On Cardinal Spline Wavelet Decomposition

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Abstract

Wavelet decomposition techniques have grown over the last two decades into a powerful tool in signal analysis. Similarly, spline functions have enjoyed a sustained high popularity in the approximation of data. The main focus of this essay is to show how cardinal B-splines with uniformly spaced knots, can be used to construct a class of cardinal spline wavelets with particularly efficient decomposition algorithms.

In this essay, after presenting some results on cardinal splines, as given in [dV06], and [Chu92], we proceed to show the explicit construction of a quasi-interpolation operator, as well as and a cardinal spline wavelets based on a local projection operator, as was done in [Mou07]. The cardinal spline wavelet decomposition algorithm in feature detection, will also be demonstrated; not only theoretically, but also graphically by means of a suitably selected numerical example.

List of symbols

$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of real numbers
$\binom{n}{k}$	$\begin{cases} \frac{n!}{k!(n-k)!}, & 0 \leq k \leq n, \\ 0, & k \notin \{0, \dots, n\}, \end{cases}$
χ_I	$\begin{cases} 1 & , \ x \in I \\ 0 & , \ x \notin I \end{cases}$
$M(\mathbb{R})$	Space of real valued functions on $\mathbb{R}$
$M_0(\mathbb{R})$	Set of functions $f \in \mathbb{R}$ such that f is finitely supported i.e there exists an interval $[a, b] \subset \mathbb{R}$ such that $f(x) = 0, \ x \notin [a, b]$
$M(\mathbb{Z})$	Space of real valued bi-finite sequences
$M_0(\mathbb{Z})$	$\{c \in M(\mathbb{R})$ such that c is finitely supported$\}$ i.e there exists $\alpha < \beta$ such that $c_j = 0, \ j \notin \{\alpha, \dots, \beta\}$
π_n	Space of polynomials of degree $\leq n$
$C(\mathbb{R})$	$\{f \in M(\mathbb{R})$ such that f is continuous on $\mathbb{R}\}$
$C^n(\mathbb{R})$	$\{f \in M(\mathbb{R})$ such that $f^{(n)} \in C(\mathbb{R})\}$
$C_0(\mathbb{R})$	$C(\mathbb{R}) \bigcap M_0(\mathbb{R})$
x_+^k	$\begin{cases} x^k & , \ x > 0, \\ 0 & , \ x \leq 0, \end{cases}$
$supp(f)$	$\overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$
$\delta_{j,k}$	the Kronecker symbol, $\delta_{j,k} = \begin{cases} 1 & , \quad j = k, \\ 0 & , \quad j \neq k, \end{cases} \ j, k \in \mathbb{Z}$

δ_j	$\delta_{j,0}, \quad j \in \mathbb{Z}$
$\lceil x \rceil$	the smallest integer larger than $x,\ x \in \mathbb{R}$
$\lfloor x \rfloor$	the largest integer smaller than $x,\ x \in \mathbb{R}$

Contents

1. Introduction

Wavelets are mathematical functions that divide data into different frequency components, as is required in signal analysis.

The goal of this essay is to show how cardinal B-splines i.e minimally supported piecewise polynomials of optimal smoothness, can be used as the basic building block for the construction of a class of spline wavelets with particularly efficient decomposition algorithms.

In Chapter 2, we present some fundamental results on m-th degree cardinal B-splines, which are piecewise polynomials with uniformly spaced breakpoints at the dyadic points $\mathbb{Z}/2^r$, for $r \in \mathbb{Z}$, and from which other cardinal spline functions of the same degree are obtained by linear combination, that is, they form a basis for the space of cardinal splines denoted by $\mathcal{S}_m^r$.

Our wavelet decomposition method starts with a quasi-interpolation operator $\mathcal{Q}_{m,r}$ which maps, for every $r \in \mathbb{Z}$, real-valued functions on $\mathbb{R}$ into $\mathcal{S}_m^r$, in such way that polynomials in π_{m-1} are reproduced. To this end, we present in Chapter 3 an explicit construction of such an operator.

Next, in Chapter 4, we characterise a local linear projector operator sequence projection $\{\mathcal{P}_{m,r} : r \in \mathbb{Z}\}$, with $\mathcal{P}_{m,r} : \mathcal{S}_m^{r+1} \to \mathcal{S}_m^r,\ r \in \mathbb{Z}$, in terms of a Laurent polynomial Λ_m solution to a certain Bezout identity. It is then shown how Λ_m can in fact be explicitly found.

Given such a linear projector operator sequence $\{\mathcal{P}_{m,r} : r \in \mathbb{Z}\}$, we define in Chapter 5 the error space sequence $W_m^r = \{f - \mathcal{P}_{m,r}f : f \in \mathcal{S}_m^{r+1}\}$, and, by solving once again a certain Bezout identity, we show that there exists a finitely supported function $\psi_m \in \mathcal{S}_m^1$, such that, for every $r \in \mathbb{Z}$, the integer shift sequence $\{\psi_m(2^r \cdot -j)\}$ spans the linear space W_m^r. We call, such a function a cardinal B-spline wavelet of order m.

Next, in Chapter 6, we explicitly construct the decomposition algorithm based on the quasi-interpolation operator $\mathcal{Q}_{m,r}$, the projection operator $\{\mathcal{P}_{m,r}\}$, and the wavelet ψ_m which can be used to obtain a decomposition of any given signal f.

Moreover, we prove that our decomposition algorithm possesses the essential property of being able to efficiently detect local irregularities (or non-smooth) behaviour in the signal f.

Finally, we graphically illustrate, by means of a specific example, this feature detection property of our algorithm.

In [CW92], the authors constructed a bi-orthogonal cardinal spline wavelet decomposition procedure with an *infinite* algorithm, so that truncation is necessary in application. In our work, we do not demand that the basis space decomposition be orthogonal as in [CW92]. By doing so, we obtain a *finite* decomposition algorithm, which is therefore an advantage above the bi-orthogonal cardinal spline algorithm of [CW92] (see also [Chu97]).

The work presented in this essay is a special case of the results of [Mou07] (see also [dV06]).
In some of the proofs the special explicit nature of the cardinal B-spline case enables us to obtain some simplifications.

2. Cardinal Splines

For $m \in \mathbb{N},\ r \in \mathbb{Z}$, we consider the linear space $\mathcal{S}_m^r$ of cardinal splines of order m defined by

$$\mathcal{S}_m^r := \mathcal{S}_m(\mathbb{Z}/2^r) = \left\{ f \in M(\mathbb{R}) :\ f|_{[\frac{j}{2^r};\frac{j+1}{2^r})} \in \pi_{m-1},\ \text{for } j \in \mathbb{Z},\ \text{and } f \in C^{m-2}(\mathbb{R}) \right\}. \quad (2.1)$$

We shall write $\mathcal{S}_m$ for $\mathcal{S}_m^0(\mathbb{Z})$.

Observe that the relation

$$\mathcal{S}_m^r \subset \mathcal{S}_m^{r+1},\ r \in \mathbb{Z} \quad (2.2)$$

holds, i.e. $\{\mathcal{S}_m^r\ :\ r \in \mathbb{Z}\}$ is a nested sequence of a linear spaces.

2.1 The cardinal B-splines

In this section,we present some well-known results on cardinal B-splines, the integer shifts of which form a basis for $\mathcal{S}_m$.

Definition 2.1. For $m \in \mathbb{N}$, let N_m denote the mth order cardinal B-spline with respect to the knot sequence $\mathbb{Z}$, as defined recursively by

$$N_m = N_{m-1} * N_1 = \int_0^1 N_{m-1}(\cdot - t)dt,\ m = 2, 3, \dots, \quad (2.3)$$

with $N_1 = \chi_{[0,1)}$.

The following properties of N_m are proved in [[Chu92],Chapter4].

Theorem 2.2. *For $m \in \mathbb{N}$, and $x \in \mathbb{R}$, the cardinal B-spline have the following properties:*

$$(i)\ N_m(\cdot - j) \in \mathcal{S}_m,\ j \in \mathbb{Z}; \quad (2.4)$$

$$(ii)\ N_m = \frac{1}{(m-1)!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(\cdot - j)_+^{m-1}; \quad (2.5)$$

$$(iii)\ N_m(x) > 0,\ x \in (0, m); \quad (2.6)$$

$$(iv)\ N_m = \frac{1}{m-1}[(\cdot)N_{m-1} + (m - \cdot)N_{m-1}(\cdot - 1)],\ m \geq 2; \quad (2.7)$$

$$(v)\ N_m(m - \cdot) = N_m,\ m \geq 2; \quad (2.8)$$

$$(vi)\ supp(N_m) = [0, m]. \quad (2.9)$$

Moreover, as proved in [Mic95], the sequence $\{N_m(2^r \cdot -j) : j \in \mathbb{Z}\}$ is a basis for $\mathcal{S}_m^r$ in the sense that, for $f \in \mathcal{S}_m^r$, there exists a unique sequence $c \in M(\mathbb{Z})$ such that

$$f = \sum_j c_j N_m(2^r \cdot -j). \quad (2.10)$$

2.2 The refinement equation

The following theorem states a property of cardinal B-spline which is of fundamental importance in this essay. Our proof is from [dV06]

Theorem 2.3. *For $m \in \mathbb{N}$, the identity*

$$N_m = \frac{1}{2^{m-1}} \sum_{j=0}^{m} \binom{m}{j} N_m(2\cdot -j) \tag{2.11}$$

holds.

Proof. First, note from (2.4), (2.2), and (2.10), with $r = 1$, that there exists a sequence $a^m \in M(\mathbb{Z})$ such that

$$\begin{aligned}
\sum_j a_j^m N_m(2\cdot -j) = N_m &= \int_0^1 N_{m-1}(\cdot - t)dt \qquad \text{(from (2.3))} \qquad (2.12)\\
&= \int_0^1 \sum_j a_j^{m-1} N_{m-1}(2\cdot -2t - j)dt \\
&= \sum_j a_j^{m-1} \int_0^1 N_{m-1}(2\cdot -2t - j)dt \\
&= \sum_j a_j^{m-1} \left[\int_0^{\frac{1}{2}} N_{m-1}(2\cdot -2t - j)dt + \int_{\frac{1}{2}}^1 N_{m-1}(2\cdot -2t - j)dt\right] \\
&= \sum_j \frac{1}{2} a_j^{m-1} \left[\int_0^1 N_{m-1}(2\cdot -t - j)dt + \int_0^1 N_{m-1}(2\cdot -t - j - 1)dt\right] \\
&= \sum_j \frac{1}{2} a_j^{m-1} \left[N_m(2\cdot -j) + N_m(2\cdot -j - 1)\right] \\
&= \sum_j \frac{1}{2} a_j^{m-1} N_m(2\cdot -j) + \sum_j \frac{1}{2} a_j^{m-1} N_m(2\cdot -j - 1) \\
&= \sum_j \frac{1}{2} \left[a_j^{m-1} N_m(2\cdot -j) + a_{j-1}^{m-1} N_m(2\cdot -j)\right] \\
&= \sum_j \left[\frac{1}{2}(a_j^{m-1} + a_{j-1}^{m-1})\right] N_m(2\cdot -j).
\end{aligned}$$

Hence

$$\sum_j \left[a_j^m - \frac{1}{2}(a_j^{m-1} + a_{j-1}^{m-1})\right] N_m(2\cdot -j) = 0. \tag{2.13}$$

But $0 \in \mathcal{S}_m^1$, so that we can use the fact that the sequence that $\{N_m(2\cdot -j) : j \in Z\}$ is a basis for $\in \mathcal{S}_m^1$ to deduce that

$$a_j^m = \frac{1}{2}(a_j^{m-1} + a_{j-1}^{m-1}), \ j \in \mathbb{Z}, m \in \mathbb{N}. \tag{2.14}$$

We proceed to show by induction that

$$a_j^m = \frac{1}{2^{m-1}}\binom{m}{j}, \; j \in \mathbb{Z}, \; m \in \mathbb{N}. \tag{2.15}$$

Since $N_1 = \chi_{[0,1)}$, thus $N_1(x) = \begin{cases} 1 & , \; x \in \; [0,1), \\ 0 & , \; x \in \mathbb{R} \setminus [0,1), \end{cases}$ we have that

$N_1 = N_1(2\cdot) + N_1(2 \cdot -1)$.

