American Options and Optimal Stopping Problems

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Abstract

American options can be exercised at any time the holder wishes to do so before its expiry date. It therefore becomes important to know when this option should be exercised to maximize the profit. American options can therefore be implemented as optimal stopping problems.

In this essay we shall discuss American options, the theory of optimal stopping problems and its applications. We shall also study American options as optimal stopping problems.

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Introduction

American options provide more opportunities to the holder of the option to exercise the option at any time a trader wishes to do so provided that the option is not yet expired. Further more some American options are perpetual. Perpetual American options are practically not traded but they can be treated with simple mathematics which help us to understand the behaviour of other American options.

It therefore becomes important to have knowledge on what is the best time to exercise American options in order to maximize the payoff. This leads us to the study of American options as optimal stopping problems.

In the first chapter we shall give the basic concepts and definitions. We shall study various types of options and show how to calculate their payoffs.

In the second chapter we shall introduce the theory of optimal stopping problems. We shall formulate a simple optimal stopping problem. The solution to this problem will be analysed systematically by applying matching value condition, smooth pasting condition, asset equilibrium condition and the boundary condition.

In the third chapter we shall study American options as optimal stopping problems. We shall formulate an optimal stopping problem and find the solution to this problem in continuous time.

1. Basic Concepts and Definitions

1.1 Brief History on Options

Options were introduced in financial markets many years ago. First options were non exchange traded because agreements were made between two traders. They were called Over The Counter (OTC) [WDH93].

In America, options were exercised in the early 19^{th} century and were simply known as privileges [opt]. The terms on the option (privileges) were made by the two trading parties. However, the rate of growth of the options was not very attractive, although such options were legally accepted in America and many other European countries [opt].

In late 1960's, it was found what was wrong within the financial markets of options and the Chicago Board of Trade (BOT) discovered many discrepancies within the markets of options. In order to eliminate the discrepancies and to strengthen the options made by traders in the financial markets, it was necessary to standardise the exercise prices, expiry dates, option prices and other important factors that came along with the financial markets [opt].

It was observed that there is importance for establishment of an intermediate organ that could guarantee performance of the financial options market. This facilitated the formation of Option Clearing Corporation (OCC) [opt]. In 1973 the financial options were officially traded by Chicago Board Options Exchange (CBOE) [WDH95]. Since then financial options have been exercised all over the world in many different ways [WDH93] [Hul97]. Moreover, the paper by Black and Scholes which appeared in 1973 had a great impact on the options markets.

1.2 Derivative Security

Definition 1.1. *A derivative security, which is also known as a contingent claim, is a financial instrument whose value is completely dependent on the price of the underlying asset in a fixed range of times within the interval [0,T], where T is the expiry date [Hul97].*

Definition 1.2. *Underlying asset refers to any market security such as stock, share, bond, commodity and currencies [BR96].*

There are different examples of derivative securities which are commonly exercised in financial markets. The major derivative securities in financial markets are forwards and futures contracts, options and swaps [Pli98].

1.2.1 Options

Options are attractive and are used in many financial markets throughout the world.

Definition 1.3. *An option is a financial instrument which gives the holder a right without being under obligation to trade the underlying asset at or by expiry date for a certain prescribed price known as exercise or strike price [WDH93].*

The option provides the right to its holder to decide whether to exercise or not to exercise the option at the acceptable time according to the type of option. If the holder of the option decides to exercise the option then the other trader is obliged to exercise the option regardless of what loss he/she is going to encounter. We call the trader who is not the holder of the option the *writer* of the option.

Since the option favours its holder to decide to exercise or not to exercise the option, the holder must purchase the option at a given price known as *option price*. The holder must pay a premium for the rights the option confers on him. This price is also referred to as the *option value*.

The writer should enquire the amount equivalent to the interest if the underlying asset or the cash equivalent to the price of the asset was to be deposited in the bank for the whole period of the option before its expiry.

1.3 Types of Options

We introduce in less details the various types of options. This is mainly for the purpose of introducing American options which will be our main area of focus in this essay. In this chapter we will include other options apart from American options in order to make some important distinguishing features.

1.3.1 European Options

Definition 1.4. *European options are options that give a right to the holder to exercise the option without being under obligation to trade the underlying asset with the exercise or the strike price at the expiry date.*

European Call Option

Definition 1.5. *A European call option is an option that gives right to the holder to buy an underlying asset at the expiry date with the strike price K.*

The holder of this option expects the price of the underlying asset to rise at the expiry date. On the other hand the writer of this option expects the price of the underlying asset to fall at the expiry date.

Let us denote by $S(T)$ the price of the underlying asset at the expiry time T and K the strike or exercise price. We define the payoff of the call option as

$$f(S,K) = \begin{cases} S(T) - K \text{ for } S(T) > K \\ 0 \quad \text{otherwise} \end{cases} \tag{1.1}$$

It is important to note that the payoff above belongs to the holder of the option and not the writer of the option. We can deduce the payoff of the writer of European call option as

$$f(S,K) = \begin{cases} K - S(T) \text{ for } S(T) > K \\ 0 \quad \text{otherwise} \end{cases} \tag{1.2}$$

From equation (1.1), it is clearly seen that the payoff of the holder of the option is never negative. This means that the holder of the call option will only exercise the option when the price of the underlying asset $S(T)$ is greater than the strike price K. At this market condition, the holder of the option will buy the underlying asset for K and immediately sell the asset for $S(T)$ in the market. If $S(T) \leq K$ the holder will not exercise the option. The holder of the option will prefer to buy the asset at the market for $S(T)$.

European put option

Definition 1.6. *European put option is the option that gives a right to the seller of the underlying asset to sell the asset at the expiry time T for the prescribed strike price without being under obligation to do so.*

The holder of this option expects the price of the underlying asset to fall at the expiry date. His payoff is

$$f(S,K) = \begin{cases} K - S(T) \quad \text{for} \quad K > S(T) \\ 0 \quad \text{otherwise} \end{cases} \tag{1.3}$$

Equation (1.3) shows that the holder will not exercise the option if $K \leq S(T)$. Because no trader could sell the underlying asset at low price K while he/she has a right to sell the asset at the market at higher price $S(T)$.

The writer of this option expects the price of the asset to increase at the expiry date and his payoff is given as

$$f(S,K) = \begin{cases} S(T) - K \quad \text{for} \quad K > S(T) \\ 0 \quad \text{otherwise} \end{cases} \tag{1.4}$$

From equation (1.4) and (1.2) we see that the writer of the option has a negative payoff whenever the holder exercises the option and zero payoff whenever the holder decides not to exercise the option.

