B$ is the relative coordinate between b and B. We use the notation $\hbar = c = 1$.
Let us denote by $\vec{q_b}$ and $\vec{q}_B$ the momenta of clusters b and B in the target, respectively. In this approach, the particles are unbound particles then, we use momentum eigenfunctions which are plane waves [8] :

$$\langle \vec{r} | \vec{k} \rangle \propto \exp(i\vec{r}.\vec{k}).$$

The initial state wave function can then be written as

$$|i\rangle = |\vec{k}_a, A\rangle \ , \tag{3.1}$$

where $|A\rangle$ represents the intrinsic state of the target. The momentum space wave function of the target is

$$\langle \vec{q_b}, \vec{q}_B | A \rangle = \phi(\vec{k_s})\delta(\vec{q_b} + \vec{q}_B) \ , \tag{3.2}$$

where $\vec{k_s} = (m_B\vec{q_b} - m_b\vec{q}_B)/m_A$ is the relative momentum of the clusters in the target. The function $\phi(\vec{k_s})$ is the Fourier transform of the relative motion wave function of the clusters

$\Psi(\vec{r}_b - \vec{r}_B)$. The δ-function represents the fact that the target is at rest in the laboratory and implies that

$$\vec{q_b} = -\vec{q}_B = \vec{k_s}. \tag{3.3}$$

The final state wave function can be written as

$$|f\rangle = |\vec{k}_{a'}, \vec{k}_b, \vec{k}_B\rangle \ , \tag{3.4}$$

and in the laboratory system, the transition matrix can be written as

$$T_{fi} = \langle f|t_{fi}|i\rangle = \langle \vec{k}_{a'}, \vec{k}_b, \vec{k}_B|t_{fi}|\vec{k}_a, A\rangle \ , \tag{3.5}$$

where t_{fi} is the complete T-operator for the reaction. At this point, we introduce the impulse approximation, i.e. the incident particle interacts only with the particle b in the nucleus. Therefore, we replace t_{fi} by t_{fi}^{2B}. The transition matrix element can now be written as

$$T_{fi} = \int d^3\vec{q_b}\langle \vec{k}_{a'}, \vec{k}_b, \vec{k}_B|t_{fi}^{2B}|\vec{k}_a\rangle|\vec{q_b}\rangle\langle\vec{q_b}|A\rangle = \int d^3\vec{q_b}\langle \vec{k}_{a'}, \vec{k}_b|t_{fi}^{2B}|\vec{k}_a, \vec{q_b}\rangle\langle\vec{q_b}, \vec{k}_B|A\rangle.$$

Using equations (3.2) and (3.4), we obtain

$$\begin{aligned} T_{fi} &= \int d^3\vec{q_b}\langle \vec{k}_{a'}, \vec{k}_b|t_{fi}^{2B}|\vec{k}_a, \vec{q_b}\rangle\phi\left(\frac{m_B\vec{q_b} - m_b\vec{k}_B}{m_A}\right)\delta(\vec{q_b} + \vec{k}_B) && (3.6)\\ &= \langle \vec{k}_{a'}, \vec{k}_b|t_{fi}^{2B}|\vec{k}_a, -\vec{k}_B\rangle\phi(-\vec{k}_B). && (3.7)\end{aligned}$$

We note that the momentum of cluster b before collision $\vec{q_b}$ is equal and opposite to the momentum of the outgoing residual nucleus $\vec{k}_B$. We rewrite the transition amplitude as

$$T_{fi} = T_{ab}\phi(-\vec{k}_B) \ , \tag{3.8}$$

where

$$T_{ab} = \langle \vec{k}_{a'}, \vec{k}_b|t_{fi}^{2B}|\vec{k}_a, -\vec{k}_B\rangle \ , \tag{3.9}$$

is the matrix element for the two-body system describing the transition from the initial state $(\vec{k}_a, -\vec{k}_B)$ to the final state $(\vec{k}_{a'}, \vec{k}_b)$.
If we transform to the centre of mass and relative momenta for the two-body system, we can write [10]

$$T_{ab} = \delta(\vec{k}_a - \vec{k}_B - \vec{k}_{a'} - \vec{k}_b)T_{ab}^{2B} \ , \tag{3.10}$$

where

$$T_{ab}^{2B} = \langle \vec{k}_f|t_{fi}^{2B}|\vec{k}_i\rangle \ , \tag{3.11}$$

is the reduced T-matrix for the two-body system, and $\vec{k}_i$ and $\vec{k}_f$ are the relative momenta of particles a and b in the initial and final states. $\vec{k}_i = \vec{k}_a$ and $\vec{k}_f = \vec{k}_{a'} + \vec{k}_b + \vec{k}_B$.

The differential cross-section for the PWIA can then be given by

$$\frac{d^3\sigma}{d\Omega_{a'}d\Omega_b dE_{a'}} \propto K_F \left(\frac{d\sigma}{d\Omega}\right)_{ab} |\phi(-\vec{k}_B)|^2, \tag{3.12}$$

where K_F is a kinematic factor calculated from kinematic values and $\left(\frac{d\sigma}{d\Omega}\right)_{ab}$ is the centre of mass cross section for elastic scattering, where $\left(\frac{d\sigma}{d\Omega}\right)_{ab} = |T_{ab}^{2B}|^2$.

3.3 The Distorted Wave Impulse Approximation

The distorted wave impulse approximation (DWIA) is a model used at intermediate energies to compensate for the effects of a mean nuclear potential. Basic scattering reaction calculations often assume that the incident nucleon behaves as a plane wave until it interacts with the nucleon in the nucleus. In fact the potential field of the nucleus, which is usually given by an optical potential, will distort the nucleon wave function.

The optical potential is typically a phenomenological potential with parameters based on elastic scattering data. It contains both real and imaginary pieces, where the imaginary part models the absorption of certain reaction channels. The form of the nucleon wave function that interacts with a constituent of the nucleus can be found by solving the Schrödinger equation with an optical potential.