It follows from (2.12) that (2.15) holds for $m = 1$.
We next assume that (2.15) holds for any $m \in \mathbb{N}$ and show that then (2.15) is also true for $m + 1$.
From (2.14) and the induction hypothesis, we have for $j \in \mathbb{Z}$ that

$$\begin{aligned} a_j^{m+1} = \frac{1}{2}(a_j^m + a_{j-1}^m) = \frac{1}{2}\left[\frac{1}{2^{m-1}}\binom{m}{j} + \frac{1}{2^{m-1}}\binom{m}{j-1}\right] \; &= \; \frac{1}{2^m}\left[\binom{m}{j} + \binom{m}{j-1}\right] \\ &= \; \frac{1}{2^m}\binom{m+1}{j}, \end{aligned}$$

so that (2.14) also holds with m replaced by $m + 1$, and thereby concluding our proof. □

According to its property (2.11), N_m is said to be a *refinable function* and the sequence $a^m \in M_0(\mathbb{Z})$ defined by (2.15) is called *the refinement mask*. We say that (a^m, N_m) is *the cardinal B-spline refinement pair* of order m.

Starting with $N_1 = \chi_{[0,1)}$,we now use the recursive porperty (2.7) of N_m to produce the following cardinal B-spline graphs.

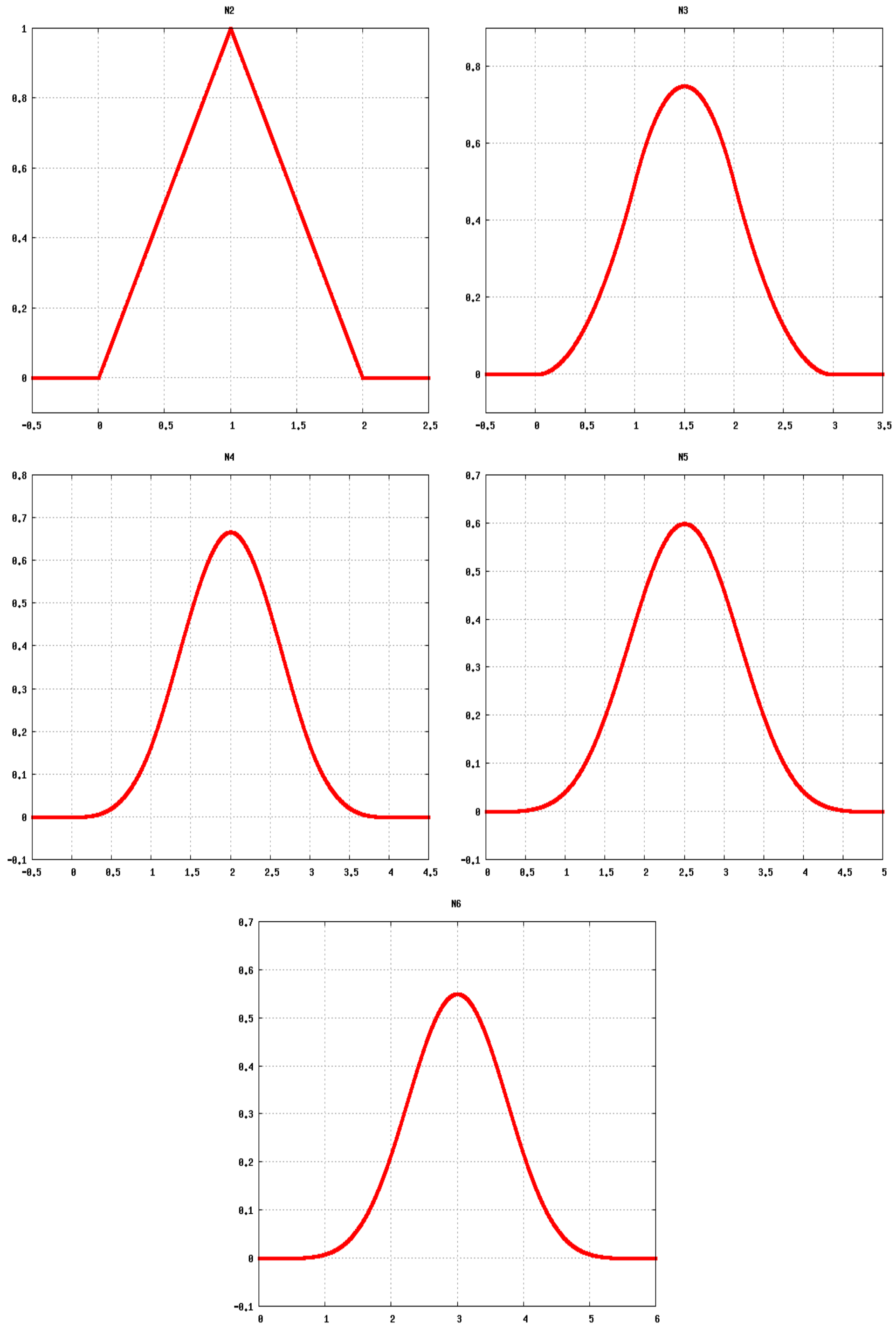

Figure 2.1: The cardinal B-splines for m=2,3,4,5,6.

3. The Quasi-interpolation Operator

For an efficient wavelet decomposition process, a good approximation is required to map a given signal $f \in M(\mathbb{R})$ into the space $\mathcal{S}_m^r$ for an appropriate value of r.
For this purpose, we define, for $m \geq 2$, an operator $\mathcal{Q}_{m,r} : M(\mathbb{R}) \to \mathcal{S}_m^r$, such that the polynomial reproduction property

$$\mathcal{Q}_{m,r}p = p, \ p \in \pi_{m-1}, \ r \in \mathbb{Z} \tag{3.1}$$

is satisfied.
Such an operator is referred to in the literature as a *quasi-interpolation operator*. To this end, we first therefore seek a function $v \ \in \mathcal{S}_m$ with finite support such that

$$\sum_j p(j+\tau)v(\cdot - j) = p, \ p \in \pi_{m-1}, \ \tau \in \mathbb{R}, \tag{3.2}$$

where

$$v = \sum_j v_j N_m(\cdot - j), \tag{3.3}$$

and $\{v_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$.

First of all, let us introduce an identity which will prove to be useful for the construction of $\mathcal{Q}_{m,r}$.

3.1 The Marsden identity

According to the definition (2.1), we have the inclusion $\pi_{m-1} \subset \mathcal{S}_m^r, \ r \in \mathbb{Z}$. Hence, since also $\{N_m(\cdot - j) : \ j \in \mathbb{Z}\}$ is a basis for $\mathcal{S}_m$, we know that, if $p \in \pi_{m-1}$, then there exits a unique sequence $c \in M(\mathbb{Z})$ such that $p = \sum\limits_j c_j N_m(\cdot - j)$.

The following so called *Marsden's identity* gives an explicit formulation of the sequence c for the case when p is the specific polynomial in (3.4) below.

Our proof is from [dV06].

Theorem 3.1. *For $m \geq 2$, we have*

$$(\cdot + t)^{m-1} = \sum_j Q_m(j+t)N_m(\cdot - j), \ t \in \mathbb{R}, \tag{3.4}$$

where Q_m is the polynomial of degree $(m-1)$ defined by

$$Q_m = \prod_{k=1}^{m-1}(\cdot + k). \tag{3.5}$$

Proof. Our proof is by induction. Using (2.5) we obtain the formula

$$N_2(x) = \begin{cases} x & , \ x \in \ [0,1), \\ 2-x & , \ x \in \ [1,2), \\ 0 & , \ x \in \mathbb{R} \setminus [0,2), \end{cases}$$

from which we see that

$$N_2(j+1) = \delta_j, \ j \in \mathbb{Z}. \tag{3.6}$$

For a fixed $t \in \mathbb{R}$, we now define polynomial $p \in \pi_1$ by

$$p(x) = x + t, \quad x \in \mathbb{R}.$$

Since $\pi_1 \subset \mathcal{S}_2$, there exists a unique sequence $c \in M(\mathbb{Z})$ such that

$$p = \sum_j c_j N_2(\cdot - j),$$

and thus

$$p(k+1) = \sum_j c_j N_2(k+1-j) = \sum_j c_j \delta_{k,j} = c_k, \ k \in \mathbb{Z},$$

from (3.6). Hence

$$p = \sum_j p(j+1) N_2(\cdot - j),$$

and thus

$$\cdot + t = \sum_j (j+1+t) N_2(\cdot - j),$$

which shows that the theorem holds for $m = 2$.
Suppose now that the theorem holds for a fixed $m \in \{2, 3, \dots, \}$. Our proof will be complete if we can show that the theorem then also holds with m replaced by $m+1$.
Using (3.5), we obtain, for $x \in \mathbb{R}$,

$$\begin{aligned} \sum_j Q_{m+1}(j+t) N_{m+1}(x-j) &= \frac{1}{m} \sum_j Q_{m+1}(j+t) \left[(x-j) N_m(x-j) \right. \\ &\quad \left. + (m+1+j-x) N_m(x-j-1) \right] \\ &= \frac{1}{m} \left[\sum_j Q_{m+1}(j+t)(x-j) N_m(x-j) \right. \\ &\quad \left. + \sum_j Q_{m+1}(j+t)(m+1+j-x) N_m(x-j-1) \right] \\ &= \frac{1}{m} \left[\sum_j Q_{m+1}(j+t)(x-j) N_m(x-j) \right. \\ &\quad \left. + \sum_j Q_{m+1}(j+t-1)(m+j-x) N_m(x-j) \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{m}\sum_j \left[Q_{m+1}(j+t)(x-j) + Q_{m+1}(j+t-1)(m+j-x)\right] N_m(x-j) \\
&= \frac{1}{m}\sum_j [Q_m(j+t)(j+t+m)(x-j) \\
&\qquad + Q_m(j+t-1)(j+t-1+m)(m+j-x)]N_m(x-j)] \\
&= \frac{1}{m}\sum_j Q_m(j+t)[(j+t+m)(x-j) + (j+t)(m+j-x)]N_m(x-j) \\
&= \frac{1}{m}\sum_j m(x+t)Q_m(j+t)N_m(x-j) \\
&= (x+t)\sum_j Q_m(j+t)N_m(x-j) \\
&= (x+t)(x+t)^{m-1} = (x+t)^m,
\end{aligned}
$$

from the inductive hypothesis, and thereby concluding our proof. □

Corollary 3.2. *For $n \geq 2$ we have*

$$
x^n = \frac{n!}{(m-1)!}\sum_j Q_m^{(m-1-n)}(j)N_m(x-j),\ x \in \mathbb{R},\ n = 0,1,\dots,m-1. \tag{3.7}
$$

Proof. Setting $t = 0$ in the n-th derivative of (3.4) with respect to t yields (3.7). □

3.2 The construction of $\mathcal{Q}_{m,r}$

To explicitly construct the quasi-interpolation operator $\mathcal{Q}_{m,r}$, we first observe that the condition (3.2) has the equivalent formulation.

$$
x^l = \sum_j (j+\tau)^l v(x-j),\ x \in \mathbb{R},\ l = 0,1,\dots,m-1. \tag{3.8}
$$

Now substitute (3.3) into (3.8) to obtain, for $l = \{0,1,\dots,m-1\}$ and $x \in \mathbb{R}$,

$$
\begin{aligned}
\sum_j (j+\tau)^l v(x-j) &= \sum_j (j+\tau)^l \sum_k v_k N_m(x-j-k) \\
&= \sum_j (j+\tau)^l \sum_k v_{k-j} N_m(x-k) \\
&= \sum_k \left[\sum_j (j+\tau)^l v_{k-j}\right] N_m(x-k).
\end{aligned} \tag{3.9}
$$