1.3.2 American Options

In this section we introduce the idea and general concepts of American options. The details in pricing American options will be discussed in the last chapter. We will show how American options differ from the previous type. We will also explain why this option is attractive to many traders and how mathematicians interpret the problems of pricing this type of option.

Definition 1.7. *American options are financial instruments that give its holder the right to trade an underlying asset at any time $t \leq T$ for prescribed price K without being obliged to do so.*

Definition 1.8. *American call option is an option that provides right to the holder to buy an underlying asset at any time $t \leq T$ for the strike price K without being under obligation to do so.*

Definition 1.9. *American put option gives the right to the holder to sell the underlying asset at time $t \leq T$ as long as the seller wishes to do so.*

The holder of the American call option will decide to buy the asset at any time before the option expires while the holder of American put option will decide to sell the asset at any time before expiration of the option.

The payoff of American call option is given by

$$f(S,K) = \begin{cases} S(t) - K, \text{if } S(t) > K, t \leq T \\ 0 \quad \text{otherwise} \end{cases} \tag{1.5}$$

Equation (1.5) means that the American call option can only be exercised when the price $S(t)$ of the underlying asset is greater than the strike price K.

Likewise the payoff of the American put option is given as

$$f(K,S) = \begin{cases} K - S(t), \quad \text{if} \quad K > S(t), \quad t \leq T \\ 0 \quad \text{otherwise} \end{cases} \tag{1.6}$$

There are American options with no expiry dates. Such options are called *perpetual American options*. We call them American options with *infinite time horizon*. American options whose expiry dates are known in advance are said to be of *finite time horizon*. The difference between perpetual American options from other American options is that they can not be traded but they are dealt as mathematical problems.

The American put options will be exercised when $K > S(t)$. If the market price does not satisfy this condition, the holder will not sell the underlying asset for K, but will sell the underlying asset at the market for $S(t) > K$.

Since American options give the right to its holder to exercise the option at any time before the option expires, traders are more attracted by this option because of these privileges. Mathematicians take advantage to interpret the option as free boundary problems in order to obtain an appropriate time to exercise the option for maximum payoff. This will be our main concern in this essay in the next chapters. Other types of options include Asian, Barrier, Chooser and Look back [Eth02] [Bja98] [WDH93] [WDH95].

2. Theory of Optimal Stopping Problems and Applications

2.1 Brief History on Optimal Stopping Problems

Results on optimal stopping were first developed in the discrete case. The formulation of optimal stopping problems for discrete stochastic processes was in sequential analysis, an area of mathematical statistics where the number of observations is not fixed in advance but is a random number determined by the behaviour of the data being observed [Ped00].

Snell was the first person to come up with results on optimal stopping theory for stochastic processes in discrete time [Ped00]. While Snell came up with results in the discrete case, Dynkin was the first person to derive general results on optimal stopping problems for continuous time Markov processes [Ped00] [Uys05].

We will introduce some important terms on optimal stopping problems which will be applied in this chapter and in the last chapter.

2.2 Brownian Motion

A Brownian motion (Wiener process) is a stochastic process $\{X(t)\}_{t\geq 0}$ with the following properties:

- $X(0) = 0$.
- The sample paths $t \mapsto X(t)$ are continuous.
- For any $0 \leq s \leq t$ the increment $X(t) - X(s)$ has the normal ditribution. The mean and variance of this increment is 0 and $t - s$, respectively.
- The Brownian motion $\{X(t)\}_{t\geq 0}$ has stationary increments.
- For any $t = t_0 \leq t_1 \leq ... \leq t_n$ the increment $X(t_1) - X(t_0), ..., X(t_n) - X(t_{n-1})$ are indepedent.
- The sample paths $\{X(t)\}_{t\geq 0}$ are of infinite total variation.
- The Brownian motion $X(t)$ is a non differentiable fo any $t \geqslant 0$.
- $dX_t dt = 0$.
- $dtdt = 0$.
- $dX_t dX_t = dt$.

2.3 Itô's formula

Itô formula is a tool for solving Stochastic Differential Equations. It states that

$$dg(t, X_t) = D_t g(\cdot, X_t)dt + D_x g(\cdot, X_t)dX_t + \frac{1}{2}D_{xx}g(\cdot, X_t)(dX_t)^2. \quad (2.1)$$

Where D_t is the derivative of the fuction g w.r.t time t, D_x is the derivative w.r.t x and D_{xx} is the second derivative of g with respective to x.

2.4 Stopping Times

Throughout this essay we will assume that we are working with a probability space (Ω, $\mathcal{F}$, $\mathcal{F}_t$, P). We denote the collection of all information available up to time t by $\mathcal{F}_t$ and (Ω, $\mathcal{F}$, $\mathcal{F}_t$, P) is a filtered probability space.

Definition 2.1. *A random variable τ defined on Ω is called a stopping time if the event $\tau \leq t$ belongs to $\mathcal{F}_t$ for all values of time t. We consider τ a stopping time if and only if the available information up to time t is sufficient to decide whether $\tau \leq t$ or not [BP06].*

An example of a stopping time is the hitting time.

Definition 2.2. *Given an adapted process X, the hitting time of the Borel set $A\epsilon\mathcal{B}$ is the first time when X reaches A. This means that [Mar07].*

$$T_A = \min\{t\epsilon\mathbb{R}_+ : X(t)\epsilon A\} \quad (2.2)$$

Definition 2.3. *Let $(X_t)_{t\geq 0}$ be a stochastic process, then the optimal stopping problem is given by*

$$V(x) = \max_{\tau} \mathbb{E}_x \left(G(X_\tau)\right) \quad (2.3)$$

where τ is a set of stopping times of X. The function $V(x)$ is called the value function, G is called the *reward function* or the gain function and $\mathbb{E}_x$ is the expectation under the probability measure P_x that the process X starts at x, i.e $X(0) = x$.

Remark 2.4. *In optimal stopping problems we often want to find out whether to continue running the process, thereby incurring more costs (in order to obtain greater reward in the future) or stopping the process with the current gain.*

In optimal stopping problems we are interested in finding the solution to the given problem such that the solution is the possible maximum payoff. Optimal stopping problems are commonly applied in statistical mathematics and have been applied in solving problems in mathematical finance [Suc92] [Uys05]. American options are good examples where optimal stopping problems are practically applied. This is because an American option holder should exercise the option only once and wants to maximise profit.