3.3.1 Formulation

Contrary to the PWIA, the interaction between the incoming and outgoing particles and the nucleus is taken into account, and the interaction between the two outgoing particles and the residual nucleus are also included.
Referring to the transition amplitude discussed in chapter 1, equation(2.42) can be expressed in more detail as

$$T_{fi} = (A!/B!b!)^{1/2}\langle\Psi_{J_B M_B T_B N_B}(\widetilde{B})\mathscr{A}_{ab}\eta_{Bab}^{(-)}\Psi_{s_a\sigma'_a t_a\nu_a}(\widetilde{a})\Psi_{s_b\sigma'_b t_b\nu_b}(\widetilde{b})|t_{fi}|\mathscr{A}_{ab}\eta_{Aa}^{(+)}$$

$$\times\Psi_{J_A M_A T_A N_A}(\widetilde{A})\Psi_{s_a\sigma_a t_a\nu_a}(\widetilde{a})\rangle, \tag{3.13}$$

where the Ψ are internal wave functions for the different particles, and $\eta_{Bab}^{(-)}$ and $\eta_{Aa}^{(+)}$ describe the relative motion of the mass centres of the particles in the entrance and exit channels, respectively. J_i (projection M_i) and T_i (projection N_i) (i=A,B) are the *angular momentum* and *isospin quantum numbers* for the target and the residual nucleus. The quantities s_j (projection σ_j) and t_j (projection ν_j) are the angular momentum and isospin quantum numbers for the incoming and outgoing particles (j=a,a',b).

Let us now rewrite the wave function of the target nucleus A. It is clearly seen that, if we project the target nucleus A onto the residual nucleus B, we will get a complicated function of the coordinates of the remaining b nucleons.
Then, to simplify this somewhat, we assume that these nucleons participate significantly to the cross section only when their relative motion and spin-isospin wave function is identical to that in the emitted cluster b.
Therefore, projecting out this wave function $\Psi_{s_b\sigma_b t_b\nu_b}(\widetilde{b})$, and integrating over the internal coordinates of B, the wave function of the target nucleus A can be rewritten as

$$\Psi_{J_AM_AT_AN_A}(\widetilde{A}) = \sum_{\alpha LJ\sigma_b\Lambda M} \mathcal{Q}_{AB}(\alpha Ls_bJt_b)(t_b\nu_bT_BN_B|T_AN_A)(JMJ_BM_B|J_AM_A)(L\Lambda s_b\sigma_b|JM)$$

$$\times\phi^{\alpha}_{L\Lambda}(\vec{r}_{bB})\Psi_{J_BM_BT_BN_B}(\widetilde{B})\Psi_{s_b\sigma_bt_b\nu_b}(\widetilde{b}), \tag{3.14}$$

where $\phi^{\alpha}_{L\Lambda}(\vec{r}_{bB})$ describes the motion of the centre of mass of b with respect to the centre of mass of B, and is normalised to unity. L and Λ are respectively the relative angular momentum quantum number of b and B and its projection. Any other quantum numbers needed to specify the motion are included in the α parameter.
$\mathcal{Q}_{AB}(\alpha Ls_bJt_b)$ is known as the "cluster coefficient of fractional parentage", which is used in describing the decomposition of the target nucleus $A \rightarrow B + b$. The quantum numbers J and M respectively denote the total angular momentum of b and its projection.
$(t_b\nu_bT_BN_B|T_AN_A), (JMJ_BM_B|J_AM_A)$ and $(L\Lambda s_b\sigma_b|JM)$ are the Clebsch-Gordan coefficients that are needed to ensure conservation of angular momentum.
The wave function for the target nucleus is now written in terms of a single-particle bound state of the particle b, $\phi^{\alpha}_{L\Lambda}(\vec{r}_{bB})$ and the core B is described by $\Psi_{J_BM_BT_BN_B}(\widetilde{B})$.
We can now write

$$T_{fi} = (A!/B!b!)^{1/2} \sum_{\alpha LJ\sigma_b\Lambda M} \mathcal{Q}_{AB}(\alpha Ls_bJt_b)(t_b\nu_bT_BN_B|T_AN_A)(JMJ_BM_B|J_AM_A)(L\Lambda s_b\sigma_b|JM)$$

$$\times\langle\mathcal{A}_{ab}\eta^{(-)}_{Bab}\Psi_{s_a\sigma'_at_a\nu_a}(\widetilde{a'})\Psi_{s_b\sigma'_bt_b\nu_b}(\widetilde{b})|t_{fi}|\mathcal{A}_{ab}\eta^{(+)}_{Aa}\phi^{\alpha}_{L\Lambda}(\vec{r}_{bB})\Psi_{s_a\sigma_at_a\nu_a}(\widetilde{a})\Psi_{s_b\sigma_bt_b\nu_b}(\widetilde{b})\rangle, \tag{3.15}$$

where

$$\langle\Psi_{J_BM_BT_BN_B}(\widetilde{B})\Psi_{s_b\sigma'_bt_b\nu_b}(\widetilde{b})|\Psi_{J_AM_AT_AN_A}(\widetilde{A})\rangle = \sum_{\alpha LJ\sigma_b\Lambda M} \mathcal{Q}_{AB}(\alpha Ls_bJt_b)(t_b\nu_bT_BN_B|T_AN_A)$$

$$\times(JMJ_BM_B|J_AM_A)(L\Lambda s_b\sigma_b|JM)\phi^{\alpha}_{L\Lambda}(\vec{r}_{bB}). \tag{3.16}$$

3.3.2 The Factorised DWIA

The impulse approximation assumes that the incident particle interacts with only one target nucleon, requiring the replacement of the t_{fi} operator by the two-body operator for the free $a+b$

scattering process $t_f^{(+)}$. Moreover the operator $\mathcal{A}_{ab}$ acting upon the wave functions to its right allows us to express the transition amplitude as

$$T_{fi} = (A!/B!b!)^{1/2} \sum_{\alpha L J \sigma_b \Lambda M} \mathcal{Q}_{AB}(\alpha L s_b J t_b)(t_b \nu_b T_B N_B | T_A N_A)(J M J_B M_B | J_A M_A)(L \Lambda s_b \sigma_b | J M)$$

$$\times \langle \eta^{(-)}_{Bab} \Psi_{s_a \sigma'_a s_b \sigma'_b t_a \nu_a t_b \nu_b}(\widetilde{a', b}) | t_f^{(+)} | \eta^{(+)}_{Aa} \phi^{\alpha}_{L\Lambda}(\vec{r}_{bB}) \Psi_{s_a \sigma_a s_b \sigma_b t_a \nu_a t_b \nu_b}(\widetilde{a, b}) \rangle. \tag{3.17}$$

If we now employ the expansion of the relative wave functions in momentum space, it becomes possible to isolate the matrix elements of $t_f^{(+)}$.[12]

If we then make use of the assumption that the reduced t matrix varies weakly with momenta, so that it may subsequently be replaced by a two-body matrix evaluated at asymptotic kinematics, this leads us to a convenient zero-range expression

$$t(\vec{r}_a - \vec{r}_b) \simeq t(q^2)\delta(\vec{r}_a - \vec{r}_b), \tag{3.18}$$

with the momentum transfer $\vec{q}$ defined as [7],

$$\vec{q} = \vec{k}_f - \vec{k}_i = (\vec{k}_{a'B} + \vec{k}_{bB}) - \left(\frac{m_B}{m_A} \vec{k}_{aA} - \vec{k}_B \right). \tag{3.19}$$