It follows from (3.9) and (3.7) that the condition (3.8) is equivalent to

$$\sum_k \left[\sum_j (j+\tau)^l v_{k-j} - \frac{l!}{(m-1)!} Q_m^{(m-1-l)}(k)\right] N_m(\cdot - k) = 0, \quad l = 0,1,\cdots m-1. \tag{3.10}$$

It follows that a sequence $\{v_k : k \in \mathbb{Z}\}$ satisfies (3.10) if and only if

$$\sum_j (j+\tau)^l v_{k-j} = \frac{l!}{(m-1)!} Q_m^{(m-1-l)}(k), \; k \in \mathbb{Z}, \; l = 0,1,\cdots m-1. \tag{3.11}$$

A necessary condition for (3.11) to hold is obtained by setting $k=0$ in (3.11) to yield

$$\sum_j (j+\tau)^l v_{-j} = \frac{l!}{(m-1)!} Q_m^{(m-1-l)}(0), \tag{3.12}$$

or, equivalently,

$$\sum_j (j-\tau)^l v_j = \frac{(-1)^l l!}{(m-1)!} Q_m^{(m-1-l)}(0), \; l = 0,1,\ldots,m-1. \tag{3.13}$$

To get a minimally supported solution $\{v_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ of (3.13), we set

$$v_j = 0, \; j \notin \{0,1,2,\cdots,m-1\}, \tag{3.14}$$

so that (3.11) becomes the $m \times m$ linear system

$$\sum_{j=0}^{m-1} (j-\tau)^l v_j = \frac{(-1)^l l!}{(m-1)!} Q_m^{(m-1-l)}(0), \quad l = 0,1,\ldots,m-1. \tag{3.15}$$

By defining

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_0 & x_1 & x_2 & \cdots & x_{m-1} \\ x_0^2 & x_1^2 & x_2^2 & \cdots & x_{m-1}^2 \\ \vdots & & & & \vdots \\ x_0^{m-1} & x_1^{m-1} & x_2^{m-1} & \cdots & x_{m-1}^{m-1} \end{pmatrix}, \tag{3.16}$$

where

$$x_j = j - \tau, \; j = 0,1,\ldots,m-1 \;, \tag{3.17}$$

$$\underline{\mathsf{v}} = [v_0, v_1, \ldots, v_{m-1}]^T, \tag{3.18}$$

and

$$\underline{\mathsf{b}} = [b_0, b_1, \ldots, b_{m-1}]^T, \text{ with } b_l = \frac{(-1)^l l!}{(m-1)!} Q_m^{(m-1-l)}(0), \tag{3.19}$$

we find the matrix-vector formulation

$$A\underline{\mathsf{v}} = \underline{\mathsf{b}} \tag{3.20}$$

of the $m \times m$ linear system (3.15).
We shall rely on the following result, for the proof of which we refer to [Mou07] (see also [dV06])

Proposition 3.3. *For* $m \in \mathbb{N}$*, suppose* $\{x_j : j = 0, 1, \ldots, m-1\}$ *are* m *distinct points in* $\mathbb{R}$*, and suppose* $\{b_l : l = 0, 1, \ldots, m-1\} \subset \mathbb{R}$*. Then the* $m \times m$ *linear system*

$$\sum_{j=0}^{m-1} x_j^l v_j = b_l, \; l = 0, 1, \ldots, m-1, \tag{3.21}$$

has the unique solution

$$v_j = \sum_{l=0}^{m-1} \frac{1}{l!} L_j^{(l)}(0) b_l, \; j = 0, 1, \ldots, m-1, \tag{3.22}$$

with

$$L_j(x) = \prod_{j \neq k=0}^{m-1} \frac{x - x_k}{x_j - x_k}, \; x \in \mathbb{R}, \quad j = 0, 1, \ldots, m-1 \tag{3.23}$$

denoting the fundamental Lagrange polynomials of degree $(m-1)$.

We proceed to show that, if the sequence $\{v_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ is defined by (3.22), (3.17), (3.19), and (3.14), then the condition (3.11) is also satisfied.
To this end, we use (3.13) to obtain, for $l \in \{0, 1, \ldots, m-1\}$ and $k \in \mathbb{Z}$,

$$\begin{aligned}
\sum_j (j+\tau)^l v_{k-j} &= \sum_j [k - (j-\tau)]^l v_j \\
&= \sum_j \sum_{n=0}^{l} \binom{l}{n} k^n (-1)^{l-n} (j-\tau)^{l-n} v_j \\
&= \sum_{n=0}^{l} \binom{l}{n} k^n (-1)^{l-n} \sum_j (j-\tau)^{l-n} v_j \\
&= \sum_{n=0}^{l} \binom{l}{n} k^n (-1)^{l-n} \frac{(-1)^{l-n}(l-n)!}{(m-1)!} Q_m^{(m-1-l+n)}(0) \\
&= \sum_{n=0}^{l} \frac{l! k^n}{n!(l-n)!} \frac{(l-n)!}{(m-1)!} Q_m^{(m-1-l+n)}(0) \\
&= \frac{l!}{(m-1)!} \sum_{n=0}^{l} \frac{Q_m^{(m-1-l+n)}(0)}{n!} k^n \\
&= \frac{l!}{(m-1)!} Q_m^{(m-1-l)}(k),
\end{aligned}$$

since $Q_m^{(m-1-l)}$ is a polynomial of degree l, and thereby showing that (3.11) is indeed satisfied.
Suppose now that the Lagrange fundamental polynomials are given by

$$L_j(x) = \sum_{k=0}^{m-1} h_{j,k} x^k, \; x \in \mathbb{R}, \quad j = 0, 1, \ldots, m-1 \tag{3.24}$$

so that

$$L_j^{(l)}(0) = l! h_{j,l}, \tag{3.25}$$

and

$$Q_m(x) = \sum_{k=0}^{m-1} q_k x^k, \; x \in \mathbb{R}, \; j = 0, 1, \ldots, m-1,$$

so that

$$Q_m^{(m-1-l)}(0) = (m-1-l)! q_{m-1-l}, \; l = 0, 1, \ldots, m-1. \tag{3.26}$$

Substituting (3.25) and (3.26) into (3.22) then yields the formula

$$v_j = \sum_{l=0}^{m-1} \frac{(-1)^l}{\binom{m-1}{l}} h_{j,l} q_{m-1-l}, \; j = 0, 1, \ldots, m-1. \tag{3.27}$$

We have therefore now established the first part of the following result.

Theorem 3.4. *The finitely supported function $v \in \mathcal{S}_m$ defined by (3.3), where the sequence $\{v_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ is, for $\tau \in \mathbb{R}$, given by (3.14) and (3.27), satisfies the polynomial reproduction property (3.2).*
Moreover

$$v(x) = 0, \quad x \notin (0, 2m-1). \tag{3.28}$$

Proof. It remains to prove the finite support property (3.28). Indeed, combining (3.3),(3.14) and (2.9), we conclude that (3.28) does indeed hold. □

Next we construct the operator sequence $\{\mathcal{Q}_{m,r} : r \in \mathbb{Z}\}$ from the function v of Theorem 3.4 such that the polynomial reproduction property (3.1) is satisfied.
For $f \in M(\mathbb{R})$ and $r \in \mathbb{Z}$, using (3.3), we have

$$\sum_j f\left(\frac{j+\tau}{2^r}\right) v(2^r \cdot -j) = \sum_j f\left(\frac{j+\tau}{2^r}\right) \sum_k v_k N_m(2^r \cdot -j-k) \tag{3.29}$$

$$= \sum_j f\left(\frac{j+\tau}{2^r}\right) \sum_k v_{k-j} N_m(2^r \cdot -k) \tag{3.30}$$

$$= \sum_k \left[\sum_j v_{k-j} f\left(\frac{j+\tau}{2^r}\right)\right] N_m(2^r \cdot -k). \tag{3.31}$$

Hence, if we choose $f \in \pi_{m-1}$ so that $g = f(\frac{\cdot}{2^r})$ i.e $f = g(2^r \cdot)$, then also $g \in \pi_{m-1}$, then (3.31) gives

$$\sum_k \left[\sum_j v_{k-j} f\left(\frac{j+\tau}{2^r}\right)\right] N_m(2^r \cdot -k) = \sum_j g(j+\tau) v(2^r \cdot -j)$$
$$= g(2^r \cdot) = f,$$

by Theorem 3.4.
The following result therefore holds.

Theorem 3.5. *For $\tau \in \mathbb{R}$, the operator sequence $\{\mathcal{Q}_{m,r} : r \in \mathbb{Z}\}$, where $\mathcal{Q}_{m,r} : M(\mathbb{R}) \to \mathcal{S}_m^r$, $\quad r \in \mathbb{Z}$, as defined by*

$$\mathcal{Q}_{m,r} f = \sum_k \left[\sum_j v_{k-j} f\left(\frac{j+\tau}{2^r}\right)\right] N_m(2^r \cdot -k),\ r \in \mathbb{Z}, f \in M(\mathbb{R}), \tag{3.32}$$

with the sequence $\{v_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ defined as in Theorem 3.4 satisfies the polynomial reproduction property (3.1).

Notice from (3.31) that the quasi-interpolation operator $\mathcal{Q}_{m,r}$, as given in (3.32), has the equivalent formulation

$$\mathcal{Q}_{m,r} f = \sum_j f\left(\frac{j+\tau}{2^r}\right) v(2^r \cdot -j),\ r \in \mathbb{Z},\ f \in M(\mathbb{R}), \tag{3.33}$$

with the function $v \in \mathcal{S}_m$ as defined in Theorem 3.4.
Based on (3.28) and (3.33), it seems natural to choose the real number τ in the definition (3.32) of the operator $\mathcal{Q}_{m,r}$ as

$$\tau = \tau_0 = m - \frac{1}{2}, \tag{3.34}$$

in which case (3.28) becomes

$$v(x) = 0,\ x \notin (0, 2\tau_0). \tag{3.35}$$

If a signal $f \in M(\mathbb{R})$ equals a polynomial $p \in \pi_{m-1}$ on a bounded interval $[\alpha, \beta]$, the polynomial reproduction property of $\mathcal{Q}_{m,r}$ yields the following result.

Theorem 3.6. *Suppose that, in Theorem 3.5, we choose $\tau = \tau_0$, as in (3.34), and suppose $f \in M(\mathbb{R})$ is such that there exist a bounded interval $[\alpha, \beta]$ and a polynomial $p \in \pi_{m-1}$ for which*

$$f(x) = p(x),\ x \in [\alpha, \beta]. \tag{3.36}$$

Then

$$(\mathcal{Q}_{m,r} f)(x) = p(x),\ x \in \left[\alpha + \frac{\tau_0}{2^r}, \beta - \frac{\tau_0}{2^r}\right], \tag{3.37}$$

for every integer r such that

$$r > 1 + \log_2 \frac{\tau_0}{\beta - \alpha}. \tag{3.38}$$

Proof. Let r denote an integer such that the inequality (3.38) is satisfied. It follows from (3.33) and (3.35) that

$$(\mathcal{Q}_{m,r} f)(x) = \sum_{j=\lceil 2^r x - 2\tau_0 \rceil}^{\lfloor 2^r x \rfloor} f\left(\frac{j+\tau}{2^r}\right) v(2^r x - j),\ x \in \mathbb{R},$$

and thus

$$(\mathcal{Q}_{m,r} f)(x) = \sum_{j=\lceil 2^r \alpha - \tau_0 \rceil}^{\lfloor 2^r \beta - \tau_0 \rfloor} f\left(\frac{j+\tau}{2^r}\right) v(2^r x - j),\ x \in \left[\alpha + \frac{\tau_0}{2^r}, \beta - \frac{\tau_0}{2^r}\right]. \tag{3.39}$$

Observe that

$$\alpha \leq \frac{j+\tau_0}{2^r} \leq \beta \text{ for } j = \lceil 2^r\alpha - \tau_0 \rceil, \ldots, \lfloor 2^r\beta - \tau_0 \rfloor. \tag{3.40}$$

The desired result (3.37) is then consequence of (3.39), (3.36), (3.40) and (3.1). □

Using (3.3), and (3.27), we graph below the function $v \in \mathcal{S}_m$ for $m = 2, 3, 4, 5, 6$.