Optimal stopping problems require the determination of an appropriate time at which the maximum payoff can be obtained. The earliest time $\tau^* \, \epsilon \, \tau$ such that the exercising yields the maximum value is called **optimal stopping time**. There might be many optimal stopping times in one process at which the maximum payoff according to the reward function is obtained. We are interested in the first optimal stopping time because we need to maximise the payoff within the shortest possible time.

The holder of the option must decide when to exercise. Such decisions should be done very carefully because [Blo03]:

1. Rewards and costs of these actions are functions of time.
2. Exercising is often irreversible or costly to reverse.

To illustrate the two points above, let us consider the example of exercising the American option. We see that the first point means that the best payoff will depend on the price of the asset at that particular time. The second point means that once you exercise the option you can not re-exercise the option unless you pay a premium to buy another option.

Note that in equation (2.3) we try to solve the problem in terms of the expected value and not over a particular sample path.

Definition 2.5. *An optimal stopping time τ is the stopping time which maximizes the value $\mathbb{E}_x$ for each initial value x of X.*

Remark 2.6. *The optimal stopping time might not exist. Therefore not all problems can be solved as optimal stopping problems.*

In addition, if the optimal stopping problem is solvable, its solution should consist of the value function V and the stopping time at which the maximum value of (2.3) is attained.

2.5 Strong Markov Property

If $X = (X_t)_{t\geq 0}$ is a process equipped with the filtration $(\mathcal{F}_t)_{t\geq 0}$ with $\mathcal{F} = \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$, then X has the strong markov property if any of the following conditions hold:

$$\mathbb{E}_x\left(f(X_{\tau+h})|\mathcal{F}\right) = \mathbb{E}_x\left(f(X_{\tau+h}|X_\tau\right). \tag{2.4}$$

$$\mathbb{E}_x\left(f(X_{\tau+h})|\mathcal{F}\right) = \mathbb{E}_{X_\tau}\left(f(X_h)|X_h\right). \tag{2.5}$$

The above conditions hold for all x, τ, $h \geq 0$ and bounded measurable functions f [Uys05].

2.6 Infinitesimal Generator

Let $(X_t)_{t\geq 0}$ be a time homogenous Itô diffusion in $\mathbb{R}^n$. We define the infinitesimal generator of X as

$$\mathbb{L}_x f = \lim_{t\to 0} \frac{\mathbb{E}_x\left(f(X_t)\right) - f(x)}{t} \tag{2.6}$$

where $x\epsilon\mathbb{R}^n$.

Definition 2.7. *Let $(X_t)_{t\geq 0}$ be an Itô diffusion with stochastic differential equation given by*

$$dX_t = aX_t dt + \sigma X_t dB_t, \tag{2.7}$$

then the infinitesimal generator is given by [Ped00]

$$\mathbb{L}_x = \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} \sigma\sigma^T_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}. \tag{2.8}$$

It is important to note that for $n = 1$ the above infinitesimal generator becomes

$$\mathbb{L}_x = a(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}. \tag{2.9}$$

Also note that for $a(x) = rx$ and $\sigma(x) = \sigma x$ then its infitesimal generator at $n = 1$ is given by

$$\mathbb{L}_x = rx\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2}{\partial x^2}. \tag{2.10}$$

2.7 Formulation of a Simple Optimal Stopping Problem

The aim of this section is to formulate an optimal stopping problem with its solution which will help us to solve the problem of American options as optimal stopping problems in the next chapter.

Consider a simple optimal stopping problem initially stated as:

$$V(x) = \max_{\tau} \mathbb{E}_x\left(e^{-\mu\tau} f(X_t)\right) \tag{2.11}$$

subject to

$$\begin{aligned} dX_t &= pX_t dt + qX_t dZ_t \\ X(0) &= x. \end{aligned} \tag{2.12}$$

We denote by τ the stopping time, τ^* the optimal stopping time and x_* is the value of the state variable at τ^*, i.e $x_* = x(\tau^*)$. μ is the discount rate and p, q are positive integers.

We call,

- dZ_t an increment of a Wiener process.

- $f(.)$ a reward function and
- $V(.)$ a value function.

The relationship between t and X is given by the equation (2.12). Therefore, the choice of τ which yields maximum value of $V(x)$ will depend on equation (2.12).

The exercising of the option should not be done by considering the time only, but we want to know whether the value function yields the maximum value at the time of exercising the option. The option is supposed to be exercised at the optimal stopping time and not otherwise. This means that we need to maximize the payoff in the shortest possible time. The solution will seek to find the optimal stopping values x_* and τ^*. It is important to note that the function f is currently unknown.

2.8 Solution of Optimal Stopping Problem

Consider the situation where we have the optimal stopping problem below as it was defined in the previous section

$$V(x) = \max_{\tau} \mathbb{E}_x \left(e^{-\mu\tau} f(X_t) \right) \tag{2.13}$$

and

$$dX_t = pX_t dt + qX_t dZ_t. \tag{2.14}$$

where $p > 0$ and $q > 0$. We call equation (2.14) **geometric Brownian motion**. This equation has the following properties:

- Given an initial value $x > 0 \Longrightarrow X_t \geq 0$ for all $t \geq 0$
- If $X_0 = 0$, It should always remain zero.

We assume that $\mu > p$ and μ>0 in order to find the solution of the problem. At each moment, the choice to exercise or not depends on whether the payoff is optimal.
We denote the market value of continuing the process by $V(x)$ and the market value of terminating the process by $f(x)$.

Throughout this problem we make the following assumptions:

1. Both f and V are continuous and smooth
2. Since f and V are both functions of asset market values $\Longrightarrow f(x)$, $V(x) \geq 0$
3. Transaction costs related to the option are not put into consideration.

In this example we need to find the optimal value x_*. We can say that while the value of the underlying process is less than x_*, we choose not to exercise. If its value is greater or equal to x_* we exercise the option. When $x = x_*$ the option is optimal to be exercised. At this value the two functions are equal. This means

$$V(x_*) = f(x_*) \tag{2.15}$$

We call equation (2.15) above, a **value matching condition**. The decision of exercising or not exercising the option is made by considering the optimality at any value of x and will depend on wealth made by comparing the two functions namely $V(x)$ and $f(x)$. This decision of exercising the option will be made according to the function ψ, where

$$\psi(x) = \max\{V(x), f(x)\}. \tag{2.16}$$

From equation (2.16), if $\psi(x) = V(x)$ we choose to continue and if $\psi(x) = f(x)$ we choose to stop. In order to find the optimal solution to this problem, we need to put into consideration the **smooth pasting condition** which states that

$$V'(x_*) = f'(x_*). \tag{2.17}$$

We can see that the smooth pasting condition shows that $V'(x)$ and $f'(x)$ are equal at optimal value x_* smoothly while the value matching condition shows that $V(x)$ and $f(x)$ are equal at optimal value x_*. If we consider the two conditions simultaneously, we can say that the function $\psi(x)$ is both smooth and continuous at the optimal value x_*.