The momenta $\vec{k}_{aA}$, $\vec{k}_{a'B}$ and $\vec{k}_{bB}$ are respectively the asymptotic relative momenta for incident particle a to the target A, and for the emitted particles a' and b to B (calculated in section 3.3.3), and $\vec{k}_i$ and $\vec{k}_f$ are the initial and final relative momenta of particles a and b (found in section 3.2). The zero -range is the approximation often employed in nucleon-transfer reactions, especially when a is a light nucleus. The interaction in this case is taken to be a delta function of the coordinate ρ ($\rho = \vec{r}_a - \vec{r}_b$) [5]. T_{fi} is then presented in factorised form as

$$T_{fi} = (A!/B!b!)^{1/2} \sum_{\alpha L J \sigma_b \Lambda M} \mathcal{Q}_{AB}(\alpha L s_b J t_b)(t_b \nu_b T_B N_B | T_A N_A)(J M J_B M_B | J_A M_A)(L \Lambda s_b \sigma_b | J M)$$

$$\times \langle \vec{k}_f, \sigma'_a, \sigma'_b | t_f^{(+)} | \vec{k}_i, \sigma_a, \sigma_b \rangle \langle \eta^{(-)}_{Bab} | \delta(\vec{r}_a - \vec{r}_b) | \eta^{(+)}_{Aa} \phi^{\alpha}_{L\Lambda}(\vec{r}_{bB}) \rangle. \tag{3.20}$$

The delta function describes only strong interaction and for simplicity, only spin projection quantum numbers were retained to write the two-body reduced t-matrix .

We define :

$$S^{1/2}_{\alpha L J} = (A!/B!b!)^{1/2} \mathcal{Q}_{AB}(\alpha L s_b J t_b), \tag{3.21}$$

and

$$T^{\alpha L \Lambda}_{fi} = (2L+1)^{-1/2} \langle \eta^{(-)}_{Bab} | \delta(\vec{r}_a - \vec{r}_b) | \eta^{(+)}_{Aa} \phi^{\alpha}_{L\Lambda}(\vec{r}_{bB}) \rangle. \tag{3.22}$$

The quantity $T^{\alpha L \Lambda}_{fi}$ is the so-called distorted momentum distribution and the isospin Clebsch-Gordan coefficient is denoted by

$$(t_b \nu_b T_B N_B | T_A N_A) = C. \tag{3.23}$$

We assume that the initial particles are unpolarised (random orientation of nuclear spins), and we must then average over the initial spin configurations (M_A and σ_a) since each is equally likely, and sum over the possible final spin configurations (M_B, σ'_a and σ'_b) in equation(2.3). By considering equations (3.21), (3.22) and (3.23), we obtain σ_{fi} in the form

$$\sigma_{fi} = \frac{2\pi}{\hbar v}\omega_f C^2 \sum_{\sigma_a\sigma'_a\sigma'_b JM} \frac{1}{(2J_A+1)(2s_a+1)}$$

$$\times \left| \sum_{\alpha L\Lambda\sigma_b} S^{1/2}_{\alpha LJ}(L\Lambda s_b\sigma_b|JM)(2L+1)^{1/2} T^{\alpha L\Lambda}_{fi} \langle \vec{k}_f, \sigma'_a, \sigma'_b|t^{(+)}_f|\vec{k}_i, \sigma_a, \sigma_b\rangle \right|^2 . \tag{3.24}$$

There are many possible assumptions we can make to simplify this expression. We might choose to apply the assumption which considers the two-body $t^{(+)}_f$ to be independent of σ_b. We then find :

$$\sigma_{fi} = \frac{2\pi}{\hbar v}\omega_f C^2 |\overline{\langle t\rangle}|^2 \sum_{LJ\Lambda} \left| \sum_{\alpha} S^{1/2}_{\alpha LJ} T^{\alpha L\Lambda}_{fi} \right|^2 , \tag{3.25}$$

where

$$|\overline{\langle t\rangle}|^2 = \frac{1}{(2s_a+1)(2s_b+1)} \sum_{\sigma_a\sigma_b\sigma'_a\sigma'_b} \left| \langle \vec{k}_f, \sigma'_a, \sigma'_b|t^{(+)}_f|\vec{k}_i, \sigma_a, \sigma_b\rangle \right|^2 . \tag{3.26}$$

Here the square of the two-body t-matrix has been averaged over initial spin projections, and then summed over final spin projections.
The expression given in Equation (3.25) is exact provided $s_b = 0,\ L = 0$ or $s_b = 1/2$ [12]. Furthermore, if we suppose that $T^{\alpha L\Lambda}_{fi}$ is independent of α, we can set

$$S_{LJ} = \left| \sum_{\alpha} S^{1/2}_{\alpha LJ} \right|^2 \tag{3.27}$$

for the different values of L and J.
The quantity $C^2 S_{LJ}$ is the *spectroscopic factor*, which represents the probability to reach a single particle hole state LJ when a nucleon is removed from the target nucleus.
Then, σ_{fi} is written in its simplest form as

$$\sigma^{LJ}_{fi} = \frac{2\pi}{\hbar v}\omega_f C^2 |\overline{\langle t\rangle}|^2 S_{LJ} \sum_{\Lambda} \left| T^{\alpha L\Lambda}_{fi} \right|^2 . \tag{3.28}$$

3.3.3 Determination of Distorted Momentum Distribution

The Hamiltonian in the entrance channel is

$$H = H_a + H_A + T_a + V_{aA} \ , \tag{3.29}$$

where the Hamiltonian H_i acts upon the internal coordinates of particle i, T_a is a kinetic energy operator and V_{aA} is the interaction of a and A.