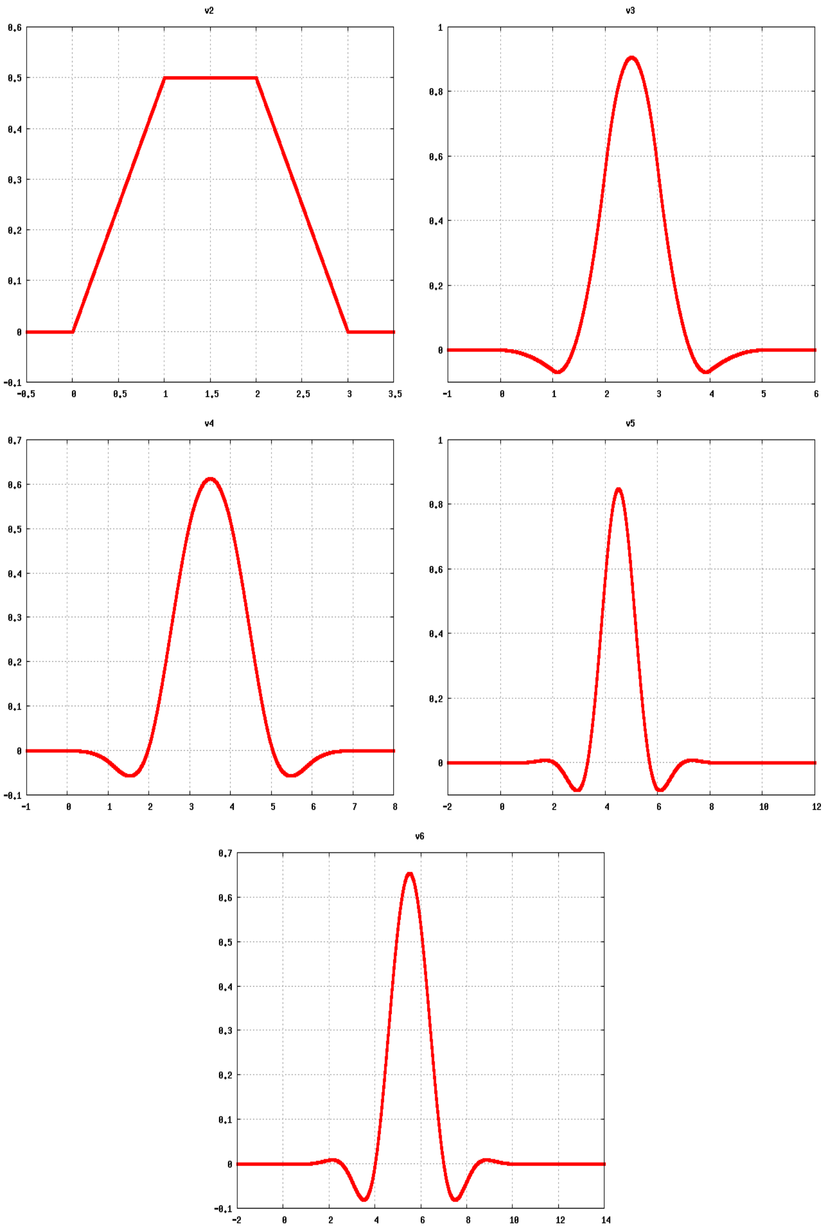

Figure 3.1: The function v for m=2,3,4,5,6.

4. The Projection Operator

For a given B-spline refinement pair (a^m, N_m), our wavelet construction method will depend on the existence of a local linear projection operator sequence $\{\mathcal{P}_{m,r} : r \in \mathbb{Z}\}$ with

$$\mathcal{P}_{m,r} : \mathcal{S}_m^{r+1} \to \mathcal{S}_m^r, \ r \in \mathbb{Z}. \tag{4.1}$$

We shall show that such a projection operator can be characterised by the solution of a Bezout identity.

4.1 The fundamental Bezout identity

For a Laurent polynomial P defined by

$$P(z) = \sum_j p_j z^j, \ z \in \mathbb{C} \setminus \{0\},$$

we define the *even part* $P^{(e)}$ and the *odd part* $P^{(o)}$ respectively by

$$P^{(e)}(z) = \sum_j p_{2j} z^{2j} \text{ and } P^{(o)}(z) = \sum_j p_{2j+1} z^{2j+1}, \ z \in \mathbb{C} \setminus \{0\}. \tag{4.2}$$

Notice that, for $z \in \mathbb{C} \setminus \{0\}$,

$$\left.\begin{array}{rcl} P(z) & = & P^{(e)}(z) + P^{(o)}(z) \\ P(-z) & = & P^{(e)}(z) - P^{(o)}(z), \end{array}\right\} z \in \mathbb{C} \setminus \{0\}, \tag{4.3}$$

and thus,

$$\left.\begin{array}{rcl} P^{(e)}(z) & = & \dfrac{P(z) + P(-z)}{2} \\ P^{(o)}(z) & = & \dfrac{P(z) - P(-z)}{2} \end{array}\right\} z \in \mathbb{C} \setminus \{0\}. \tag{4.4}$$

The polynomial A_m defined by

$$A_m(z) = \sum_{j=0}^{m} a_j^m z^j = \frac{1}{2^{m-1}}(1+z)^m, \ z \in \mathbb{C}, \tag{4.5}$$

with the sequence a^m defined by (2.15), is called the *cardinal B-spline refinement mask symbol* of order m.

Theorem 4.1. *For a sequence $\{\lambda_{m,j} : \ j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$, the local linear operator sequence $\{\mathcal{P}_{m,r} : \ r \in \mathbb{Z}\}$, where $\mathcal{P}_{m,r} : \mathcal{S}_m^{r+1} \to \mathcal{S}_m^r$, as defined by*

$$\mathcal{P}_{m,r} f = \sum_j \left[\sum_k \lambda_{m,2j-k} c_k \right] N_m(2^r \cdot -j), \text{ for } f = \sum_j c_j N_m(2^{r+1} \cdot -j), \tag{4.6}$$

satisfies the reproduction property

$$\mathcal{P}_{m,r}f = f, \quad f \in \mathcal{S}_m^r, \quad r \in \mathbb{Z}, \tag{4.7}$$

according to which $\mathcal{P}_{m,r}$ *is a projection on* $\mathcal{S}_m^r$ *for every* $r \in \mathbb{Z}$*, if and only if the Laurent polynomial* Λ_m *given by*

$$\Lambda_m(z) = \sum_j \lambda_{m,j} z^j, \quad z \in \mathbb{C} \setminus \{0\}, \tag{4.8}$$

satisfies the Bezout identity

$$(1+z)^m \Lambda_m(z) + (1-z)^m \Lambda_m(-z) = 2^m, \; z \in \mathbb{C} \setminus \{0\}. \tag{4.9}$$

Proof. Let $r \in \mathbb{Z}$ be fixed, and suppose $f \in \mathcal{S}_m^r$, so that there exists a sequence $\tilde{c} \in M(\mathbb{Z})$ such that $f = \sum\limits_j \tilde{c}_j N_m(2^r \cdot - j)$. Using (2.11), we deduce that

$$\begin{aligned} f = \sum_j \tilde{c}_j \sum_k a_k^m N_m(2^{r+1} \cdot -2j - k) &= \sum_j \tilde{c}_j \sum_k a_{k-2j}^m N_m(2^{r+1} \cdot - k) \\ &= \sum_k \left[\sum_j a_{k-2j}^m \tilde{c}_j \right] N_m(2^{r+1} \cdot - k). \end{aligned} \tag{4.10}$$

For a sequence $\{\lambda_{m,j} : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$, it follows from (4.6) and (4.10) that

$$\begin{aligned} \mathcal{P}_{m,r}f &= \sum_j \left[\sum_k \lambda_{m,2j-k} \left[\sum_l a_{k-2l}^m \tilde{c}_l \right] \right] N_m(2^r \cdot - k) \\ &= \sum_j \left[\sum_l \left[\sum_k a_{k-2l}^m \lambda_{m,2j-k} \right] \tilde{c}_l \right] N_m(2^r \cdot - k). \end{aligned} \tag{4.11}$$

Hence $\mathcal{P}_{m,r}f = f$ if and only if

$$\sum_j \left\{ \tilde{c}_j - \sum_l \left[\sum_k a_{k-2l}^m \lambda_{m,2j-k} \right] \tilde{c}_l \right\} N_m(2^r \cdot - k) = 0,$$

or, equivalently,

$$\sum_j \left\{ \sum_l \left[\delta_{j,l} - \sum_k a_{k-2l}^m \lambda_{m,2j-k} \right] \tilde{c}_l \right\} N_m(2^r \cdot - k) = 0. \tag{4.12}$$

Hence the reproduction property (4.7) is satisfied if and only if

$$\delta_{j,l} = \sum_k a_{k-2l}^m \lambda_{m,2j-k}, \quad j, l \in \mathbb{Z}. \tag{4.13}$$

The proof will be complete if we can show that $\{\lambda_{m,j} : j \in \mathbb{Z}\}$ is a sequence in $M_0(\mathbb{Z})$ satisfying (4.13) if and only if the corresponding Laurent polynomial Λ_m satisfies the Bezout identity (4.9).

Using (4.8), (4.2) and (4.4), we find, for a sequence $\{\lambda_{m,j} : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$, and for $j \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \{0\}$, that

$$
\begin{aligned}
\sum_l \sum_k a^m_{k-2l} \lambda_{m,2j-k} z^{2l} &= \sum_l \sum_k a^m_{2k-2l} \lambda_{m,2j-2k} z^{2l-2k} z^{2k} \\
&\quad + \sum_l \sum_k a^m_{2k-2l+1} \lambda_{m,2j-2k-1} z^{2l-2k-1} z^{2k+1} \\
&= \sum_k \lambda_{m,2j-2k} \left[\sum_l a^m_{2k-2l} z^{2l-2k} \right] z^{2k} \\
&\quad + \sum_k \lambda_{m,2j-2k-1} \left[\sum_l a^m_{2k-2l+1} z^{2l-2k-1} \right] z^{2k+1} \\
&= \sum_k \lambda_{m,2j-2k} \left[\sum_l a^m_{2l} z^{-2l} \right] z^{2k} \\
&\quad + \sum_k \lambda_{m,2j-2k-1} \left[\sum_l a^m_{2l+1} z^{-(2l+1)} \right] z^{2k+1} \\
&= A^{(e)}_m(z^{-1}) \left[\sum_k \lambda_{m,2j-2k} z^{2k-2j} \right] z^{2j} \\
&\quad + A^{(o)}_m(z^{-1}) \left[\sum_k \lambda_{m,2j-2k-1} z^{2k-2j+1} \right] z^{2j} \\
&= A^{(e)}_m(z^{-1}) \left[\sum_k \lambda_{m,2k} z^{-2k} \right] z^{2j} \\
&\quad + A^{(o)}_m(z^{-1}) \left[\sum_k \lambda_{m,2k+1} z^{-(2k+1)} \right] z^{2j} \\
&= z^{2j} \left[A^{(e)}_m(z^{-1}) \Lambda^{(e)}_m(z^{-1}) + A^{(o)}_m(z^{-1}) \Lambda^{(o)}_m(z^{-1}) \right] \\
&= z^{2j} \left[\frac{A_m(z^{-1}) + A_m(-z^{-1})}{2} \, \frac{\Lambda_m(z^{-1}) + \Lambda_m(-z^{-1})}{2} \right. \\
&\quad \left. + \frac{A_m(z^{-1}) - A_m(-z^{-1})}{2} \, \frac{\Lambda_m(z^{-1}) - \Lambda_m(-z^{-1})}{2} \right] \\
&= \frac{1}{2} z^{2j} \left[A_m(z^{-1}) \Lambda_m(z^{-1}) + A_m(-z^{-1}) \Lambda_m(-z^{-1}) \right], \qquad (4.14)
\end{aligned}
$$

whereas

$$
\sum_l \delta_{j,l} z^{2l} = z^{2j}, \; j \in \mathbb{Z}, \; z \in \mathbb{C}. \qquad (4.15)
$$

It follows from (4.15) and (4.14) that, for $j \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \{0\}$, we have

$$\sum_l \left[\sum_k a^m_{k-2l} \lambda_{m,2j-k} - \delta_{j,l} \right] z^{2l} = z^{2j} \left[\frac{A_m(z^{-1})\Lambda_m(z^{-1}) + A_m(-z^{-1})\Lambda_m(-z^{-1})}{2} - 1 \right], \tag{4.16}$$

from which we observe that a sequence $\{\lambda_{m,j} : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ satisfies (4.13) if and only if the corresponding Laurent polynomial Λ_m defined by (4.8) satisfies the Bezout identity

$$A_m(z^{-1})\Lambda_m(z^{-1}) + A_m(-z^{-1})\Lambda_m(-z^{-1}) = 2, \; z \in \mathbb{C} \setminus \{0\},$$

which is equivalent to the Bezout identity

$$A_m(z)\Lambda_m(z) + A_m(-z)\Lambda_m(-z) = 2, \; z \in \mathbb{C} \setminus \{0\}, \tag{4.17}$$

Our proof is then completed by substituting (4.5) into (4.17). □

We proceed to explicitly construct a Laurent polynomial Λ_m satisfying the Bezout identity (4.9).