When $f(x) \geq V(x)$ we do exercise because the stopping value is greater than the value of continuing value. However, it is optimal to exercise when $f(x) = V(x)$. This is because it will take more time to wait for $f(x)>V(x)$. On the other hand if $f(x)<V(x)$ it is optimal to continue. This leads us to the definition of stopping and continuing region.

The stopping region is given by

$$S_{rg} = \{x : f(x) \geq V(x)\}. \tag{2.18}$$

Likewise the continuing region is given by

$$C_{rg} = \{x : V(x)>f(x)\}. \tag{2.19}$$

From this fact we can express the optimal stopping time as

$$\tau^* = \inf\{t \geq 0 : V(X_t) \leq f(X_t)\}. \tag{2.20}$$

From (2.20) we see that the optimal stopping time is the first instant at which the value function is not greater than the gain function.

From equation (2.20) and (2.18) the optimal stopping region becomes $S_{rg} = \{x : f(x) = V(x)\}$. This is known as the **optimal stopping boundary**. Optimal stopping boundary divides the state space into the stopping region and the continuation region. At this boundary the two functions are equal. Though the optimal stopping boundary divides the two regions, it is a part of stopping region at which it is optimal to exercise the option. It is important to note that at the optimal stopping boundary $x = x_*$.

Suppose it is still optimal to continue for a small time δt after the optimal stopping time. Then

$$V(x) = e^{-\mu\delta t}\mathbb{E}\left(V\left(x+\delta x\right)\right). \tag{2.21}$$

If we expand equation (2.21) using Taylor expansion we get

$$V(x+\delta x) \approx V(x) + \delta x V'(x) + \frac{1}{2}(\delta x)^2 V''(x) + O(\delta x)^3. \tag{2.22}$$

If we neglect the higher order terms and substitute equation (2.22) in equation (2.21) we get

$$V(x) \approx e^{-\mu\delta t}\mathbb{E}\left[V(x) + \delta x V'(x) + \frac{1}{2}(\delta x)^2 V''(x)\right]. \tag{2.23}$$

Applying $\mathbb{E}$ over (δx) in equation (2.23) above we get

$$V(x) \approx e^{-\mu\delta t}\left[V(x) + V'(x)\mathbb{E}(\delta x) + \frac{1}{2}V''(x)\mathbb{E}(\delta x)^2\right]. \tag{2.24}$$

Subtracting $e^{-\mu\delta t}V(x)$ from both sides we get

$$(1-e^{-\mu\delta t})V(x) \approx e^{-\mu\delta t}\left[V'(x)\mathbb{E}(\delta x) + \frac{1}{2}V''(x)\mathbb{E}(\delta x)^2\right]. \tag{2.25}$$

If we multiply equation (2.25) by $\frac{1}{\delta t}$ we get

$$\frac{(1-e^{-\mu\delta t})V(x)}{\delta t} \approx e^{-\mu\delta t}(\delta t)^{-1}\left[V'(x)\mathbb{E}(\delta x) + \frac{1}{2}V''(x)\mathbb{E}(\delta x)^2\right]. \tag{2.26}$$

We can further simplify our expression as $\delta t \longrightarrow 0$. This can be done by taking the derivative and replacing δx with dx and δt with dt. Then equation (2.26) becomes

$$\mu V(x) = (dt)^{-1}\left[V'(x)\mathbb{E}dx + \frac{1}{2}V''(x)\mathbb{E}(dx)^2\right]. \tag{2.27}$$

From Stochastic equation of motion that given

$$\begin{aligned} &dx = pxdt + qxdz \quad \text{then} \\ &\mathbb{E}(dx) = pxdt \quad \text{and} \quad \mathbb{E}(dx^2) = [qx]^2 dt. \end{aligned} \tag{2.28}$$

For proof of the above condition see [Blo03] Chapter 5 p.$63-66$.

If we substitute the conditions of equation (2.28) in equation (2.27) we get

$$\mu V(x) = pxV'(x) + \frac{1}{2}V''(x)[qx]^2. \tag{2.29}$$

We call the above equation the **asset equilibrium condition**. The asset equilibrium condition means that the measure of the return that can be obtained if the asset is traded in the market and the income obtained if the asset was to be invested as the risk free asset should be the same. If the condition is not satisfied then the asset was traded in a way that allowed arbitrage against the trader. This means that the option was exercised with no optimality consideration.

In equation (2.29), $\mu V(x)$ is the return that could be obtained by trading the asset at its market value while $pxV'(x) + \frac{1}{2}V''(x)[qx]^2$ is the return if the asset was invested as a risk free asset.

Moreover if equation (2.29) is not satisfied then the asset is said to be either undervalued or overvalued [Blo03].

If $\mu V(x) < pxV'(x) + \frac{1}{2}V''(x)[qx]^2$, we say the asset is undervalued. This means that a higher payoff is received if the asset is invested as a risk free asset rather than trading the asset at its market value. On the other hand if it happen that $\mu V(x) > pxV'(x) + \frac{1}{2}V''(x)[qx]^2$ it means a higher payoff would be obtained if the asset was sold at the market at its market value on that particular time than when it was traded as the risk free asset.