Consider the relative coordinates of a which respect to the coordinates of centre of mass in the entrance channel :

$$\vec{r}_{aA} = \vec{r}_a - \vec{R}_{a+A} \ , \tag{3.30}$$

the centre of mass coordinates

$$\vec{R}_{a+A} = \frac{m_a\vec{r}_a + m_A\vec{r}_A}{m_a + m_A} \ , \tag{3.31}$$

and the conjugate momentum operators

$$\vec{p}_{aA} = \frac{m_A\vec{p}_a}{m_a + m_A} \ ,$$

$$\vec{p}_{a+A} = \vec{p}_a. \tag{3.32}$$

Writing $V_{aA} = V_{aB} + V_{ab}$ and noting that $T_a = p_a^2/2m_a$ where m_i is the mass of particle i, it is easy to obtain the unperturbed Hamiltonian, i.e. in the absence of V_{ab},

$$H - V_{ab} = H_a + H_A + T_{aA} + V_{aB} + \frac{2m_A + m_a}{2(m_a + m_A)^2}p_{a+A}^2 \ , \tag{3.33}$$

where $T_{ij} = p_{ij}^2/2\mu_{ij}$ is the relative kinetic energy and $\mu_{ij} = m_i m_j/(m_i + m_j)$ is a reduced mass. Hence, the unperturbed initial state wave function may be written as,

$$\Phi = \Phi(\widetilde{A})\Phi(\widetilde{a})\chi_{aA}^{(+)}(\vec{k}_{aA}, \vec{r}_{aA})e^{i\vec{k}_{a+A}.\vec{R}_{a+A}} \ , \tag{3.34}$$

where $\hbar\vec{k}_{aA} = \vec{p}_{aA}, \hbar\vec{k}_{a+A} = \vec{p}_{a+A}$ and we have assumed $V_{aB} \approx V_{aB}(\vec{r}_{aA})$.
$\chi_{aA}^{(+)}$ is then a solution of

$$\left(T_{aA} + V_{aB} - \epsilon_{aA}\right)\chi_{aA}^{(+)}(\vec{k}_{aA}, \vec{r}_{aA}) = 0 \ , \tag{3.35}$$

with $\epsilon_{aA} = p_{aA}^2/2\mu_{aA}$.

The unperturbed Hamiltonian for the exit channel may be written

$$H - V_{a'b} = H_{a'} + H_b + H_B + T_{a'} + T_b + T_B + V_{a'B} + V_{bB}. \tag{3.36}$$

Writing the relative coordinates of a' and b with respect to the centre of mass in the exit channel as

$$\vec{r}_{a'B} = \vec{r}_{a'} - \vec{R}_{a'+b+B} \ ,$$

$$\vec{r}_{bB} = \vec{r}_b - \vec{R}_{a'+b+B} \ , \tag{3.37}$$

$$\vec{R}_{a'+b+B} = \frac{m_a\vec{r}_{a'} + m_b\vec{r}_b + m_B\vec{r}_B}{m_a + m_b + m_B} \ ,$$

leads to conjugate momentum operators

$$\begin{aligned}
\vec{p}_{a'B} &= \vec{p}_{a'} - \frac{m_a}{m_a+m_b+m_B}(\vec{p}_{a'}+\vec{p}_B+\vec{p}_B) = \vec{p}_{a'} - \frac{m_a}{m_a+m_b+m_B}\vec{p}_a, \\
\vec{p}_{bB} &= \vec{p}_b - \frac{m_b}{m_a+m_b+m_B}(\vec{p}_{a'}+\vec{p}_b+\vec{p}_B) = \vec{p}_b - \frac{m_b}{m_a+m_b+m_B}\vec{p}_a, \\
\vec{p}_{a'+b+B} &= \vec{p}_{a'}+\vec{p}_b+\vec{p}_B = \vec{p}_a.
\end{aligned} \tag{3.38}$$

Hence, the total kinetic energy of the final state in the laboratory system is

$$T_{a'}+T_b+T_B = \frac{p_{a'}^2}{2m_a}+\frac{p_b^2}{2m_b}+\frac{p_B^2}{2m_B} = \frac{1}{2m_a}\left[\vec{p}_{a'B}+\frac{m_a}{m_a+m_b+m_B}\vec{p}_{a'+b+B}\right]^2$$

$$+\frac{1}{2m_b}\left[\vec{p}_{bB}+\frac{m_b}{m_a+m_b+m_B}\vec{p}_{a'+b+B}\right]^2+\frac{1}{2m_B}\left[\frac{m_B}{m_a+m_b+m_B}\vec{p}_{a'+b+B}-\vec{p}_{a'b}-\vec{p}_{bB}\right] ,$$

which gives

$$T_{a'}+T_b+T_B = T_{a'b}+T_{bB}+\frac{p_{a'+b+B}^2}{2(m_a+m_b+m_B)}+\frac{\vec{p}_{a'B}\cdot\vec{p}_{bB}}{m_B}. \tag{3.39}$$

In order to simplify the three-particle wave functions, we neglect the last term in equation (3.39), denoted the coupling term. This is the so-called "kinematic energy approximation". It is clearly seen that this will have little effect provided m_B is large compared to $m_{a'}$ and m_b. In addition the omission of this term has no effect in the plane wave limit.
Using the kinetic energy approximation, the unperturbed final state wave function may be written as

$$\Phi = \Phi(\widetilde{B})\Phi(\widetilde{a'})\Phi(\widetilde{b})\chi^{(-)}_{a'B}(\vec{k}_{a'B},\vec{r}_{a'B})\chi^{(-)}_{bB}(\vec{k}_{bB},\vec{r}_{bB})e^{i\vec{k}_{a'+b+B}\cdot\vec{R}_{a'+b+B}} , \tag{3.40}$$

where $\chi^{(-)}_{a'B}$ is the solution of

$$\left(T_{a'B}+V_{a'B}-\epsilon_{a'B}\right)\chi^{(-)}_{a'B}(\vec{k}_{a'B},\vec{r}_{a'B}) = 0 , \tag{3.41}$$

with $\epsilon_{a'B} = p_{a'B}^2/2\mu_{a'B}$ and $\vec{p}_{a'B} = \hbar\vec{k}_{a'B}$. Similarly $\chi^{(-)}_{bB}$ is a solution of

$$\left(T_{bB}+V_{bB}-\epsilon_{bB}\right)\chi^{(-)}_{bB}(\vec{k}_{bB},\vec{r}_{bB}) , \tag{3.42}$$

with $\epsilon_{bB} = p_{bB}^2/2\mu_{bB}$ and $\vec{p}_{bB} = \hbar\vec{k}_{bB}$. Also $\hbar\vec{k}_{a'+b+B} = \vec{p}_{a'+b+B}$.
We can verify the relative motion wave functions as

$$\eta^{(+)}_{Aa} = \chi^{(-)}_{bB}(\vec{k}_{aA},\vec{r}_{aA}) \tag{3.43}$$

and

$$\eta^{(-)}_{Ba'b} = \chi^{(-)}_{a'B}(\vec{k}_{a'B},\vec{r}_{a'B})\chi^{(-)}_{bB}(\vec{k}_{bB},\vec{r}_{bB}). \tag{3.44}$$

The bound wave function $\phi^{\alpha}_{L\Lambda}(\vec{r}_{bB})$ of b is found by introducing a phenomenological real (Woods-Saxon) potential V_{WS} which is adjusted to reproduce the empirical $A \rightarrow B + b$ separation energy S_{bB}

$$(T_{bB} + V_{WS} - S_{bB})\phi^{\alpha}_{L\Lambda}(\vec{r}_{bB}) = 0. \tag{3.45}$$

The distorted waves depend on the relative coordinates, which $\vec{r}_{a'B}$ and $\vec{r}_{aA}$ are related by

$$\vec{r}_{a'B} = \vec{r}_{a'b} + \vec{r}_{bB} \quad \text{and} \quad \vec{r}_{aA} = \vec{r}_{ab} + \frac{B}{B+b}\vec{r}_{bB}, \tag{3.46}$$

where in the zero-range approximation $\vec{r}_{a'B} = \vec{r}_{bB}$ and $\vec{r}_{aA} = \dfrac{B}{B+b}\vec{r}_{bB}$ with $\vec{r}_{ab} = \vec{r}_{a'b} \longrightarrow 0$, the interaction is supposed to be strong.