Theorem 4.2. *Let the function sequence $\{S_m : m = 2, 3, \dots\}$ be defined by*

$$\left.\begin{aligned} S_2(z) &= \frac{1}{2} \\ S_{2k+1}(z) &= \frac{2S_{2k}(z) - 2^{1-2k} S_{2k}(-1)(1-z)^{2k}}{1+z} \\ S_{2k+2}(z) &= \frac{2z^2 S_{2k+1}(z) - 2^{-2k} S_{2k+1}(-1)(1-z)^{2k+1}}{1+z} \end{aligned}\right\} z \in \mathbb{C}, k \in \mathbb{N}. \tag{4.18}$$

Then $S_m \in \pi_{m-2}, \; m = 2, 3, \dots$, and the Laurent polynomial Λ_m defined by

$$\Lambda_m(z) = 2z^{-\mu} S_m(z), \; z \in \mathbb{C} \setminus \{0\}, \tag{4.19}$$

where

$$\mu = \begin{cases} m-1 & \text{if } m \text{ is even}, \\ m-2 & \text{if } m \text{ is odd}, \end{cases} \tag{4.20}$$

satisfies the Bezout identity (4.9).

Proof. Since the numerators in the right-hand-sides of the second and third line of (4.18) both equals to zero if we set $z = -1$, we see that S_m is indeed a polynomial for $m = 2, 3, \dots$. The fact that, moreover, $S_m \in \pi_{m-2}, \; m = 2, 3, \dots$, follows inductively from (4.18).
According to (4.9) and (4.19), and noting also from (4.20) that μ is an odd integer, we see that the result of the theorem will follow if we can prove that the polynomial S_m satisfies the Bezout identity

$$(1+z)^m S_m(z) - (1-z)^m S_m(-z) = 2^{m-1} z^{\mu}, \; z \in \mathbb{C} \; \{0\}. \tag{4.21}$$

For $m = 2$, and using the top line of (4.18), we have

$$(1+z)^2 S_2(z) - (1-z)^2 S_2(-z) = 2z, \ z \in \mathbb{C},$$

i.e. (4.21) holds for $m = 2$.
Our inductive proof will be complete if we can show that, if, for a fixed $k \in \mathbb{N}$, the Bezout identity (4.21) holds for $m = 2k$, then it holds for $m = 2k+1$ and for $m = 2k+2$.
To this end, we first use (4.18) and (4.20), together with the inductive hypothesis, to obtain, for $m = 2k$, and for $z \in \mathbb{C}$,

$$\begin{aligned}
(1+z)^{m+1} S_{m+1}(z) - (1-z)^{m+1} S_{m+1}(-z) &= (1+z)^{2k+1} S_{2k+1}(z) - (1-z)^{2k+1} S_{2k+1}(-z) \\
&= (1+z)^{2k+1} \left[\frac{2S_{2k}(z) - 2^{1-2k} S_{2k}(-1)(1-z)^{2k}}{1+z} \right] \\
&\quad - (1-z)^{2k+1} \left[\frac{2S_{2k}(-z) - 2^{1-2k} S_{2k}(-1)(1+z)^{2k}}{1-z} \right] \\
&= (1+z)^{2k} \left[2S_{2k}(z) - 2^{1-2k} S_{2k}(-1)(1-z)^{2k} \right] \\
&\quad - (1-z)^{2k} \left[2S_{2k}(-z) - 2^{1-2k} S_{2k}(-1)(1+z)^{2k} \right] \\
&= 2 \left[(1+z)^{2k} S_{2k}(z) - (1-z)^{2k} S_{2k}(-z) \right] \\
&= 2(2^{m-1}) z^{2k-1} = 2^{(m+1)-1} z^{(m+1)-2},
\end{aligned}$$

and it follows that (4.21) holds for $m = 2k+1$. But then, similarly, for $m = 2k+1$, and $z \in \mathbb{C}$, we get

$$\begin{aligned}
(1+z)^{m+1} S_{m+1}(z) - (1-z)^{m+1} S_{m+1}(-z) &= (1+z)^{2k+2} S_{2k+2}(z) - (1-z)^{2k+2} S_{2k+2}(-z) \\
&= (1+z)^{2k+2} \left[\frac{2z^2 S_{2k+1}(z) - 2^{-2k} S_{2k+1}(-1)(1-z)^{2k+1}}{1+z} \right] \\
&\quad - (1-z)^{2k+2} \left[\frac{2z^2 S_{2k+1}(-z) - 2^{-2k} S_{2k+1}(-1)(1+z)^{2k+1}}{1-z} \right] \\
&= 2z^2 \left[(1+z)^{2k+1} S_{2k+1}(z) - (1-z)^{2k+1} S_{2k+1}(-z) \right] \\
&= 2z^2 (2^{m-1} z^{2k-1}) \\
&= 2^m z^{2k+1} = 2^{(m+1)-1} z^{(m+1)-1},
\end{aligned}$$

and it follows that (4.21) also holds for $m = 2k+2$, threreby completing our inductive proof. □

Using (4.2), we obtained the following table

	$S_m(z)$
$m=2$	$\frac{1}{2}$
$m=3$	$-\frac{1}{4}z+\frac{3}{4}$
$m=4$	$-\frac{1}{4}z^2+z-\frac{1}{4}$
$m=5$	$\frac{3}{32}z^3-\frac{15}{32}z^2+\frac{25}{32}z-\frac{5}{32}$

$\lambda_{m,j}$	$m=2$	$m=3$	$m=4$	$m=5$
$j=0$	X	$-\frac{1}{2}$	X	$\frac{3}{8}$
$j=-1$	1	$\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{15}{8}$
$j=-2$	X	X	2	$\frac{25}{8}$
$j=-3$	X	X	$-\frac{1}{2}$	$-\frac{5}{8}$

5. Wavelets

In this chapter, we shall proceed to generate cardinal B-spline wavelets by using the local linear projection operator sequences $\{\mathcal{P}_{m,r},\ r \in \mathbb{Z}\}$ derived previously.

For a given B-spline refinement pair (a^m, N_m), and its local linear projection operator sequence $\{\mathcal{P}_{m,r},\ r \in \mathbb{Z}\}$, as given in Theorems 4.1 and 4.2, we define the error linear space sequence $\{W_m^r : r \in \mathbb{Z}\}$ by

$$W_m^r = \{f - \mathcal{P}_{m,r} f :\ f \in \mathcal{S}_m^{r+1}\},\ r \in \mathbb{Z}. \tag{5.1}$$

Since $\mathcal{P}_{m,r} : \mathcal{S}_m^{r+1} \to \mathcal{S}_m^r$, we observe from (5.1) that

$$W_m^r \subset \mathcal{S}_m^{r+1},\ r \in \mathbb{Z}. \tag{5.2}$$

A function $\psi_m \in \mathcal{S}_m^1$ which is such that

$$W_m^r = \left\{ \sum_j c_j \psi_m(2^r \cdot - j) :\ c \in M(\mathbb{Z}) \right\},\ r \in \mathbb{Z}, \tag{5.3}$$

is called the *cardinal B-spline m-th order wavelet, as generated by the local projection operator sequence* $\{\mathcal{P}_{m,r} :\ r \in \mathbb{Z}\}$.

5.1 The wavelet Bezout identity

First, we prove the following result.

Proposition 5.1. *The linear space sequence $\{W_m^r :\ r \in Z\}$ defined by (5.1) satisfies*

$$W_m^r = \{f \in \mathcal{S}_m^{r+1} :\ \mathcal{P}_{m,r} f = 0\},\ r \in \mathbb{Z}. \tag{5.4}$$

Proof. Let r fixed, and let us suppose that $f \in W_m^r$. According to the definition (5.1), there then exists a function $g \in \mathcal{S}_m^{r+1}$ such that $f = g - \mathcal{P}_{m,r} g$.

From the reproduction property (4.7) we then have

$$\mathcal{P}_{m,r} f = \mathcal{P}_{m,r} g - \mathcal{P}_{m,r}(\mathcal{P}_{m,r} g) = \mathcal{P}_{m,r} g - \mathcal{P}_{m,r} g = 0,$$

i.e. $f \in \{g \in \mathcal{S}_m^{r+1} :\ \mathcal{P}_{m,r} g = 0\}$.
Next, we suppose $f \in \mathcal{S}_m^{r+1}$ is such that $\mathcal{P}_{m,r} f = 0$, and thus

$$f = f - 0 = f - \mathcal{P}_{m,r} f,$$

which implies $f \in \{g - \mathcal{P}_{m,r} g :\ g \in \mathcal{S}_m^{r+1}\}$. □

Our goal in this section is therefore to seek a minimally supported function $\psi_m \in \mathcal{S}_m^1$, i.e. a function for which there exists a sequence $\gamma_m \in M_0(\mathbb{Z})$ such that

$$\psi_m = \sum_j \gamma_{m,j} N_m(2 \cdot -j), \tag{5.5}$$

so that the property

$$\mathcal{P}_{m,0}\psi_m = 0 \tag{5.6}$$

is satisfied.
Our next result gives an equivalent Bezout identity formulation of the condition (5.6).

Theorem 5.2. *A function ψ_m of the form (5.5), with $\gamma_m \in M_0(\mathbb{Z})$, satisfies the condition (5.6) if and only if the Laurent polynomial Γ_m defined by*

$$\Gamma_m(z) = \sum_j \gamma_{m,j} z^j, \;\; z \in \mathbb{C} \setminus \{0\} \tag{5.7}$$

satisfies the Bezout identity

$$\Lambda_m(z)\Gamma_m(z) + \Lambda_m(-z)\Gamma_m(-z) = 0, \;\; z \in \mathbb{C} \setminus \{0\}. \tag{5.8}$$

Proof. Using (4.6) and (5.5), we get

$$\mathcal{P}_{m,0}\psi_m = \sum_j \left[\sum_k \lambda_{m,2j-k}\gamma_{m,k} \right] N_m(\cdot - j), \tag{5.9}$$

according to which the function ψ_m given by (5.5) satisfies the condition (5.6) if and only if

$$\sum_k \lambda_{m,2j-k}\gamma_{m,k} = 0, \;\; j \in \mathbb{Z}. \tag{5.10}$$

It remains to show that the condition (5.10) holds for a sequence $\gamma_m \in M_0(\mathbb{Z})$ if and only if the Laurent polynomial Γ_m defined by (5.7) satisfies the Bezout identity (5.8). To this end, we use (4.4) to deduce that, for a sequence $\gamma_m \in M_0(\mathbb{Z})$, we have, for $z \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned}
\sum_j \left[\sum_k \lambda_{m,2j-k}\gamma_{m,k} \right] z^{2j} &= \sum_j \left[\sum_k \lambda_{m,2j-2k}\gamma_{m,2k} \right] z^{2j} + \sum_j \left[\sum_k \lambda_{m,2j-2k-1}\gamma_{m,2k+1} \right] z^{2j} \\
&= \sum_k \left[\sum_j \lambda_{m,2j-2k} z^{2j-2k} \right] \gamma_{m,2k} z^{2k} \\
&\quad + \sum_k \left[\sum_j \lambda_{m,2j-2k-1} z^{2j-2k-1} \right] \gamma_{m,2k+1} z^{2k+1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_k \left[\sum_j \lambda_{m,2j} z^{2j} \right] \gamma_{m,2k} z^{2k} + \sum_k \left[\sum_j \lambda_{m,2j+1} z^{2j+1} \right] \gamma_{m,2k+1} z^{2k+1} \\
&= \Lambda_m^{(e)}(z)\Gamma_m^{(e)}(z) + \Lambda_m^{(o)}(z)\Gamma_m^{(o)}(z) \\
&= \frac{\Lambda_m(z)+\Lambda_m(-z)}{2}\frac{\Gamma_m(z)+\Gamma_m(-z)}{2} + \frac{\Lambda_m(z)-\Lambda_m(-z)}{2}\frac{\Gamma_m(z)-\Gamma_m(-z)}{2} \\
&= \frac{1}{2}\left[\Lambda_m(z)\Gamma_m(z) + \Lambda_m(-z)\Gamma_m(-z)\right],
\end{aligned}
$$

thereby establishing the desired equivalence of (5.8) and (5.10). □

The following result gives a polynomial minimal degree satisfying the Bezout identity (5.8).