From equation(2.29) we have

$$pxV'(x) + \frac{1}{2}q^2x^2V''(x) - \mu V(x) = 0. \tag{2.30}$$

Our task is to solve the differential equation

$$\frac{1}{2}q^2x^2V''(x) + pxV'(x) - \mu V(x) = 0. \tag{2.31}$$

The above homogeneous differential equation is a Cauchy-Euler equation and its solution can be obtained by taking a guess. Let

$$V(x) = kx^w \Longrightarrow xV'(x) = wkx^w \quad \text{and} \quad x^2V''(x) = w(w-1)kx^w. \tag{2.32}$$

If we substitute the expressions in (2.32) in equation (2.31) and simplify we get

$$kx^w\left[q^2w(w-1) + 2pw - 2\mu\right] = 0. \tag{2.33}$$

We can further simply equation (2.33) and get

$$kx^w\left[q^2w^2 + (2p - q^2)w - 2\mu\right] = 0. \tag{2.34}$$

Remark 2.8. *Both k, p, q are constants and $x>0$.*

If this is the case, equation (2.34) is a quadratic equation in w whose roots are

$$w_{\pm} = \frac{(q^2 - 2p) \pm \sqrt{(2p - q^2)^2 + 8q^2\mu}}{2q^2}. \tag{2.35}$$

From the above fact, it follows that both $V(x) = kx^{w_\pm}$ are solutions to the equation (2.31). This implies that its general solution will be given by linear combinations of the respective solutions. Hence

$$V(x) = k_1x^{w_+} + k_2x^{w_-}. \tag{2.36}$$

If we substitute the solution given in equation (2.36) in equation (2.15) and in equation (2.17), we will have three unknowns, namely k_1, k_2 and x_* in only two equations. We will need a third equation to obtain a unique solution.

Since we obtained the two equations in (2.36) and (2.15) by considering the value matching and smooth condition respectively, we need to include a third condition in order to get a third equation. The condition which we will consider is the so called **boundary condition.**

From the properties of geometric Brownian motion, we have seen that whenever $x = 0$, it should remain zero forever and hence the optimal value x_* will never be reached. We can show from equation (2.36) that

$$V(0) = 0. \tag{2.37}$$

Remark 2.9. *From (2.35) it is clear that $w_+ > 0$ and $w_- < 0$, since $\mu > 0$.*

Applying Remark (2.9) to equation (2.36) we can see that as $x \longrightarrow 0$ then $k_2 x^{w_-} \longrightarrow +\infty$ for $k_2 > 0$. Likewise we can see that as $x \longrightarrow 0$ then $k_2 x^{w_-} \longrightarrow -\infty$ for $k_2 < 0$. This implies that condition (2.37) can only hold when $k_2 = 0$.

Remark 2.10. *The first term of (2.36) goes to zero when $x \longrightarrow 0$. This satisfies the condition in equation (2.37).*

We can therefore write equation (2.36) as

$$V(x) = k_1 x^{w_+} \quad \text{since} \quad k_2 = 0. \tag{2.38}$$

Applying equation (2.38) to equation (2.15) and equation (2.17) we get that

$$k_1 x_*^{w_+} = f(x_*) \qquad \text{and} \tag{2.39}$$

$$(w_+)\, k_1 x_*^{[(w_+)-1]} = f'(x_*). \tag{2.40}$$

Using equation (2.39) in equation (2.40) we get

$$\frac{(w_+) f(x_*)}{x_*} = f'(x_*). \tag{2.41}$$

Hence the optimal value to this particular problem is

$$x_* = \frac{(w_+) f(x_*)}{f'(x_*)}. \tag{2.42}$$

Since we have obtained the optimal value x_*, we can deduce the optimal stopping time τ^* as

$$\begin{aligned} \tau^* &= \min\{t \geq 0 : X_t = x_*\} \\ &= \min\left\{t \geq 0 : X_t = \frac{(w_+) f(x_*)}{f'(x_*)}\right\} \end{aligned} \tag{2.43}$$

The value function at these optimal stopping values becomes

$$V(x) = E_x\left(e^{-\mu\tau^*} f(x_*)\right) \tag{2.44}$$

We can extend the solution x_* if the function $f(x)$ is known. The function $f(x)$ depends on the nature of the problem someone is dealing with.

Example

Let us consider the case where we have a commodity whose price depends on its weight as the underlying asset (e.g a cattle). We want to find the optimal weight to sell the asset. Let a be the price in Rand per unit weight of the asset. If we consider b to be the transaction costs we can formulate

$$f(x) = ax - b. \tag{2.45}$$

From equation (2.45) we get

$$f'(x) = a. \tag{2.46}$$

Substituting equation (2.45) and (2.46) in equation (2.42) we get

$$x_* = \frac{(ax_* - b)\, w_+}{a}, \tag{2.47}$$

Which gives

$$\begin{aligned} x_* &= \frac{-bw_+}{[1 - (w_+)]\, a} \\ &= \frac{bw_+}{[(w_+) - 1]\, a}. \end{aligned} \tag{2.48}$$

The optimal stopping time for this specific example becomes

$$\tau_* = \min \left\{ t \geq 0 : X_t = \frac{bw_+}{[(w_+) - 1]\, a} \right\}. \tag{2.49}$$

Remark 2.11. *The assumption we made that $\mu{>}p$ is intended to make $w_+ > 1$ in order to make $x_* > 0$.*

The above strategies can be used in solving many optimal stopping problems. However, it is important to note that the function $f(x)$ is not unique but it varies from one problem to another.

Remark 2.12. *In this optimal stopping problem the gain function f and the value function V are merely functions of x and not both t and x. Therefore the problem solved in this chapter is much simpler than the general problem.*

2.9 Infinitesmal Generator in a Boundary Condition

We have already seen that we need many boundary conditions in order to solve an optimal stopping problem, because such problem might have more than one solution. In the next chapter we shall apply the infinitesimal generator to obtain the asset equilibrium condition.

Definition 2.13. *Given a gain function* $G{=}e^{-\eta t}f(X)$ *where* $\eta{>}0$ *is a constant then for a value function*

$$V(x) = \max_{\tau} \mathbb{E}_x \left(e^{-\eta\tau} f(X_\tau)\right) \quad \textit{the asset equilibrium condition is given by}$$

$$\mathbb{L}_x V(x) = \eta V(x) \quad \textit{for } x \textit{ in domain of continued observation } [\textit{Ped00}]. \tag{2.50}$$

We shall see that where the problem is specified this definition gives an equation which is similar to equation (2.31) which was obtained after the asset equilibrium condition.

3. American Options as Optimal Stopping Problems

3.1 Brief History on American Options as Optimal Stopping Problems

Bensoussan and Karatzas were the first to apply the method of no arbitrage to show that the price of an American put option could be treated as optimal stopping problems. McKean then derived a free boundary problem for the discounted American call option with gain function $f(x) = e^{-\beta t}(x - K)^+$. He was able to express the option price V as the function of optimal stopping boundary in a countable nonlinear integral equation. However Mckean did not prove existence and uniqueness of solutions to this system of equations [Uys05].