Equation(3.22) can be rewritten now, after using the relative coordinates in the zero-range approximation and integrating over $\vec{r}_{ab}$:

$$T^{\alpha L\Lambda}_{fi} = \frac{1}{(2L+1)^{1/2}} \int \chi^{(-)^*}_{a'B}(\vec{k}_{a'B}, \vec{r})\chi^{(-)^*}_{bB}(\vec{k}_{bB}, \vec{r})\chi^{(+)}_{aA}(\vec{k}_{aA}, \gamma\vec{r})\phi^{\alpha}_{L\Lambda}(\vec{r})d^3\vec{r}, \tag{3.47}$$

where $\gamma = B/(B+b)$.

Partial Wave expansion

The general Schrödinger equation describing the relative motion of two particles is

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\Psi(r) = E\Psi(r). \tag{3.48}$$

In spherical coordinates the Laplacien operator ∇^2 may be written as

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}L^2(\theta, \phi), \tag{3.49}$$

where

$$L^2(\theta, \phi) = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]. \tag{3.50}$$

The operator L^2 given by Equation (3.50) is just

$$L^2 = (\vec{r}\times\vec{p})\,.\,(\vec{r}\times\vec{p}) = -\hbar^2\left(\vec{r}\times\vec{\nabla}\right).\left(\vec{r}\times\vec{\nabla}\right), \tag{3.51}$$

where $\vec{r}$ and $\vec{p}$ are respectively the position vector and the momentum and L^2 represents the square of the orbital angular momentum.
The spherical harmonics $Y^m_l(\theta, \phi)$ are defined to be eigenfunctions of L^2 and the z component of $\vec{L}$,

$$L_z = (\vec{r}\times\vec{p})\,.\widehat{z} = -i\hbar\frac{\partial}{\partial\phi}, \tag{3.52}$$

such that

$$L^2 Y_l^m(\theta,\phi) = l(l+1)\hbar^2 Y_l^m(\theta,\phi), \tag{3.53}$$

and

$$L_z Y_l^m(\theta,\phi) = m\hbar Y_l^m(\theta,\phi). \tag{3.54}$$

The function

$$\Psi_{lm}(r) = r^{-1} u_l(r) Y_l^m(\theta,\phi) \tag{3.55}$$

is then solution of Equation (3.48), provided $u_l(r)$ is a solution of the radial equation

$$\frac{d^2 u_l(r)}{dr^2} + \left[k^2 - U(r) - \frac{l(l+1)}{r^2}\right] u_l(r) = 0, \tag{3.56}$$

where $k^2 = (2m/\hbar^2)E$ and $U(r) = (2m/\hbar^2)V(r)$.
As we have defined the optical potential, $U(r)$ is assumed to vanish with increasing r so that it may be neglected beyond some radius $r = R$. We shall also require that $u_l(r)$ vanish at the origin. This boundary condition at the origin follows from the requirement that the wave function $\Psi(r)$ and its gradient be finite everywhere, in particular at $r = 0$. It is evident that any linear combination of the functions $\Psi_{lm}(r)$, for different l and m, is also a solution of Equation (3.48). The solution is

$$\Psi_{lm}(r) = \frac{1}{kr}\sum_l i^l u_l(k,r) Y_l^0(\omega) = \frac{1}{kr}\sum_l (2l+1) i^l u_l(k,r) P_l(\cos\omega), \tag{3.57}$$

or more specifically

$$\Psi_{lm}(r) = \frac{4\pi}{kr}\sum_{l,m} i^l u_l(k,r) Y_l^m(\widehat{r}) Y_l^m(\widehat{k}), \tag{3.58}$$

where $\cos\omega = \widehat{r}.\widehat{k} = \vec{r}.\vec{k}/rk$.

The spherical harmonic addition theorem states that

$$P_l(\cos\omega) = 4\pi/(2l+1)\sum_m Y_l^m(\theta,\phi) Y_l^m(\theta',\phi'), \tag{3.59}$$

where ω is the angle between vectors pointing, respectively, in the directions (θ,ϕ) and (θ',ϕ'). We often denote the angle θ, ϕ by $\widehat{r}$ and θ', ϕ' by $\widehat{k}$, it being understood that θ, ϕ and θ', ϕ' are respectively polar coordinates of the vectors $\vec{r}$ and $\vec{r}$ [10] :

$$\widehat{r} = \sin\theta\cos\phi\widehat{x} + \sin\theta\sin\phi\widehat{y} + \cos\theta\widehat{z}\ ,$$

$$\widehat{k} = \sin\theta'\cos\phi'\widehat{x} + \sin\theta'\sin\phi'\widehat{y} + \cos\theta'\widehat{z}. \tag{3.60}$$

The solutions of equations (3.35), (3.41), and (3.42) can then be given by ,

$$\chi^{(+)}_{aA}(\vec{k}_{aA},\gamma\vec{r}) = \frac{4\pi}{\gamma k_{aA} r}\sum_{l_a\lambda_a} u_{l_a}(k_{aA},\gamma r) i^{l_a} Y_{l_a\lambda_a}(\widehat{r}) Y^*_{l_a\lambda_a}(\widehat{k}_{aA}), \tag{3.61}$$

$$\chi^{(-)*}_{a'B}(\vec{k}_{a'B},\vec{r}) = \frac{4\pi}{k_{a'B} r}\sum_{l'_a\lambda'_a} u_{l'_a}(k_{a'B}, r) i^{-l'_a} Y_{l'_a\lambda'_a}(\widehat{r}) Y^*_{l'_a\lambda'_a}(\widehat{k}_{a'B}), \tag{3.62}$$

$$\chi^{(-)*}_{bB}(\vec{k}_{bB},\vec{r}) = \frac{4\pi}{k_{bB} r}\sum_{l_b\lambda_b} u_{l_b}(k_{bB}, r) i^{-l_b} Y_{l_b\lambda_b}(\widehat{r}) Y^*_{l_b\lambda_b}(\widehat{k}_{bB}), \tag{3.63}$$

and the solution of equation(3.45) is written as

$$\phi^{\alpha}_{L\Lambda}(\vec{r}) = R_{\alpha L}(r) i^L Y_{L\Lambda}(\widehat{r}). \tag{3.64}$$