Theorem 5.3. *The polynomial*

$$\Gamma_m(z) = 2S_m(-z), \ z \in \mathbb{C}, \tag{5.11}$$

is a polynomial of minimal degree satisfying the Bezout identity (5.8), with S_m denoting the polynomial of Theorem 4.2

Proof. First, we write the Bezout identity (5.8) in the form

$$\Lambda_m(z)\Gamma_m(z) = -\Lambda_m(-z)\Gamma_m(z), \ z \in \mathbb{C} \setminus \{0\}. \tag{5.12}$$

Since (4.9) shows that $\Lambda_m(z)$ and $\Lambda_m(-z)$ can have no common factors, it follows from (5.12) that Γ_m is a Laurent polynomial of shortest possible length satisfying (5.12) if and only if Γ_m has the form

$$\Gamma_m(z) = Kz^{2n_0+1}\Lambda_m(-z), \quad z \in \mathbb{C} \setminus \{0\}, \tag{5.13}$$

with $K \in \mathbb{R}$ and $n_0 \in \mathbb{Z}$.
But then (4.19) gives

$$\Gamma_m(z) = 2Kz^{2n_0+1-\mu}S_m(-z), \ z \in \mathbb{C} \setminus \{0\}. \tag{5.14}$$

Recalling from (4.20) that μ is an odd integer, we now choose $K = 1$ and $n_0 = \frac{\mu-1}{2}$ to obtain the desired result (5.11). □

Combining the results of the Theorems 5.2 and 5.3, we have therefore established the first part of the following result.

Corollary 5.4. *The function ψ_m defined by*

$$\psi_m = 2\sum_{j=0}^{m-2}(-1)^j s_{m,j} N_m(2\cdot -j), \tag{5.15}$$

where the sequence $\{S_{m,j} : \ j = 0,1,2,\ldots,m-2\}$ is defined by

$$S_m(z) = \sum_{j=0}^{m-2} s_{m,j} z^j, \quad z \in \mathbb{C} \setminus \{0\}, \tag{5.16}$$

with S_m denoting the polynomial given in Theorem 4.2, is a minimally supported function in $\mathcal{S}_m^1$ satisfying the condition (5.6). Moreover,

$$\psi_m = 0,\ x \notin (0, m-1). \tag{5.17}$$

Proof. It remains to prove the finite support property (5.17). Using (2.9) in (5.15) we see that (5.17) does indeed holds. □

5.2 The basic decomposition result

The goal of this section is to show that the function ψ_m in Corollary 5.4 does satisfy the condition (5.3), and is therefore indeed a cardinal B-spline wavelet as generated by the operator sequence $\{\mathcal{P}_{m,r} :\ r \in \mathbb{Z}\}$

Theorem 5.5. *The function $\psi_m \in \mathcal{S}_m^1$ of Corollary 5.4 satisfies*

$$f - \mathcal{P}_{m,r} f = \frac{1}{2^{m-1}} \sum_j \left[\sum_k (-1)^k \binom{m}{2j-k+\mu} c_k \right] \psi_m(2^r \cdot - j) \quad \text{for} \quad f = \sum_j c_j N_m(2^{r+1} \cdot - j),\ r \in \mathbb{Z}, \tag{5.18}$$

with the odd integer μ given as in (4.20).

Proof. Our result will be proved if we can show that a sequence $\omega \in M_0(\mathbb{Z})$ satisfies the condition

$$f - \mathcal{P}_{m,r} f = \sum_j \left[\sum_k \omega_{2j-k} c_k \right] \psi_m(2^r \cdot - j) \quad \text{for} \quad f = \sum_j c_j N_m(2^{r+1} \cdot - j),\ r \in \mathbb{Z}, \tag{5.19}$$

if and only if

$$\omega_j = (-1)^{j+1} a^m_{j+\mu},\ j \in \mathbb{Z}, \tag{5.20}$$

where the sequence $a^m \in M_0(\mathbb{Z})$ is defined as (2.11).
Let $r \in \mathbb{Z}$ be fixed, and suppose $f = \sum_j c_j N_m(2^{r+1} \cdot - j)$ for a sequence $c \in M(\mathbb{Z})$.
Using (4.6) and (2.12), we obtain

$$\begin{aligned} f - \mathcal{P}_{m,r} f &= \sum_j c_j N_m(2^{r+1} \cdot - j) - \sum_j \sum_k \lambda_{m,2j-k} c_k \left[\sum_l a^m_l N_m(2^{r+1} \cdot - 2j - l) \right] \\ &= \sum_j c_j N_m(2^{r+1} \cdot - j) - \sum_j \sum_k \lambda_{m,2j-k} c_k \sum_l a^m_{l-2j} N_m(2^{r+1} \cdot - l) \\ &= \sum_j c_j N_m(2^{r+1} \cdot - j) - \sum_l \left[\sum_k \lambda_{m,2l-k} c_k \right] \sum_j a^m_{j-2l} N_m(2^{r+1} \cdot - j) \end{aligned}$$

$$= \sum_j c_j N_m(2^{r+1} \cdot -j) - \sum_j \sum_l \left[\sum_k \lambda_{m,2l-k} c_k \right] a^m_{j-2l} N_m(2^{r+1} \cdot -j)$$

$$= \sum_j \left\{ \sum_k \left[\delta_{j,k} - \sum_l \lambda_{m,2l-k} a^m_{j-2l} \right] c_k \right\} N_m(2^{r+1} \cdot -j). \tag{5.21}$$

Let $\{\omega_j : j \in \mathbb{Z}\}$ denote a sequence in $M_0(\mathbb{Z})$. Then, using (5.15), we obtain

$$\sum_j \left[\sum_k \omega_{2j-k} c_k \right] \psi_m(2^r \cdot -j) = \sum_j \sum_k \omega_{2j-k} c_k \left[\sum_l \gamma_{m,l} N_m(2^{r+1} \cdot -2j - l) \right]$$

$$= \sum_j \sum_k \omega_{2j-k} c_k \left[\sum_l \gamma_{m,l-2j} N_m(2^{r+1} \cdot -l) \right]$$

$$= \sum_l \sum_k \omega_{2l-k} c_k \sum_j \gamma_{m,j-2l} N_m(2^{r+1} \cdot -j)$$

$$= \sum_j \left\{ \sum_k \left[\sum_l \gamma_{m,j-2l} \omega_{2l-k} \right] c_k \right\} N_m(2^{r+1} \cdot -j). \tag{5.22}$$

It follows from (5.21) and (5.22) that $\{\omega_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ satisfies (5.18) if and only if

$$\sum_j \left\{ \sum_k \left[\delta_{j,k} - \sum_l a^m_{j-2l} \lambda_{m,2l-k} - \sum_l \gamma_{m,j-2l} \omega_{2l-k} \right] c_k \right\} N_m(2^{r+1} \cdot -j) = 0, \ r \in \mathbb{Z}. \tag{5.23}$$

or, equivalently

$$\delta_{j,k} = \sum_l a^m_{j-2l} \lambda_{m,2l-k} + \sum_l \gamma_{m,j-2l} \omega_{2l-k}, \quad j,k \in \mathbb{Z}. \tag{5.24}$$

We proceed to show that the condition (5.24) is equivalent to pair of Bezout identities. To this end, we define the Laurent polynomial Ω by

$$\Omega(z) = \sum_j \omega_j z^j, \quad z \in \mathbb{C} \setminus \{0\}. \tag{5.25}$$

Now use (4.8) and (5.25) in (5.24) to deduce that, for $j \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \{0\}$, we have

$$z^j = \sum_k \delta_{j,k} z^k = \sum_k \left[\sum_l a^m_{j-2l} \lambda_{m,2l-k} + \sum_l \gamma_{m,j-2l} \omega_{2l-k} \right] z^k$$

$$= \sum_l \left[\sum_k \lambda_{m,2l-k} z^{k-2l} \right] a^m_{j-2l} z^{2l} + \sum_l \left[\sum_k \omega_{2l-k} z^{k-2l} \right] \gamma_{m,j-2l} z^{2l}$$

$$= z^j \left\{ \sum_l \left[\sum_k \lambda_{m,k} (z^{-1})^k \right] a^m_{j-2l} z^{2l-j} + \sum_l \left[\sum_k \omega_k (z^{-1})^k \right] \gamma_{m,j-2l} z^{2l-j} \right\}$$

$$= z^j \left\{ \sum_l \left[a^m_{j-2l} z^{2l-j} \right] \Lambda_m(z^{-1}) + \sum_l \left[\gamma_{m,j-2l} z^{2l-j} \right] \Omega(z^{-1}) \right\}. \tag{5.26}$$

It follows from (5.26) that the condition (5.24) is satisfied if and only if

$$\left[\sum_l a^m_{j-2l} z^{2l-j}\right] \Lambda_m(z^{-1}) + \sum_l \left[\sum_l \gamma_{m,j-2l} z^{2l-j}\right] \Omega(z^{-1}) = 1, \quad z \in \mathbb{C} \setminus \{0\},\ j \in \mathbb{Z},$$

or, equivalently,

$$\left[\sum_l a^m_{j-2l} z^{j-2l}\right] \Lambda_m(z) + \sum_l \left[\gamma_{m,j-2l} z^{j-2l}\right] \Omega(z) = 1, \quad z \in \mathbb{C} \setminus \{0\}, \quad j \in \mathbb{Z}, \tag{5.27}$$

holds. But (5.27) holds for $j \in \mathbb{Z}$ if and only if it holds for even integers j and for odd integers j. Hence (5.27) is equivalent to certain pair of Bezout identities

$$\left.\begin{array}{rcl} A^{(e)}_m(z)\Lambda^{(e)}_m(z) + \Gamma^{(e)}_m(z)\Omega(z) &=& 1, \\ A^{(o)}_m(z)\Lambda^{(o)}_m(z) + \Gamma^{(o)}_m(z)\Omega(z) &=& 1, \end{array}\right\} z \in \mathbb{C} \setminus \{0\},$$

or, equivalently,

$$\left.\begin{array}{rcl} [A_m(z) + A_m(-z)]\Lambda_m(z) + [\Gamma_m(z) + \Gamma_m(-z)]\Omega(z) &=& 2, \\ [A_m(z) - A_m(-z)]\,\Lambda_m(z) + [\Gamma_m(z) - \Gamma_m(-z)]\Omega(z) &=& 2, \end{array}\right\} z \in \mathbb{C} \setminus \{0\}. \tag{5.28}$$