Moerbeke then derived a nonlinear integral equation for the boundary. He was able to prove the existence and uniqueness of the solution to this equation for the general optimal stopping problems. However, his work was only restricted to discounted American call options [Uys05].

Steven E. Shreve reviewed stopping times of American put options in discrete time. He also proved that American call options do not satisfy optimal stopping if the stock does not pay dividends [Shr05].

In 2006, Goran Peskir and Albert Shiryaev wrote a book which discussed how various options are solved as optimal stopping problems. They wrote about American put options and Russian options as optimal stopping problems in both infinite and finite horizons. They also showed how Asian options can be considered as optimal stopping problems in finite horizons. They dealt the optimal stopping problems only in the aspect of continuous time [PS06].

3.2 Optimal Stopping Problems in Continuous Time

In this section we shall see how American options can be considered as optimal stopping problems. Throughout this chapter we shall make the following assumptions:

- There is no arbitrage opportunity in the market.
- The underlying asset (stock) pays no dividends.
- There is one riskless bank account and one risky underlying asset.

3.2.1 Volatility

Definition 3.1. *Volatility is a statistical measure of the dispersion of return for a derivative security which determines the extent to which the return fluctuates before the expiry date.*

The volatility can be determined by using standard deviation or the variance between the returns of the same derivative security. If the volatility is higher it means the return of a given security fluctuates dramatically within a short period. For lower volatility it means the value of the security does not fluctuate dramatically. We shall denote volatility by σ in this section.

3.3 Formulation of the Problem

We consider a perpetual American put option. A perpetual American put is an American put with no maturity or expiry date. This means that the expiry date is at *infinity*. It is also known as American put with infinite time horizon. The reason we solve the optimal stopping problem on perpetual American put is that its price is the function of the asset price only and its optimal stopping boundary is a constant function. This makes the problem easy to solve.

Finite time horizon options are difficult to solve. Option price of finite time American options are functions of both time t and stock price price x. The optimal stopping boundary of finite American put is not a constant function but a function of time [PS06]. Moreover the exact form of the continuation region of finite American put option is still unknown as a result the analytic solution becomes very hard (possibly impossible) [Oks02].

We need to find the optimal free arbitrage price of the option and the optimal exercising time of the perptual American put option.

Let $B = (B_t)_{t\geq 0}$ be the standard Brownian motion started at time $t_0 = 0$. We consider $X = (X_t)_{t\geq 0}$ to be the geometric Brownian motion that governs the price of the asset in the market. In addition $\sigma>0$ and $r>0$ are the volatility coefficient and risk free interest rate of the bank account, respectively.

Then the stock price process $X = (X_t)_{t\geq 0}$ satisfies

$$dX_t = rX_t dt + \sigma X_t dB_t, \quad \text{with} \quad X_0 = x > 0. \tag{3.1}$$

Proposition 3.2. *Given the Stochastic Differential Equation in equation (3.1), the solution is given by*

$$X_t = X_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma B_t\right). \tag{3.2}$$

Proof:
We can write the equation (3.1) as

$$\frac{dX_t}{X_t} = rdt + \sigma dB_t. \tag{3.3}$$

If we use Itô formula with $g(x,t) = \text{log}x$, It follows that

$$d(\text{log}X_t) = \frac{\partial g}{\partial t}dt\Big|_{x=X_t} + \frac{\partial g}{\partial x}dX_t\Big|_{x=X_t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX_t)^2\Big|_{x=X_t}. \tag{3.4}$$

It is obvious that the function $g(x,t) = \text{log}x$ has no term in t, this means the term in equation (3.4) differentiated w.r.t t vanishes. Hence equation (3.4) above reduces to

$$d(\text{log}X_t) = \frac{\partial g}{\partial x}dX_t\Big|_{x=X_t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX_t)^2\Big|_{x=X_t}. \tag{3.5}$$

Given $g(x,t) = \text{log}x \implies \dfrac{\partial g}{\partial x} = \frac{1}{x}$. We can therefore substitute the required terms in equation (3.5) and get

$$d(\text{log}X_t) = \frac{1}{X_t}dX_t - \frac{1}{2X_t^2}(dX_t)^2. \tag{3.6}$$

From equation (3.3) it follows that

$$\left(\frac{dX_t}{X_t}\right)^2 = r^2(dt)^2 + 2rdtdB_t + \sigma^2 dB_t dB_t. \tag{3.7}$$

By applying the given properties of Brownian motion, equation (3.7) becomes

$$\left(\frac{dX_t}{X_t}\right)^2 = \sigma^2 dt. \tag{3.8}$$

This simplifies equation (3.6) above to

$$d(\text{log}X_t) = \frac{1}{X_t}dX_t - \frac{1}{2}\sigma^2 dt. \tag{3.9}$$

Combining equation (3.3) above and equation (3.9) we get

$$d(\text{log}X_t) = rdt + \sigma dB_t - \frac{1}{2}\sigma^2 dt. \tag{3.10}$$

We simplify equation (3.10) above and get

$$d(\text{log}X_t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dB_t. \tag{3.11}$$

The integral form of equation (3.11) is

$$\int_0^t d(\text{log}X_s) = \int_0^t (r - \frac{1}{2}\sigma^2)ds + \int_o^t \sigma dB_s. \tag{3.12}$$

From the above equation we have

$$[logX_s]_0^t = (r - \frac{1}{2}\sigma^2)t + \sigma B_t. \tag{3.13}$$

Finally we have

$$X_t = X_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma B_t\right). \tag{3.14}$$

Remark 3.3. *The logX_t in the above proof represents log$_e X_t$. It is also important to note that X_0=x under P measure.*

The arbitrage free price of the perpetual American put is given by [PS06]

$$V(x) = \max_{\tau} \mathbb{E}_x \left[e^{-r\tau}(K - X_\tau)^+ \right], \tag{3.15}$$

where K is the strike price, τ is the stopping time and X_t is the asset price process at time t according to equation (3.14). It is important to note that the gain function of equation (3.15) is $f(x) = (K - x)^+$ and X_τ in the gain function is the solution X_t of equation (3.14) at $t = \tau$.

The problem we are facing is to determine the optimal arbitrage free price and the optimal stopping time say τ_* to exercise the option which yields the maximum value according to equation (3.15). Using no arbitrage arguments, the optimal stopping time becomes the best time to stop and that if you continue, you will in fact be loosing money even if the stock price drops further.
We shall need to find the price of the stock according to equation (3.14) which will help us to proceed.