We can write spherical harmonics in terms of reduced rotation matrix elements

$$Y^m_l(\theta,\phi) = \left(\frac{2l+1}{4\pi}\right)^{1/2} d^l_{mo}(\theta) e^{im\phi}, \tag{3.65}$$

where

$$d^l_{mo}(\theta) = (-1)^m \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} P^m_l(\theta),\ m \geq 0, \tag{3.66}$$

are the reduced rotation matrices.
The reduced rotation matrices $d^j_{mn}(\beta)$ can be expressed explicitly as

$$d^j_{mn}(\beta) = \sum_t (-1)^t \frac{[(j+m)!(j-m)!(j+n)!(j-n)!]^{1/2}}{(j+m-t)!(j-n-t)!(t+n-m)!}$$

$$\times (\cos\beta/2)^{2j+m-n-2t} (\sin\beta/2)^{2t+n-m} \tag{3.67}$$

where the sum is taken over all values of t which lead to non-negative factorials [4].

Choosing the z-axis along $\vec{k}_{aA}$ and y-axis along $\vec{k}_{aA}\times\vec{k}_{a'B}$ ($\widehat{k}_{aA} = (0,0)$, $\widehat{k}_{a'B} = (\theta_a, 0)$ and $\widehat{k}_{bB} = (\theta_b,\phi_b)$), equation(3.47) becomes

$$T^{\alpha L\Lambda}_{fi} = \frac{(4\pi)^3}{(2L+1)^{1/2}\gamma k_{aA} k_{a'B} k_{bB}} \sum \int_0^{\pi}\int_0^{2\pi} \left(\frac{(2l_a+1)(2l'_a+1)(2l_b+1)}{(4\pi)^3}\right)^{1/2} d^{l'_a}_{\lambda'_a 0}(\theta_a) d^{l_b}_{\lambda_b 0}(\theta_b) e^{-i\lambda_b\phi_b}$$

$$\times \delta_{\lambda_a 0} i^{l_a+L-l'_a-l_b} I^L_{l_a l'_a l_b} Y_{l_a\lambda_a}(\widehat{r}) Y_{l'_a\lambda'_a}(\widehat{r}) Y_{l_b\lambda_b}(\widehat{r}) Y_{L\Lambda}(\widehat{r}) \sin\theta d\theta d\phi, \tag{3.68}$$

where

$$I^L_{l_a l'_a l_b} = \int_0^\infty u_{l_a}(k_{aA}, \gamma r) u_{l'_a}(k_{a'B}, r) u_{l_b}(k_{bB}, r) R_{\alpha L}(r) dr/r. \tag{3.69}$$

$(\theta_a, 0)$ and (θ_b, ϕ_b) are the centre-of-mass angles and are found by using the relative momenta. They represent the angles under which the particles are detected in the exit channel.
We make use of the relations

$$Y^m_l(\theta, \phi) = \left(\frac{2l+1}{4\pi}\right)^{1/2} C_{lm}(\theta, \phi), \tag{3.70}$$

and

$$Y^{m^*}_l(0,0) = \sqrt{\frac{(2l+1)}{4\pi}} \left(P_l(\cos\theta)\right)_{\cos\theta=1} \delta_{m0} = \sqrt{\frac{(2l+1)}{4\pi}} \delta_{m0}, \tag{3.71}$$

which vanish for $m \neq 0$ ($P_l(1) = 1$) [9], where

$$C_{lm}(\theta, \phi) = (-1)^m \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} P^m_l(\theta) e^{im\phi}, \ m \geq 0, \tag{3.72}$$

with the properties

$$C_{a\alpha}(\theta,\phi) C_{b\beta}(\theta,\phi) = \sum_c C_{c\gamma}(\theta,\phi)(2c+1)(-1)^\gamma \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ \alpha & \beta & \text{-}\gamma \end{pmatrix} \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 0 & 0 & 0 \end{pmatrix}, \ 0 \leq \gamma \leq c$$

and

$$\int C_{a\alpha}(\theta,\phi) C_{b\beta}(\theta,\phi) C_{c\gamma}(\theta,\phi) \sin\theta d\theta d\phi = 4\pi \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ \alpha & \beta & \text{-}\gamma \end{pmatrix} \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 0 & 0 & 0 \end{pmatrix}$$

The Clebsch-Gordan coefficients are defined by

$$(j_1 m_1 j_2 m_2 | j_3 m_3) = (-1)^{j_1 - j_2 + m_3} \sqrt{2j_3 + 1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & \text{-}m_3 \end{pmatrix}, \tag{3.73}$$

they are real and are non-vanishing only if

$$m_1 + m_2 = m_3 \ \text{ and } \ |j_1 - j_2| \leq j_3 \leq j_1 + j_2 \ , \tag{3.74}$$

and are calculated by using *Racah's General Formula* :

$$\langle ab\alpha\beta | c\gamma \rangle = \delta(\alpha+\beta, \gamma) \Delta(abc) \times \left[(2c+1)!(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!(c+\gamma)!(c-\gamma)!\right]^{1/2}$$

$$\times \sum_\nu (-1)^\nu \left[(a-\alpha-\nu)!(c-b+\alpha+\nu)!(b+\beta-\nu)!(c-a-\beta+\nu)!\nu!(a+b-c-\nu)!\right]^{-1}$$

where

$$\Delta(abc) = \left[\frac{(a+b-c)!(a+c-b)!(b+c-a)!}{(a+b+c+1)!}\right]^{1/2},$$

and ν runs over all values which do not led to negative factorials.

The distorted momentum distribution $T_{fi}^{L\Lambda}$ may be expressed as

$$T_{fi}^{\alpha L\Lambda} = \frac{\sqrt{4\pi}}{\gamma k_{aA} k_{a'B} k_{bB}} \sum_{l_a l'_a l_b \lambda_b k q} i^{l_a+L-l'_a-l_b} \frac{(2l_a+1)(2l'_a+1)(2l_b+1)}{2k+1}$$

$$\times (l_b \lambda_b L\Lambda|kq)(l_b 0 L0|k0)(l_a 0 l'_a q|kq)(l_a 0 l'_a 0|k0) I^L_{l_a l'_a l_b} d^{l'_a}_{q0}(\theta_a) d^{l_b}_{\lambda_b 0}(\theta_b) e^{-i\lambda_b \phi_b}, \tag{3.75}$$

where $Y_l^m(0,0)$ and Clebsch-Gordan coefficients condition allow us to find $\lambda_a = 0$, $\lambda'_a = q$ and the Clebsch-Gordan property (3.74) gives $|l_a - l_{a'}| \leq k \leq l_a - l_{a'}$.