Now observe that (5.28), and therefore also the condition (5.27), is satisfied if and only if the pair of Bezout identities

$$\left.\begin{array}{rcl} A_m(z)\Lambda_m(z) + \Gamma_m(z)\Omega(z) &=& 2, \\ A_m(-z)\Lambda_m(z) + \Gamma_m(-z)\Omega(z) &=& 0, \end{array}\right\} z \in \mathbb{C} \setminus \{0\}. \tag{5.29}$$

hold. Observe next from (5.13) that

$$\Gamma_m(-z) = -Kz^{2n_0+1}\Lambda_m(z)\ z \in \mathbb{C} \setminus \{0\}. \tag{5.30}$$

It now follows from (5.30) that (5.29), and therefore also the condition (5.24), holds if only if the pair of Bezout identities

$$\left.\begin{array}{rcl} A_m(z)\Lambda_m(z) + Kz^{2n_0+1}\Lambda_m(-z)\Omega(z) &=& 2, \\ \Lambda_m(z)[A_m(-z) - Kz^{2n_0+1}\Omega(z)] &=& 0, \end{array}\right\} z \in \mathbb{C} \setminus \{0\}, \tag{5.31}$$

are satisfied. The second line of (5.31) is satisfied by a Laurent polynomial Ω if and only if

$$\Omega(z) = \frac{1}{K} z^{-2n_0-1} A_m(-z),\ z \in \mathbb{C} \setminus \{0\}, \tag{5.32}$$

since $K \neq 0$. Inserting (5.32) into the left-hand side of the first line in (5.31) then yields

$$A_m(z)\Lambda_m(z) + Kz^{2n_0+1}\Gamma_m(-z)\Omega(z) = A_m\Lambda_m(z) + A_m(-z)\Lambda_m(-z),\ z \in \mathbb{C} \setminus \{0\},$$

which, together with the Bezout identity (5.8) satisfied by A_m and Λ_m, shows that the choice (5.32) of Ω also satisfies the first line of (5.31). We have therefore shown that the condition (5.24) is satisfied by a sequence $\omega \in M_0(\mathbb{Z})$ if only if the corresponding Laurent polynomial Ω is given by the formula (5.32); which can, according to (5.25) and (4.5), be rewritten as

$$\sum_j \omega_j z^j = \frac{1}{K}\sum_j (-1)^j a_j^m z^{j-2n_0-1} = \frac{1}{K}\sum_j (-1)^{j+1} a_{j+2n_0+1}^m z^j, \; z \in \mathbb{Z}\setminus\{0\},$$

where $K = 1$ and $n_0 = \dfrac{\mu - 1}{2}$, which, in turn, holds if and only if the sequence $\omega \in M_0(\mathbb{Z})$ is given by (5.20).

It follows that, if we choose the sequence $\omega \in M_0(\mathbb{Z})$ as in (5.20), then (5.21), and therefore also the desired result (5.18), hold.

□

Using (4.18) and (5.15) we graph below the cardinal B-spline wavelet ψ_m for $m = 2, 3, 4, 5, 6$.

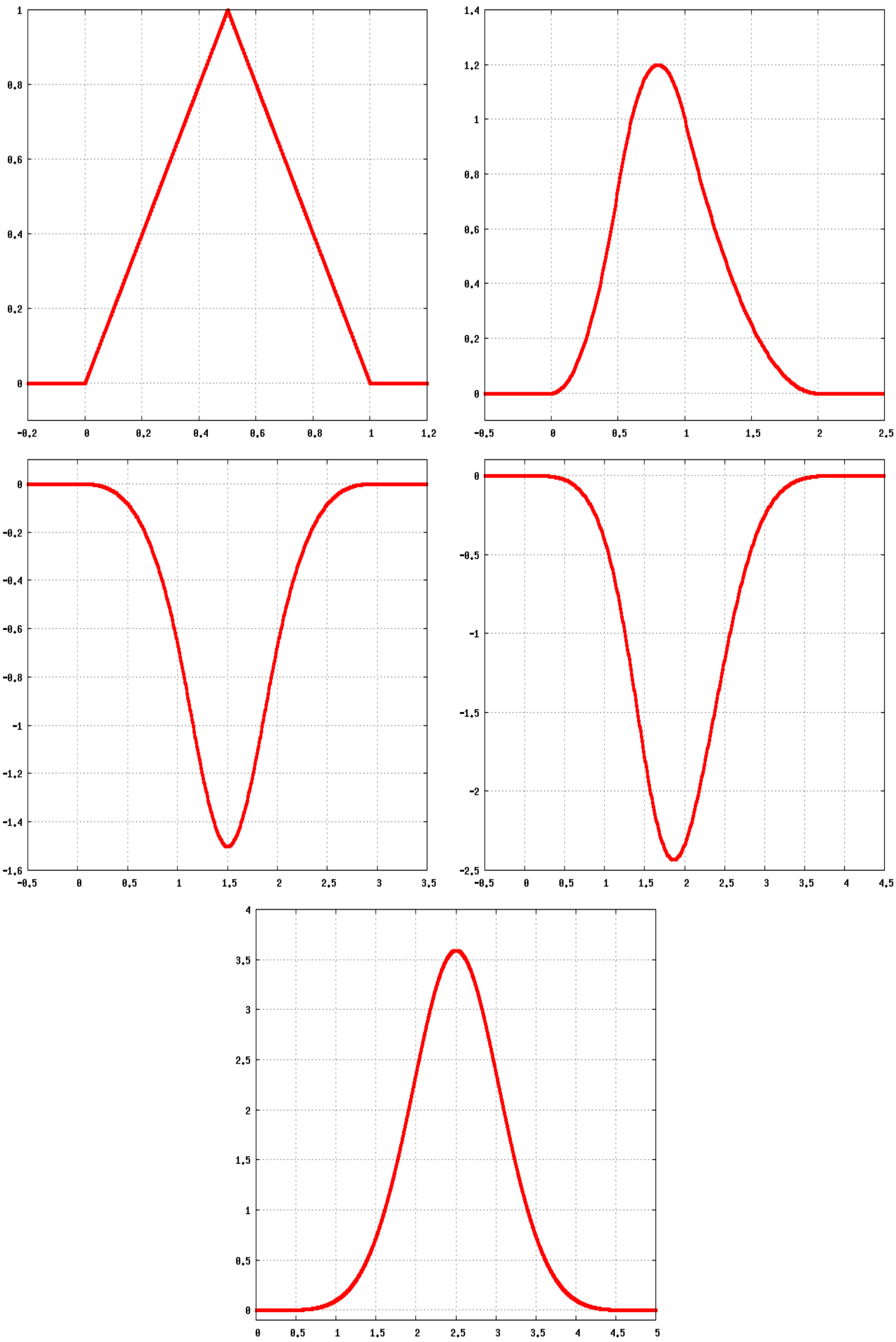

Figure 5.1: The cardinal B-spline wavelet for m=2,3,4,5,6.

6. Decomposition Algorithms

6.1 The decomposition algorithm

Given a signal $f \in M(\mathbb{R})$, the purpose of this chapter is to describe the cardinal B-spline wavelet decomposition of f into detail components.
For a sufficiently large integer N, we set

$$f_N = \mathcal{Q}_{m,N} f, \tag{6.1}$$

where $\mathcal{Q}_{m,r}$ is defined in Theorem 3.5.
Then

$$f_N = \sum_j c_k^N N_m(2^N \cdot - j), \tag{6.2}$$

with the sequence $c^N \in M(\mathbb{Z})$ given by

$$c_k^N = \sum_j v_{k-j} f\left(\frac{j+\tau_0}{2^N}\right), \tag{6.3}$$

where τ_0 is defined by (3.34).
For a sufficiently large integer M, we define the function sequences

$$\left.\begin{aligned} f_r &= \sum_j c_j^r N_m(2^r \cdot - j) \\ g_r &= \sum_j d_j^r \psi_m(2^r \cdot - j) \end{aligned}\right\} r = N-1, \dots, N-M, \tag{6.4}$$

where the sequences $c^r, d^r \in M(\mathbb{Z})$ are obtained recursively from (6.3), and the equations

$$\left.\begin{aligned} c_j^r &= \sum_k \lambda_{m,2j-k} c_k^{r+1} \\ d_j^r &= \sum_k (-1)^k a_{2j+\mu-k}^m c_k^{r+1} \end{aligned}\right\} j \in \mathbb{Z},\ r = N-1, \dots, N-M \tag{6.5}$$

with μ defined in (4.20), so that $f_r \in \mathcal{S}_m^r,\ r = N, \dots, N-M$, and
$g_r \in W^r, r = N-1, \dots, N-M$.

Then, according to Theorem 5.5, the decomposition result

$$f_{r+1} = f_r + g_r, \quad r = N-1, \dots, N-M, \tag{6.6}$$

holds.

It follows from (6.6) that

$$f_N = f_{N-M} + \sum_{r=N-M}^{N-1} g_r. \tag{6.7}$$

For every $r \in \{N-1, \dots, N-M\}$, the function g_r is called the *wavelet component* at the resolution level r, and the sequence d^r the *wavelet coefficients* at the resolution level r.
Together, the equations (6.3) and (6.5) are known as the *cardinal B-spline wavelet decomposition algorithm*.
The figures below represent graphically the procedure for wavelet decomposition algorithm.

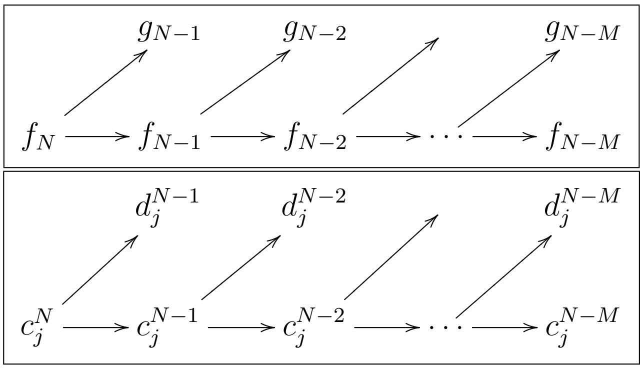

Figure 6.1: The wavelet decomposition algorithm.

6.2 The singularity detection property

We proceed to prove that the decomposition algorithm defined by (6.3) and (6.5) possesses the property to locally detect singularities in a given signal f.

Proposition 6.1. *The sequence $a^m \in M_0(\mathbb{Z})$ defined by (2.15) satisfies the condition*

$$\sum_k (-1)^k a^m_{2j+\mu-k} p(k) = 0, \; j \in \mathbb{Z}, \; p \in \pi_{m-1}, \tag{6.8}$$

with μ denoting the odd integer defined by (4.20).

Proof. First, observe that (6.8) has the equivalent formulation

$$\sum_k (-1)^k a^m_{2j+\mu-k} k^l = 0, \; j \in \mathbb{Z}, \; l = 0, 1, \dots, m-1. \tag{6.9}$$

Now note that, for $j \in \mathbb{Z}$, and using also the fact that μ is an odd integer, we have

$$\begin{aligned}
\sum_k (-1)^k a^m_{2j+\mu-k} k^l &= \sum_k (-1)^{2j+\mu-k} a^m_k (2j+\mu-k)^l \\
&= -\sum_k (-1)^k a^m_k \sum_{q=0}^{l} \binom{l}{q} (-1)^q k^q (2j+\mu)^{l-q} \\
&= -\frac{1}{2^{m-1}} \sum_{q=0}^{l} (-1)^q \binom{l}{q} (2j+\mu)^{l-q} \sum_k (-1)^k \binom{m}{k} k^q,
\end{aligned} \tag{6.10}$$

after using also (2.15). Our proof of (6.9) will be complete if we can prove the identity

$$\sum_k (-1)^k \binom{m}{k} k^q = 0, \ q = 0, 1, \dots, m-1, \tag{6.11}$$

which, together with (6.10), then yield the desired result (6.9).
We prove (6.11) by induction on the integer m.
Noting that (6.11) holds for $m = 1$, we next suppose that (6.11) is true for a fixed $m \in \mathbb{N}$. Now define

$$a_{m,q} = \sum_k (-1)^k \binom{m}{k} k^q, \ q \in \mathbb{Z}_+.$$

Our inductive proof will be complete if we can show that

$$a_{m+1,q} = 0, \ q = 0, 1, \dots, m. \tag{6.12}$$

First, note that

$$a_{m+1,0} = \sum_k (-1)^k \binom{m+1}{k} = (1-1)^{m+1} = 0.$$

Next, for $q \in \{1, 2, \dots, m\}$, we see that

$$\begin{aligned}
a_{m+1,q} &= \sum_k (-1)^k \binom{m+1}{k} k^q \\
&= \sum_k (-1)^k \frac{(m+1)!}{k!(m+1-k)!} k^q \\
&= \sum_k (-1)^k \frac{(m+1)m!}{(k-1)!(m+1-k)!} k^{q-1} \\
&= -(m+1) \sum_k (-1)^k \frac{m!}{(k)!(m-k)!} (k+1)^{q-1} \\
&= -(m+1) \sum_k (-1)^k \binom{m}{k} \sum_{r=0}^{q-1} \binom{q-1}{r} k^r \\
&= -(m+1) \sum_{r=0}^{q-1} \binom{q-1}{r} \sum_k (-1)^k \binom{m}{k} k^r \\
&= -(m+1) \sum_{r=0}^{q-1} \binom{q-1}{r} a_{m,r} = 0,
\end{aligned}$$

from the inductive hypothesis, and thereby concluding our inductive proof of the desired identity (6.11). □

Proposition 6.1 enables to prove the following fundamental property of the decomposition algorithm (6.3),(6.5).