3.4 Solution to the Problem

We know that the optimal time to exercise the American put is when the stock price falls as much as possible. However we also need to consider the time and the stock price in order to exercise the option. The option should be exercised at the possible minimum time of duration from when the option was made but with the possible maximum payoff. From equation (3.15) and (3.14) as X becomes very small, if the option is not exercised then less likely the payoff will not increase upon continuation. This is because the payoff is maximum when the price of the stock falls in the market if we do not consider other factors.

Therefore we assume that there exists a point $p\epsilon(0, K)$ such that

$$\tau_p = min \left\{ t \geq 0 : X_t \leq p \right\}. \tag{3.16}$$

We need to find the point p which will give the optimal price in order to find optimal value of $V(x)$ and τ_p.

The perpetual American put is a special case of the general optimal stopping problem solved in Chapter 2. We can apply the results of chapter 2 to this problem. It is important to note that p is similar to x_* given in equation (2.43).

It is also important to note that p stands for a certain price which is between 0 and K. This is because the option can not be exercised if the stock price exceeds or is equal K. The stock price can not be zero.

Based on the standard argument of strong Markov property for the value function $V(x)$ and the unknown point p, we get the following boundaries ([PS06] $p.376$).

$$\mathbb{L}_x V = rV \quad \text{for} \quad x > p \quad \text{(asset equilibrium condition)} \tag{3.17}$$

$$V(x) = (K-x)^{+} \quad \text{for} \quad x = p \quad \text{(value matching condition)}. \tag{3.18}$$

$$V'(x) = -1 \quad \text{for} \quad x = p \quad \text{(smooth pasting condition)}. \tag{3.19}$$

$$V(x) > (K-x)^{+} \quad \text{for} \quad x > p \quad \text{(continuation region)}. \tag{3.20}$$

$$V(x) = (K-x)^{+} \quad \text{for} \quad 0 < x < p \quad \text{(value matching condition)}. \tag{3.21}$$

It is important to note that the condition (3.18) and (3.21) are all value matching conditions. They are separated in order to make clear the boundary of the option but they all represent the stopping region. Moreover (3.18) is a value of the option on the optimal stopping boundary. At this optimal stopping boundary, the price of the option is equal to the payoff. The condition in equation (3.17) is a definition in equation (2.50).

Optimal Stopping Boundary for Perpetual American Put

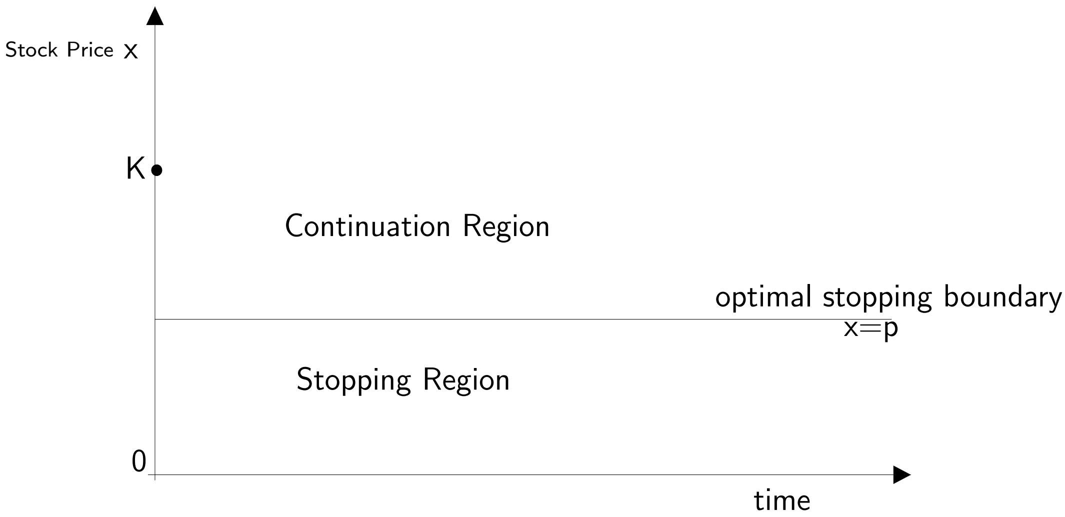

Figure 3.1: The optimal stopping boundary for perpetual American put as a constant function

From Figure 3.1, while in the continuation region we do wait until when the price falls to $x = p$. The constant function $x = p$ is the optimal stoping boundary. At the optimal stopping boundary it is optimal to exercise the options. We can stop at any other point within the stopping region but it is not optimal to wait until when the price falls below the optimal stopping boundary. This is because we don't know when the price will fall as much as possible and it might take too long to wait.

We are interested to find this optimal value in this problem. This will be the value which will help us to find other important optimal values. We shall consider the conditions stated in equation (3.17-3.21) in order to find these values.

From equation (3.17) we get

$$rx\frac{\partial V}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 V}{\partial^2 x} = rV. \tag{3.22}$$

Equation (3.22) can be modified to be

$$\frac{\sigma^2}{2}x^2V''(x) + rxV'(x) - rV(x) = 0. \tag{3.23}$$

Equation (3.23) above is a Cauchy-Euler equation similar to equation (2.31). Its general solution is therefore given by equation (2.36) as

$$V(x) = k_1x^{w_+} + k_2x^{w_-}. \tag{3.24}$$

We need to know the values of w_+ and w_- as far as this problem is concerned. To do this we consider equation (2.35). Comparing equation (2.31) and (3.23), we see that $q = \sigma$, $p = r$ and $\mu = r$. Hence from (2.35) we get

$$\begin{aligned} w_\pm &= \frac{(\sigma^2 - 2r) \pm \sqrt{(2r-\sigma^2)^2 + 8\sigma^2 r}}{2\sigma^2} \\ &= \frac{(\sigma^2 - 2r) \pm \sqrt{4r^2 + 4r\sigma^2 + \sigma^4}}{2\sigma^2} \\ &= \frac{(\sigma^2 - 2r) \pm (2r + \sigma^2)}{2\sigma^2} \\ &= 1, \frac{-2r}{\sigma^2} \end{aligned} \tag{3.25}$$

$w_+ = 1$ and $w_- = \frac{-r}{A}$, where $A = \frac{\sigma^2}{2}$.

Remark 3.4. *It is clear that* $w_-{<}0$*, since both* $r > 0$ *and* $\sigma > 0$.