3.3.4 The two-body t-matrix

We have introduced the two-body t-matrix for the $b(a,a')b$ reaction by applying the impulse approximation to the quasi-free $A(a,a'b)B$.
Let us consider the free $b(a,a')b$ reaction to evaluate a value of this quantity. In the centre of mass the momenta are related by

$$\begin{array}{rcccl} \vec{p}_a & = & -\vec{p}_b & = & \vec{p}_i \ , \\ \vec{p}_{a'} & = & -\vec{p}'_b & = & \vec{p}_f \ , \end{array} \tag{3.76}$$

and the energy density is

$$\omega = \frac{d^3\vec{p}_{a'}}{(2\pi\hbar)^3} \frac{1}{dE}. \tag{3.77}$$

The conservation of energy of this reaction is

$$E = E_{a'} + E'_b = \sqrt{p^2_{a'}c^2 + m^2_a c^2} + \sqrt{p'^2_b c^2 + m^2_b c^2} \ , \tag{3.78}$$

which gives

$$dE = dE_{a'} + dE'_b = \frac{c^2 p_{a'} dp_{a'}}{E_{a'}} + \frac{c^2 p'_b dp'_b}{E_b} \ , \tag{3.79}$$

and using equation (3.76), we can write

$$dE = \frac{(E_{a'} + E'_b)}{E_{a'} E'_b} c^2 p_f dp_f. \tag{3.80}$$

Thus

$$\omega = \frac{p_f}{(2\pi\hbar)^3} \frac{E_{a'} E'_b}{(E_{a'} + E'_b)c^2} d\Omega \ , \tag{3.81}$$

where we have replaced $d^3\vec{p}_{a'}$ by $p_f^2 dp_f d\Omega$.
As before, the velocity of the particle is $\vec{v} = \vec{p}c^2/E$ and then the initial relative velocity of the interacting particle is

$$\vec{v} = c^2\left(\frac{\vec{p}_a}{E_a} - \frac{\vec{p}_b}{E_b}\right) = \vec{p}_i\frac{(E_a + E_b)}{E_a E_b}c^2. \tag{3.82}$$

Hence, using Fermi's Golden rule and averaging over initial spin projections and summing over final spin projections, we obtain

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi\hbar^2c^2)^2}.\frac{p_f}{p_i}\frac{E_a E_{a'} E_b E'_b}{(E_a + E_b)(E_{a'} + E'_b)}\frac{1}{(2s_a+1)(2s_b+1)}$$

$$\times \sum_{\sigma_a\sigma_b\sigma_{a'}} |\langle\vec{k}_f, \sigma'_a, \sigma'_b|\ t_f^{(+)}|\vec{k}_i, \sigma_a, \sigma_b\rangle|^2\ , \tag{3.83}$$

where $\hbar\vec{k}_i = \vec{p}_i$ and $\hbar\vec{k}_f = \vec{p}_f$. Let us introduce the amplitude

$$M = \sqrt{E_a E_b}\langle\vec{k}_f, \sigma'_a, \sigma'_b|\ t_f^{(+)}|\vec{k}_i, \sigma_a, \sigma_b\rangle\sqrt{E_{a'}E'_b}. \tag{3.84}$$

Introducing by explicit superscripts, quantities which are evaluated in the centre of mass for the free two-body reaction and introducing $\overline{M}^2$ which indicates the sums and averages over spin variables, we may write

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi\hbar^2c^2)^2}.\frac{p_f^{CM}}{p_i^{CM}}\frac{\overline{M}^2}{(E^{CM})^2}\ , \tag{3.85}$$

where $E^{CM} = E_a^{CM} + E_b^{CM} = E_{a'}^{CM} + E_b'^{CM}$.

We have then

$$\overline{\langle t\rangle}^2 = \left(2\pi\hbar^2c^2\right)^2\frac{p_i^{CM}}{p_f^{CM}}\frac{E_{CM}^2}{E_a E_{a'} E_b E'_b}\frac{d\sigma}{d\Omega}(E_{CM}, \theta_{CM}). \tag{3.86}$$

In the quasi-free two-body reaction reaction centre of mass, the energy is just the total energy in the $a' + b$ rest system (in the exit channel)

$$E_{a'+b}^{CM} = \sqrt{(E_{a'} + E'_b)^2 - (\vec{p}_{a'} + \vec{p'_b})^2} = \left\{m_a^2 + m_b^2 + 2E_{a'}E'_b - 2p_{a'}p_b\cos\theta_{a'b}\right\}^{1/2}. \tag{3.87}$$

The centre of mass scattering angle is the angle between the three-momenta $\vec{p}_a$ and $\vec{p}_{a'}$ in the rest $a' + b$ rest frame . We can write then

$$\cos\theta_{a'a}^{CM} = \frac{\vec{p}_a^{CM}.\vec{p}_{a'}^{CM}}{2p_a^{CM}p_{a'}^{CM}}, \tag{3.88}$$

and

$$\left(E_{a+a'}^{CM}\right)^2 = \left(E_{a'}^{CM} + E_a^{CM}\right)^2 = 2m_a^2 + 2E_a E_{a'} - 2p_a p_{a'}\cos\theta_{aa'}\ , \tag{3.89}$$

which gives

$$-\left[\left(\vec{P_a}^{\,CM}+\vec{P_{a'}}^{\,CM}\right)^2-2\vec{P_a}^{\,CM}\vec{P_{a'}}^{\,CM}\right]=2E_{a'}^{CM}E_a^{CM}-2E_aE_{a'}+2p_{a'}p_a\cos\theta_{aa'}\ ,$$

and then

$$\cos\theta_{aa'}^{CM}=\frac{E_{a'}^{CM}E_a^{CM}-E_aE_{a'}+p_{a'}p_a\cos\theta_{aa'}}{[(E_a^{CM})^2-m_a^2]^{1/2}\,[(E_{a'}^{CM})^2-m_{a'}^2]^{1/2}}\ , \tag{3.90}$$

where E_a^{CM} and $E_{a'}^{CM}$ are calculated using the following relations

$$\left(E_{a+a'}^{CM}\right)^2=m_a^2+m_{a'}^2+2E_aE_{a'}-2p_ap_{a'}\cos\theta_{aa'},$$

$$\left(E_{a+b}^{CM}\right)^2=m_a^2+m_b^2+2E_aE_b'-2p_ap_b'\cos\theta_{ab}'\ ,$$

$$\left(E_{a'+b}^{CM}\right)=m_{a'}^2+m_b^2+2E_{a'}E_b'-2p_{a'}p_b'\cos\theta_{a'b}'. \tag{3.91}$$