Theorem 6.2. *If, in (6.3), we choose* $f \in \pi_{m-1}$*, then the wavelet coefficient sequences* $\{d_j^r : j \in \mathbb{Z}\}$, $r = N-1, \ldots, N-M$ *obtained from (6.3) and (6.5), satisfy*

$$d_j^r = 0, \ j \in \mathbb{Z}, \ r = N-1, \ldots, N-M. \tag{6.13}$$

Proof. Let $n \in \{0, 1, \ldots, m-1\}$ be fixed. It will suffice to prove our theorem for the choice

$$f(x) = x^n, \ x \in \mathbb{R}. \tag{6.14}$$

From (3.11), (6.5), (6.3), and Proposition 6.1, we have

$$\begin{aligned} d_j^{N-1} &= \sum_j (-1)^k a_{2j+\mu-k}^m \left[\sum_i v_{k-i} (\frac{i+\tau}{2^N})^n \right] \\ &= \frac{n!}{2^{Nn}(m-1)!} \sum_j (-1)^k a_{2j+\mu-k}^m Q_m^{(m-1-n)}(k) = 0, \end{aligned}$$

since $Q_m \in \pi_{m-1}$ implies $Q_m^{(m-1-n)} \in \pi_{m-1}$.
Similarly, with the polynomial $\tilde{Q}_m \in \pi_{m-1}$ defined by

$$\tilde{Q}_m = \frac{n!}{2^{Nn}(m-1)!} Q_m,$$

we deduce that

$$\begin{aligned} d_j^{N-2} &= \sum_k (-1)^k a_{2j+\mu-k}^m c_k^{N-1} \\ &= \sum_k (-1)^k a_{2j+\mu-k}^m \sum_l \lambda_{m,2k-l} c_l^N \\ &= \sum_k (-1)^k a_{2j+\mu-k}^m \sum_l \lambda_{m,2k-l} \tilde{Q}_m^{(m-1-n)}(l) \\ &= \sum_k (-1)^k a_{2j+\mu-k}^m \left[\sum_l \lambda_{m,2k-2l} \tilde{Q}_m^{(m-1-n)}(2l) + \sum_l \lambda_{m,2k-2l-1} \tilde{Q}_m^{(m-1-n)}(2l+1) \right] \\ &= \sum_k (-1)^k a_{2j+\mu-k}^m \left[\sum_l \lambda_{m,2l} \tilde{Q}_m^{(m-1-n)}(2k-2l) + \sum_l \lambda_{m,2l+1} \tilde{Q}_m^{(m-1-n)}(2k-2l-1) \right] \\ &= \sum_l \lambda_{m,2l} \left[\sum_k (-1)^k a_{2j+\mu-k}^m \tilde{Q}_m^{(m-1-n)}(2k-2l) \right] \\ &\quad + \sum_l \lambda_{m,2l+1} \left[\sum_k (-1)^k a_{2j+\mu-k}^m \tilde{Q}_m^{(m-1-n)}(2k-2l-1) \right] \\ &= 0, \end{aligned}$$

since, if, for $l \in \mathbb{Z}$, we define $p_l = \tilde{Q}_m^{(m-1-n)}(2 \cdot -2l-1)$, then $p_l \in \pi_{m-1}$.
Repeated use of this procedure yields

$$d_j^r = 0, \quad j \in \mathbb{Z}, \quad r = N-3, \ldots, N-M.$$

□

The result of the Theorem 6.2 has the following important with respect to the cardinal B-spline wavelet decomposition algorithm (6.3),(6.5). If the signal f is C^m-smooth in a certain region, so that, according to Taylor's theorem, f is locally well approximated by a polynomial in π_{m-1} in that region, it follows from Theorem 6.2 that the wavelet coefficients d_j^r can be expected to be relatively small in the support interval $\left[\frac{j}{2^r}, \frac{j+m-1}{2^r}\right]$, as implied by (5.17), of the corresponding wavelet $\psi_m(2^r \cdot -j)$ overlaps with this C^m-smooth region of f, thereby providing localised information, at each resolution level r, on the smoothness (or the lack thereof) of f.

6.3 Example

Let consider the signal $f \in C_0(\mathbb{R})$ defined by

$$f(x) = \begin{cases} \frac{1}{2}x^2 & , x \in \ [0,1), \\ \frac{1}{2}(-2x^2+6x-3) & , x \in \ [1,2), \\ \frac{1}{2}(3-x)^2 & , x \in \mathbb{R}-[2,3) \\ 0 & , x \in \mathbb{R}-[0,3) \end{cases}$$

Then $f = N_3 \in C_0^1(\mathbb{R}) \setminus C_0^2(\mathbb{R})$, and possesses discontinuities in the second derivative at x $\in \{0,1,2,3\}$.
We now use the decomposition algorithm (6.3), (6.5) based on the cardinal B-spline pair of order 4, i.e. (a^4, N_4) to plot, for $N = 10$, the quasi-interpolant approximation $f_{10} = \mathcal{Q}_{4,10}f$ which is shown in Figure 6.2

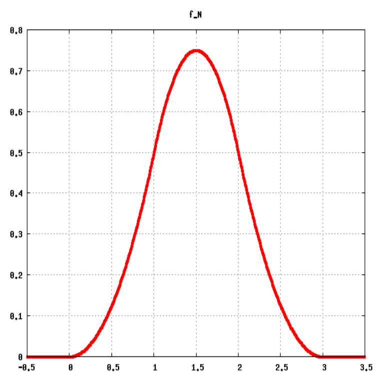

Figure 6.2: The approximation $f_{10} = \mathcal{Q}_{4,10}f$.

In our graphs in Figures 6.3 and 6.4 the singularities in the second derivatives f'' at $x \in \{0,1,2,3\}$ are efficiently detected by our decomposition algorithm, with sharply defined localisation.

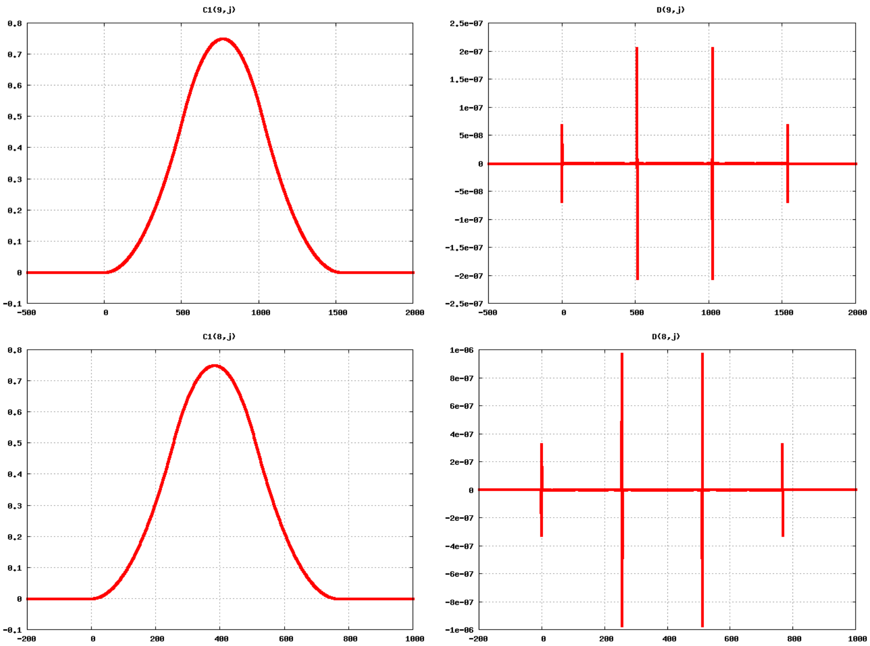

Figure 6.3: The coefficients $\{c_j^r : j \in \mathbb{Z}\}$ and $\{d_j^r : j \in \mathbb{Z}\}$ at level $r = 9, 8$

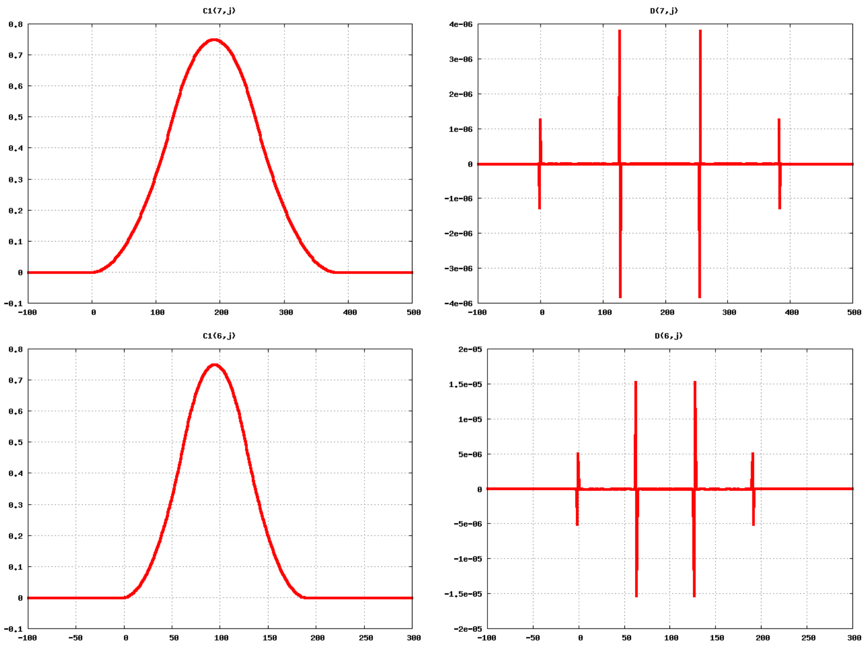

Figure 6.4: The coefficients $\{c_j^r : j \in \mathbb{Z}\}$ and $\{d_j^r : j \in \mathbb{Z}\}$ at level $r = 7, 6$.

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Bibliography

[Chu92] C.K. Chui, *An introduction to wavelets*, MA, Academic Press, Boston, 1992.

[Chu97] ———, *A mathematical tool for signal analysis*, SIAM, 1997.

[CW92] C.K Chui and J.Z Wang, *On compactly supported spline wavelets and the duality principle*, Trans.Amer.Soc.330 (1992), 903–915.

[dV06] Johan de Villiers, *Subdivision, wavelets and splines*, Second semester 2006, Lecture notes, Department of Mathematical Sciences, University of Stellenbosch.

[Mic95] Micchelli, *Mathematical aspects of geometric modeling*, SIAM, 1995.

[Mou07] Desiree V Moubandjo, *Polynomial containment in refinement spaces and wavelets based on local projection operators*, Phd thesis, University of Stellenbosch, 2007.