The general solution in equation (3.24) becomes

$$V(x) = k_1x + k_2x^{\frac{-r}{A}}. \tag{3.26}$$

In an arbitrage free market the option price of American options is $V(x) \leq K$ and $x{>}0$ [CZ03]. This implies that the solution in equation (3.24) should be bounded. This means $k_1x + k_2x^{\frac{-r}{A}} \leq K$. When x gets very large the first term $\longrightarrow \infty$ for $k_1 > 0$ and $-\infty$ for $k_1 < 0$. This implies that the function in equation (3.26) is no longer bounded. For it to be bounded k_1 must be zero.

Remark 3.5. *The second term is finite for all values of* x.

From the above fact, the solution in equation (3.26) becomes

$$V(x) = k_2x^{\frac{-r}{A}}, \tag{3.27}$$

where k_2 is a constant to be determined.

Since the term with the root w_+ cancels, the optimal value p is obtained by replacing w_+ with w_- from equation (2.42). Hence

$$\begin{aligned} p &= \frac{(K-x)^+(w_-)}{\frac{d}{dx}(K-x)^+} \\ &= -(w_-)(K-x)^+. \end{aligned} \tag{3.28}$$

Applying a condition in equation (3.18) we get

$$\begin{aligned} p = -(w_-)(K-p) &= \frac{-Kw_-}{1-(w_-)} \\ &= \frac{K\frac{r}{A}}{1+\frac{r}{A}} \\ &= \frac{K}{1+\frac{A}{r}}. \end{aligned} \tag{3.29}$$

Remark 3.6. *It was assumed that $p\epsilon\,(0,K)$. This is the optimal stock price to exercise the option. This shows that the optimal stock price depends on the strike price K and the volatility σ since $A{=}\dfrac{\sigma^2}{2}$. If the volatility σ and the strike K in the perpetual American put option are known in advance we could be able to find the specific value of p.*

From equation (3.16) and (3.29), the optimal stopping time becomes

$$\tau_* = min\left\{t \geq 0 : X_t \leq \frac{K}{1+\frac{A}{r}}\right\}. \tag{3.30}$$

We need to find the value of $V(x)$ at this point. To do this, we need to find the value of k_2 at this particular value of p.
From equation (3.27) we have

$$V'(x) = -\frac{r}{A}k_2 x^{(\frac{-r}{A}-1)}. \tag{3.31}$$

Applying (3.19) in equation (3.31) we get

$$\frac{r}{A}k_2 p^{(\frac{-r}{A}-1)} = 1. \tag{3.32}$$

This gives

$$\begin{aligned} k_2 &= \frac{A}{r}p^{(\dfrac{r}{A}+1)} \\ &= \frac{A}{r}\left(\frac{K}{1+\frac{A}{r}}\right)^{(1+\frac{r}{A})}. \end{aligned} \tag{3.33}$$

Substituting the value of k_2 in equation (3.27) we get

$$V(x) = x^{\frac{-r}{A}}\frac{A}{r}\left(\frac{K}{1+\frac{A}{r}}\right)^{(1+\frac{r}{A})}. \tag{3.34}$$

We have applied the first three boundary conditions so far to reach the solution in (3.34). The first three conditions result into $x \geq p$. From the last two boundary conditions represented by equation (3.20) and (3.21) we see that $V(x) = K - x$ for $0 < x \leq p$.

If we consider all five boundaries we see that

$$V(x) = \begin{cases} x^{\frac{-r}{A}} \frac{A}{r} \left(\frac{K}{1+\frac{A}{r}} \right)^{(1+\frac{r}{A})} & \text{for} \quad x \geq p \\ K - x \qquad \text{for} \quad 0 < x \leq p, \end{cases} \tag{3.35}$$

where $p = \dfrac{K}{1+\frac{A}{r}}$ and $A = \dfrac{\sigma^2}{2}$.

From equation (3.35) we see that at the optimal stopping boundary $x = p$, the function $V(x)$ should be continuous. This is true if

$$x^{\frac{-r}{A}} \frac{A}{r} \left(\frac{K}{1+\frac{A}{r}} \right)^{(1+\frac{r}{A})} = K - x \quad \text{at} \quad x = p. \tag{3.36}$$

To show this we consider the first case where

$$V(x) = K - x. \tag{3.37}$$

At $x = p$, we have

$$\begin{aligned} V &= K - p \\ &= K - \frac{K}{1+\frac{A}{r}} \\ &= K\left(1 - \frac{1}{1+\frac{A}{r}}\right) \\ &= K\left(1 - \frac{r}{r+A}\right) \\ &= \frac{AK}{r+A}. \end{aligned} \tag{3.38}$$

In the second case we have

$$V(x) = x^{\frac{-r}{A}} \frac{A}{r} \left(\frac{K}{1+\frac{A}{r}} \right)^{(1+\frac{r}{A})}. \tag{3.39}$$

At $x = p$, we have

$$
\begin{aligned}
V &= p^{\frac{-r}{A}} \frac{A}{r} \left(\frac{K}{1+\frac{A}{r}} \right)^{(1+\frac{r}{A})} \\
&= \left(\frac{K}{1+\frac{A}{r}} \right)^{\frac{-r}{A}} \frac{A}{r} \left(\frac{K}{1+\frac{A}{r}} \right)^{\left(1+\frac{r}{A}\right)} \\
&= \frac{A}{r} \left(\frac{K}{1+\frac{A}{r}} \right) \\
&= \frac{A}{r} \left(\frac{rK}{r+A} \right) \\
&= \frac{AK}{r+A}.
\end{aligned} \tag{3.40}
$$

Clearly we see that the results in equation (3.40) and equation (3.38) are exactly the same. This proves that the function V is continuous at the point $x = p$. Hence the price of the option at the optimal stopping boundary is given by $\frac{AK}{r+A}$ which is equal to the payoff.

Conclusion

We have seen that American put options can be treated as optimal stopping problems. We require boundary conditions to find the optimal solutions. We saw that perpetual American put is always an optimal stopping problem. This is not the case with the American call options which can be considered as optimal stopping problems only when the stock pays dividend. However, we saw that American put option can be of infinite or finite horizon.

We use related strategies in solving the optimal stopping problems in other securities. But in every option the boundaries conditions changes. It takes several aspects to consider in doing so. As we have already seen in the perpetual American put, we considered both option price, stopping time and strike price to find the solution of optimal stopping problems.

Other derivative securities which can be considered as optimal stopping problems are Asian options, Russian options and finite American put option. We can also consider American call option as the optimal stopping problem if the stock pays dividend. This essay does not include any work on these other options. It could be considered to be the future work.

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