Then, E_a^{CM} and $E_{a'}^{CM}$ are given by

$$E_a^{CM}=\frac{E_a(E_{a'}+E_b')-p_a(p_{a'}\cos\theta_{aa'}+p_b'\cos\theta_{ab}')}{E_{a'+b}^{CM}}\ , \tag{3.92}$$

and

$$E_{a'}^{CM}=\frac{m_{a'}^2+E_{a'}E_b'-p_{a'}p_b'\cos\theta_{a'b}'}{E_{a'+b}^{CM}}. \tag{3.93}$$

Using Equations (2.19) and (3.28), and the distorted momentum distribution expression found, the triple differential cross section is given by

$$\frac{d^3\sigma_{fi}^{LJ}}{d\Omega_{a'}d\Omega_b dE_{a'}}=C^2|\overline{\langle t\rangle}|^2S_{LJ}F_K\sum_{\Lambda}|T_{fi}^{\alpha L\Lambda}|^2. \tag{3.94}$$

3.4 Analysing Power

The analysing power is the effect on the differential cross section of having the particles i in the entrance channel polarised, or alternatively it may be regarded as a measure of the effect on scattering cross sections of changes in the polarisation of the beam or target nuclei. The word polarisation is used here to mean the non-random orientation of nuclear spins [6].
The analysing power is defined by

$$A_y=\frac{1}{p}\left(\frac{\sigma^{3\uparrow}-\sigma^{3\downarrow}}{\sigma^{3\uparrow}+\sigma^{3\downarrow}}\right), \tag{3.95}$$

where $\sigma^{3\uparrow(\downarrow)}$ denotes the triple differential cross section for an incident beam of spin-up (spin-down) polarised particles and p represents the degree of polarisation of the projectile beam, defined by

$$p = \frac{N_u - N_d}{N_u + N_d} \ , \tag{3.96}$$

where N_u is the number of nucleons in the beam with spin upward and N_d with spin downward.

4. Summary

This work describes the formulation of the differential cross section of the knockout reaction. Two approaches of calculations are presented: the DWIA and the PWIA.

The simplest theory for knockout reactions is the PWIA or spectator model. Although this theory is a huge oversimplification, it clearly demonstrates the ideas and uselfuness of the knockout reactions. The differential cross section in the PWIA is written as the product of the kinematic factor which can be calculated, the differential cross section for the interaction of particles a and b, and the momentum-space wave function describing the relative motion of cluster b in the target, which is the probability of finding the proton in the nucleus with a momentum $-\vec{k}_B$.

In the DWIA, the form of the differential cross section in the PWIA is preserved. However, the momentum-space wave function describing the relative momentum of the cluster becomes the distorted momentum distribution involving three distorted waves, one for the incident particle and two for emitted particles. These are calculated in the usual way from optical potentials that reproduce appropriate elastic scattering data.

Certain major approximations are employed in the DWIA. We have made the impulse approximation that leads to the replacement of the exact operator t_{fi} by the operator $t_f^{(+)}$ for the free scattering process. We have employed this factorisation so that the matrix elements of $t_f^{(+)}$ may be taken outside the integration in t_{fi}, and evaluated for the asymptotic particle momenta. We do this by considering $t_f^{(+)}$ to vary little with momentum and/or the momentum spreading arising from the distorting potentials to be small, at least for the regions of configurations space which contribute strongly to the cross section. The coupling term was omitted to obtain a simple product form for the three-body final state wave function.

The major approximation made was the projection of the target wave function not only onto the residual state B, but also onto the emitted cluster ground state wave function. This corresponds to the assumption that only that part of the target wave function resembling a "preformed" cluster b will contribute appreciably to the cross section. An additional approximation included in our calculations was that the projected target wave function should be replaced by the bound wave function generated in a Woods-Saxon well .

In each of the two approaches PWIA and DWIA, we have assumed the incident beam of particles to be unpolarised, and we have further introduced the concept of analysing power to treat the polarised case.

Some interesting questions remain outstanding. A future project could conceivably entail a qualitative study using data to calculate the differential cross section, and interpreting the results. Predictably, a comparison will follow from there for estimating these two methods, and developing other physical aspects which we did not have the opportunity to investigate within the limited scope and time frame of this essay.

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Bibliography

[1] C A Bertulani and P Danielewicz. *Introduction to Nuclear Reactions*. IOP Publishing Ltd, 2004.

[2] B.R.Martin. *Nuclear and Particle Physics*. John Wiley and Sons, Ltd, 2006.

[3] A.A. Cowley. Research Proposal to iTHEMBA LABS : Analysing power distribution in the $^{12}C(p, p\alpha)^{8}Be$ reaction at an incident energy of 100 Mev. 2006.

[4] D.M.Brink and G.R.Satchler. *Angular momentum Third Edition*. Claredon Press-Oxford, 1993.

[5] Norman K. Glendenning. *Direct Nuclear Reactions*. ACADEMIC PRESS, 1983.

[6] G.R.Satchler. *Direct Nuclear Reactions*. Claredon Press-Oxford, 1983.

[7] Yasuhiko Ikebata. (p,2p) reaction in zero-range and finite-range relativistic distorted-wave impulse approximation. *Physical Review C*, 52, 1995.

[8] Mahavir Jain, P.G.Roos, H.G.pugh, and H.D.Holmgren. The $(p, p\alpha)$ and $(\alpha, 2\alpha)$ reactions on ^{6}Li and ^{7}Li at 60 Mev. *Nuclear Physics A153*, pages 49–81, 1970.

[9] J.J.Sakurai. *Modern Quantum Mechanics Revised Edition*. Addison-Wesley Publishing Company, 1994.

[10] L.S.Rodberg and R.M.Thaler. *Introduction to the quantum theory of scattering*. Academic Press INC., 1967.

[11] N.S.Chant and P.G.Roos. Spin orbit in quasifree knockout reaction. *Physical Review C*, 27(3):1060–1072, 1983.

[12] N.S.Chant and R.G.Roos. Distorted-wave impulse-approximation calculations for quasifree cluster knockout reactions. *Physical Review C*, 15(1):57–68, 1977.